REAL GROMOV–WITTEN THEORY

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ABSTRACT

In this note we survey some of the recent developments in real Gromov–Witten theory. In particular, we discuss the main difficulties of the construction and important structural results.

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1. INTRODUCTION

The Gromov–Witten invariants can be viewed as a modern counterpart of the classical enumeration of curves in projective varieties. They arise from integration over the moduli spaces of pseudoholomorphic maps into a symplectic manifold introduced in the seminal work of Gromov [21]. An influential perspective proposed by Witten interprets them as the coefficients of a partition function of a topological string theory. As such they play a central role in striking dualities relating them to mathematical objects of completely different nature. Understanding these relations has and continues to generate substantial amount of high-level research.

The real Gromov–Witten invariants arise in a similar way from integration over moduli spaces of pseudoholomorphic maps. In the real case these maps are required to be equivariant with respect to an antisymplectic involution on the target and one on the domain. The antisymplectic involution corresponds to an intrinsic symmetry of the theory and is preserved under dualities thus providing conjectures relating the real Gromov–Witten invariants with the dual equivariant objects. In particular, relations with SO/Sp gauge theory and the Gaussian orthogonal/symplectic ensembles are expected.

In this note we present an overview of the construction of the real Gromov–Witten invariants, based on a joint work [18] with Aleksey Zinger, and discuss structural results for the local real Gromov–Witten theory, based on a joint work [16] with Eleny Ionel.

2. REAL GROMOV–WITTEN INVARIANTS

The foundations of (complex) Gromov–Witten theory, i.e., of counts of J-holomorphic curves in symplectic manifolds, were established in the 1990s and have been spectacularly applied ever since. On the other hand, the progress in establishing the foundations of real GW theory, i.e., of counts of J-holomorphic curves in symplectic manifolds preserved by antisymplectic involutions, has been much slower. The two main difficulties in developing real GW theory are the potential nonorientability of the moduli space $\mathfrak{M}_{g,l}(X, B; J)^{\phi,\sigma}$, defined in (2.2), and the fact that its virtual boundary strata have real codimension 1. This is in contrast with the complex GW theory, where the moduli spaces have canonical orientations and the “boundary” strata have real codimension of at least 2. These two ingredients are crucial for the construction of a (virtual) fundamental class, integration upon which defines the invariants.

The difficulty arising from the existence of a real codimension 1 boundary strata can be resolved by considering the larger moduli space (2.3) that is a union over all topological types of involutions on the domain. As explained in Section 2.1, inside this space all codimension 1 strata form a hypersurface rather than boundary and the definition of the invariants becomes a question about the orientability of this moduli space. We introduce the notion of real orientation on a symplectic manifold in Section 2.2 – these are topological conditions on the symplectic manifold which ensure the orientability of the real moduli space (2.3). We define the primary and descendant real GW invariants in Section 2.3 and give examples of large collections of real-orientable symplectic manifolds.

Invariant counts of real curves were first constructed by Welschinger [38,39] following a different approach. They are defined in genus 0, for real symplectic 4- and 6-folds, and under certain topological conditions ruling out maps from type (E) nodal symmetric surfaces (later removed in [13]). In the Gromov–Witten-style approach to these counts developed in [11, 35], the invariance corresponds to the relevant moduli spaces being orientable outside of (virtual) hypersurfaces which are shown not to be crossed by the paths of stable maps induced by paths between two generic almost complex structures and two generic collections of constraints.

Many methods have been developed for the computation of the real invariants, notably by employing methods from tropical geometry [5–8, 23, 24, 29, 34], establishing WDVV-type formulas [10, 17, 36], and through localization techniques [19, 30, 33]. In particular, the result of [33] provides the first instance of a real mirror symmetry phenomenon and that of [30] the first real enumerative bounds in higher genus.

2.1. Moduli spaces of real maps

A real symplectic manifold is a triple (X, ω, ϕ) consisting of a symplectic manifold (X, ω) and an antisymplectic involution ϕ. For such a triple, denote by $\mathcal{J}_{\omega}^{\phi}$ the space of ω-compatible almost complex structures J on X such that $\phi^* J = -J$. The fixed locus X^{ϕ} of ϕ is then a Lagrangian submanifold of (X, ω) which is totally real with respect to any $J \in \mathcal{J}_{\omega}^{\phi}$.

