EMERGENT COMPLEX GEOMETRY

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ABSTRACT

This is a double exposure of the probabilistic construction of Kähler–Einstein metrics on a complex projective algebraic variety X – where the Kähler–Einstein metric emerges from a canonical random point process on X – and the variational approach to the Yau–Tian–Donaldson conjecture, highlighting their connections. The final section is a report on joint work in progress with Sébastien Boucksom and Mattias Jonsson on how the non-Archimedean geometry of X (with respect to the trivial absolute value) also emerges from the probabilistic framework.

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1. INTRODUCTION

A recurrent theme in geometry is the quest for canonical metrics on a given manifold X. The prototypical case is when X is a compact orientable two-dimensional surface, which can be endowed with a metric of constant scalar curvature, essentially uniquely determined by a complex structure J on X. On the other hand, from a physical point of view, geometrical shapes – as we know them from everyday experience – are, of course, not fundamental physical entities. They merely arise as macroscopic *emergent* features of ensembles of microscopic point particles in the limit as the number N of particles tends to infinity. In mathematical terms such microscopical ensembles are *random point processes*, i.e., they are represented by a probability measure on the configuration space of N points on X or, equivalently, a symmetric probability measure $\mu^{(N)}$ on the N-fold product X^N. One is thus led to ask whether a given manifold X may be endowed with a canonical random point process – defined without reference to any metric – from which a canonical metric g emerges as $N \to \infty$? Here we shall focus on Kähler metrics with constant Ricci curvature. From the physics perspective, these arise as solutions to Einstein's equations in vacuum (with Euclidean signature). The Kähler condition means that X is compatible with an integrable complex structure J on X (in that parallel translation preserves the complex structure J). Such metrics – known as *Kähler–Einstein metrics* – play a central role in current complex geometry and the study of complex algebraic varieties, in particular in the context of the Yau–Tian–Donaldson conjecture [38] and the Minimal Model Program in birational algebraic geometry [45]. When a projective algebraic variety X admits a Kähler–Einstein metric, it is essentially unique, i.e., canonically attached to X and can thus be leveraged to probe X using differential-geometric techniques (as, for example, in the construction of moduli spaces [61]).

One virtue of the probabilistic approach is that it leads to essentially explicit period type integral formulas for canonical Kähler metrics converging towards the Kähler–Einstein metric as $N \to \infty$ (see formula (2.7)). These formulas are reminiscent of the few explicit formulas for Kähler–Einstein metrics that are available on special complex curves, involving hypergeometric integrals (notably the modular curve, the Klein curve, and Fermat curves; see [6, SECTION 2.1]). The probabilistic approach also generates new connections between Kähler geometry and algebraic geometry in the context of the Yau–Tian–Donaldson conjecture on Fano varieties, through the concept of Gibbs stability and the related stability threshold (δ-invariant) [19, 41]. The present contribution to the 2022 ICM proceedings attempts a double exposure of the probabilistic approach in [2, 4, 5] and the variational approach to the Yau–Tian–Donaldson conjecture in [14], highlighting their connections. For more details and background, the reader is referred to the survey [6]. See also [15] for connections between the present probabilistic approach to Kähler geometry and quantum gravity in the context of the AdS/CFT correspondence, and [7, 39] for connections to polynomial approximation theory and pluripotential theory in $\mathbb{C}^n$.

2. EMERGENT KÄHLER GEOMETRY

Let X be a compact complex manifold, whose dimension over $\mathbb{C}$ will be denoted by n. The existence of a Kähler–Einstein metric ω_{KE} on X, i.e., a Kähler metric with constant Ricci curvature,

$$\operatorname{Ric}\omega = -\beta\omega, \tag{2.1}$$

implies that the *canonical line bundle* K_X of X (the top exterior power of the cotangent bundle of X) has a definite sign, when $\beta \neq 0$,

$$\operatorname{sign}(K_X) = \operatorname{sign}(\beta). \tag{2.2}$$

We will be using the standard terminology of positivity in complex geometry: a line bundle is said to be *positive*, $L > 0$, if L carries some Hermitian metric with strictly positive curvature (or, equivalently, L is ample in the algebro-geometric sense). The standard additive notation for tensor products of line bundles will be adopted. Accordingly, the dual of L is expressed as $-L$, and L is thus said be negative, $L < 0$, if $-L > 0$. In general, when $\beta \neq 0$, the manifold X is automatically a complex projective algebraic manifold, and after a rescaling of the Kähler–Einstein metric we may as well assume that $\beta = \pm 1$. For example, in the case when X is a hypersurface in $\mathbb{P}^{n+1}_{\mathbb{C}}$, cut out by a homogeneous polynomial of degree d, $K_X > 0$ when $d > n+2$, and $-K_X > 0$ when $d < n+2$.

Remark 2.1. In the more general "logarithmic" setup, X is replaced by a *log pair* (X, Δ) consisting of a $\mathbb{Q}$-divisor Δ on a normal variety X and K_X is replaced by $K_X + \Delta$, assumed to be a $\mathbb{Q}$-line bundle. The corresponding log Kähler–Einstein equation (2.1) is obtained by replacing $\operatorname{Ric}\omega$ with $\operatorname{Ric}\omega - [\Delta]$, where $[\Delta]$ denotes the current of integration corresponding to Δ. For simplicity we will stick to the case when X is nonsingular and Δ is trivial (but all the results surveyed in this and the following section generalize to the logarithmic setting, assuming that (X, Δ) is klt (Kawamata log terminal) [5, 8, 13]).

Coming back to the question of emergence of geometry, discussed in the introduction, a Kähler–Einstein metric g_{KE} has the crucial property that it can be readily recovered from its volume form dV_{KE}, in the case $\beta \neq 0$. Indeed, in local terms g_{KE} is proportional to the complex Hessian of the logarithm of the local density of dV_{KE} (see formula (3.4)). Thus in order to probalistically construct the Kähler–Einstein metric, one just needs to construct a random point process on X with N particles such that the empirical measure

$$\delta_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, \tag{2.3}$$

viewed as a random discrete probability measure on X, converges in probability to dV_{KE}, as $N \to \infty$.

2.1. The case $K_X > 0$ ($\beta = 1$)

The starting point for the probabilistic approach is the observation that there is a canonical symmetric probability measure $\mu^{(N)}$ on the N-fold product X^N of X. More precisely, the integers N are taken to be of the special form

$$N = N_k := \dim_{\mathbb{C}} H^0(X, kK_X),$$

where $H^0(X, kK_X)$ denotes the complex vector space of all holomorphic section of the kth tensor power of the canonical line bundle $K_X \to X$. Recall that the elements $s^{(k)}$ of $H^0(X, kK_X)$ are called pluricanonical forms and may be represented by local holomorphic functions transforming as $dz^{\otimes k}$, in terms of local holomorphic coordinates $z \in \mathbb{C}^n$ on X. As a consequence, $|s^{(k)}(z)|^{2/k}$ transforms as a local density on X and thus defines a global measure on X. Replacing X with X^{N_k}, the canonical symmetric probability measure $\mu^{(N_k)}$ on X^{N_k} is now defined by

$$\mu^{(N_k)} = \frac{1}{\mathcal{Z}_{N_k}} \left|\det S^{(k)}\right|^{2/k}, \quad \mathcal{Z}_{N_k} := \int_{X^{N_k}} \left|\det S^{(k)}\right|^{2/k}, \tag{2.4}$$

where $\det S^{(k)}$ is the holomorphic section of the line bundle $(kK_{X^{N_k}}) \to X^{N_k}$, expressed as the Slater determinant

$$\left(\det S^{(k)}\right)(x_1, x_2, \dots, x_N) := \det\left(s_i^{(k)}(x_j)\right), \tag{2.5}$$

in terms of a given basis $s_i^{(k)}$ in $H^0(X, kK_X)$. Under a change of bases, the section $\det S^{(k)}$ only changes by a multiplicative complex constant (the determinant of the change of bases matrix on $H^0(X, kK_X)$). As a consequence, $\mu^{(N_k)}$ is independent of the choice of bases in $H^0(X, kK_X)$ and, since $\det S^{(k)}$ is antisymmetric, this means that the probability measure $\mu^{(N_k)}$ indeed defines a canonical symmetric probability measure on X^{N_k}. Moreover, it is completely encoded by algebro-geometric data in the following sense: realizing X as a projective algebraic subvariety, the section $\det S^{(k)}$ can be identified with a homogeneous polynomial, determined by the coordinate ring of X.

The assumption that $K_X > 0$ ensures that $N_k \to \infty$ as $k \to \infty$. To simplify the notation, we will often drop the subindex k on N_k and consider the large-N limit. The following convergence result was shown in [4]:

Theorem 2.2. *Let X be a compact complex manifold with positive canonical line bundle K_X. Then the empirical measures δ_N of the corresponding canonical random point processes on X (formula (2.3)) converge in probability, as $N \to \infty$, towards the normalized volume form dV_{KE} of the unique Kähler–Einstein metric ω_{KE} on X.*

In fact, the proof shows that the convergence holds at an exponential rate, in the sense of large deviation theory: for any given $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\operatorname{Prob}\left(d\left(\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, dV_{\mathrm{KE}} \right) > \varepsilon \right) \le C_\varepsilon e^{-N\varepsilon}, \tag{2.6}$$

where d denotes any metric on the space $\mathcal{P}(X)$ of probability measures on X compatible with the weak topology. The convergence in probability in the previous theorem implies, in particular, that the measures dV_k on X, defined by the expectations $\mathbb{E}(\delta_{N_k})$ of the empirical measure δ_{N_k}, converge to dV_{KE} in the weak topology of measures on X. Concretely, dV_k

is obtained by integrating $\mu^{(N_k)}$ over the fibers of the projection from X^{N_k} onto the first factor X, that is,

$$dV_k := \int_{X^{N_k-1}} \mu^{(N_k)} \to dV_{\mathrm{KE}}, \quad k \to \infty.$$

For k sufficiently large (ensuring that kK_X is very ample), the measures dV_k are, in fact, volume forms on X and induce a sequence of canonical Kähler metrics ω_k on X [5, PROP. 5.3]:

$$\omega_k := \frac{i}{2\pi}\partial\bar{\partial}\log dV_k = \frac{i}{2\pi}\partial\bar{\partial}\log \int_{X^{N_k-1}} \left|\det S^{(k)}\right|^{2/k}. \tag{2.7}$$

The convergence above also implies that the canonical Kähler metrics ω_k converge, as $k \to \infty$, towards the Kähler–Einstein metric ω_{KE} on X, in the weak topology. More generally, as shown in [5], the convergence holds on any variety X of positive Kodaira dimension (i.e., such that $N_k \to \infty$, as $k \to \infty$) if dV_{KE} and ω_{KE} are replaced by the canonical measure and current on X, respectively, introduced by Song-Tian and Tsuji in different geometric contexts [5] (in the case when X is singular it is assumed that X is klt and k is assumed to be sufficently divisible to ensure that kK_X is a bona fide line bundle).

