# LAGRANGE MULTIPLIER FUNCTIONALS AND THEIR APPLICATIONS IN SYMPLECTIC GEOMETRY AND STRING TOPOLOGY

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**ABSTRACT**

This note discusses the role of Lagrange multiplier functionals in mathematics and physics. The main focus is on Rabinowitz' action functional and its usage in symplectic geometry, as well as recent applications in string topology and the study of closed geodesics.

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## 1. INTRODUCTION

The purpose of this note is to tell the story of how an old and simple idea—Lagrange multipliers—has led to new insights in symplectic geometry and loop space topology.

The beginning of our story is the observation by Joseph-Louis Lagrange in 1804 [58] that critical points of a functional $f(x)$ subject to a constraint $h(x) = 0$ correspond to unconstrained critical points of the function $F(x, \lambda) = f(x) - \lambda h(x)$ depending on a Lagrange multiplier $\lambda$. In modern terms, $f : X \to \mathbb{R}$ and $h : X \to V$ should be sufficiently smooth maps, where $X$ is a Banach manifold and $V$ a Banach space. Denoting by $\langle \cdot, \cdot \rangle$ the canonical pairing between $V$ and its topological dual $V^*$, we consider the *Lagrange multiplier functional*

$$F : X \times V^* \to \mathbb{R}, \quad F(x, \lambda) = f(x) - \big\langle \lambda, h(x) \big\rangle.$$

Then $(x, \lambda)$ is a critical point of $F$ if and only if

$$df(x) = \big\langle \lambda, dh(x) \big\rangle \quad \text{and} \quad h(x) = 0.$$

Assuming that 0 is a regular value of $h$, so that $Z = h^{-1}(0) \subset X$ is a Banach submanifold, this is equivalent to $x$ being a critical point of the restriction $f|_Z$. The Lagrange multiplier $\lambda$ at $x$ is uniquely determined by the first equation. Although it was introduced as an auxiliary parameter, the Lagrange multiplier often has mathematical or physical meaning.

**Example 1.1** (Eigenvalues). Let $X$ be a complex Hilbert space and $A : X \to X$ a self-adjoint bounded linear operator. Consider the functions

$$f, h : X \to \mathbb{R}, \quad f(x) = \langle x, Ax \rangle, \quad h(x) = \|x\|^2 - 1.$$

Then the critical points of the restriction of $f$ to the unit sphere $S = h^{-1}(0)$ correspond to solutions $(x, \lambda) \in X \times \mathbb{R}$ of the equations $\|x\| = 1$ and $Ax = \lambda x$, so the Lagrange multiplier $\lambda \in \mathbb{R}$ is an eigenvalue of $A$ with eigenvector $x$. If $A$ is compact (e.g., if $X$ is finite dimensional), then $f$ attains its maximum and minimum on $S$ and it follows that $\|A\|$ or $-\|A\|$ is an eigenvalue.

The Hessian of $F$ at a critical point $(x, \lambda)$ is given by

$$\operatorname{Hess} F(x, \lambda) = \begin{pmatrix} \operatorname{Hess} f(x) & dh(x)^* \\ dh(x) & 0 \end{pmatrix}.$$

If $X$ and $V$ are finite dimensional, it follows that the Hessians $\operatorname{Hess} F(x, \lambda)$ and $\operatorname{Hess} f|_Z(x)$ have the same nullity and signature (the number of positive minus the number of negative eigenvalues). These relations also hold in some infinite-dimensional cases where nullity and signature can be defined; one such case arises for Hamiltonian systems where the role of the signature is played by the Conley–Zehnder index, see Section 2.

We see in particular that the Hessian of $F$ is never positive or negative definite, so its critical points cannot be detected by direct maximization or minimization methods and one needs to resort to indirect variational methods. Of particular relevance for this note will be *Morse homology* (see, e.g., [70, 77]). This is the homology of the chain complex whose generators are critical points of $F$ and whose differential counts gradient trajectories $(x,\lambda):\mathbb{R}\to X\times V^*$ between critical points. Based on the preceding discussion, we expect that the Morse homology of $F$ equals the Morse homology of $f|_Z$ if both are graded by the signature rather than the Morse index. However, even in finite dimensions it is not obvious that both Morse homologies are defined, and in addition equal, due to the possible escape of gradient trajectories to infinity. This issue will be a recurring theme in this note, which is structured as follows.

Section 2 focuses on a specific Lagrange multiplier functional, the Rabinowitz action functional, and its applications in symplectic geometry. Section 3 presents some recent applications of the ideas from Section 2 in string topology. Section 4 discusses some further occurrences of Lagrange multiplier functionals in mathematics and physics. Besides established results, I will also discuss some work in progress, as well as open questions.

## 2. RABINOWITZ FLOER HOMOLOGY

In this section we will focus on one particular Lagrange multiplier functional, the Rabinowitz action functional, and discuss properties and applications of the corresponding Floer homology. For more details and background, see the original references or the survey by P. Albers and U. Frauenfelder [9].

### 2.1. Definition and basic properties

Let $(W,\lambda)$ be a *Liouville manifold* of dimension $2n$, i.e., a connected manifold with a 1-form such that $\omega=d\lambda$ is symplectic and $W$ is exhausted by compact sets $W_k$ with smooth boundary such that $\lambda|_{\partial W_k}$ is a positive contact form. Examples of Liouville manifolds include $\mathbb{C}^n$, cotangent bundles, and, more generally, Stein and Weinstein manifolds (see [18]).

To a 1-periodic time-dependent Hamilton function $H:S^1\times W\to\mathbb{R}$, we associate its Hamiltonian vector field $X_{H_t}$ by $dH_t=\omega(\cdot,X_{H_t})$, where $H_t=H(t,\cdot)$. Then 1-periodic solutions $x:S^1\to W$ of the Hamiltonian system $\dot x=X_{H_t}(x)$ are the critical points of the Hamiltonian action

$$\mathcal{A}_H: C^\infty(S^1,W)\to\mathbb{R},\quad \mathcal{A}_H(x)=\int_x\lambda-\int_0^1 H(t,x)dt.$$

Assume now that $H:W\to\mathbb{R}$ is time-independent. Then we have conservation of energy and it is natural to look for solutions of prescribed energy rather than prescribed period. For this, suppose that 0 is a regular value of $H$ and consider the *Rabinowitz action functional*

$$\mathcal{A}^H: C^\infty(S^1,W)\times\mathbb{R}\to\mathbb{R},\quad \mathcal{A}^H(x,\eta)=\int_x\lambda-\eta\int_0^1 H(x)dt.$$

Its critical points satisfy the equations

$$\dot x=\eta X_H(x),\qquad \int_0^1 H(x)dt=0.$$

By the first equation, $H(x(t))$ is constant and, by the second equation, this constant equals zero, so the critical point equations become

$$\dot x=\eta X_H(x),\qquad H\big(x(t)\big)\equiv 0.$$

Critical points of $\mathcal{A}^H$ thus correspond to orbits $t \mapsto x(t/\eta)$ of $X_H$ of period $\eta$ and energy 0 (if $\eta > 0$), such orbits run backwards (if $\eta < 0$), or to constant loops on $H^{-1}(0)$ (if $\eta = 0$). As we will see in Section 3, the appearance of solutions with negative $\eta$ is responsible for an additional symmetry of the corresponding Floer homology.[1] In 1978, P. Rabinowitz used this functional to prove existence of periodic orbits on star-shaped energy hypersurfaces in $\mathbb{C}^n$ [66].[2]

To define the Floer homology of $\mathcal{A}^H$, we pick an $\omega$-compatible almost complex structure $J$ on $W$ and equip $C^\infty(S^1, W) \times \mathbb{R}$ with the metric

$$m_{(x,\eta)}\big((\hat{x}_1, \hat{\eta}_1), (\hat{x}_2, \hat{\eta}_2)\big) = \int_0^1 \omega(\hat{x}_1, J\hat{x}_2)dt + \hat{\eta}_1\hat{\eta}_2.$$

Then gradient flow lines of $\mathcal{A}^H$ are maps $(u, \eta) : \mathbb{R} \to C^\infty(S^1, W) \times \mathbb{R}$, satisfying

$$\partial_s u + J(u)\big(\partial_t u - \eta X_H(u)\big) = 0, \quad \partial_s \eta + \int_0^1 H(u)dt = 0, \tag{2.1}$$

where $(s, t)$ are the coordinates on $\mathbb{R} \times S^1$. This is a coupled system of an elliptic PDE and a nonlocal ODE. Its solutions exhibit three potential sources of noncompactness: explosion of the gradient of $u$, which is excluded by exactness of $\omega$; escape of $u$ to infinity, which can be prevented by suitable conditions on $J$; and escape of the Lagrange multiplier $\eta$ to $\pm\infty$. To prevent the latter, we need to impose some geometric condition on the hypersurface $\Sigma = H^{-1}(0)$.