Example 2.1. An example of a real Kähler manifold (X, ω, ϕ, J) is the complex projective space $\mathbb{P}^{n-1}$. The maps

$$\begin{aligned} \tau_n &: \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}, & [z_1, \dots, z_n] &\to [\bar{z}_1, \dots, \bar{z}_n], \\ \eta_{2m} &: \mathbb{P}^{2m-1} \to \mathbb{P}^{2m-1}, & [z_1, z_2, \dots, z_{2m-1}, z_{2m}] &\to [-\bar{z}_2, \bar{z}_1, \dots, -\bar{z}_{2m}, \bar{z}_{2m-1}], \end{aligned}$$

are antisymplectic involutions with respect to the standard Fubini–Study symplectic form ω_n on $\mathbb{P}^{n-1}$. Another important example is a real quintic threefold X_5, i.e., a smooth hypersurface in $\mathbb{P}^4$ cut out by a real equation.

A symmetric surface (Σ, σ) is a connected oriented, possibly nodal, surface Σ with an orientation-reversing involution σ. There are $\lfloor \frac{3g+4}{2} \rfloor$ topological types of smooth symmetric genus g surfaces; the type is determined by the number of fixed components and the orientability of the quotient. A symmetric Riemann surface $(\Sigma, \sigma, \mathfrak{j})$ is a symmetric surface (Σ, σ) with an almost complex structure $\mathfrak{j}$ on Σ such that $\sigma^* \mathfrak{j} = -\mathfrak{j}$. We denote by $\mathcal{J}_{\Sigma}^{g}$ the space of such complex structures.

A continuous map

$$u: (\Sigma, \sigma) \to (X, \phi)$$

is called real if $u \circ \sigma = \phi \circ u$; see Figure 1. It is said to be of degree $B \in H_2(X; \mathbb{Z})$ if $u_*[\Sigma] = B$. We denote the space of such maps by $\mathfrak{B}_g(X)^{\phi,\sigma}$, with g denoting the genus of the domain Σ of σ.

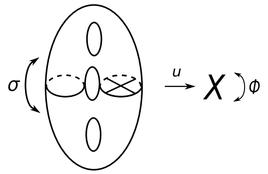


FIGURE 1
Here the domain Σ has 1 fixed circle and 1 cross-cap circle; the quotient Σ/σ is a nonorientable surface with 1 boundary and 1 cross-cap.

For $J \in \mathcal{J}_\omega^\phi$, $\mathfrak{j} \in \mathcal{J}_\Sigma^\sigma$, and $u \in \mathfrak{B}_g(X)^{\phi,\sigma}$, let

$$\bar{\partial}_{J,\mathfrak{j}} u = \frac{1}{2}(\mathrm{d}u + J \circ \mathrm{d}u \circ \mathfrak{j})$$

be the $\bar{\partial}_J$-operator on $\mathfrak{B}_g(X)^{\phi,\sigma}$.

Let $g, l \in \mathbb{Z}^{\geq 0}$, (Σ, σ) be a genus g symmetric surface, $B \in H_2(X;\mathbb{Z}) - 0$, and $J \in \mathcal{J}_\omega^\phi$. Let $\Delta^{2l} \subset \Sigma^{2l}$ be the big diagonal, i.e., the subset of $2l$-tuples with at least two coordinates equal. Denote by

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} = \big\{\big(u, (z_1^+, z_1^-), \dots, (z_l^+, z_l^-), \mathfrak{j}\big) \in \mathfrak{B}_g(X)^{\phi,\sigma} \times (\Sigma^{2l} - \Delta^{2l}) \times \mathcal{J}_\Sigma^\sigma : z_i^- = \sigma\big(z_i^+\big)\ \forall\, i = 1, \dots, l,\ u_*[\Sigma]_{\mathbb{Z}} = B,\ \bar{\partial}_{J,\mathfrak{j}} u = 0\big\}/\sim \quad (2.1)$$

the (uncompactified) moduli space of equivalence classes of degree B real J-holomorphic maps from (Σ, σ) to (X, ϕ) with l conjugate pairs of marked points. Two marked J-holomorphic (ϕ, σ)-real maps determine the same element of this moduli space if they differ by an orientation-preserving diffeomorphism of Σ commuting with σ. We denote by