2.2. The Fano case, $K_X < 0$ ($\beta = -1$)

When $-K_X$ is positive, which means that X is a Fano manifold, any Kähler–Einstein metric on X has positive Ricci curvature. However, not all Fano manifolds X carry Kähler–Einstein metrics; according to the Yau–Tian–Donaldson conjecture (discussed in Section 4) a Fano manifold admits a Kähler–Einstein-metric if and only if X is K-polystable. In the probabilistic approach, a new type of stability assumption naturally appears, as is explained next. First note that when $-K_X > 0$ the spaces $\dim H^0(X, kK_X)$ are trivial for all positive integers k. On the other hand, the dimensions tend to infinity as $k \to -\infty$. Thus it is natural to replace k with $-k$ in the previous constructions. In particular, given a positive integer k, we set

$$N_k := \dim H^0(X, -kK_X)$$

and attempt to define a probability measure on X^{N_k} as

$$\mu^{(N_k)} := \frac{|\det S^{(k)}|^{-2/k}}{\mathcal{Z}_{N_k}}, \quad \mathcal{Z}_{N_k} := \int_{X^{N_k}} \left|\det S^{(k)}\right|^{-2/k},$$

where the numerator defines a measure on the complement in X^{N_k} of the zero-locus of $\det S^{(k)}$. However, it may happen that the normalizing constant $\mathcal{Z}_{N_k}$ diverges, since the integrand of $\mathcal{Z}_{N_k}$ blows-up along the zero-locus in X^{N_k} of $\det S^{(k)}$. Accordingly, a Fano manifold X is called *Gibbs stable at level* k if $Z_{N_k} < \infty$ and *Gibbs stable* if it is Gibbs stable at level k for k sufficiently large. We thus arrive at the following probabilistic analog of the Yau–Tian–Donaldson conjecture posed in [5]:

Conjecture 2.3. *Let X be Fano manifold. Then*

- *X admits a unique Kähler–Einstein metric ω_{KE} if and only if X is Gibbs stable.*
- *If X is Gibbs stable, the empirical measures δ_N of the corresponding canonical point processes converge in probability to the normalized volume form of ω_{KE}.*

It should be stressed that the Gibbs stability of X implies that the group $\mathrm{Aut}(X)$ of automorphisms of X is finite [5, PROP. 6.5]. Accordingly, when comparing Conjecture 2.3 with the Yau–Tian–Donaldson conjecture, one should view Gibbs stability as the analog of K-stability. There is also a natural analog of the stronger notion of uniform K-stability [24, 36]. To see this, first note that Gibbs stability can be given a purely algebro-geometric formulation, saying that the $\mathbb{Q}$-divisor D_{N_k} in X^{N_k} cut out by the (multivalued) holomorphic section $(\det S^{(k)})^{1/k}$ of $-K_{X^{N_k}}$ has mild singularities in the sense of the Minimal Model Program [44]. More precisely, X is Gibbs stable at level k iff D_{N_k} is *klt (Kawamata log terminal)*. This means that the *log canonical threshold (lct)* of D_{N_k} satisfies $\mathrm{lct}(D_{N_k}) > 1$, as follows directly from the standard analytic representation of the log canonical threshold of a $\mathbb{Q}$-divisor as an integrability threshold [44]. Accordingly, X is called *uniformly Gibbs stable* if the there exists $\varepsilon > 0$ such that, for k sufficiently large, $\mathrm{lct}(D_{N_k}) > 1 + \varepsilon$. One is thus led to pose the following purely algebro-geometric conjecture:

Conjecture 2.4. *Let X be a Fano manifold. Then X is (uniformly) K-stable iff X is (uniformly) Gibbs stable.*

One direction of the uniform version of the previous conjecture was established in [40, 41], using techniques from the Minimal Model Program:

Theorem 2.5 ([41]). *Uniform Gibbs stability implies uniform K-stability.*

Let us briefly recall the elegant argument in [41], introducing the invariant $\delta(X)$, which has come to play a key role in recent developments around the Yau–Tian–Donaldson conjecture. First, by [41, THM. 2.5],

$$\mathrm{lct}(D_{N_k}) \leq \delta_k(X) := \inf_{\Delta_k} \mathrm{lct}(\Delta_k), \tag{2.8}$$

where the inf is taken over all anticanonical $\mathbb{Q}$-divisors Δ_k on X of k-basis type, i.e., Δ_k is the normalized sum of the N_k zero-divisors on X defined by the members of a given basis in $H^0(X, -kK_X)$. Finally, by [41, THM. 0.3], if the invariant $\delta(X)$ defined as

$$\delta(X) := \limsup_{k\to\infty} \delta_k(X) \tag{2.9}$$

satisfies $\delta(X) > 1$, then X is uniformly K-stable [40] and thus admits a unique Kähler–Einstein metric by the solution of the (uniform) Yau–Tian–Donaldson conjecture recalled in Section 4.2. In particular, this means that uniform Gibbs stability implies the existence of a Kähler–Einstein metrics (in line with Conjecture 2.3). For a direct analytic proof of this implication see [9]. However, the converse implication, that we shall come back to in Section 5, is still open. Anyhow, even if confirmed, it is a separate analytic problem to prove the convergence in Conjecture 2.3. “Tropicalized” analogs of Conjecture 2.3 are established on toric varieties in [18] and on tori in [43].

In [6] a variational approach to the convergence problem was introduced, further developed in [8], where the convergence was settled on log Fano curves. In the general case the approach yields, in particular, the following conditional convergence result:

Theorem 2.6 ([6,8]). *Let X be a Fano manifold and assume that X admits a Kähler–Einstein metric ω_{KE}. Take the basis $s_i^{(k)}$ in formula (2.5) to be orthonormal with respect to the Hermitian metric on $H^0(X, -kK_X)$ induced by ω_{KE} and assume that*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathcal{Z}_N = 0. \tag{2.10}$$

Then $\mathrm{Aut}(X)$ is finite and the empirical measures δ_N converge in probability to the normalized volume form dV_{KE} of the unique Kähler–Einstein-metric ω_{KE} on X.

In [6] two different types of hypotheses were put forth, ensuring that the convergence (2.10) holds, one of which will be recalled in Section 2.3.1. The other assumes, in particular, that the partition function $\mathcal{Z}_N(\beta)$, discussed in the following section, is zero-free in some N-independent neighborhood Ω of $]-1,0]$ in $\mathbb{C}$ (when $\mathcal{Z}_N(\beta)$ is analytically continued to a holomorphic function on Ω). This allows one to "analytically continue" the convergence when $\beta > 0$ to $\beta < 0$. This is discussed in detail in [8], where some intriguing connections between this zero-free hypothesis and the zero-free property of the local L-functions appearing in the Langlands program are also pointed out.

2.3. The statistical mechanical formalism and outlines of the proofs

Theorem 2.2 (or more precisely, the exponential convergence in formula (2.6)) is deduced from a large deviation principle (LDP), which may be symbolically expressed as

$$\mathrm{Prob}\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \in B_\varepsilon(\mu)\right) \sim e^{-NR(\mu)}, \quad N \to \infty,\ \varepsilon \to 0, \tag{2.11}$$

where $B_\varepsilon(\mu)$ denotes the ball of radius ε centered at a given μ in the space $\mathcal{P}(X)$ of all probability measures on X, endowed with a metric d compatible with the weak topology. In probabilistic terminology, the functional $R(\mu)$ is called the *rate functional*. By general principles, any rate functional of an LDP is lower-semicontinuous and its infimum vanishes. In the present setup, the volume form dV_{KE} of the Kähler–Einstein metric is the unique minimizer of $R(\mu)$, which yields the exponential convergence in formula (2.6).

As next explained, the proof of the LDP is inspired by statistical mechanics. Fix a Kähler metric on X. It induces a volume form dV on X and a Hermitian metric $\|\cdot\|$ on K_X. The canonical probability measure (2.4) may then be decomposed as

$$\mu^{(N)} = \frac{1}{\mathcal{Z}_{N_k}} \left\|\det S^{(k)}\right\|^{2/k} dV^{\otimes N},$$

where the basis $s_i^{(k)}$ in formula (2.5) is taken to be orthonormal with respect to the Hermitian metric on $H^0(X, kK_X)$ induced by dV and $\|\cdot\|$. Introducing the *energy per particle* as

$$E^{(N)}(x_1,\dots,x_N) := -\frac{1}{kN} \log\left\|\det S^{(k)}(x_1,\dots,x_{N_k})\right\|^2, \tag{2.12}$$

we can thus express $\mu^{(N)}$ as the following *Gibbs measure,* at *inverse temperature* $\beta = 1$:

$$\mu_\beta^{(N)} = \frac{e^{-\beta N E^{(N)}}}{\mathcal{Z}_N(\beta)} dV^{\otimes N}, \quad \mathcal{Z}_N(\beta) := \int_{:X^N} e^{-\beta N E^{(N)}} dV^{\otimes N}. \tag{2.13}$$

In statistical mechanical terms, the Gibbs measures represent the microscopic thermal equilibrium state of N interacting identical particles on X. The normalizing constant $\mathcal{Z}_N(\beta)$ is called the *partition function.*

The starting point of the proof of the LDP (2.11) is a classical result of Sanov in probability, going back to Boltzmann, saying that in the "noninteracting case" $\beta = 0$ (where the positions x_i define independent random variables on X) the LDP holds with rate functional given by the entropy $\mathrm{Ent}(\mu)$ of μ relative to dV, i.e., the functional on $\mathcal{P}(X)$ defined by

$$\mathrm{Ent}(\mu) := \int_X \log\left(\frac{\mu}{dV}\right)\mu,$$

if μ is absolutely continuous with respect to dV, and otherwise $\mathrm{Ent}(\mu) := +\infty$.[1] The strategy to handle the "interacting case" $\beta \neq 0$ is to first show that there exists a functional $E(\mu)$ on $\mathcal{P}(X)$ such that the energy per particle, $E^{(N)}(x_1, \dots, x_N)$, may be approximated as

$$E^{(N)}(x_1, \dots, x_N) \to E(\mu), \tag{2.14}$$

when $\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \to \mu$, in an appropriate sense, as $N \to \infty$. Formally combining this result with Sanov's LDP suggests that, for any $\beta > 0$, the corresponding rate functional is given by

$$R_\beta(\mu) = F_\beta(\mu) - \inf_{\mathcal{P}(X)} F_\beta, \quad F_\beta(\mu) = \beta E(\mu) + \mathrm{Ent}(\mu) \in]0, \infty], \tag{2.15}$$

In thermodynamical terms, the functional $F_\beta(\mu)$ is the *free energy,* at inverse temperature β (strictly speaking, it is $\beta^{-1}F_\beta$ which is the free energy, i.e., the energy that is free to do work once the disordered thermal energy has been subtracted). In the present setting the role of the "macroscopic" energy $E(\mu)$ is played by the *pluricomplex energy* of the measure μ (introduced in **[12]** and discussed in Section 3). Briefly, it is first shown in **[4]** that the convergence (2.14) holds in the sense of *Gamma-convergence*. This means that

$$\frac{1}{N_j}\sum_{i=1}^{N_j}\delta_{x_i} \to \mu \implies \liminf_{N_j\to\infty} E^{(N_j)}(x_1, \dots, x_{N_j}) \geq E(\mu) \tag{2.16}$$

and, for any μ, there exists some sequence of configurations in X^N saturating the previous inequality. The Gamma-convergence is deduced from the convergence of weighted transfinite diameters established in **[11]** using a duality argument (where $E(\mu)$ arises as a Legendre–Fenchel transform; compare formula (3.12)). The combination with Sanov's theorem is then made rigorous using an effective submean inequality on small balls in the Riemannian orbifold X^N/S_N, established using geometric analysis.

The free energy functional F_β has a unique minimizer μ_β in $\mathcal{P}(X)$ for any $\beta > 0$ (as discussed in Section 3.3). As a consequence, the empirical measures δ_N converge in probability to μ_β, as $N \to \infty$. The LDP proved in **[4]** also implies that for $\beta > 0$,

$$\lim_{N\to\infty} -\frac{1}{N}\log \mathcal{Z}_N(\beta) = \inf_{\mathcal{P}(X)} F_\beta. \tag{2.17}$$

Incidentally, the free energy functional F_β on $\mathcal{P}(X)$ may be identified with the (twisted) Mabuchi functional in Kähler geometry, as explained in Section 3.4.

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1 In the physics literature, the opposite sign convention for $\mathrm{Ent}(\mu)$ is used.