A hypersurface $\Sigma \subset W$ is of *restricted contact type* if it admits a contact form $\alpha$ such that $\alpha - \lambda|_\Sigma$ is exact (for $H^1(\Sigma; \mathbb{R}) = 0$ this agrees with A. Weinstein's contact type condition [78]). We also assume that $\Sigma$ is connected and bounds a compact subset. Then it admits a (nonunique) *defining Hamiltonian*, i.e., a smooth function $H : W \to \mathbb{R}$ which is constant outside a compact set such that $H^{-1}(0) = \Sigma$ and $X_H = R$ along $\Sigma$, where $R$ is the Reeb vector field of $\alpha$.

**Theorem 2.1** ([19]). (a) *Given a hypersurface $\Sigma \subset W$ of restricted contact type and a defining Hamiltonian $H$, the Floer homology* $\mathrm{FH}_*(\mathcal{A}^H)$ *is well defined; it is independent of the defining Hamiltonian and called the* Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma)$.

(b) *For a smooth family of hypersurfaces $\Sigma_s$, $s \in [0, 1]$, of restricted contact type there is a canonical isomorphism* $\mathrm{RFH}_*(\Sigma_0) \cong \mathrm{RFH}_*(\Sigma_1)$.

(c) *If a hypersurface $\Sigma \subset W$ of restricted contact type is displaceable from itself by a Hamiltonian isotopy, then* $\mathrm{RFH}_*(\Sigma) = 0$.

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1 Though of course unrelated, this phenomenon is reminiscent of the appearance of negative energy solutions in Dirac's equation.

2 As H. Hofer pointed out, the functional appeared already in a 1976 article by J. Moser [61], where he concluded that the corresponding variational principle "is certainly not suitable for an existence proof."

**Remark 2.2** (Grading and coefficients). (i) For simplicity, we will assume throughout this note that *the first Chern class of $TW$ vanishes* and we have made choices so that $\mathrm{RFH}_*(\Sigma)$ and all other Floer homologies below are $\mathbb{Z}$-graded by their Conley–Zehnder indices.

(ii) Coefficients are in a principal ideal domain $R$, which will sometimes be specialized to $\mathbb{Z}$ or a field or generalized to twisted coefficients.

(iii) The notation $\mathrm{RFH}_*(\Sigma)$ is chosen to emphasize the dependence on $\Sigma$, but it a priori depends also on the ambient Liouville manifold $(W, \lambda)$; we will return to this question below.

If $\Sigma$ carries no periodic orbits, then the only generators of $\mathrm{RFH}_*(\Sigma)$ are the constant loops on $\Sigma$ with Lagrange multiplier $\eta = 0$ and it follows that $\mathrm{RFH}_*(\Sigma) \cong H^{n-*}(\Sigma)$. In view of Theorem 2.1(c), such a hypersurface cannot be displaceable. This implies the Weinstein conjecture for hypersurfaces of restricted contact type in Liouville manifolds where all compact sets are displaceable such as $\mathbb{C}^n$, subcritical Stein manifolds, or products of a Liouville manifold with $\mathbb{C}$ (see [76,78]).

### 2.2. Stability and Mañé's critical values

One may wonder whether Theorem 2.1 can be extended to a larger class of hypersurfaces $\Sigma$. The preceding discussion shows that some condition on $\Sigma$ is needed: for example, it cannot apply to closed hypersurfaces in $\mathbb{C}^n$ without periodic orbits as constructed in [44,46].

In [21], Theorem 2.1 is generalized to the case that $(W, \omega)$ is a geometrically bounded symplectic manifold with $\omega|_{\pi_2(W)} = 0$, and the hypersurface $\Sigma \subset W$ and homotopy $\Sigma_s$ are *tame and stable*. Here *stability* was introduced by Hofer and Zehnder [53] as a condition, generalizing contact type, under which existence results for periodic orbits continue to hold; it appeared again in [14,29] as the hypothesis for compactness in symplectic field theory.

An intriguing class of Hamiltonian systems is given by Hamiltonians $H(q,p) = \frac{1}{2}|p|^2 + U(q)$ on a cotangent bundle $W = T^*M$ with a twisted symplectic form

$$\omega = dp \wedge dq + \tau^*\sigma,$$

where $\tau : T^*M \to M$ is the projection and $\sigma$ is a closed 2-form on $M$ whose physical significance is that of a *magnetic field*. It has long been known that the dynamics on a level set $\Sigma_k = H^{-1}(k)$ can change drastically with the level $k$, even in the case $U = 0$ where all level sets are diffeomorphic (see [45] and the references therein). A famous example is that of a hyperbolic surface $M$ with its area form $\sigma$ and $U = 0$: Here $\Sigma_k$ is foliated by contractible periodic orbits for $k < 1/2$, all periodic orbits on $\Sigma_k$ are noncontractible for $k > 1/2$, and $\Sigma_{1/2}$ (the horocycle flow) does not possess any periodic orbits. The value $1/2$ at which the dynamics changes is the *Mañé critical value*, and at this value also the geometric type of the hypersurfaces $\Sigma_k$ changes: above $1/2$ they are of contact type, below $1/2$ they are stable and tame but not of contact type, and $\Sigma_{1/2}$ is unstable. The Rabinowitz Floer homology of $\Sigma_k$ is well defined and zero for $k < 1/2$, it is well defined and nonzero for $k > 1/2$, and it is

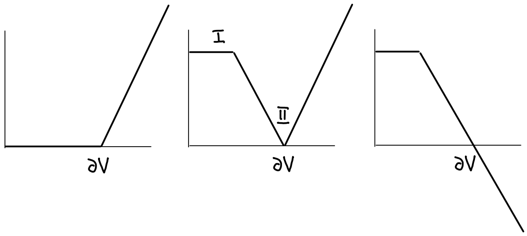


**FIGURE 1**
Three shapes of Hamiltonans.

undefined for $k = 1/2$. Using the generalization of Theorem 2.1, it is shown in [21] that this picture persists for large classes of magnetic systems in arbitrary dimension.

### 2.3. Relation to symplectic homology

Let us return to the setup in Section 2.1, so $\Sigma$ is a hypersurface of restricted contact type in a Liouville manifold $(W, \lambda)$. Recall that, by assumption, $\Sigma = \partial V$ for a compact subdomain $V \subset W$. After modifying $\lambda$ near $\Sigma$, we may assume that $\lambda|_{\partial V}$ is a contact form, so that $(V, \lambda)$ is a *Liouville domain*. It is shown in [20] that $\mathrm{RFH}_*(\partial V)$ depends only on the completion $\hat{V} = V \cup [1, \infty) \times \partial V$ of $V$ (which is a Liouville manifold with the 1-form $\widehat{\lambda}$ that equals $\lambda$ on $V$ and $r\lambda|_{\partial V}$ on $[1, \infty) \times \partial V$, with $r$ the coordinate on $[1, \infty)$). Moreover, $\mathrm{RFH}_*(\partial V)$ is closely related to another invariant of $V$ that we now recall.

*Symplectic homology* was introduced in 1994 by A. Floer and H. Hofer [43]. We will use the version defined by C. Viterbo [76] as the direct limit

$$\mathrm{SH}_*(V) = \varinjlim \mathrm{FH}_*(H)$$

over Hamiltonians $H : \hat{V} \to \mathbb{R}$ that are zero on $V$ and linearly increasing in $r$ outside a compact set, as shown (up to smoothing) on the left of Figure 1. Dualizing, we obtain symplectic cohomology as the inverse limit

$$\mathrm{SH}^*(V) = \varprojlim \mathrm{FH}^*(H) = \varprojlim \mathrm{FH}_{-*}(-H).$$

These groups have refinements where the action is restricted to some interval $(a, b)$,

$$\mathrm{SH}_*^{(a,b)}(V) = \varinjlim \mathrm{FH}_*^{(a,b)}(H), \quad \mathrm{SH}^*_{(a,b)}(V) = \varprojlim \mathrm{FH}^*_{(a,b)}(H) = \varprojlim \mathrm{FH}_{-*}^{(-b,-a)}(-H).$$

In [20], a new *V-shaped symplectic homology* was introduced as the direct–inverse limit

$$\check{\mathrm{SH}}_*(V) = \varinjlim_b \varprojlim_a \check{\mathrm{SH}}_*^{(a,b)}(V), \quad \check{\mathrm{SH}}_*^{(a,b)}(V) = \varinjlim_H \mathrm{FH}_*^{(a,b)}(H),$$

where the second direct limit is taken over "V-shaped" Hamiltonians $H : \hat{V} \to \mathbb{R}$ as shown (up to smoothing) in the middle of Figure 1. For any given $-\infty < a < b < \infty$ and sufficiently large $H$, the orbits in group I have action outside $(a, b)$, so $\check{\mathrm{SH}}_*(V)$ is generated by the orbits in group II, which are in one-to-one correspondence with generators of $\mathrm{RFH}_*(\partial V)$. This observation combined with a technical tour de force leads to

**Theorem 2.3** ([20]). *For each Liouville domain, we have*

$$\mathrm{S\check{H}}_*(V) = \mathrm{RFH}_*(\partial V).$$

*Moreover, this group fits into a commuting diagram with exact row*

$$\begin{tikzcd}
\dots \mathrm{SH}^{-*}(V) \arrow[r, "\varepsilon"] \arrow[d] & \mathrm{SH}_*(V) \arrow[r, "\iota"] & \mathrm{RFH}_*(\partial V) \arrow[r, "\pi"] & \mathrm{SH}^{1-*}(V) \dots \\
H^{n-*}(V, \partial V) \arrow[r, "e"] & H^{n-*}(V). \arrow[u] & &
\end{tikzcd} \tag{2.2}$$

In this diagram, $e$ is the canonical map in the long exact sequence of the pair $(V, \partial V)$ and the vertical arrows correspond to the action zero (constant loop) part. It allows the computation of Rabinowitz Floer homology in terms of symplectic homology and singular cohomology. The following example will play a fundamental role in Section 3.