$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma} \supset \mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} \quad (2.2)$$

Gromov's convergence compactification of $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ obtained by including stable real maps from nodal symmetric surfaces. The (virtually) codimension-one boundary strata of

$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma} - \mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} \subset \overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$$

consist of real J-holomorphic maps from one-nodal symmetric surfaces to (X, ϕ). Each stratum is either a (virtual) hypersurface in $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ or a (virtual) boundary. The existence of boundary is what prevents us from defining invariants for each topological type of involutions σ. However, one-nodal symmetric surfaces can always be smoothed out in (real) one-parameter family to symmetric surfaces. Thus, each boundary stratum appears in the compactification of precisely two of the moduli spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ corresponding to two different topological types of orientation-reversing involutions σ on Σ. This means that the union over all topological types of involutions on Σ forms a space without boundary. Let

$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi} = \bigcup_{\sigma} \overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma} \quad (2.3)$$

denote the union of the compactified real moduli spaces taken over all topological types of orientation-reversing involutions σ on Σ. Furthermore, denote by

$$\mathbb{R}\overline{\mathcal{M}}_{g,l} \equiv \overline{\mathfrak{M}}_{g,l}(\mathrm{pt}, 0)^{\mathrm{id}}$$

the Deligne–Mumford moduli space of marked real curves. If $g + l \geq 2$, there is a natural forgetful morphism

$$\mathfrak{f}\colon \overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi} \to \mathbb{R}\overline{\mathcal{M}}_{g,l} \equiv \overline{\mathfrak{M}}_{g,l}(\mathrm{pt}, 0)^{\mathrm{id}}. \tag{2.4}$$

In order to study the orientability of these spaces, it is crucial to understand their codimension one strata which consist of maps from one-nodal domains. As described in [27, SECTION 3], there are four types of nodes a one-nodal symmetric surfaces (Σ, σ) may have:

(E) the node is an isolated point of the fixed locus $\Sigma^{\sigma} \subset \Sigma$;

(H) the node is a nonisolated point of the fixed locus Σ^{σ} and

- (H1) the topological component of Σ^{σ} containing the node is algebraically irreducible (its normalization is connected);
- (H2) the topological component of Σ^{σ} containing the node is algebraically reducible, but Σ is algebraically irreducible;
- (H3) Σ is algebraically reducible.

In the genus 0 case, the degenerations (E) and (H3) are known as the codimension 1 sphere bubbling and disk bubbling, respectively; the degenerations (H1) and (H2) cannot occur in the genus 0 case.

As a one-nodal symmetric surface is smoothed out in one-parameter family of symmetric surfaces, we observe the transition of a smooth symmetric surface through one-nodal degeneration. A transition through a degeneration (H3) does not change the topological type of the involution. Thus, each stratum of morphisms from a one-nodal symmetric surface of type (H3) to (X, ϕ) is a hypersurface inside of $\overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi,\sigma}$ for some genus g involution σ.

A transition through a degeneration (H2) also does not change the number of fixed components. The transformation of the real locus is the same as in the (H3) case, but an (H2) transition also inserts or removes two cross-caps. This transition may or may not change the topological type of the involution. The former occurs when the fixed locus is separating in which case the transition changes the topological type of the involution and thus each stratum of morphisms from such one-nodal surfaces to (X, ϕ) is a boundary of the spaces $\overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi,\sigma}$ for precisely two topological types of genus g involutions σ. If the fixed locus is nonseparating, then the transition does not change the topological type of the involution and each stratum of morphisms from such one-nodal surfaces to (X, ϕ) is a hypersurface inside of $\overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi,\sigma}$ for some genus g involution σ. A degeneration (H2) cannot occur in genus 0 or 1, but does occur in genus 2 and higher.

A transition through a degeneration (E) or (H1) changes the number of fixed components by one. In particular, each stratum of morphisms from a one-nodal symmetric surface of type (E) or (H1) to (X,ϕ) is a boundary of the spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ for precisely two topological types of genus g involutions σ. A degeneration (H1) cannot occur in genus 0, but does occur in genus 1 and higher.