2.3.1. The case $\beta < 0$

The Gibbs measure $\mu_\beta^{(N)}$ can, alternatively, be viewed as a Gibbs measure at *unit* temperature, if $E^{(N)}$ is replaced with with the rescaled energy $\beta E^{(N)}$ (thus treating β as a coupling constant). For $\beta > 0$, this energy is *repulsive,* since it tends to ∞ as any two particle positions merge (due to the vanishing of the determinant $\det S^{(k)}(x_1, \dots, x_{N_k})$). However, when β changes sign, the rescaled energy $\beta E^{(N)}$ becomes *attractive*; it tends to $-\infty$ as any two points merge, which leads to subtle concentration phenomena and various new technical difficulties. For example, one reason that the proof of the LDP does not generalize to $\beta < 0$ is that the Gamma-convergence in formula (2.14) is not preserved when $E^{(N)}$ is replaced by $-E^{(N)}$. In order to bypass this difficulty, a variational approach was introduced in [6]. The starting point is the classical Gibbs variational principle, which yields

$$-\frac{1}{N}\log \mathcal{Z}_N(\beta) = \inf_{\mathcal{P}(X^N)} F_\beta^{(N)}, \quad F_\beta^{(N)}(\cdot) := \beta\langle E^{(N)}, \cdot\rangle + N^{-1}\mathrm{Ent}(\cdot), \tag{2.18}$$

where the functional $F_\beta^{(N)}$ on $\mathcal{P}(X^N)$ is called the *N-particle mean free energy* and $\mathrm{Ent}(\cdot)$ denotes the entropy relative to $dV^{\otimes N}$. When its infimum is finite, it is uniquely attained at the corresponding Gibbs measure $\mu_\beta^{(N)}$. In [6, 8] this variational formulation is leveraged to show that, if X admits a Kähler–Einstein metric dV_{KE}, then δ_N converge in probability to dV_{KE}, under the assumption that the convergence of the partition functions (2.17) holds at $\beta = -1$. In particular, when the fixed metric on X is taken to be a Kähler–Einstein metric, this proves Theorem 2.6, since $F_{-1}(dV_{\mathrm{KE}}) = 0$. Moreover, the convergence (2.17) of the partition functions at $\beta = -1$ is shown to be implied by the following hypothesis:

$$\lim_{N_j\to\infty} (\delta_{N_j})_*\mu_{-1}^{(N_j)} = \Gamma \in \mathcal{P}\big(\mathcal{P}(X)\big) \implies \limsup_{N_j\to\infty}\langle E^{(N_j)}, \mu_{-1}^{(N_j)}\rangle \le \langle E, \Gamma\rangle, \tag{2.19}$$

where $(\delta_N)_*\mu_{-1}^{(N)}$ is the probability measure on the infinite-dimensional $\mathcal{P}(X)$, defined as the pushforward of the canonical probability measure $\mu_{-1}^{(N)}$ on X^N to $\mathcal{P}(X)$ under the map δ_N (the reversed inequality holds for *any* sequence μ_N in $\mathcal{P}(X^N)$, as follows from the inequality (2.16)). If the hypothesis holds, then it follows that Γ is the Dirac mass at dV_{KE}, which is equivalent to the convergence in Theorem 2.6. In fact, as shown in [8], the previous hypothesis is "almost" equivalent to the convergence in Conjecture 2.3.

Finally, we note that the conjectural extension of formula (2.17) to any $\beta < 0$ also suggests the following conjecture posed in [4] (the definition of the log canonical threshold $\mathrm{lct}(D_N)$ was discussed after Conjecture 2.3):

Conjecture 2.7. *For any Fano manifold X,*

$$\lim_{N\to\infty} \mathrm{lct}(D_N) = \Gamma(X), \quad \Gamma(X) := \sup_{\beta<0}\Big\{-\beta : \inf_{\mathcal{P}(X)} F_\beta > -\infty\Big\}. \tag{2.20}$$

3. THE THERMODYNAMICAL FORMALISM AND PLURIPOTENTIAL THEORY

The pluricomplex energy $E(\mu)$, appearing as the "energy part" of the free energy functional $F_\beta(\mu)$ in formula (2.15), may be defined as the greatest lower semicontinuous extension to the space $\mathcal{P}(X)$ of the functional whose first variation on the subspace of volume forms is given by

$$dE(\mu) = -u_\mu, \tag{3.1}$$

with $u_\mu \in C^\infty(X)$ denoting the solution to the following complex Monge–Ampère equation (known as the *Calabi–Yau equation*)

$$\mathrm{MA}(u) = \mu, \tag{3.2}$$

expressed in terms of the complex Monge–Ampère measure $\mathrm{MA}(u)$, whose definition we next recall.

3.1. Kähler geometry recap

Assume that we are given a line bundle L endowed with a Hermitian metric $\|\cdot\|$ (in the present setup, $L = \pm K_X$ and $\|\cdot\|$ is the metric on L induced by a fixed Kähler metric on X). Then any smooth function u on X induces a metric $\|\cdot\|e^{-u/2}$ on L, whose curvature form, multiplied by $i/2\pi$, will be denoted by ω_u; it is a real closed two-form on X, representing the first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ of L. Concretely,

$$\omega_u = \omega_0 + \frac{i}{2\pi}\partial\bar{\partial}u_\beta, \quad \partial\bar{\partial}u := \sum_{i,j\leq n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_i} dz_i \wedge d\bar{z}_j, \tag{3.3}$$

in terms of local holomorphic coordinates, where ω_0 is the normalized curvature form of the fixed metric $\|\cdot\|$ on L. The complex Monge–Ampère measure $\mathrm{MA}(u)$ is the normalized volume form on X defined by

$$\mathrm{MA}(u) := \omega_u^n / V, \quad V := \int_X \omega_u^n = \int_X \omega_0^n.$$

By the Calabi–Yau theorem, there exists a smooth solution u_μ to the Calabi–Yau equation (3.2), uniquely determined up to an additive constant. It has the property that ω_{u_μ} is a *Kähler form*. Recall that a J-invariant closed real form ω on X is said to be Kähler if $\omega > 0$ in the sense that the corresponding symmetric two-tensor

$$g := \omega(\cdot, J\cdot)$$

is positive definite, i.e., defines a Riemannian metric (where J denotes the complex structure on X). In practice, one then identifies the Kähler form ω with the corresponding *Kähler metric* g. Likewise, the Ricci curvature of a Kähler metric ω may be identified with the two-form

$$\mathrm{Ric}\,\omega = -\frac{i}{2\pi}\partial\bar{\partial}\log dV, \tag{3.4}$$

where dV denotes the volume form of ω. In other words, $\mathrm{Ric}\,\omega$ is the curvature of the metric on $-K_X$ induced by ω. If the Kähler form ω is of the form ω_u (as in formula (3.3)), then u is said to be a *Kähler potential* for ω (relative to ω_0). We will denote by $\mathcal{H}(X, \omega_0)$ the space of all Kähler potentials relative to ω_0, and by $\mathcal{H}(X, \omega_0)_0$ the subspace of all sup-normalized u, $\sup_X u = 0$. The map

$$u \mapsto \omega_u, \quad \mathcal{H}(X, \omega_0)_0 \hookrightarrow c_1(L)$$

yields a one-to-one correspondence between $\mathcal{H}(X,\omega_0)_0$ and the space of all Kähler forms in the first Chern class $c_1(L)$ of L. Similarly, the Calabi–Yau theorem yields the "Calabi–Yau correspondence"

$$u \mapsto \mathrm{MA}(u), \quad \mathcal{H}(X,\omega_0)_0 \hookrightarrow \mathcal{P}(X) \tag{3.5}$$

between $\mathcal{H}(X,\omega_0)_0$ and the space of all volume forms in $\mathcal{P}(X)$, where u corresponds to the normalized volume form of the Kähler metric ω_u. The one-form on $\mathcal{H}(X,\omega_0)$ induced by MA is exact, i.e., there exists a functional $\mathcal{E}$ on $\mathcal{H}(X,\omega_0)$ such that

$$d\mathcal{E} = \mathrm{MA}, \quad \text{i.e.,} \quad \frac{d\mathcal{E}(u+t\dot{u})}{dt}\Big|_{t=0} = \langle \mathrm{MA}(u), \dot{u} \rangle.$$

(this functional is often denoted by E in the literature **[22]**, but here we shall reserve capital letters for functionals defined on $\mathcal{P}(X)$). The functional $\mathcal{E}(u)$ is uniquely determined up to an additive a constant and may be explicitly defined by

$$\mathcal{E}(u) := \frac{1}{V(n+1)} \sum_{j=0}^{n} \int_X u\omega_u^j \wedge \omega_0^{n-j}. \tag{3.6}$$

3.2. Pluripotential theory recap

The analysis of the minimizers of F_β involves some pluripotential theory that we briefly recall. The space $\mathrm{PSH}(X,\omega)$ of all ω_0-psh functions on X may be defined as the closure of $\mathcal{H}(X,\omega_0)$ in $L^1(X)$ (more precisely, any $u \in \mathrm{PSH}(X,\omega)$ is the decreasing limit of elements $u_k \in \mathcal{H}(X,\omega_0)$). The corresponding sup-normalized subspace $\mathrm{PSH}(X,\omega_0)_0$ is compact in $L^1(X,\omega_0)$. By **[12]**, the "Calabi–Yau correspondence" (3.5) extends to a correspondence between the subspace of probability measures μ with finite energy and a subspace of $\mathrm{PSH}(X,\omega_0)$ denoted by $\mathcal{E}^1(X,\omega_0)$, that is,

$$\mathrm{MA}: \quad \mathcal{E}^1(X,\omega_0)_0 \leftrightarrow \{\mu \in \mathcal{P}(X) : E(\mu) < \infty\}, \tag{3.7}$$

where $\mathrm{MA}(u)$ is defined on $\mathcal{E}^1(X,\omega_0)$ using the notion of nonpluripolar products introduced in **[22]**. The space $\mathcal{E}^1(X,\omega_0)$ was originally introduced in **[42]**, but, as shown in **[12]**, it may also be defined as the space of all $u \in \mathrm{PSH}(X,\omega_0)$ such that $\mathcal{E}(u) > -\infty$, where $\mathcal{E}$ denotes the smallest upper semicontinuous extension of $\mathcal{E}$ to $\mathrm{PSH}(X,\omega_0)$.

3.3. Back to the free energy functional F_β

The free energy functional F_β, defined in formula (2.15), $F_\beta = \beta E + \mathrm{Ent}$, is lsc and convex on $\mathcal{P}(X)$ when $\beta > 0$ (since both terms are). In the case when $\beta < 0$, we define $F_\beta(\mu)$ by the same expression when $E_{\omega_0}(\mu) < \infty$ and otherwise we set $F_\beta(\mu) = \infty$. The definition is made so that we still have $F_\mu(\mu) \in]-\infty, \infty]$ with $F_\mu(\mu) < \infty$ iff both $E(\mu) < \infty$ and $\mathrm{Ent}(\mu) < \infty$.

The following lemma follows readily from the first variation (3.1) and formula (3.4) for Ricci curvature of a Kähler metric.

Lemma 3.1. *A volume form μ on X is a critical point of the functional F_β on $\mathcal{P}(X)$ iff the function*

$$u_\beta := \frac{1}{\beta} \log \frac{\mu}{dV}$$

solves the complex Monge–Ampère equation

$$\mathrm{MA}(u) = e^{\beta u} dV \tag{3.8}$$

iff $\omega_\beta := \omega_{u_\beta}$ *is a Kähler form solving the twisted Kähler–Einstein equation*

$$\mathrm{Ric}\,\omega + \beta\omega = \theta, \quad \theta := (\beta \mp 1)\omega_0. \tag{3.9}$$

In the Fano case, the previous equation coincides with Aubin's continuity equation with "time-parameter" $t := -\beta$. When $\beta > 0$, it follows directly from the lower semicontinuity of F_β on the compact space $\mathcal{P}(X)$ that F_β admits a minimizer.

Theorem 3.2 ([2]). *The following are true:*

(regularity) *Any minimizer μ_β of the functional F_β on $\mathcal{P}(X)$ is a volume form and thus of the form in Lemma 3.1.*

(existence) *If F_{β_0} is bounded from below for some $\beta_0 < 0$, then for any $\beta > \beta_0$ the functional F_β on $\mathcal{P}(X)$ admits a minimizer. In other words, if F_β is* coercive *(with respect to E) in the sense that there exists $\varepsilon > 0$ and $C > 0$ such that*

$$F_\beta \geq \varepsilon E + C, \tag{3.10}$$

then F_β admits a minimizer.