**Example 2.4** (Cotangent bundles). Let $T^*M$ be the cotangent bundle of a closed manifold $M$ with its canonical Liouville form $\lambda = p\,dq$. Its unit disk bundle $D^*M = \{(q,p) \in T^*M \mid |p| \le 1\}$ with respect to some Riemannian metric is a Liouville domain with boundary $S^*M = \{(q,p) \in T^*M \mid |p| = 1\}$. Viterbo's isomorphism (proved by a joint effort of many people, see [2, 4, 55, 68, 75])

$$\mathrm{SH}_*(D^*M) \cong H_*(\Lambda) \tag{2.3}$$

expresses its symplectic homology in terms of the singular homology (with suitably twisted coefficients) of the loop space $\Lambda = C^\infty(S^1, M)$. Hence diagram (2.2) becomes

$$\begin{tikzcd}
\dots H^{-*}(\Lambda) \arrow[r, "\varepsilon"] \arrow[d] & H_*(\Lambda) \arrow[r, "\iota"] & \mathrm{RFH}_*(S^*M) \arrow[r, "\pi"] & H^{1-*}(\Lambda) \dots \\
H^{-*}(M) \arrow[r, "e"] & H^{n-*}(M), \arrow[u] & &
\end{tikzcd} \tag{2.4}$$

where $e$ is the canonical map in the Gysin sequence of the sphere bundle $S^*M \to M$. We see that the map $e$ (and therefore $\varepsilon$) lives only in degree zero and multiplies the class of a basepoint $q_0 \in M$ by the Euler characteristic $\chi$ of $M$. So

$$\mathrm{RFH}_*(S^*M) \cong H_*(\Lambda, \chi q_0) \oplus H^{1-*}(\Lambda, \chi q_0)$$

is the direct sum of "reduced" loop space homology $H_*(\Lambda, \chi q_0) = \operatorname{coker} \varepsilon$ (in degrees $\ge 0$) and cohomology $H^{1-*}(\Lambda, \chi q_0) = \ker \varepsilon$ (in degrees $\le 1$).

### 2.4. Applications in symplectic topology

Over the past ten years, Rabinowitz Floer homology has found numerous applications in symplectic topology and Hamiltonian dynamics. One circle of applications was touched in Section 2.2, and three more are discussed in this subsection. I apologize for the omission, due to space constraints, of many other beautiful applications, such as [7, 10], that would also have deserved to be included.

**Leafwise intersections.** The proof of Theorem 2.1(c) is based on more general action functionals

$$\mathcal{A}_F^{\chi H}(x,\eta) = \int_x \lambda - \eta \int_0^1 \chi(t)H(x)dt - \int_0^1 F(t,x)dt,$$

where $H : W \to \mathbb{R}$ is a defining Hamiltonian for a hypersurface $\Sigma = H^{-1}(0)$ of restricted contact type, $\chi \in C^\infty(S^1, \mathbb{R})$ has support in $(0, 1/2)$ and integral 1, and $F : S^1 \times W \to \mathbb{R}$ has compact support and vanishes for $t \in [0, 1/2]$. Critical points of $\mathcal{A}_F^{\chi H}$ correspond to *leafwise intersections*, i.e., points on $\Sigma$ whose image under the time-one-map of $X_F$ lands on the same $X_H$-orbit on $\Sigma$. P. Albers and U. Frauenfelder [8] have proved that the Floer homology of any such functional equals $\mathrm{RFH}_*(\Sigma)$. Applied to a Hamiltonian $F$ whose time-one-map displaces $\Sigma$ from itself, and which therefore has no leafwise intersections, this implies Theorem 2.1(c). Conversely, it proves the existence of leafwise intersections for any $F$ if $\mathrm{RFH}_*(\Sigma) \neq 0$. See [8, 9] for further results in this direction.

**Exact contact embeddings.** We now return to the question of dependence of $\mathrm{RFH}_*(\Sigma)$ on the ambient Liouville manifold $W$. Using neck-stretching from symplectic field theory, independence of $W$ is proved in [20] if $\pi_1(\Sigma) = 0$ and all periodic orbits on $\Sigma$ have Conley–Zehnder index $> 3 - n$. For example, this holds if $\Sigma$ is the unit cotangent bundle $S^*M$ of a closed simply connected manifold with $\dim M > 3$ with its standard contact structure. Since $\mathrm{RFH}_*(S^*M) \neq 0$ by Example 2.4, it follows that the image of an *exact contact embedding* $S^*M \hookrightarrow (W, \lambda)$ (i.e., an embedding such that the pullback of $\lambda$ defines the standard contact structure) cannot be displaceable. Thus no such embedding exists if all compact sets are displaceable in $W$ (e.g., for $\mathbb{C}^n$, subcritical Stein manifolds, or products of a Liouville manifold with $\mathbb{C}$), and if $W$ is a cotangent bundle the image of such an embedding must intersect each fiber. Since an exact Lagrangian embedding $M \hookrightarrow W$ gives rise to an exact contact embedding $S^*M \hookrightarrow W$, these results generalize Gromov's theorem [49] that there are no closed exact Lagrangian submanifolds $M \subset \mathbb{C}^n$, under the assumptions that $M$ is simply connected of dimension $> 3$. The nonexistence of exact contact embeddings $S^*M \hookrightarrow \mathbb{C}^n$ without these assumptions on $M$ appears to be unknown.

**Periodic Reeb flows.** In many examples for which symplectic homology has been explicitly computed, such as Brieskorn manifolds (see, e.g., [42, 56]), it exhibits some kind of periodicity. P. Uebele [74] has found a beautiful explanation of this phenomenon in terms of Rabinowitz Floer homology. It uses the graded commutative associative products on symplectic homology and Rabinowitz Floer homology that will be discussed in the next section.

**Theorem 2.5** (P. Uebele [74]). *Let $V$ be a Liouville domain such that $\partial V$ is simply connected and all periodic orbits on $\partial V$ have Conley–Zehnder index $> 3 - n$. Assume that the Reeb flow on $\partial V$ is periodic with minimal common period $T > 0$. Let $s \in \mathrm{RFH}_*(\partial V)$ be the class of a principal orbit (corresponding to the maximum on the Bott manifold of orbits of period $T$). Its Conley–Zehnder index has the form $n + 2b$ for some $b \in \mathbb{Z}$. If $b \neq 0$, then multiplication* *with $s$ makes* $\mathrm{RFH}_{*+n}(\partial V)$ *with coefficients in a field* $\mathbb{K}$ *a free and finitely generated module over the ring of Laurent polynomials* $\mathbb{K}[s, s^{-1}]$.[3]

Besides establishing periodicity, this theorem allows the computation of the ring structure on $\mathrm{RFH}_*(\partial V)$ for many such Liouville domains. It also implies that the symplectic homology of such a Liouville domain with $\mathbb{K}$-coefficients is finitely generated as a $\mathbb{K}$-algebra. This finiteness result does not hold for all Liouville domains, counterexamples arising, e.g., from unit disk cotangent bundles of closed hyperbolic manifolds of dimension $\geq 3$. It would be interesting to understand for which Liouville domains the finiteness result holds.

## 3. POINCARÉ DUALITY FOR LOOP SPACES

Let $M$ be a closed connected manifold of dimension $n$ and $\Lambda = C^\infty(S^1, M)$ its free loop space. Let us assume for simplicity that $M$ is oriented, although everything in this section remains true in the unoriented case with suitable twisted coefficients. In their 1999 article **[17]** and its sequels, M. Chas and D. Sullivan introduced a wealth of operations on the homology of $\Lambda$ that gave rise to a whole new research area named *string topology*. We will focus on the following two operations:

- the *loop product* $\mu = \bullet$ on $H_*\Lambda$, which is graded commutative, associative, and unital of degree $-n$ **[17]**, and
- the *loop coproduct* $\lambda$ on the homology $H_*(\Lambda, \Lambda_0)$ relative to the subspace $\Lambda_0 \subset \Lambda$ of constant loops, which is graded cocommutative and coassociative of degree $1 - n$ **[73]**.

The loop coproduct is dual to a product $\circledast$ of degree $n - 1$ on cohomology $H^*(\Lambda, \Lambda_0)$ that was extensively studied in **[48]** and is often referred to as the *Goresky–Hingston product*. Subsequent studies of these products led to a number of puzzles, including the following two (see **[25]** for more details):

(a) Sullivan **[73]** has conjectured the following relation which we will refer to as *Sullivan's relation*:

$$\lambda\mu = (1 \otimes \mu)(\lambda \otimes 1) + (\mu \otimes 1)(1 \otimes \lambda). \tag{3.1}$$

How and on which space is this relation to be interpreted and proved?

(b) Many results concerning $\bullet$ and $\circledast$ arise in dual pairs. For example, the *critical levels* $\mathrm{Cr}(X)$ for $X \in H_*\Lambda$ and $\mathrm{cr}(x)$ for $x \in H^*(\Lambda, \Lambda_0)$ defined in **[48]** satisfy

3 In **[74]**, the result is stated with $\mathbb{Z}_2$-coefficients, but the extension to an arbitrary field $\mathbb{K}$ is straightforward. The restriction to field coefficients is essential because the proof uses the fact that $\mathbb{K}[s, s^{-1}]$ is a principal ideal domain.

the dual inequalities

$$\mathrm{Cr}(X \bullet Y) \leq \mathrm{Cr}(X) + \mathrm{Cr}(Y), \quad \mathrm{cr}(x \circledast y) \geq \mathrm{cr}(x) + \mathrm{cr}(y).$$

Can these phenomena be explained by some kind of "Poincaré duality"?

We will see in this section that these puzzles get naturally resolved in terms of the Rabinowitz Floer homology of the unit sphere cotangent bundle $S^*M$, which we will call the *Rabinowitz loop homology* and denote by

$$\check{H}_*\Lambda = \mathrm{RFH}_*(S^*M) = \mathrm{S\check{H}}_*(D^*M).$$

### 3.1. Product and coproduct on symplectic homology

We begin by describing the analogues of $\mu$ and $\lambda$ on symplectic homology of a Liouville domain $V$. They are based on topological quantum field theory (TQFT) operations that were introduced by M. Schwarz [71] on Floer homology over closed symplectic manifolds, and extended by P. Seidel [72] to symplectic homology. Let us recall the construction, following the exposition of A. Ritter [67]. It takes the following inputs: a nonnegative asymptotically linear Hamiltonian $H : \hat{V} \to \mathbb{R}_{\geq 0}$; a Riemann surface $(S, j)$ with $q$ positive ends modeled over $\mathbb{R}_+ \times S^1$ and $p$ negative ends modeled over $\mathbb{R}_- \times S^1$; and a 1-form $\beta$ on $S$ with $d\beta \leq 0$ which equals $A_k dt$ in canonical coordinates $s + it$ on the negative ends and $B_\ell dt$ on the positive ends, for some positive weights $A_k, B_\ell$. By Stokes' theorem, such a $\beta$ exists if and only if the weights satisfy

$$\sum_{k=1}^{p} A_k \geq \sum_{\ell=1}^{q} B_\ell. \tag{3.2}$$

The algebraic count of maps $u : S \to \hat{V}$ satisfying $(du - \beta \otimes X_H)^{0,1} = 0$ gives a map

$$\psi_S : \bigotimes_{\ell=1}^{q} \mathrm{FH}_*(B_\ell H) \to \bigotimes_{k=1}^{p} \mathrm{FH}_*(A_k H).$$

(Here we need to use field coefficients in order to have a Künneth formula). Applied to Hamiltonians as on the left of Figure 1, these maps induce maps on symplectic homology

$$\overline{\psi}_S : \mathrm{SH}_*(V)^{\otimes q} \to \mathrm{SH}_*(V)^{\otimes p}$$

which depend only on the topological type of $S$ and satisfy the usual TQFT composition rules. Note, however, that (3.2) forces $p \geq 1$, so we only get a *noncompact TQFT structure*. As part of this structure, we get on $\mathrm{SH}_*(V)$ the unital, graded commutative and associative *pair-of-pants product* $\mu$ of degree $-n$.

The TQFT structure on $\mathrm{SH}_*(V)$ also includes a coproduct of degree $-n$, which is, however, not very interesting. Namely, by deforming the weight at one of the outputs to 0, we can force the corresponding output to land in the action zero (constant loop) part $\mathrm{SH}_*^{=0}(V)$, hence to vanish in the quotient $\mathrm{SH}_*^{>0}(V)$. Since the coproduct vanishes in two different ways, interpolating the weights at the two outputs gives rise to a *secondary pair-of-pants coproduct* $\lambda$ on $\mathrm{SH}_*^{>0}(V)$ of degree $1 - n$ (this was first pointed out by P. Seidel and further explored by T. Ekholm and A. Oancea [38]).

The following theorem relates the operations on symplectic homology to those in string topology. Here the assertion concerning the products is due to A. Abbondandolo and M. Schwarz [3], and that concerning the coproduct is proved in [24].

**Theorem 3.1** (Relation to string topology operations [3, 24]). *Viterbo's isomorphism* (2.3) *intertwines the pair-of-pants product with the loop product (both denoted $\mu$). It descends to an isomorphism* $\mathrm{SH}_*^{>0}(D^*M) \cong H_*(\Lambda, \Lambda_0)$ *which intertwines the secondary pair-of-pants coproduct with the loop coproduct (both denoted $\lambda$).*

### 3.2. Product and coproduct on Rabinowitz Floer homology

It was observed in [30] that applying the arguments of the previous subsection to Hamiltonians as in the middle of Figure 1 also equips Rabinowitz Floer homology $\mathrm{RFH}_*(V)$ with a noncompact TQFT structure. In particular, it carries a unital, graded commutative and associative product of degree $-n$ which we will denote by $\boldsymbol{\mu}$. Moreover, the map $\iota$ in diagram (2.2) is a ring homomorphism.

It turns out that Rabinowitz Floer homology also carries a canonical coproduct. To describe the resulting algebraic structure, let us introduce the degree shifted (co)homology groups[4]

$$\mathrm{S}\mathbb{H}_*(V) = \mathrm{SH}_{*+n}(V), \quad \mathrm{S}\mathbb{H}^*(V) = \mathrm{SH}^{*+n}(V), \quad \mathrm{RF}\mathbb{H}_*(\partial V) = \mathrm{RFH}_{*+n}(\partial V).$$

With respect to the shifted gradings, the products $\mu$ and $\boldsymbol{\mu}$ have degree 0. Let us call *involutive infinitesimal bialgebra*[5] the structure consisting of a graded commutative associative product and a graded cocommutative coassociative coproduct satisfying $\mu\lambda = 0$ and

$$\lambda\mu = (1 \otimes \mu)(\lambda \otimes 1) + (\mu \otimes 1)(1 \otimes \lambda) - (\mu \otimes \mu)(1 \otimes \lambda(1) \otimes 1). \tag{3.3}$$

**Theorem 3.2** (Involutive infinitesimal bialgebra structure on Rabinowitz Floer homology [25]). *There exists a degree* $1-2n$ *coproduct* $\boldsymbol{\lambda}$ *on* $\mathrm{RF}\mathbb{H}_*(\partial V)$ *making* $(\mathrm{RF}\mathbb{H}_*(\partial V), \boldsymbol{\mu}, \boldsymbol{\lambda})$ *an involutive infinitesimal bialgebra.*

To define $\boldsymbol{\lambda}$, we modify the construction of the secondary pair-of-pants coproduct $\lambda$ described above by deforming the weights at the outputs to *negative weights* $-1$ rather than 0. This has the effect of splitting off the chain-level continuation map $\varepsilon : SC^{-*}(V) \to SC_*(V)$ at the corresponding output. Since the continuation map is induced by monotone homotopies from $-H$ to $H$ as on the left of Figure 1, which factor through the zero Hamiltonian which has only constant orbits, this shows again that $\lambda$ induces an operation on positive symplectic homology $\mathrm{SH}_*^{>0}(V)$. Applying the same reasoning on the Rabinowitz Floer complex we are in for a pleasant surprise: the chain-level continuation map is now induced by monotone homotopies from $-H$ to $H$ with $H$ as in the middle of Figure 1, which factor through the Hamiltonian on the right of Figure 1 which has no 1-periodic orbits at all in a given

4 This degree shift is also common in string topology, see [17].

5 The structure has further properties which will not be discussed here. Similar structures have appeared in the work of Aguiar [6], Joni-Rota [54], Ehrenborg-Readdy [36], and Loday-Ronco [60].

action interval! Thus the secondary pair-of-pants coproduct induces an operation $\boldsymbol{\lambda}$ on all of $\mathrm{RFH}_*(\partial V)$.

**Remark 3.3.** The operations $\boldsymbol{\mu}, \boldsymbol{\lambda}$ are defined using the interpretation of Rabinowitz Floer homology as V-shaped symplectic homology. It would be interesting to find a definition in terms of the original definition of Rabinowitz Floer homology.