Moreover, by the Bando–Mabuchi theorem, if $\beta > -1$, the minimizer is uniquely determined and, if $\beta = -1$, it is uniquely determined iff the automorphism group $\mathrm{Aut}(X)$ of X is finite (see [10] for generalizations). The proof of the previous theorem employs a duality argument, which fits naturally into the thermodynamical formalism, when combined with pluripotential theory and the variational approach to complex Monge–Ampère equation developed in [12]. The strategy is to show that any minimizer satisfies the Monge–Ampère equation (3.8) in the weak sense of pluripotential theory, so that the regularity theory for Monge–Ampère equations (going back to Aubin and Yau) can be invoked. In the case when $\beta > 0$, the proof of Theorem 3.2 follows from the strict convexity of F_β, resulting from the convexity of $E(\mu)$ and the strict convexity of $\mathrm{Ent}(\mu)$ on $\mathcal{P}(X)$, combined with the Aubin–Yau theorem [1, 69] (showing that there exists a unique smooth solution to equation (3.8)). The proof in the case when $\beta < 0$ exploits the *Legendre–Fenchel transform.* Recall that, in general, this transform yields a correspondence between lsc convex functions on a locally convex topological vector space V and its dual V^*. In order to facilitate the comparison to the standard functionals in Kähler geometry (discussed in the following section), it will, however, be convenient to use a slightly nonstandard sign convention where an lsc convex function f on V corresponds to the usc concave function f^* on V^* defined by

$$f^*(w) := \inf_{v \in V} \big(\langle v, w\rangle + f(v)\big). \tag{3.11}$$

Conversely, if Λ is a functional on V^*, we define $\Lambda^*(v)$ as the lsc convex function

$$\Lambda^*(v) = \sup_{w \in V^*} \big(-\langle v, w\rangle + f(w)\big).$$

We take V to be the space of all signed measures μ on X, so that $V^* = C^0(X)$. We can then view E and Ent as convex lsc functions on V, which, by definition, are equal to ∞ on the complement of $\mathcal{P}(X)$ in V. Under the Legendre–Fenchel transform, these correspond to the usc convex functions E^* and Ent^*, respectively, on $C^0(X)$, which turn out to be Gateaux differentiable. Indeed, by a classical result (which follows from Jensen's inequality),

$$\text{Ent}^*(u) = -\log \int e^{-u} dV.$$

Moreover, as shown in [11,12], the functional E^* on $C^0(X)$ is Gateaux differentiable and

$$E^*(u) = \mathcal{E}(u), \quad dE^*|_u = \text{MA}(u), \quad \text{for } u \in \mathcal{H}(X, \omega_0). \tag{3.12}$$

Now consider, for simplicity, the case $\beta = -1$ (the general case is obtained by a simple scaling). It follows directly from the fact that the Legendre–Fenchel transform is increasing and involutive that

$$\inf_{\mathcal{P}(X)} F_{-1} := \inf_{\mathcal{P}(X)} (-E + \text{Ent}) = \inf_{C^0(X)} (-E^* + \text{Ent}^*). \tag{3.13}$$

Moreover, it readily from the definitions that

$$F_{-1}\big(\text{MA}(u)\big) = (-E + \text{Ent})\big(dE^*|_u\big) \geq (-E^* + \text{Ent}^*)(u).$$

Hence, if μ minimizes F_{-1} and we express $\mu = \text{MA}(u_\mu)$, then u_μ minimizes the functional $-E^* + \text{Ent}^*$ on $C^0(X)$. However, in the present setup u_μ is not, a priori, in $C^0(X)$, but only in $\mathcal{E}^1(X, \omega_0)$. This problem is circumvented using a simple approximation argument to deduce that u_μ minimizes the extension of the functional $(-E^* + \text{Ent}^*)$ to $\mathcal{E}^1(X, \omega_0)$. Finally, by the Gateaux differentiability of the functional $-E^* + \text{Ent}^*$ on $C^0(X)$ (or more precisely, on $\{u\} + C^0(X)$ for any given $u \in \mathcal{E}^1(X, \omega_0)$), it then follows that u_μ is a critical point of the functional $-E^* + \text{Ent}^*$. Thus, after perhaps adding a constant to u_μ, it satisfies the complex Monge–Ampère equation (3.8) in the weak sense of pluripotential theory.

The proof of the first point in Theorem 3.2 can now be concluded by invoking the regularity results for pluripotential solutions to Monge–Ampère equations (which, by [13, APPENDIX B], hold in the general setup of log Fano varieties). As for the second point, it is shown in [2] by proving that any minimizing sequence μ_j in $\mathcal{P}(X)$ (i.e., a sequence μ_j such that $F_\beta(\mu_j)$ converges to the infimum of F_β) converges (after perhaps passing to a subsequence) to a minimizer of F_β. This is shown using a duality argument, as above. Alternatively, as shown in [13] in a more general singular context (including singular log Fano varieties), the existence of a minimizer for $F_\beta(\mu)$ follows from the following result in [13]:

Theorem 3.3 (energy/entropy compactness). *The functional $E(\mu)$ is continuous on any sublevel set $\{\text{Ent} \leq C\} \subset \mathcal{P}(X)$. As a consequence, if F_β is coercive on $\mathcal{P}(X)$, then it is lower semicontinuous and thus admits a minimizer.*

This result has come to play a prominent role in recent developments in Kähler geometry, as discussed in Section 4.1.1.

3.4. The Mabuchi and Ding functionals

Under the "Calabi–Yau correspondence" (3.5), the free energy functional F_β on $\mathcal{P}(X)$ corresponds to a functional $\mathcal{M}_\beta(u)$ on $\mathcal{E}^1(X,\omega_0)$ defined by

$$\mathcal{M}_\beta(u) := F_\beta\big(\mathrm{MA}(u)\big). \tag{3.14}$$

Also, the functional $E(\mu)$ on $\mathcal{P}(X)$ corresponds to the functional $E(\mathrm{MA}(u))$ on $\mathrm{PSH}(X,\omega_0)$ which induces an exhaustion function on $\mathcal{E}^1(X,\omega_0)_0$, comparable to $-\mathcal{E}(u)$, defining a notion of coercivity on $\mathcal{E}^1(X,\omega_0)$ (in terms of the standard functionals I and J in Kähler geometry $E(\mathrm{MA}(u)) = (I-J)(u)$).

As is turns out, when restricted to $\mathcal{H}(X,\omega_0)$ the functional $\mathcal{M}_\beta(u)$ coincides with the *(twisted) Mabuchi functional.* The Mabuchi functional $\mathcal{M}$ associated to a general polarized manifold (X,L) was originally defined (up to normalization) by the property that its first variation is proportional to the scalar curvature of the Kähler metric ω_u minus the average scalar curvature **[53]**. An "energy+entropy" formula for $\mathcal{M}$, similar to formula (3.14), holds for a general polarized manifold, as first discovered in **[29, 64]**. Likewise, the functional on $\mathcal{E}^1(X,\omega_0)$ induced by $-E^* + \mathrm{Ent}^*$ coincides with the *Ding functional* $\mathcal{D}(u)$ in Kähler geometry, extended to $\mathcal{E}^1(X,\omega_0)$ in **[12]**. For a general β, the corresponding twisted Ding functional $\mathcal{D}_\beta$ on $\mathcal{E}^1(X,\omega_0)$ is given by

$$\mathcal{D}_\beta(u) := -\mathcal{E}(u) + \frac{1}{\beta}\log\int e^{\beta u}\,dV.$$

An extension of the argument used to prove formula (3.13) (concerning the boundedness statement) now gives

Theorem 3.4 (**[2]**). *The functional $\mathcal{M}_\beta$ is bounded from below (coercive) on $\mathcal{E}^1(X,\omega_0)_0$ iff $\mathcal{D}_\beta$ is bounded from below (coercive) on $\mathcal{E}^1(X,\omega_0)_0$. Moreover, by the regularization result in* **[16]**, *these properties are equivalent to the corresponding boundedness/coercivity properties on the dense subspace $\mathcal{H}(X,\omega_0)_0$ of $\mathcal{E}^1(X,\omega_0)_0$.*

For $\beta = -1$, the first statement was first established in **[46, 57]**. The proof in **[46]** shows that the difference $\mathcal{M}_\beta - \mathcal{D}_\beta$ is bounded along the Kähler–Ricci flow, thanks to Perelman's estimates, while the proof in **[57]** utilizes the Ricci iteration. In the case $\beta = -1$, the coercivity of $\mathcal{M}_\beta$ is, in fact, equivalent to the existence of unique Kähler–Einstein metric, as first shown in **[65]**, using Aubin's method of continuity (discussed above in connection to Lemma 3.1). More recently, this result has been given a new proof using the notion of geodesics in $\mathcal{E}^1(X)$ and extended in various directions, as discussed in Section 4.1.1.

4. THE YAU–TIAN–DONALDSON CONJECTURE

4.1. The Yau–Tian–Donaldson conjecture for polarized manifolds (X, L)

Let (X,L) be a polarized projective algebraic manifold, i.e., L is a holomorphic line bundle over X whose first Chern class $c_1(L)$ contains some Kähler form.

Conjecture 4.1 (Yau–Tian–Donaldson, YTD). *There exists a Kähler metric in $c_1(L)$ with constant scalar curvature iff (X,L) is K-polystable.*

We will briefly recall the notion of K-polystability (see the survey [38] for more background on the Yau–Tian–Donaldson conjecture and its relation to geometric invariant theory (GIT)). The notion of K-polystability can be viewed as a "large-N_k limit" of the classical notion of Chow polystability in GIT with respect to the action of complex reductive group $\mathrm{GL}(N_k, \mathbb{C})$ on the Chow variety, induced from the action of $\mathrm{GL}(N_k, \mathbb{C})$ on the N_k-dimensional complex vector space $H^0(X, kL)$. Recall that in GIT the stability in question is equivalent to the positivity of the GIT-weight of all one-parameter subgroups (by the Mumford–Hilbert criterion). In the definition of K-polystability, the role of a one-parameter subgroup ρ_k of $\mathrm{GL}(N_k, \mathbb{C})$ is played by a *test configuration* ρ for (X, L). In a nutshell, this is a $\mathbb{C}^*$-equivariant embedding

$$\rho : (X \times \mathbb{C}^*, L) \hookrightarrow (\mathcal{X}, \mathcal{L})$$

of the polarized trivial fibration $(X \times \mathbb{C}^*, L)$ over $\mathbb{C}^*$ into a normal variety $\mathcal{X}$ fibered over $\mathbb{C}$ endowed with a relatively ample $\mathbb{Q}$-line bundle $\mathcal{L}$. To any test configuration ρ is attached an invariant, called the *Donaldson–Futaki invariant* $\mathrm{DF}(\rho) \in \mathbb{R}$, and (X, L) is said to be *K-semistable* if $\mathrm{DF}(\rho) \geq 0$ for any test configuration, *K-polystable* if, moreover, equality only holds when $\mathcal{X}$ is biholomorphic to $X \times \mathbb{C}$, and *K-stable* if the equality only holds when $\mathcal{X}$ is equivariantly biholomorphic to $X \times \mathbb{C}$. The Donaldson–Futaki invariant of ρ may be defined as a limit of the GIT-weights of a sequence of one-parameter subgroups ρ_k of $\mathrm{GL}(N_k, \mathbb{C})$ induced by ρ. But it may also be expressed directly as an intersection number [54, 66]:

$$\mathrm{DF}(\rho) = \frac{1}{L^n(n+1)}\big(a\mathcal{L}^{n+1} + (n+1)K_{\overline{\mathcal{X}}/\mathbb{P}^1} \cdot \mathcal{L}^n\big), \quad a := -nK_X \cdot L^{n-1}/L^n,$$

where we have identified a test configuration $(\mathcal{X}, \mathcal{L})$ with its $\mathbb{C}^*$-equivariant compactification over $\mathbb{P}^1$ (obtained by replacing the base $\mathbb{C}$ of $\mathcal{X}$ with $\mathbb{P}^1$) and the intersection numbers are computed on the compactification $\overline{\mathcal{X}}$ of the total space $\mathcal{X}$.