The next result relates the operations on Rabinowitz Floer homology to those on symplectic homology. Here we denote by $\mu^\vee$ the coproduct on $\mathrm{SH}^*(V)$ dual to $\mu$, and by $\lambda^\vee$ the product on $\mathrm{SH}^*_{>0}(V)$ dual to $\lambda$.

**Theorem 3.4** (Almost splitting [25]). *The long exact sequence* (2.2) *fits into the canonical commuting diagram*

$$
\begin{tikzcd}
 & & & (S\mathbb{H}^{1-2n-*}_{>0}(V), \lambda^\vee) \arrow[dl, "i"'] \arrow[d, "j"] & \\
{} \arrow[r, "\varepsilon"] & (S\mathbb{H}_*(V), \mu) \arrow[r, "\iota"] \arrow[d, "q"] & (\mathrm{RF}\mathbb{H}_*(\partial V), \boldsymbol{\mu}, \boldsymbol{\lambda}) \arrow[r, "\pi"] \arrow[dl, "p"'] & (S\mathbb{H}^{1-2n-*}(V), \mu^\vee) \arrow[r, "\varepsilon"] & {} \\
 & (S\mathbb{H}_*^{\geq 0}(V), \lambda) & & &
\end{tikzcd}
$$

*in which the maps $\iota$ and $i$ intertwine the products $\mu$, $\boldsymbol{\mu}$, and $\lambda^\vee$, and the maps $p$ and $\pi$ intertwine the coproducts $\lambda$, $\boldsymbol{\lambda}$, and $\mu^\vee$.*

We can interpret this as saying that the long exact sequence (2.2) "almost splits" in the sense that it splits up to some discrepancy in the action zero part. Note that, while the map $\iota$ preserves the products, the corresponding splitting map $p$ preserves the coproducts, and similarly for $\pi$ and $i$.

To apply the preceding discussion to string topology, we introduce the corresponding degree shifted (co)homology groups

$$
\mathbb{H}_*\Lambda = H_{*+n}\Lambda, \quad \mathbb{H}^*\Lambda = H^{*+n}\Lambda, \quad \check{\mathbb{H}}_*\Lambda = \check{H}_{*+n}\Lambda.
$$

Then Theorems 3.2 and 3.4 yield

**Corollary 3.5** (Involutive infinitesimal bialgebra structure on Rabinowitz loop homology [25]). *There exists a degree $1-2n$ coproduct $\boldsymbol{\lambda}$ on $\check{\mathbb{H}}_*\Lambda$ making $(\check{\mathbb{H}}_*\Lambda, \boldsymbol{\mu}, \boldsymbol{\lambda})$ an involutive infinitesimal bialgebra. Moreover, the long exact sequence* (2.4) *fits into the canonical commuting diagram*

$$
\begin{tikzcd}
 & & & (\mathbb{H}^{1-2n-*}(\Lambda, \Lambda_0), \lambda^\vee = \circledast) \arrow[dl, "i"'] \arrow[d, "j"] & \\
{} \arrow[r, "\varepsilon"] & (\mathbb{H}_*\Lambda, \mu = \bullet) \arrow[r, "\iota"] \arrow[d, "q"] & (\check{\mathbb{H}}_*\Lambda, \boldsymbol{\mu}, \boldsymbol{\lambda}) \arrow[r, "\pi"] \arrow[dl, "p"'] & (\mathbb{H}^{1-2n-*}\Lambda, \mu^\vee) \arrow[r, "\varepsilon"] & {} \\
 & (\mathbb{H}_*(\Lambda, \Lambda_0), \lambda) & & &
\end{tikzcd}
$$

*in which the maps $\iota$ and $i$ intertwine the products $\mu = \bullet$, $\boldsymbol{\mu}$ and $\lambda^\vee = \circledast$, and the maps $p$ and $\pi$ intertwine the coproducts $\lambda$, $\boldsymbol{\lambda}$ and $\mu^\vee$.*

In this case the maps $i, j$ are injective and the maps $p, q$ are surjective. The map $\iota$ becomes injective after replacing $H_*\Lambda$ by its "reduced" version $H_*(\Lambda, \chi q_0)$ from Example 2.4, and the map $\pi$ becomes surjective after replacing $H^*\Lambda$ by $H^*(\Lambda, \chi q_0)$.

This provides the full solution to puzzle (a): By the left triangle, $\mu$ and $\lambda$ extend to operations $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ on the common domain $\check{\mathbb{H}}_*\Lambda$ satisfying the generalized form (3.3) of Sullivan's relation. Note that the right triangle gives the same conclusion for the dual operations $\lambda^\vee = \circledast$ and $\mu^\vee$.

On the other hand, the products $\bullet$ and $\circledast$ both appear as components of the product $\boldsymbol{\mu}$ on $\check{H}_*\Lambda$. This provides an unexpected alternative interpretation of Sullivan's relation for $\mu$ and $\lambda$, as part of associativity for the product $\boldsymbol{\mu}$ on $\check{H}_*\Lambda$! See [27] for further discussion of this topic.

### 3.3. Poincaré duality for Rabinowitz Floer homology

The main motivation for introducing Rabinowitz Floer homology into string topology was that it satisfies a form of Poincaré duality. This was proved in [30] on the level of vector spaces, and in [25] with the additional algebraic structure. To formulate it, note that the operations $\boldsymbol{\lambda}^\vee, \boldsymbol{\mu}^\vee$ dual to $\boldsymbol{\lambda}, \boldsymbol{\mu}$ define again the structure of an involutive infinitesimal bialgebra on the degree shifted Rabinowitz Floer cohomology $\mathrm{RF}\mathbb{H}^*(\partial V) = \mathrm{RFH}^{*+n}(\partial V)$ (the notion of an involutive infinitesimal bialgebra is "self-dual").

**Theorem 3.6** (Poincaré duality for Rabinowitz Floer homology [25]). *With field coefficients, there exists for each Liouville domain $V$ a canonical isomorphism of involutive infinitesimal bialgebras*

$$\mathrm{PD} : \big(\mathrm{RF}\mathbb{H}_*(\partial V), \boldsymbol{\mu}, \boldsymbol{\lambda}\big) \xrightarrow{\cong} \big(\mathrm{RF}\,\mathbb{H}^{1-2n-*}(\partial V), \boldsymbol{\lambda}^\vee, \boldsymbol{\mu}^\vee\big).$$

In the zero action range, the left-hand side is $H^{-*}(\partial V)$ with $\boldsymbol{\mu}$ the cup product, and the right-hand side is $H_{2n-1+*}(\partial V)$ with $\boldsymbol{\lambda}^\vee$ the intersection product. These are related by Poincaré duality on $\partial V$, and the other two operations are their algebraic duals. Theorem 3.6 thus extends classical Poincaré duality on $\partial V$ to Rabinowitz Floer homology.

On the level of vector spaces, Poincaré duality is most transparent in the original definition of Rabinowitz Floer homology via a Lagrange multiplier functional: it arises from the simple observation that, under the canonical involution

$$(x, \eta) \mapsto (\bar{x}, \bar{\eta}), \quad \bar{x}(t) = x(-t), \quad \bar{\eta} = -\eta,$$

the Rabinowitz action functional changes sign,

$$\mathcal{A}^H(\bar{x}, \bar{\eta}) = -\mathcal{A}^H(x, \eta).$$

It follows that the involution maps positive gradient lines of $\mathcal{A}^H$ to negative gradient lines of $\mathcal{A}^H$, and thus induces an isomorphism from Floer homology to Floer cohomology which implies Poincaré duality on the level of vector spaces. To show compatibility with the involutive infinitesimal bialgebra structures, one needs to reprove Poincaré duality using the description of Rabinowitz Floer homology as V-shaped symplectic homology (which is less intuitive).

### 3.4. Applications in Riemannian geometry

Applied to degree shifted Rabinowitz loop homology and cohomology $\check{\mathbb{H}}^*\Lambda = \check{H}^{*+n}\Lambda = \mathrm{RFH}^{*+n}(S^*M)$, Theorem 3.6 becomes

**Corollary 3.7** (Poincaré duality for free loop spaces [25]). *With field coefficients, there exists a canonical isomorphism of involutive infinitesimal bialgebras*

$$\mathrm{PD} : (\check{\mathbb{H}}_*\Lambda, \boldsymbol{\mu}, \boldsymbol{\lambda}) \xrightarrow{\simeq} \big(\check{\mathbb{H}}^{1-2n-*}\Lambda, \boldsymbol{\lambda}^\vee, \boldsymbol{\mu}^\vee\big).$$

It is shown in [25] that Corollary 3.7 resolves puzzle (b): each classical pair of theorems for the products $\bullet$ and $\circledast$ on loop (co)homology extends to a pair of theorems for the products $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}^\vee$ on Rabinowitz loop (co)homology which are related via Poincaré duality. This leads to unified proofs for each classical pair of theorems. While the original proofs were topological, the unified proofs are always symplectic. More specifically, the following applications are discussed in [25,26,28]:

- the behavior of critical levels of the length and action functionals with respect to products;
- the computation of the Rabinowitz loop homology ring of manifolds all of whose geodesics are closed using Uebele's theorem [74], with applications to the question of string point invertibility of constant rank one symmetric spaces, resonances, and a conjecture of Viterbo concerning spectral norms;
- a duality between index and index $+$ nullity for closed geodesics as a consequence of an iteration formula due to Liu and Long;
- the characterization of level-potent (co)homology classes in terms of symplectically degenerate maxima and minima, with dynamical implications for the existence of infinitely many closed geodesics and the Conley conjecture.