4.1.1. The uniform YTD and geodesic stability

The "only if" direction of the YTD conjecture was established in [60] in the case when the group $\mathrm{Aut}(X, L)$ of all automorphisms of X that lift to L is finite and in [16], in general. However, for the converse implication, there are indications that the notion of K-polystability needs to be strengthened, in general. Here we will, for simplicity, focus on the case when $\mathrm{Aut}(X, L)$ is finite. Then K-polystability is equivalent to K-stability and, moreover, if $c_1(L)$ contains a Kähler metric with constant curvature then it is uniquely determined [10, 37]. Following [24, 36], (X, L) is said to be *uniformly K-stable* (in the L^1-sense) if there exists $\varepsilon > 0$ such that

$$\mathrm{DF}(\rho) \geq \varepsilon \|\rho\|_{L^1}, \tag{4.1}$$

where the L^1-norm $\|\rho\|_{L^1}$ is defined as the normalized limit of the l^1-norms of the weights of the $\mathbb{C}^*$-action on the central fiber of $(\mathcal{X}, \mathcal{L})$. The "only if" direction of the *"uniform YTD conjecture"* – where K-stability is replaced by uniform K-stability (in the L^1-sense) – was established in [24], by leveraging the connection to the "metric space analog" of the uniform

YTD conjecture, to which we next turn. Denote by d_1 the metric on $\mathcal{H}(X,\omega_0)$ induced by the intrinsic L^1-Finsler metric

$$\int_X |\dot{u}|^1 \omega_{u_0}^n, \quad \dot{u} := \left.\frac{du}{dt}\right|_{t=0}, \quad u_0 \in \mathcal{H}.$$

As shown in [32], the metric space completion $(\overline{\mathcal{H}(X,\omega_0)_0}, d_1)$ may be identified with the space $\mathcal{E}^1(X,\omega_0)_0$ (discussed in Section 3.2) and $d_1(u,0)$ is comparable to $-\mathcal{E}(u)$, which, equivalently, means that there exists a constant c such that

$$-c + c^{-1} d_1(u,0) \leq E\big(\mathrm{MA}(u)\big) \leq c d_1(u,0) + c. \tag{4.2}$$

The relevant constant speed geodesics u_t in the metric space $(\mathcal{E}^1(X,\omega_0)_0, d_1)$ have the property that

$$U(x,\tau) := u_{-\log|\tau|}(x) \in \mathrm{PSH}(X \times D^*, \omega_0), \tag{4.3}$$

where we are using the same notation ω_0 for the pullback of ω_0 to the product $X \times D^*$ of X with the punctured unit-disc D^* in $\mathbb{C}$. In fact, u_t may be characterized by a maximality property of the corresponding ω_0-psh function U [14]. Any test configuration ρ induces a geodesic ray u_t in $\mathcal{E}^1(X,\omega_0)_0$, emanating from $0 \in \mathcal{H}(X,\omega_0)$ (such that U extends, after removing divisorial singularities, to a bounded function on $\mathcal{X}$) [32,55]. Moreover,

$$\|\rho\|_{L^1} = \frac{d}{dt} d_1(u_t,0) = t^{-1} d(u_t,0)$$

for any $t > 0$. As conjectured in [29], and confirmed in [10], the Mabuchi functional $\mathcal{M}$ (Section 3.4) is convex along geodesic u_t such that $\omega_U \in L^\infty_{\mathrm{loc}}$. More generally, the extension of $\mathcal{M}$ to $\mathcal{E}^1(X,\omega_0)$ is also convex along geodesics u_t [16]. In particular, its (asymptotic) slope

$$\dot{\mathcal{M}}(u_t) := \lim_{t\to\infty} t^{-1}\mathcal{M}(t) \in \left]-\infty, \infty\right]$$

is well defined. In the case when u_t is the geodesic ray attached to a test configuration ρ the slope $\dot{\mathcal{M}}(u_t)$ is closely related to $\mathrm{DF}(\rho)$ (the two invariants coincide after a base change [49,59]).

Theorem 4.2 ([17,30,33]). *Let (X,L) be a polarized manifold. The following are equivalent:*

(1) *(X,L) admits a unique Kähler metric with constant scalar curvature.*

(2) *(X,L) is geodesically stable, i.e., $\dot{\mathcal{M}}(u_t) > 0$ for any nontrivial geodesic ray u_t in $\mathcal{E}^1(X,\omega_0)_0$.*

(3) *$\mathcal{M}$ is coercive on $\mathcal{E}^1(X,\omega_0)_0$ (or, equivalently, on $\mathcal{H}(X,\omega_0)_0 \subset \mathcal{E}^1(X,\omega_0)_0$).*

The equivalence "2 $\iff$ 3" is implicit in [33] (see [14, THM. 2.16] for a generalization). It can be seen as an analog of the classical fact that a convex function on Euclidean $\mathbb{R}^n$ is comparable to the distance to the origin iff all its slopes are positive. In the proof of "2 $\iff$ 3" a substitute for the compactness of the unit-sphere in $\mathbb{R}^n$ (parametrizing all unit speed geodesics) is provided by the energy–entropy compactness in Theorem 3.3. The implication "1 $\implies$ 3" follows directly from the convexity of $\mathcal{M}$ combined with the weak-strong uniqueness result in [17], showing, in particular, that if (X,L) admits a unique Kähler metric with constant scalar curvature ω, then any minimizer of $\mathcal{M}$ in $\mathcal{E}^1$ coincides with the Kähler potential of ω. The final implication "3 $\implies$ 1" was recently settled in [30], using a new a priori estimate for a generalization of Aubin's continuity method for constant scalar curvature metrics (bounding the C^0-norm of the solutions by the entropy of the corresponding Monge–Ampère measures, which, in turn, is uniformly bounded under the coercivity assumption).

4.2. The variational approach to the uniform YTD conjecture in the "Fano case"

The "Fano case" of the YTD conjecture, i.e., the case when X is Fano and $L = -K_X$, was settled in [31], by establishing Tian's partial C^0-estimate [63] along a singular version of Aubin's continuity method. Here we will focus on the variational proof of the uniform YTD conjecture on Fano manifolds in [14], which, in particular, exploits the notion of Ding stability originating in [3] (as further developed in [14,24]; see the survey [20] for more background).

Theorem 4.3 ([14]). *Let X be a Fano manifold. The following are equivalent:*

(1) *X admits a unique Kähler–Einstein metric.*

(2) *X is uniformly Ding stable.*

(3) *X is uniformly K-stable.*

The implication "1 $\implies$ 2" follows from the convexity of the Ding functional along geodesics, as in [3] – here we shall focus on the converse implication. By Theorem 4.2, it is enough to show that if X is uniformly Ding stable, then X it geodesically stable. This is achieved in [14], using a valuative (non-Archimedean) language. For simplicity, it may be helpful to briefly first describe the argument with the non-Archimedean language stripped away. The starting point is the observation that the function U on $X \times D^*$ corresponding to a geodesic u_t in $\mathcal{E}^1(X, \omega_0)_0$ (formula (4.3)) extends to a sup-normalized ω_0-psh function U on $X \times D$, which, however, is highly singular on $X \times \{0\}$, unless u_t is trivial. But employing Demailly's approximation procedure [35] (involving the multiplier ideal sheaves $\mathfrak{J}(kU)$, whose definition is recalled in the following section) the function U may be expressed as a decreasing limit of S^1-invariant ω_0-psh functions U_k with analytic (algebraic) singularities, which define $\mathbb{C}^*$-invariant ideals $\mathfrak{J}_k$ supported in $X \times \{0\}$. Accordingly, by the standard resolution of singularities, there exists a $\mathbb{C}^*$-equivariant holomorphic surjection π_k from a nonsingular variety $\mathcal{X}_k$ to $X \times \mathbb{C}$ such that $E_k := \pi_k^* \mathfrak{J}_k$ is a principal ideal, i.e., defines a divisor on $\mathcal{X}_k$. This procedure yields a sequence of test configurations $\rho_k = (\mathcal{X}_k, \mathcal{L}_k)$ where $\mathcal{L}_k$ is the pullback to $\mathcal{X}_k$ of $L \to X$ with an appropriate multiple of $\mathcal{O}(E_k)$ subtracted. To show that "3 $\implies$ 1," it would, essentially, be enough show that the slope $\mathcal{M}(u_t)$ dominates the Donaldson–Futaki invariants $\mathrm{DF}(\rho_k)$. However, this leads to technical problems that are bypassed by exploiting that $\mathcal{M} \geq \mathcal{D}$, where $\mathcal{D}$ is the Ding functional on $\mathcal{H}_0$ (discussed in

Section 3.4) which behaves better under the approximation procedure above, giving

$$\dot{\mathcal{D}}(u_t) \geq \liminf_{k\to\infty} \mathcal{D}(\rho_k), \tag{4.4}$$

where $\mathcal{D}(\rho_k)$ is the "Ding invariant" originating in [3] (that we shall come back to in Section 4.3.2). Assuming that $\mathcal{X}$ is uniformly Ding stable this shows that "2 $\implies$ 1" (after a twist of the argument which amounts to replacing $\mathcal{D}$ with $\mathcal{D}_\beta$ for $\beta = -(1+\varepsilon)$).

Finally, the equivalence "2 $\iff$ 3" is shown in the first preprint version of [14], using techniques from the Minimal Model Program, inspired by [51] (the proof can, loosely speaking, be interpreted as a non-Archimedean analog of the Kähler–Ricci flow argument in [46] mentioned in connection to Theorem 3.4). The equivalence "2 $\iff$ 3" in the general setup of log Fano varieties is established in [40].

4.2.1. Twisted Kähler–Einstein metrics

The results in [14] apply more generally to Kähler–Einstein metrics twisted by a positive klt current θ, showing that such a metric exists iff $\delta_\theta(X) > 1$, where $\delta_\theta(X)$ is a twisted generalization of the invariant $\delta(X)$ appearing in formula (2.9). This part of the proof does not need any results from the Minimal Model Program (as discussed in the following section). As a corollary, it is also shown that

$$\min\{1, \delta(X)\} = \min\{1, \Gamma(X)\} = R(X), \tag{4.5}$$

where $\Gamma(X)$ is the invariant appearing in Conjecture 2.7 and $R(X)$ denotes the greatest lower bound on the Ricci curvature (independently shown in [28]).

4.3. Non-Archimedean pluripotential theory and the variational formula for $\delta(X)$

The only properties of the geodesic u_t that actually entered into the proof outlined above concerned the multiplier ideal sheaves $\mathfrak{J}(kU)$ of the ω_0-psh function U on $X \times D$, whose stalks consist of all germs of holomorphic functions f such that $|f|^2 e^{-2kU}$ is locally integrable. In turn, the multiplier ideal sheaves $\mathfrak{J}(kU)$ only depend on the Lelong numbers of U on all modifications (blow-ups) of $X \times \mathbb{C}$ (see [23, THM. A] and [14, THM. B.5]). The Lelong numbers in question can be packaged into a function $U(v)$ on the space $[X \times \mathbb{C}]_{\mathrm{div}}$ of all divisorial valuations v on $X \times \mathbb{C}$, as follows. First recall that, by definition, a divisorial valuation v on variety Y is encoded by a positive number c and a prime divisor E_v over Y, i.e., a prime divisor on some blow-up of Y (which may be assumed to be a nonsingular hypersurface). Such a valuation v acts on rational (meromorphic) function $f \in \mathbb{C}(Y)$ by $v(f) := c \operatorname{ord}_{E_v}(f) \in \mathbb{R}$, where $\operatorname{ord}_{E_v}(f)$ denotes the order of vanishing at a generic point of E_v of the pullback of f. Now, if U is, locally, of the form $U = \log|f| + O(1)$ for a holomorphic function, one defines

$$U(v) := -v(f) := -c \operatorname{ord}_{E_v}(f).$$

In the general definition of $U(v)$, one replaces $\operatorname{ord}_{E_v}(f)$ with the Lelong number of U at a generic point p of E_v (i.e., the sup of all $\lambda \in [0, \infty[$ such that $f \leq \lambda \log|z| + O(1)$

with respect to local holomorphic coordinates z centered at p). In this context, Demailly's approximation procedure yields

$$U_k(v) := k^{-1} \max_i \big(-\mathrm{ord}_{E_v}\big(f_i^{(k)}\big)\big) \to U(v), \tag{4.6}$$

where $f_i^{(k)}$ denote local generators of the multiplier ideal sheaf $\mathfrak{J}(kU)$. In fact, after passing to a subsequence (replacing k with 2^k), the sequence U_k is decreasing in k (by the subaditivity of multiplier ideals).