The question of homotopy invariance of Rabinowitz loop homology will be addressed in upcoming work. This question has become particularly interesting due to the recent discovery by Naef [63] that, in contrast to the loop product, the loop coproduct is *not* homotopy invariant.

### 3.5. Topological descriptions of Rabinowitz loop homology

Rabinowitz loop homology $\check{H}_*\Lambda$ was defined above as Rabinowitz Floer homology $\mathrm{RFH}_*(S^*M)$ and the operations $\boldsymbol{\mu}, \boldsymbol{\lambda}$ were constructed Floer theoretically. This subsection outlines four purely topological constructions of $\check{H}_*\Lambda$ and its operations which are the subject of joint work in progress with A. Oancea, N. Hingston, M. Abouzaid, and T. Kragh.

**Construction via cones.** In [31], Rabinowitz Floer homology $\mathrm{RFH}_*(\partial V)$ of a Liouville domain $V$ is described in terms of the cones of chain-level continuation maps $\varepsilon : \mathrm{FC}_*(-H) \to$

$\mathrm{FC}_*(H)$ for Hamiltonians $H$ as on the left of Figure 1. Moreover, the operations $\boldsymbol{\mu}, \boldsymbol{\lambda}$ are derived from an $A_2^+$*-structure* on the Floer chain complex $\mathrm{FC}_*(H)$, which consists of a chain-level product and coproduct satisfying suitable compatibility conditions with the continuation map. This description carries over to the Morse chain complex $\mathrm{MC}_*(S)$ of a Lagrangian action functional of the form $S(\gamma) = \int_0^1 (|\dot{\gamma}|^2 - V(t,\gamma))dt$ on $H^1(S^1, M)$. In view of the discussion in Example 2.4, we define the chain-level continuation map $\varepsilon : \mathrm{MC}^{-*}(S) \to \mathrm{MC}_*(S)$ to live only in degree 0 and send the basepoint $q_0$ to $\chi q_0$. It is proved in [24] that suitable chain-level versions of the loop product and coproduct give rise to an $A_2^+$-structure on $\mathrm{MC}_*(S)$, and the homology of the cone of $\varepsilon$ is isomorphic to $\check{H}_*\Lambda$ as an infinitesimal bialgebra.

**Construction via spectra.** Since Rabinowitz loop homology generally lives in arbitrarily positive and negative degrees, it cannot be the homology of a topological space. It can, however, be obtained as the homology of a *spectrum* (see, e.g., [5] for background on spectra). The construction uses the Spanier–Whitehead dual and a cone construction on the level of spectra.

**Construction via constant speed loops.** It was suggested by N. Hingston that the cohomology product $\circledast$ should correspond to a "Chas–Sullivan product on cochains of constant speed loops." This leads to the following conjectural description of $\check{H}_*\Lambda$. Let $\mathcal{CS} \subset H^1(S^1, M)$ be the subspace of *constant speed loops*, i.e., loops parametrized with constant speed. For an action functional $S : \mathcal{CS} \to \mathbb{R}$ as above, we consider the chain complex generated by the stable and unstable manifolds of its critical points, viewed as chains of finite dimension resp. finite codimension, with a differential such that its homology equals $\check{H}_*\Lambda$. Chas–Sullivan-type products and coproducts between these chains should then recover its infinitesimal bialgebra structure. One difficulty in making this approach rigorous is that $\mathcal{CS}$ does not appear to be a Hilbert manifold, but only (away from constant loops) an sc-manifold in the sense of [52] (see [64]).

**Construction via a Lagrange multiplier functional.** The space of constant speed loops is defined by the constraint $|\dot{\gamma}(t)| = \text{const}$. Morse homology with such a constraint can be described by a Lagrange multiplier functional with an infinite-dimensional space of Lagrange multipliers. Imitating the construction of Rabinowitz Floer homology, we can replace this by an integrated constraint with a 1-dimensional Lagrange multiplier. More precisely, we fix an $\epsilon > 0$ and define the *Rabinowitz energy functional*

$$\check{E} : \Lambda \times \mathbb{R} \to \mathbb{R}, \quad \check{E}(\gamma, \eta) = \eta \int_0^1 |\dot{\gamma}|^2 - \frac{\eta^3}{3} + \eta\epsilon^2.$$

Its critical points are pairs $(\gamma, \pm\eta_\gamma)$ with $\gamma$ a (possibly constant) closed geodesic and $\eta_\gamma = (\int_0^1 |\dot{\gamma}|^2 + \epsilon^2)^{1/2}$, so they are in one-to-one correspondence with the generators of the complex computing $\check{H}_*\Lambda$. It should not be too hard to prove that the Morse homology of $\check{E}$ equals Rabinowitz loop homology, but it remains unclear how to recover its product and coproduct from this description.

## 4. OTHER LAGRANGE MULTIPLIER FUNCTIONALS

This section is devoted to some further examples of Lagrange multiplier functionals.

**Example 4.1** (Constrained Lagrangian systems). Let $M$ be a Riemannian manifold and $L : TM \to \mathbb{R}$ a smooth Lagrange function. For $a < b$ and $A, B \in M$, consider the path space

$$X = \{x \in C^\infty([a,b], M) \mid x(a) = A, x(b) = B\}$$

and the Lagrangian action

$$S_L : X \to \mathbb{R}, \quad S_L(x) = \int_a^b L(x(t), \dot{x}(t)) dt.$$

Given a smooth function $k : M \to W$ to a vector space $W$ with 0 as a regular value, consider the Lagrange multiplier functional

$$\hat{S}_L : X \times C^\infty([a,b], W^*) \to \mathbb{R}, \quad \hat{S}_L(x, \lambda) = S_L(x) - \int_a^b \langle \lambda(t), k(x(t)) \rangle dt.$$

Its critical points are solutions of the equations

$$k(x(t)) \equiv 0, \quad \frac{\partial L}{\partial x} - \nabla_t \frac{\partial L}{\partial \dot{x}} = \langle \lambda, \nabla k(x) \rangle.$$

and correspond to critical points of $S_L$ subject to the pointwise constraint $k(x(t)) \equiv 0$. Here $\frac{\partial L}{\partial x} - \nabla_t \frac{\partial L}{\partial \dot{x}}$ is the familiar term from the Euler–Lagrange equation of $L$ and $-\langle \lambda, \nabla k(x) \rangle$ is the constraint force.

**Example 4.2** (Euler's equation for rigid bodies). Let us now specialize Example 4.1 to the case that $M$ is a Lie group $G$, and $H = k^{-1}(0)$ is a subgroup defined by a function $k : G \to W$ satisfying $k(gh) = k(g)$ for all $g \in G, h \in H$. We also specialize the Lagrange function to $L(g, \dot{g}) = \frac{1}{2}|\dot{g}|_g^2$ for a right-invariant Riemannian metric on $G$. Then the critical point equation for $(g, \lambda) \in C^\infty([a,b], G) \times C^\infty([a,b], W^*)$ is equivalent to the first-order equations

$$v(t) \in \mathfrak{h}, \quad \dot{v} + B(v, v) + \langle \lambda, \nabla k(e) \rangle = 0 \tag{4.1}$$

on the Lie algebra $\mathfrak{g} = T_e G$ (see [12, 13]). Here $v(t) = \dot{g}(t) g(t)^{-1}$ is the "body angular velocity," $e \in G$ is the unit, $\mathfrak{h}$ is the Lie algebra of $H$, and $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the bilinear form defined by $\langle B(c, a), b \rangle = \langle [a, b], c \rangle$. In the case $G = H = SO(3)$, this becomes Euler's equation for the motion of a free rigid body [41].