4.3.1. Pluripotential theory on the Berkovich space X_{NA}

In the present setup, the valuative procedure above is initially applied to $Y = X \times \mathbb{C}$. However, exploiting that we are only interested in the value $U(w)$ at a divisorial valuation w on $X \times \mathbb{C}$ which is $\mathbb{C}^*$-invariant, we can identify $[U](w)$ with the function on $u(v)$ on X_{div}, defined by

$$u(v) := U(w), \quad v \in X_{\mathrm{div}},\ w \in (X \times \mathbb{C})_{\mathrm{div}},$$

where w is the Gauss extension of v, defining a $\mathbb{C}^*$-equivariant valuation over $X \times \mathbb{C}$ normalized by $w(\tau) = 1$ (where τ denotes the coordinate on the factor $\mathbb{C}$) [24, SECTION 4.1]. Next, by identifying a valuation v on X with the corresponding non-Archimedean absolute value on $\mathbb{C}(X)$, i.e., with $|\cdot|_v := e^{-v(\cdot)}$, the space X_{div} injects as a dense subspace of the Berkovich analytification X_{NA} of the projective variety X over the field $\mathbb{C}$, induced by the trivially valued absolute value on the ground field $\mathbb{C}$ (locally consisting of all multiplicative seminorms extending the trivially valued absolute value, $|\cdot|_v \equiv 1$, on the field $\mathbb{C}$). The notation X_{NA} (with NA a shorthand for non-Archimedean) is used here to distinguish X_{NA} from X which is the Berkovich analytification in the "Archimedean case," i.e., the case of the standard absolute value $|\cdot|$ on the ground field $\mathbb{C}$.

The topological space X_{NA} has the virtue of being both compact and connected. Moreover, the function $u(v)$ on X_{div} extends to a plurisubharmonic (psh) function on X_{NA} in the sense of [25], denoted by u_{NA}. Indeed, in analogy to the Archimedean case, one can first define $\mathcal{H}(X_{\mathrm{NA}})_0$ to be the space of all functions u_{NA} on X_{NA} induced by test configurations ρ as above, and then define $\mathrm{PSH}(X_{\mathrm{NA}})$ as the space of all functions that can be written as decreasing nets of functions in $\mathcal{H}(X_{\mathrm{NA}})_0$ plus constants (functions in $\mathrm{PSH}(X_{\mathrm{NA}})$ are called L-psh in [25] to emphasize their global dependence on L). There is a Monge–Ampère operator MA on $\mathcal{H}(X_{\mathrm{NA}})$ taking values in the space of probability measures on X_{NA} [24, 25] (which, in a very general setup can be defined in terms of the non-Archimedean generalization of exterior products of curvature forms introduced in [27]). Concretely, $\mathrm{MA}(u_{\mathrm{NA}})$ is a discrete probability measure supported on the valuations $v_i \in X_{\mathrm{div}}$ induced by irreducible components of the central fiber of the test configuration corresponding to u_{NA} [24, SECTION 6.7]. Anyhow, in the present setup, one may directly define MA on $\mathcal{H}(X_{\mathrm{NA}})$ as the differential of the functional

$$\mathcal{E}_{\mathrm{NA}}(u_{\mathrm{NA}}) := \frac{\mathcal{L}^{n+1}}{(n+1)L^n},$$

whose definition mimics formula (3.6) (with $\omega_0 = 0$); this analogy becomes more clear when both $\mathcal{E}$ and $\mathcal{E}_{\mathrm{NA}}$ are expressed in terms of Deligne pairings **[21]**. As in the usual Archimedean setup (Section 3.2), the function $\mathcal{E}_{\mathrm{NA}}$ on $\mathcal{H}(X_{\mathrm{NA}})$ has a unique smallest usc extension to $\mathrm{PSH}(X_{\mathrm{NA}})$; the subspace $\{\mathcal{E}_{\mathrm{NA}} > -\infty\}$ of $\mathrm{PSH}(X_{\mathrm{NA}})$ is denoted by $\mathcal{E}^1(X_{\mathrm{NA}})$ and MA extends to $\mathcal{E}^1(X_{\mathrm{NA}})$, as the differential of the functional $\mathcal{E}^1_{\mathrm{NA}}$.

Remark 4.4. The map $u_t \mapsto u_{\mathrm{NA}}$ from geodesic rays in $\mathcal{E}^1(X,\omega_0)_0$ to the space $\mathcal{E}^1(X_{\mathrm{NA}})_0$, described above, has the property that $\dot{\mathcal{E}}(u_t) \leq \mathcal{E}(u_{\mathrm{NA}})$ and is, in general, not injective. The geodesic rays satisfying $\dot{\mathcal{E}}(u_t) = \mathcal{E}(u_{\mathrm{NA}})$ are precise those called *maximal* in **[14, SECTION 6.4]** and they are in one-to-one correspondence with $\mathcal{E}^1(X_{\mathrm{NA}})$.

4.3.2. The thermodynamical formalism

The non-Archimedean formalism naturally ties in with the thermodynamical formalism (discussed in Section 3). For example, as shown in **[24–26]**, up to a base change of ρ,[2]

$$\mathrm{DF}(\rho) = \mathcal{M}_{\mathrm{NA}}(U_{\mathrm{NA}}) := F_{\mathrm{NA}}\big(\mathrm{MA}(U_{\mathrm{NA}})\big), \tag{4.7}$$

where F_{NA} is the non-Archimedean analog on $\mathcal{P}(X_{\mathrm{NA}})$ of the free energy functional F on $\mathcal{P}(X)$ defined by

$$F_{\mathrm{NA}}(\mu) = -E_{\mathrm{NA}}(\mu) + \mathrm{Ent}_{\mathrm{NA}}(\mu),$$

where the non-Archimedean energy $E_{\mathrm{NA}}(\mu)$ may be defined as a Legendre–Fenchel transform of the functional $\mathcal{E}_{\mathrm{NA}}$ and the non-Archimedean entropy $\mathrm{Ent}_{\mathrm{NA}}(\mu)$ is defined by

$$\mathrm{Ent}_{\mathrm{NA}}(\mu) := \int_{X_{\mathrm{NA}}} A(v)\mu, \quad A(v) := c\big(1 + \mathrm{ord}_{E_v}(K_{Y_v/X})\big) v \in X_{\mathrm{div}}$$

where $A(v)$ is the *log discrepancy*, defined as the greatest lsc extension to X_{NA} of the function on X_{div} defined above. Thus, in contrast to the usual entropy functional on $\mathcal{P}(X)$, the non-Archimedean entropy is a linear functional. Likewise, the "Ding invariant" appearing in formula (4.4) may be expressed as follows in terms of the Legendre–Fenchel transform

$$\mathcal{D}(\rho) = \mathcal{D}_{\mathrm{NA}}(u_{\mathrm{NA}}) := -E^*_{\mathrm{NA}}(u_{\mathrm{NA}}) + \mathrm{Ent}^*_{\mathrm{NA}}(u_{\mathrm{NA}})$$

in analogy with the usual Archimedean setup in Section 3.4. Inequality (4.4) is then obtained by showing that the slope $\dot{\mathcal{D}}(u_t)$ is bounded from below by $\mathcal{D}(u_{\mathrm{NA}})$, which, in turn, equals the limit of $\mathcal{D}(\rho_k)$ (where ρ_k is the test configuration corresponding to U_k defined by formula (4.6)).

As shown in **[26]** (and **[14]** in the general twisted setting) the thermodynamical formalism can be leveraged to prove the following theorem ("1 $\iff$ 3" is shown in **[40]** using the Minimal Model Program):

2 The base change is needed as the righ-hand side in formula (4.7) is one-homogeneous under the natural action of $\mathbb{R}_{>0}$ on X_{NA}, corresponding to a base change of ρ.

Theorem 4.5 ([26]). *Let X be a Fano manifold. The following are equivalent:*

(1) $\delta(X) > 1$.

(2) *X is uniformly K-stable on $\mathcal{E}^1(X_{\mathrm{NA}})$ (i.e., inequality (4.1) extends from $\mathcal{H}(X_{\mathrm{NA}})$ to $\mathcal{E}^1(X_{\mathrm{NA}})$).*

(3) *X is uniformly Ding stable.*

The starting point of the proof of ``1 $\iff$ 2" is the following variational formula for $\delta(X)$ established in [19,26], realizing $\delta(X)$ as a "stability threshold" (where δ_v denotes the Dirac measure at a point v in X_{NA}):

$$\delta(X) = \inf_{v \in X_{\mathrm{div}}} \frac{\mathrm{Ent}_{\mathrm{NA}}(\delta_v)}{E_{\mathrm{NA}}(\delta_v)} = \inf_{v \in X_{\mathrm{NA}}} \frac{\mathrm{Ent}_{\mathrm{NA}}(\delta_v)}{E_{\mathrm{NA}}(\delta_v)} = \inf_{\mu \in \mathcal{P}(X_{\mathrm{NA}})} \frac{\mathrm{Ent}_{\mathrm{NA}}(\mu)}{E_{\mathrm{NA}}(\mu)} \tag{4.8}$$

using, in the second equality, that X_{div} is dense in X_{NA} (together with a semicontinuity argument) and in the last equality (shown in [26]) that $\mathrm{Ent}_{\mathrm{NA}}(\mu)$ and $E_{\mathrm{NA}}(\mu)$ are linear and convex, respectively, on $\mathcal{P}(X_{\mathrm{NA}})$. The function $v \mapsto E_{\mathrm{NA}}(\delta_v)$ is usually denoted by $S(v)$ and can be shown to coincide with the "expected order of vanishing along v" [19]. In terms of the non-Archimedean version of the free energy functional at inverse temperature β, denoted by $F_{\mathrm{NA},\beta}(\mu)$, formula (4.8) yields

$$\delta(X) \geq 1 + \varepsilon \iff \inf_{\mu \in \mathcal{P}(X_{\mathrm{NA}})} F_{\mathrm{NA},-1-\varepsilon}(\mu) \geq 0 \iff \inf_{\mu \in \mathcal{P}(X_{\mathrm{NA}})} \frac{F_{\mathrm{NA}}(\mu)}{E_{\mathrm{NA}}(\mu)} \geq \varepsilon.$$

Finally, expressing $\mu = \mathrm{MA}(U_{\mathrm{NA}})$ for $U_{\mathrm{NA}} \in \mathcal{E}^1(X_{\mathrm{NA}})$, using the non-Archimedean version of the "Calabi–Yau correspondence" (3.5), and invoking the non-Archimedean version of inequalities (4.2) (established in [24]) proves the equivalence ``1 $\iff$ 2". Next, using the Legendre–Fenchel transform, just as in the proof of Theorem 3.4, one sees that uniform K-stability on $\mathcal{E}^1(X_{\mathrm{NA}})$ is equivalent to uniform Ding stability on $\mathcal{E}^1(X_{\mathrm{NA}})$. Finally, ``2 $\iff$ 3" follows from the fact that $\mathcal{D}_{\mathrm{NA}}$ is continuous under approximation of $U_{\mathrm{NA}} \in \mathcal{E}^1(X_{\mathrm{NA}})$ by a decreasing sequence in $\mathcal{H}(X_{\mathrm{NA}})$ (e.g., using multiplier ideal sheaves as in formula (4.6)).

In order to deduce the equivalence ``2 $\iff$ 3" in Theorem 4.3 from the previous theorem, it would be enough to prove the following non-Archimedean analog of the regularization property shown in [16, SECTION 3].

Conjecture 4.6 ([26]). *Given any $u \in \mathcal{E}^1(X_{\mathrm{NA}})$, there exists a sequence of $u_j \in \mathcal{H}(X_{\mathrm{NA}})$ converging weakly to u such that $E_{\mathrm{NA}}(\mathrm{MA}(u_j))$ and $\mathrm{Ent}_{\mathrm{NA}}(\mathrm{MA}(u_j))$ converge to $E_{\mathrm{NA}}(\mathrm{MA}(u))$ and $\mathrm{Ent}_{\mathrm{NA}}(\mathrm{MA}(u))$, respectively.*

Remark 4.7. Combining Theorem 4.3 and Theorem 4.5 reveals that a Fano manifold X is uniformly K-stable iff $\delta(X) > 1$, as first shown in [19,40,41]. More precisely, the "if statement" was shown in [41], where the "only if" statement was also conjectured. The conjecture was then settled in [19]. It should also be pointed out that if one defines δ(X) as a stability threshold (see the first equality in formula (4.8)), then the equivalence between the uniform K-stability of X and the criterion $\delta(X) > 1$ is essentially equivalent to the valuative criterion for uniform K-stability established in [40]. A closely related valuative criterion for K-semistability was established in [47].

4.4. Recent developments

Recently there has been an explosion of exciting further developments. In [48, 50], Theorem 4.3 and its variational proof were extended to general singular (log) Fano varieties using, in particular, the singular version of Theorem 3.2 established in [13]. Moreover, very recently it was shown in [52], using techniques from the Minimal Model Program, that the infimum over X_{div} in formula (4.8) is (when $\delta(X) \leq 1$) attained at some $v \in X_{\mathrm{div}}$. Moreover, any such minimizing divisorial valuation v has the property that associated graded ring is finitely generated and defines a special test configuration ρ for $(X, -K_X)$. In particular, the central fiber of ρ is irreducible (the relation between test configurations, filtrations, and finitely graded rings originates in [62, 67]). In non-Archimedean terms, the result in [52] can be formulated as a regularity result for the minimizer in question, saying that $\delta_v = \mathrm{MA}(U_{\mathrm{NA}})$ for some $U_{\mathrm{NA}} \in \mathcal{H}(X_{\mathrm{NA}})$ (in analogy to the regularity result in Theorem 3.2; cf. the appendix in [40]). As a corollary it is shown in [52] that uniform K-stability is equivalent to K-stability. In fact, these results are shown to hold in the general setup of (log) Fano varieties. When combined with the aforementioned results in [48, 50] this settles the YTD conjecture in the general setting of (log) Fano varieties (the "only if" implication was previously shown in [3]). In another direction, a new variational proof of the uniform YTD conjecture in the nonsingular Fano case is given in [70], using the quantized Ding-functional (leveraging the result in [58] saying that the algebro-geometric invariant $\delta_k(X)$ in formula (2.8) coincides with coercivity threshold of the quantized Ding-functional). More generally, the results in [70] imply that the first equality in formula (4.5) holds without taking the minimum with 1 (by combining [70] with Theorem 3.4)

The variational/non-Archimedean approach is extended to polarized manifolds (X, L) in [49] to show that, if X is uniformly K-stable on $\mathcal{E}^1(X_{\mathrm{NA}})$ (as in Theorem 4.5), then X is geodesically stable and thus by Theorem 4.2 (i.e., by [30]) (X, L) admits a Kähler metric with constant scalar curvature. The converse statement is, however, still open. The complete solution of the uniform YTD conjecture for (X, L) is thus reduced to Conjecture 4.6. An important ingredient in [49] is the notion of maximal geodesic rays u_t introduced in [14] (see Remark 4.4). The theory of maximal geodesic rays is further developed in in [34] and related to singularity types of quasi-psh functions and the Legendre transform construction of geodesic rays introduced in [56]. In [68], analytic variants of stability thresholds are introduced, expressed in terms of singularity types of quasi-psh functions.

5. A NON-ARCHIMEDEAN APPROACH TO GIBBS STABILITY

This final section is a report on joint work in progress with Sébastien Boucksom and Mattias Jonsson to prove the converse of Theorem 2.5 or, more generally, to prove that

$$\lim_{N \to \infty} \mathrm{lct}(D_N) = \delta(X) \tag{5.1}$$

(which, when combined with results in [70], would also settle Conjecture 2.7). The strategy is to adapt the variational approach to the convergence in Conjecture 2.3, discussed in Section 2.3.1, to the non-Archimedean setup. The starting point is the standard valuative expression for the log canonical threshold of a divisor that yields (using the notation in Section 4.3)

$$\mathrm{lct}\,(D_N) = \inf_{v^{(N)}\in[X^N]_{\mathrm{div}}} \frac{A(v^{(N)})}{k^{-1}(v^{(N)}(\det S^{(k)}))} := \frac{N^{-1}A(v^{(N)})}{E^{(N)}_{\mathrm{NA}}(v^{(N)})}, \tag{5.2}$$

where we have introduced the *non-Archimedean energy per particle* as the following function on $[X^N]_{\mathrm{div}}$:

$$E^{(N)}_{\mathrm{NA}}(v^{(N)}) := N^{-1}k^{-1}\big(v^{(N)}(\det S^{(k)})\big) =: -N^{-1}k^{-1}\log\big|\det S^{(k)}\big|_{v^{(N)}}$$

(which is proportional to the negative of the psh function on $[X^N]_{\mathrm{NA}}$ induced by the quasi-psh function $\log\|\det S^{(k)}\|^2$ on X^N). In this notation, formula (5.2) can be viewed as a non-Archimedean analog of Gibbs variational principle (2.18) (since $\mathrm{lct}\,(D_N)-1$ is equal to the one-homogeneous non-Archimedean "N-particle free energy" $-E^{(N)}_{\mathrm{NA}}+N^{-1}A$, normalized by $E^{(N)}_{\mathrm{NA}}$). There are standard inclusions i_N and surjections π_N,

$$i_N : (X_{\mathrm{NA}})^N \hookrightarrow \big[X^N\big]_{\mathrm{NA}}, \quad \pi_N : \big[X^N\big]_{\mathrm{NA}} \twoheadrightarrow (X_{\mathrm{NA}})^N.$$

(the map i_N is, however, not surjective). The non-Archimedean version of the empirical measure δ_N mapping $(X_{\mathrm{NA}})^N$ to $\mathcal{P}(X_{\mathrm{NA}})$ (obtained by replacing X with X_{NA} in formula (2.3)) thus induces a map

$$\pi_N^*\delta_N : \big[X^N\big]_{\mathrm{div}} \to \mathcal{P}(X_{\mathrm{NA}}), \quad v^{(N)} \mapsto N^{-1}\sum_{i=1}^{N}\delta_{(\pi_N(v^{(N)}))_i}.$$

It follows from the results in **[70]** (which are non-Archimedean versions of results in **[11]**) that the restriction of $E^{(N)}_{\mathrm{NA}}$ to $(X_{\mathrm{NA}})^N$ Gamma-converges towards $E_{\mathrm{NA}}(\mu)$ (in analogy with the convergence (2.14)). In particular,

$$\lim_{N\to\infty}\delta_N(v_1,\dots,v_N) = \mu \in \mathcal{P}\big([X]_{\mathrm{NA}}\big) \implies \liminf_{N\to\infty} E^{(N)}_{\mathrm{NA}}\big(i_N(v_1,\dots,v_N)\big) \geq E_{\mathrm{NA}}(\mu). \tag{5.3}$$

Moreover, $N^{-1}A(i_N(v_1,\dots,v_N)) = \int_{X_{\mathrm{NA}}} A(v)\delta_N(v_1,\dots,v_N)$, as follows readily from the definitions. Hence, restricting the inf in formula (5.2) to v_N of the form $v_N = i_N(v,\dots,v)$ for $c \in X_{\mathrm{div}}$ reveals that the lim sup of $\mathrm{lct}\,(D_N)$ is bounded from above by $A(v)/E(\delta_v)$, proving the upper bound in formula (5.1). This proof essentially amounts to a reformulation of the proof of Theorem 2.5 in **[41]** into a non-Archimedean language. But the main point of the non-Archimedean formulation is that it opens the door for a non-Archimedean approach to the missing lower bound. Indeed, it can be shown that

$$\lim_{N_j\to\infty}\big(\pi_{N_j}^*\delta_{N_j}\big)\big(v^{(N_j)}\big) = \mu \in \mathcal{P}\big([X]_{\mathrm{NA}}\big) \implies \liminf_{N_j\to\infty} N_j^{-1}A(v_{N_j}) \geq \mathrm{Ent}_{\mathrm{NA}}(\mu).$$

Hence, all that remains is to establish the following hypothesis for any valuation $v_*^{(N)}$ realizing the infimum in formula (5.2) (which is a non-Archimedean analog of hypothesis (2.19)):

$$\lim_{N_j\to\infty}\big(\pi_{N_j}^*\delta_{N_j}\big)\big(v_*^{(N_j)}\big) = \mu_* \in \mathcal{P}(X_{\mathrm{NA}}) \implies \limsup_{N_j\to\infty} E^{(N_j)}_{\mathrm{NA}}\big(v_*^{(N_j)}\big) \leq E_{\mathrm{NA}}(\mu_*), \tag{5.4}$$

(by (5.3) the opposite inequality holds). Indeed, if the hypothesis holds then we get

$$\inf_{\mu\in\mathcal{P}([X]_{\mathrm{NA}})}\frac{\mathrm{Ent}_{\mathrm{NA}}(\mu)}{E_{\mathrm{NA}}(\mu)}\leq\liminf_{N\to\infty}\mathrm{lct}\,(D_N)\leq\limsup_{N\to\infty}\mathrm{lct}\,(D_N)\leq\inf_{v\in[X]_{\mathrm{div}}}\frac{\mathrm{Ent}_{\mathrm{NA}}(\delta_v)}{E_{\mathrm{NA}}(\delta_v)},\quad(5.5)$$

which, when combined with identity (4.8), yields the desired formula (5.1).

It remains to verify the inequality in the hypothesis above. It would be enough to establish the following *"restriction hypothesis"*: the minimizer $v_*^{(N)}$ can, asymptotically, be taken to be of the form $i_N(v_*, v_*, \dots, v_*)$ for a fixed divisorial valuation v_* on X:

$$\exists v_*\in X_{\mathrm{div}}\quad\text{such that }\liminf_{N\to\infty}\mathrm{lct}\,(D_N)=\liminf_{N\to\infty}\frac{N^{-1}A(i_N(v_*,v_*,\dots,v_*))}{E^{(N)}_{\mathrm{NA}}(i_N(v_*,v_*,\dots,v_*))}.$$

Indeed, it follows from the convergence of Fekete points on X_{NA} in **[21]** that

$$\lim_{N\to\infty}E^{(N)}_{\mathrm{NA}}\big(i_N(v,v,\dots,v)\big)=E(\delta_v)\quad(5.6)$$

for *any* divisorial valuation v on X (or, more generally, for any nonpluripolar point v in X_{NA}). In particular, it then follows that any v_* satisfying the "restriction hypothesis" above computes $\delta(X)$. For instance, it can be verified that the "restriction hypothesis" does hold for log Fano curves (X,Δ). Anyhow, for any given divisorial valuation v on X, formula (5.6) yields a "microscopic" formula for the non-Archimedean free-energy $F_{\mathrm{NA}}(\delta_v)$ (coinciding with the invariant $\beta(v)$ introduced in **[40]**) of independent interest:

$$F_{\mathrm{NA}}(\delta_v):=-E(\delta_v)+A(\delta_v)=\lim_{N\to\infty}\big(-E^{(N)}_{\mathrm{NA}}\big(i_N(v,v,\dots,v)\big)+N^{-1}A\big(i_N(v,v,\dots,v)\big)\big).$$

In particular, if ρ is a given test configuration, whose central fiber $\mathcal{X}_0$ is irreducible, this gives a new formula for the Donaldson–Futaki invariant $\mathrm{DF}(\rho)$, using that $\mathrm{DF}(\rho)=F_{\mathrm{NA}}(\delta_v)$, where v is the divisorial valuation on X corresponding to $\mathcal{X}_0$. Comparing with the formula for $\mathrm{DF}(\rho)$ in terms of Chow weights thus suggests that the divisorial valuation $i_N(v,v,\dots,v)$ on X^N, attached to v, plays the role of the one-parameter subgroup of $\mathrm{GL}(N,\mathbb{C})$ attached to ρ. Accordingly, the "restriction hypothesis" is an analog of the Hilbert–Mumford criterion for stability in Geometric Invariant Theory.

Finally, coming back to the statistical mechanical point of view discussed in Section 2.3, it may be illuminating to point out that the "restriction hypothesis" essentially amounts to a concentration phenomenon which may be pictured as follows. Let us decrease the inverse temperature β from a given positive value towards the critical negative inverse temperature β_N where $\mathcal{Z}_N(\beta)=\infty$. As β changes sign from positive to negative, all the particles start to mutually attract each other and, as $\beta\to\beta_N$, a large number of particles concentrate along the subvariety of X defined by the center of the valuation v_*.