**Example 4.3** (Euler's equations of hydrodynamics). We also owe L. Euler the equations of motion for the velocity field $v$ and the pressure $p$ of an inviscous incompressible fluid [40], namely

$$\operatorname{div} v = 0, \qquad \dot{v} + \nabla_v v + \nabla p = 0. \tag{4.2}$$

The general setup for these equations is a closed Riemannian manifold $M$ equipped with a volume form vol, so that $v$ is a vector field and $p$ a function on $M$ (both time dependent). In 1966, V. I. Arnold [11] derived these equations by formally applying Example 4.2 to the diffeomorphism group $G = \mathrm{Diff}(M)$ and its subgroup $H = \mathrm{Diff}(M, \mathrm{vol})$ of volume preserving diffeomorphisms. Here the right-invariant metric on $\mathrm{Diff}(M)$ is defined for $g \in \mathrm{Diff}(M)$ and vector fields $v, w \in \mathfrak{g} = \mathfrak{X}(M)$ by

$$\langle v \circ g, w \circ g\rangle = \int_M \langle v, w\rangle \,\mathrm{vol},$$

and the subgroup $\mathrm{Diff}(M, \mathrm{vol})$ is the zero set of the function

$$k : \mathrm{Diff}(M) \to \Omega^n(M), \quad k(g) = (g^{-1})^* \mathrm{vol} - \mathrm{vol}\,.$$

The Lagrange multiplier $\lambda$, now denoted by $p$, is a function from $[a, b]$ to $\Omega^n(M)^* = C^\infty(M, \mathbb{R})$. Short computations (see [13]) yield $B(v, v) = \nabla_v v$ and $\langle p, \nabla k(e)\rangle = \nabla p$, so that equation (4.1) becomes equation (4.2).

A famous open problem (unfortunately not worth a million dollars) concerns the existence for all times of smooth solutions of (4.2) with smooth initial conditions. This is only known in dimension 2 where it was first proved by O. Ladyzhenskaya [57]. In 1970, D. Ebin and J. Marsden [34] reproved this result using Arnold's geometric interpretation of (4.2) as rigid body motion on the group $\mathrm{Diff}(M, \mathrm{vol})$. This interpretation also allows us to consider Euler equations on other subgroups of $\mathrm{Diff}(M)$. For example, long-time existence has been proved for the Euler equations on groups of symplectomorphisms (Ebin [33]) and contactomorphisms (Ebin and Preston [35]). See also [50] for results on the group of diffeomorphisms preserving a stable Hamiltonian structure. It would be interesting to investigate the interaction of the Euler equations with other geometric structures on these groups such as Hofer's metric on symplectomorphisms, or partial orders on contactomorphisms.

**Example 4.4** (Gauge theories). One often encounters the situation that a function $f : X \to \mathbb{R}$ is invariant under the action of a Lie group $G$ on $X$, and we are interested in its critical $G$-orbits. Suppose there exists a function $h : X \to \mathfrak{g}$ such that $Z = h^{-1}(0)$ meets all $G$-orbits and for each $x \in Z$ the map

$$\mathfrak{g} \to \mathfrak{g}, \quad \xi \mapsto dh(x) \cdot X_\xi(x)$$

is an isomorphism, where $\xi \mapsto X_\xi(x) = \frac{d}{dt}|_{t=0} \exp(t\xi)x$ denotes the infinitesimal action of the Lie algebra. Then $Z$ is a slice for the $G$-action and the critical $G$-orbits correspond to critical points of the Lagrange multiplier functional $F(x, \lambda) = f(x) - \langle\lambda, h(x)\rangle$ with $\lambda \in \mathfrak{g}^*$. While such a slice $Z$ usually does not exist globally, it often exists at least locally near a given $G$-orbit.

For example, let $G$ be a compact connected simple Lie group and consider the Chern–Simons action

$$S(A) = \frac{1}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

on connections $\nabla = d + A$, $A \in \Omega^1(M, \mathfrak{g})$ on the trivial principal $G$-bundle over a closed 3-manifold $M$. Its critical points are the flat connections, and it is invariant up to integer multiples of $2\pi$ under the action of the gauge group $\mathcal{G} = C^\infty(M, G)$ (see [69,79]). Let us fix a flat connection $A$ and consider the complex

$$\Omega^0(M, \mathfrak{g}) \xrightarrow{d_A} \Omega^1(M, \mathfrak{g}) \xrightarrow{d_A} \Omega^2(M, \mathfrak{g}).$$

Here the first map is the infinitesimal action of the Lie algebra Lie $\mathcal{G} = \Omega^0(M, \mathfrak{g})$ and the second map is the linearization of $S$ at $A$. Let $h = d_A^* : \Omega^1(M, \mathfrak{g}) \to \Omega^0(M, \mathfrak{g})$ be the adjoint of $d_A$ with respect to some Riemannian metric on $M$. This function satisfies the condition above and thus defines a slice $Z = h^{-1}(0)$ for the $\mathcal{G}$-action iff the $d_A$-cohomology $H^0_A(M, \mathfrak{g})$ vanishes, and the restriction $S|_Z$ has a nondegenerate critical point at $A$ iff $H^1_A(M, \mathfrak{g})$ vanishes. Writing a general connection as $A + \beta$, $\beta \in \Omega^1(M, \mathfrak{g})$, the Lagrange multiplier functional

$$F(\beta, \lambda) = S(A + \beta) - \langle \lambda, d_A^* \beta \rangle = \frac{1}{4\pi} \int_M \operatorname{Tr}\Big( \beta \wedge d_A \beta + \frac{2}{3} \beta \wedge \beta \wedge \beta + d_A^* \beta \wedge *\lambda \Big)$$

corresponds to the first three terms of the Fadeev–Popov action, where $\lambda \in \Omega^*(M, \mathfrak{g})$ is the gauge fixing boson. With an additional fermionic term, this becomes the relevant functional for the perturbative expansion of the Chern–Simons partition function at the flat connection $A$ (see [69,79]).

**Example 4.5** (Symplectic vortex equations). This example follows [23]; it was the original motivation leading to Rabinowitz Floer homology. Consider a Hamiltonian action of a compact connected Lie group $G$ on a symplectic manifold $(M, \omega)$ with an equivariant moment map $\mu : M \to \mathfrak{g}^*$. Let $\Lambda = C^\infty(S^1, M)$, where $S^1 = \mathbb{R}/\mathbb{Z}$, and denote by $\Lambda^{\text{contr}} \subset \Lambda$ the subspace of contractible loops. Consider the action

$$\mathcal{A} : \Lambda^{\text{contr}} \to \mathbb{R}, \quad \mathcal{A}(x) = \int_{\bar{x}} \omega,$$

where $\bar{x} : D \to M$ is an extension of $x$ to the closed unit disk. This is independent of the choice of $\bar{x}$ if $\omega$ vanishes on $\pi_2(M)$, which we will assume for simplicity. Suppose that 0 is a regular value of $\mu$ and consider the Lagrange multiplier functional

$$\mathcal{A}^\mu : \Lambda^{\text{contr}} \times C^\infty(S^1, \mathfrak{g}) \to \mathbb{R}, \quad \mathcal{A}^\mu(x, \eta) = \mathcal{A}(x) - \int_0^1 \langle \mu(x), \eta \rangle dt$$

with a Lagrange multiplier $\eta \in C^\infty(S^1, \mathfrak{g})$. To describe its gradient flow, we pick a compatible almost complex structure $J$ on $(M, \omega)$ and an Ad-invariant inner product on $\mathfrak{g}$, and define a metric $m$ on $\Lambda^{\text{contr}} \times C^\infty(S^1, \mathfrak{g})$ by

$$m_{(x,\eta)}\big((\hat{x}_1, \hat{\eta}_1), (\hat{x}_2, \hat{\eta}_2)\big) = \int_0^1 \big(\omega\big(\hat{x}_1, J(x)\hat{x}_2\big) + \langle \hat{\eta}_1, \hat{\eta}_2 \rangle\big) dt.$$

Then gradient flow lines of $\mathcal{A}^\mu$ are maps $(u, \eta) : \mathbb{R} \times S^1 \to M \times \mathfrak{g}$, satisfying

$$\frac{\partial u}{\partial s} + J(u)\Big( \frac{\partial u}{\partial t} + X_\eta(u) \Big) = 0, \quad \frac{\partial \eta}{\partial s} + \mu(u) = 0,$$

where $(s, t)$ are the coordinates on $\mathbb{R} \times S^1$ and $X_\eta(x) = \frac{d}{dt}|_{t=0} \exp(t\eta)x$. To interpret these equations more geometrically, we view the Lagrange multiplier as a connection

$$A = \eta(s, t) dt \in \Omega^1(Z, \mathfrak{g})$$

on the cylinder $Z = \mathbb{R} \times S^1$. Its curvature is $F_A = \frac{\partial \eta}{\partial s} ds \wedge dt$, which can be converted to a function $Z \to \mathbb{R}$ using the Hodge $*$ operator $*(ds \wedge dt) = 1$. Moreover, the connection induces a covariant derivative

$$d_A u = du + X_A(u) = du + X_\eta(u) dt : TZ \to TM$$

and its complex antilinear part

$$\bar{\partial}_{J,A}(u) = \frac{1}{2}\big(d_A u + J(u) \circ d_A u \circ j\big)$$

with respect to the standard complex structure $j$ on $\mathbb{R} \times S^1$ sending $\partial_s$ to $\partial_t$. Then the above equations for gradient flow lines of $\mathcal{A}^\mu$ become the *symplectic vortex equations*

$$\bar{\partial}_{J,A}(u) = 0, \quad *F_A + \mu(u) = 0.$$

These equations were discovered independently by D. Salamon and I. Mundet i Riera [23,62] for an arbitrary Riemann surface in place of the cylinder $Z$. They give rise to invariants of Hamiltonian group actions and a quantum Kirwan map, with applications to the quantum cohomology of symplectic quotients [22,32,47,65]. Applied to suitable infinite-dimensional symplectic manifolds, the symplectic vortex equations comprise other well-known equations of mathematical physics such as the anti-self-dual Yang–Mills equations and the Seiberg–Witten equations [23].