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REFERENCES

[1] T. Aubin, Equations du type Monge–Ampère sur les varietes Kahleriennes compactes. *Bull. Sci. Math. (2)* **102** (1978), no. 1, 63–95.

[2] R. J. Berman, A thermodynamical formalism for Monge–Ampère equations, Moser–Trudinger inequalities and Kähler–Einstein metrics. *Adv. Math.* **248** (2013), 1254.

[3] R. J. Berman, K-polystability of Q-Fano varieties admitting Kähler–Einstein metrics. *Invent. Math.* **203** (2016), no. 3, 973–1025.

[4] R. J. Berman, Large deviations for Gibbs measures with singular Hamiltonians and emergence of Kähler–Einstein metrics. *Comm. Math. Phys.* **354** (2017), no. 3, 1133–1172.

[5] R. J. Berman, Kähler–Einstein metrics, canonical random point processes and birational geometry. In *Algebraic Geometry, Salt Lake City 2015 (Part 1)*, pp. 29–74, Proc. Sympos. Pure Math. 97.1, Amer. Math. Soc., Providence, RI, 2018.

[6] R. J. Berman, An invitation to Kähler–Einstein metrics and random point processes. *Surv. Differ. Geom.* **23** (2018), 35–87.

[7] R. J. Berman, Statistical mechanics of interpolation nodes, pluripotential theory and complex geometry. *Ann. Polon. Math.* **123** (2019), 71–153.

[8] R. J. Berman, Kähler–Einstein metrics and Archimedean zeta functions. 2021, arXiv:2112.04791.

[9] R. J Berman, The probabilistic vs the quantization approach to Kähler–Einstein geometry. 2021, arXiv:2109.06575.

[10] R. J. Berman and B. Berndtsson, Convexity of the K-energy on the space of Kähler metrics. *J. Amer. Math. Soc.* **30** (2017), 1165–1196.

[11] R. J. Berman and S. Boucksom, Growth of balls of holomorphic sections and energy at equilibrium. *Invent. Math.* **181** (2010), no. 2, 337.

[12] R. J. Berman, S. Boucksom, V. Guedj, and A. Zeriahi, A variational approach to complex Monge–Ampère equations. *Publ. Math. Inst. Hautes Études Sci.* **117** (2013), 179–245.

[13] R. J. Berman, P. Eyssidieu, S. Boucksom, V. Guedj, and A. Zeriahi, Kähler–Einstein metrics and the Kähler–Ricci flow on log Fano varieties. *J. Reine Angew. Math.* (2016), published online.

[14] R. J. Berman, S. Boucksom, and M. Jonsson, A variational approach to the Yau–Tian–Donaldson conjecture. *J. Amer. Math. Soc.* **34** (2021), 605–652. arXiv:1509.04561.

[15] R. J. Berman, T. Collins, and D. Persson, The AdS/CFT correspondence and emergent Sasaki–Einstein metrics. *Nat. Commun.* (to appear), arXiv:2008.12004.

[16] R. J. Berman, T. Darvas, and C. H. Lu, Convexity of the extended K-energy and the long time behavior of the Calabi flow. *Geom. Topol.* **21** (2017), no. 5, 2945–2988.

[17] R. J. Berman, T. Darvas, and C. H. Lu, Regularity of weak minimizers of the K-energy and applications to properness and K-stability. *Ann. Sci. Éc. Norm. Supér.* **53** (2020), no. 2, 267–289.

[18] R. J. Berman and M. Önnheim, Propagation of chaos, Wasserstein gradient flows and toric Kähler–Einstein metrics. *Ann. PDE* **11** (2018), no. 6, 1343–1380.

[19] H. Blum and M. Jonsson, Thresholds, valuations, and K-stability. *Adv. Math.* **365** (2020). pp. 57

[20] S. Boucksom, Variational and non-Archimedean aspects of the Yau–Tian–Donaldson conjecture. In *Proc. of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pp. 591–617, World Sci. Publ., Hackensack, NJ, 2018.

[21] S. Boucksom and D. Eriksson, Spaces of norms, determinant of cohomology and Fekete points in non-Archimedean geometry. *Adv. Math.* **378** (2021), 12.

[22] S. Boucksom, P. Essidieux, V. Guedj, and A. Zeriahi, Monge–Ampère equations in big cohomology classes. *Acta Math.* **205** (2010), no. 2, 199–262.

[23] S. Boucksom, C. Favre, and M. Jonsson, Valuations and plurisubharmonic singularities. *Publ. Res. Inst. Math. Sci.* **44** (2008), no. 2, 449–494.

[24] S. Boucksom, T. Hisamoto, and M. Jonsson, Uniform K-stability, Duistermaat–Heckman measures and singularities of pairs. *Ann. Inst. Fourier* **67** (2017), 743–841.

[25] S. Boucksom and M. Jonsson, Global pluripotential theory over a trivially valued field. 2018, arXiv:1801.08229.

[26] S. Boucksom and M. Jonsson, A non-Archimedean approach to K-stability. 2018, arXiv:1805.11160v1.

[27] A. Chambert-Loir and A. Ducros, Formes différentielles réelles et courants sur les espaces deBerkovich. 2012, arXiv:1204.6277.

[28] I. A. Cheltsov, Y. A. Rubinstein, and K. Zhang, Basis log canonical thresholds, local intersection estimates, and asymptotically log del Pezzo surfaces. *Selecta Math. (N.S.)* **25** (2019), 34. arXiv:1807.07135v2.

[29] X. X. Chen, On the lower bound of the Mabuchi energy and its application. *Int. Math. Res. Not.* **2000** (2000), no. 12, 607–623.

[30] X. X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics (II)—Existence results. 2018, arXiv:1801.00656.

[31] X. X. Chen, S. Donaldson, and S. Sun, Kähler–Einstein metrics on Fano manifolds, I, II, III. *J. Amer. Math. Soc.* **28** (2015).

[32] T. Darvas, The Mabuchi geometry of finite energy classes. *Adv. Math.* **285** (2015), 182–219.

[33] T. Darvas and Y. Rubinstein, Tian's properness conjectures and Finsler geometry of the space of Kähler metrics. *J. Amer. Math. Soc.* **30** (2017), 347–387.

[34] T. Darvas and M. Xia, The closures of test configurations and algebraic singularity types. 2020, arXiv:2003.04818.

[35] J.-P. Demailly, Regularization of closed positive currents and intersection theory. *J. Algebraic Geom.* **1** (1992), 361–409.

[36] R. Dervan, Uniform stability of twisted constant scalar curvature Kähler metrics. *Int. Math. Res. Not. IMRN* (2016), no. 15, 4728–4783.

[37] S. K. Donaldson, Scalar curvature and projective embeddings. I. *J. Differential Geom.* **59** (2001), no. 3, 479–522.

[38] S. K. Donaldson, Stability of algebraic varieties and Kähler geometry. In *Algebraic geometry: Salt Lake City 2015*, pp. 199–221, Proc. Sympos. Pure Math. 97.1, Amer. Math. Soc., Providence, RI, 2018.

[39] R. Dujardin, Theorie globale de pluripotentiel, equidistributions et processes ponctuels [d'après Berman, Boucksom, Witt Nyström,...]. *Séminaire Bourbaki* 2018–2019, no. 1153. http://www.bourbaki.ens.fr/TEXTES/Exp1153-Dujardin.pdf

[40] K. Fujita, A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties. *J. Reine Angew. Math.* **751** (2019), 309–338.

[41] K. Fujita and Y. Odaka, On the K-stability of Fano varieties and anticanonical divisors. *Tohoku Math. J. (2)* **70** (2018), no. 4, 511–521.

[42] V. Guedj and A. Zeriahi, The weighted Monge–Ampère energy of quasiplurisubharmonic functions. *J. Funct. Anal.* **250** (2007), 442–482.

[43] J. Hultgren, Permanental point processes on real tori, theta functions and Monge–Ampère equations. *Ann. Fac. Sci. Toulouse Math. (6)* **28** (2019), no. 1, 11–65.

[44] J. Kollár, Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, pp. 221–287, Proc. Sympos. Pure Math. 62, Part 1, Amer. Math. Soc., Providence, RI, 1997.

[45] J. Kollár, The structure of algebraic varieties. In *Proceedings of ICM, Seoul, 2014, Vol. I.*, pp. 395–420, Kyung Moon SA, 2014, http://www.icm2014.org/en/vod/proceedings.html.

[46] H. Li, On the lower bound of the K-energy and F-functional. *Osaka J. Math.* **45** (2008), no. 1, 253–264.

[47] C. Li, K-semistability is equivariant volume minimization. *Duke Math. J.* **166** (2017), no. 16, 3147–3218.

[48] C. Li, G-uniform stability and Kähler–Einstein metrics on Fano varieties, 2019. arXiv:1907.09399.

[49] C. Li, Geodesic rays and stability in the cscK problem. *Ann. Sci. Éc. Norm. Supér.* (to appear), arXiv:2001.01366.

[50] C. Li, G. Tian, and F. Wang, The uniform version of Yau–Tian–Donaldson conjecture for singular Fano varieties. 2019, arXiv:1903.01215.

[51] C. Li and C. Xu, Special test configuration and K-stability of Fano varieties. *Ann. of Math.* **180** (2014), no. 1, 197–232.

[52] Y. Liu, C. Xu, and Z. Zhuang, Finite generation for valuations computing stability thresholds and applications to K-stability. 2021, arXiv:2102.09405.

[53] T. Mabuchi, K-energy maps integrating Futaki invariants. *Tohoku Math. J. (2)* **38** (1986), no. 4, 575–593.
[54] Y. Odaka, A generalization of the Ross–Thomas slope theory. *Osaka J. Math.* **50** (2013), no. 1, 171–185.
[55] D. H. Phong and J. Sturm, Test configurations for K-stability and geodesic rays. *J. Symplectic Geom.* **5** (2007), no. 2, 221–247.
[56] J. Ross and D. Witt Nyström, Analytic test configurations and geodesic rays. *J. Symplectic Geom.* **12** (2014), no. 1, 125–169.
[57] Y. A. Rubinstein, Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics. *Adv. Math.* **218** (2008), 1526–1565.
[58] Y. A. Rubinstein, G. Tian, and K. Zhang, Basis divisors and balanced metrics. *J. Reine Angew. Math.* **778** (2021), 171–218. 2020, arXiv:2008.08829.
[59] Z. Sjöström Dyrefelt, K-semistability of cscK manifolds with transcendental cohomology class. *J. Geom. Anal.* **28** (2018), 2927–2960.
[60] J. Stoppa, K-stability of constant scalar curvature Kähler manifolds. *Adv. Math.* **221** (2009), no. 4, 1397–1408.
[61] S. Sun, Degenerations and moduli spaces in Kähler geometry. In *Proceedings of the International Congress of Mathematicians (ICM 2018)*, pp. 993–1012, World Sci. Publ., Hackensack, NJ, 2018.
[62] G. Székelyhidi, Filtrations and test configurations. With an appendix by S. Boucksom. *Math. Ann.* **362** (2015), 451–484.
[63] G. Tian, On Calabi's conjecture for complex surfaces with positive first Chern class. *Invent. Math.* **101** (1990), no. 1, 101–172.
[64] G. Tian, Kähler-Einstein metrics on algebraic manifolds. In *Transcendental methods in algebraic geometry (Cetraro, 1994)*, pp. 143–185, Lecture Notes in Math. 1646, Fond. CIME/CIME Found. Subser., Springer, Berlin, 1996.
[65] G. Tian, Kähler–Einstein metrics with positive scalar curvature. *Invent. Math.* **130** (1997), no. 1, 1–37.
[66] X. Wang, Height and GIT weight. *Math. Res. Lett.* **19** (2012), no. 04, 909–926.
[67] D. Witt Nyström, Test configurations and Okounkov bodies. *Compos. Math.* **148** (2012), 1736–1756.
[68] M. Xia, Pluripotential-theoretic stability thresholds. 2020, arXiv:2012.12039.
[69] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Comm. Pure Appl. Math.* **31** (1978), no. 3, 339–411.
[70] K. Zhang, A quantization proof of the uniform Yau–Tian–Donaldson conjecture. 2021, arXiv:2102.02438.

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