The functional $\mathcal{A}^\mu$ is still invariant under the action of the gauge group $\mathcal{G} = C^\infty(S^1, G)$. If this action has a global slice, we can remove the gauge symmetry as in Example 4.4 by introducing another Lagrange multiplier in the dual Lie algebra $C^\infty(S^1, \mathfrak{g}^*)$. One situation where such a slice exists is an $\mathbb{R}$-action generated by a single Hamiltonian $\mu = H$ such that $\Sigma = H^{-1}(0)$ is of contact type, in which case the functional $\mathcal{A}^\mu$ restricts on the slice to the Rabinowitz action functional [9]. On the other hand, considering the action $\mathcal{A}$ on the loop space $C^\infty(S^1, \Sigma)$ and removing the gauge symmetry leads to the equations in Example 4.6 below. It would be interesting to further explore the various Lagrange multiplier functionals and their Floer homologies in this situation.

**Example 4.6** (Symplectic field theory). Let $(\omega, \lambda)$ be a stable Hamiltonian structure on a closed $(2n-1)$-manifold $M$ with Reeb vector field $R$ (see [14]).[6] Assume for simplicity that $\omega = d\theta$ is exact, so that we have a well-defined action functional

$$\mathcal{A} : \Lambda = C^\infty(S^1, M) \to \mathbb{R}, \quad x \mapsto \int_x \theta.$$

This functional is invariant under the group $G = \mathrm{Diff}_+(S^1)$ of orientation-preserving diffeomorphisms of the circle. One can break this symmetry by imposing the gauge fixing condition $\lambda(\dot{x}) = \text{const}$. (The constant should not be fixed because we cannot expect closed Reeb orbits of prescribed period). Critical points of $\mathcal{A}$ subject to this constraint are critical points of the Lagrange multiplier functional

$$\hat{\mathcal{A}} : \Lambda \times \mathfrak{g}_0 \to \mathbb{R}, \quad \hat{\mathcal{A}}(x, \eta) = \mathcal{A}(x) + \int_0^1 \eta(t)\lambda\big(\dot{x}(t)\big)dt,$$

with the codimension-one subspace $\mathfrak{g}_0$ of the Lie algebra of $G$ given by

$$\mathfrak{g}_0 = \left\{\eta \in C^\infty(S^1, \mathbb{R}) \mid \int_0^1 \eta(t)dt = 0\right\}.$$

6 Even in the contact case $\omega = d\lambda$, separating the roles of $\omega$ and $\lambda$ clarifies the discussion.

The derivative of $\hat{\mathcal{A}}$ is given by

$$d\hat{\mathcal{A}}(x,\eta)(\hat{x},\hat{\eta}) = \int_0^1 \big[\omega(\hat{x},\dot{x}) + \eta\, d\lambda(\hat{x},\dot{x}) - \dot{\eta}\lambda(\hat{x}) + \hat{\eta}\lambda(\dot{x})\big]dt,$$

so its critical points are solutions of the equations

$$\lambda(\dot{x}) = T = \text{const}, \quad i_{\dot{x}}(\omega + \eta\, d\lambda) + \dot{\eta}\lambda = 0, \quad \int_0^1 \eta = 0.$$

Inserting $R$ in the second equation yields $\dot{\eta} = 0$ and thus $\eta \equiv 0$, so critical points are pairs $(x, 0)$ where $\dot{x} = TR(x)$ for some $T \in \mathbb{R}$. Now consider, for an $\omega$-compatible complex structure $J$ on $\xi = \ker\lambda$, the "metric" on $\Lambda^{\text{contr}} \times \mathfrak{g}_0$ defined by

$$m_{(x,\eta)}\big((\hat{x}_1,\hat{\eta}_1),(\hat{x}_2,\hat{\eta}_2)\big) = \int_0^1 \big[(\omega + \eta\, d\lambda)(\pi\hat{x}_1, J\pi\hat{x}_2) + \lambda(\hat{x}_1)\lambda(\hat{x}_2) + \hat{\eta}_1\hat{\eta}_2\big]dt,$$

where $\pi : TM \to \xi$ is the projection along $R$. Note that in the contact case $\omega = d\lambda$, the bilinear form $m_{(x,\eta)}$ is symmetric, but it is only positive definite as long as $\eta > -1$. Nevertheless, $\hat{\mathcal{A}}$ has a well-defined "gradient" with respect to $m$ given by

$$\nabla_m \hat{\mathcal{A}}(x,\eta) = \left(-J(x)\pi\dot{x} - \dot{\eta}R(x), \lambda(\dot{x}) - \int_x \lambda\right),$$

where the term $-\int_x \lambda$ comes from projecting $\lambda(\dot{x})$ onto $\mathfrak{g}_0$. So a gradient flow line $u = (\eta, f) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ of $\hat{\mathcal{A}}$ satisfies the equations

$$\begin{cases} \partial_s\eta - \lambda(\partial_t f) + \int_0^1 f(s,\cdot)^*\lambda = 0, \\ \lambda(\partial_s f) + \partial_t\eta = 0, \\ \pi\partial_s f + J(f)\pi\partial_t f = 0. \end{cases} \tag{4.3}$$

Replacing $\eta$ by $a(s,t) = \eta(s,t) + \alpha(s)$ for a function $\alpha : \mathbb{R} \to \mathbb{R}$ (unique up to a constant) satisfying $\alpha'(s) = \int_0^1 f(s,\cdot)^*\lambda$, we obtain the familiar equations

$$\begin{cases} \partial_s a - \lambda(\partial_t f) = 0, \\ \lambda(\partial_s f) + \partial_t a = 0, \\ \pi\partial_s f + J(f)\pi\partial_t f = 0, \end{cases} \tag{4.4}$$

for $\hat{J}$-holomorphic curves $u = (a, f) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ with respect to the almost complex structure $\hat{J}$ restricting to $J$ on $\xi$ and mapping the unit vector in $\mathbb{R}$ to $R$. On the plane $\mathbb{C}$ instead of the cylinder $\mathbb{R} \times S^1$, these equations were introduced by H. Hofer in his 1993 paper **[51]** on the Weinstein conjecture in dimension three. They make sense for a domain being any punctured Riemann surface, giving rise to *symplectic field theory (SFT)* **[39]**, a general theory of punctured holomorphic curves in symplectic cobordisms which has found numerous applications in contact and symplectic topology.

The description via Lagrange multipliers raises some interesting questions concerning symplectic field theory. Let us begin with a brief comparison of equations (4.3) and (4.4). Note first that $\eta \to 0$ as $s \to \pm\infty$, whereas $a$ grows linearly with slope the asymptotic periods as $s \to \pm\infty$. Moreover, shifting $a$ by a constant yields again a solution, which is not the case for $\eta$. The Floer energy of $(\eta, f)$ equals the $\omega$-energy $\int_{\mathbb{R}\times S^1} f^*\omega = \mathcal{A}(x^+) - \mathcal{A}(x^-)$, which in the contact case equals the difference $T_+ - T_-$ of the asymptotic periods. Moreover, in the contact case the action $\mathcal{A}(x^+)$ at $+\infty$ is equivalent to the Hofer energy (see [29]).

It would be interesting to give a direct proof of compactness modulo breaking of solutions of (4.3) (say, in the absence of finite energy planes) without appealing to the SFT compactness theorem [14,29]. Generalizing this proof to $\lambda$ being replaced by a loop of contact forms $\lambda_t$ may lead to a description of nonequivariant contact homology (see [15,38]) in terms of such loops, analogous to the definition of Hamiltonian Floer homology in terms of loops of Hamiltonians. In a different direction, this may also shed some light on the variant of (4.4) introduced in [1] where the first two equations are replaced by a harmonic 1-form $(f^*\lambda) \circ j$, for which the compactness question is still wide open.

Another interesting feature of (4.3) is the fact that the asymptotic periods $T_\pm$ can also be negative or zero. This parallels the corresponding feature in Rabinowitz Floer homology (see Section 2) and suggests that suitable counts of solutions of (4.3) (or equivalently (4.4)) compute the equivariant Rabinowitz Floer homology of the symplectization $\mathbb{R} \times M$ whenever the latter is defined. This should lead to an interpretation of algebraic structures on Rabinowitz Floer homology such as its involutive infinitesimal bialgebra structure and Poincaré duality in terms of symplectic field theory. In the Lagrangian setting, N. Legout has constructed an $A_\infty$-structure on the corresponding SFT-type complex ([59], see also [16, 37]), whose relation to Rabinowitz Floer homology is still conjectural.


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