2.2. Real orientations

Let (X,ϕ) be a topological space with an involution. A conjugation on a complex vector bundle $V\to X$ lifting an involution ϕ is a vector bundle homomorphism $\varphi\colon V\to V$ covering ϕ (or equivalently a vector bundle homomorphism $\varphi\colon V\to\phi^*V$ covering id_X) such that the restriction of φ to each fiber is anticomplex linear and $\varphi\circ\varphi=\mathrm{id}_V$.

A real bundle $(V,\varphi)\to(X,\phi)$ consists of a complex vector bundle $V\to X$ and a conjugation φ on V lifting ϕ.

Example 2.2. (1) If X is a smooth manifold with a smooth involution ϕ, then $(TX,\mathrm{d}\phi)$ is a real bundle over (X,ϕ).

(2) If $L\to X$ is a complex vector bundle then $L\oplus\phi^*\bar{L}\to X$ with the conjugation $\widetilde{\phi}:(x,v,w)\mapsto(\phi(x),w,v)$ is also a real bundle over (X,ϕ).

For any real bundle (V,φ) over (X,ϕ), the fixed locus

$$V^{\varphi}\to X^{\phi}$$

of φ is a real vector bundle over X^{ϕ}. We denote by

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\varphi)=\big(\Lambda^{\mathrm{top}}_{\mathbb{C}}V,\Lambda^{\mathrm{top}}_{\mathbb{C}}\varphi\big)$$

the top exterior power of V over $\mathbb{C}$ with the induced conjugation. Direct sums, duals, and tensor products over $\mathbb{C}$ of real bundles over (X,ϕ) are again real bundles over (X,ϕ).

Definition 2.3 ([15,18]). Let (X,ϕ) be a topological space with an involution and (V,φ) be a real bundle over (X,ϕ). A real orientation on (V,φ) consists of

(RO1) a complex line bundle $L\to X$ such that

$$w_2\big(V^{\varphi}\oplus L^*_{|X^{\phi}}\big)=0\quad\text{and}\quad\Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\varphi)\approx\Lambda^{\mathrm{top}}_{\mathbb{C}}(L\oplus\phi^*\bar{L},\widetilde{\phi}),\qquad(2.5)$$

(RO2) a homotopy class of isomorphisms of real bundles in (2.5), and

(RO3) a spin structure on the real vector bundle $V^{\varphi}\oplus L^*$ over X^{ϕ} compatible with the orientation induced by (RO2).

An isomorphism in (2.5) restricts to an isomorphism $\Lambda^{\mathrm{top}}_{\mathbb{R}}V^{\varphi}\approx\Lambda^{\mathrm{top}}_{\mathbb{R}}L$ of real line bundles over X^{ϕ}. Since L is a complex vector bundle it is canonically oriented, and thus (RO2) determines orientations on V^{φ} and $V^{\varphi}\oplus L^*$. By the first assumption in (2.5), the real vector bundle $V^{\varphi}\oplus L^*$ over X^{ϕ} admits a spin structure.

A real orientation on a real symplectic manifold (X,ω,ϕ) is a real orientation on the real bundle $(TX,\mathrm{d}\phi)$. We call a real symplectic manifold (X,ω,ϕ) real-orientable if it admits a real orientation. As established in [18] a real orientation on (X, ϕ) determines a canonical orientation of the uncompactified moduli spaces when X is of odd complex dimension. This orientation extends across the codimension 1 boundary strata of types (H2) and (H3) and changes across the codimension 1 boundary strata of types (E) and (H1). The parity of $|\pi_0(\Sigma^\sigma)|$ behaves in the same way. This allows us to readjust this canonical orientation by the parity of the number of fixed components of the domain and thus obtain an orientation on the compactified moduli space.

Theorem 2.4 ([18, THEOREM 1.3]). *Let (X, ω, ϕ) be a real-orientable $2n$-manifold, $g, l \in \mathbb{Z}^{\geq 0}$, $B \in H_2(X;\mathbb{Z})$, and $J \in \mathcal{J}_\omega^\phi$.*

(1) *If $n \notin 2\mathbb{Z}$, a real orientation on (X, ω, ϕ) orients $\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi$.*

(2) *If $n \in 2\mathbb{Z}$ and $g + l \geq 2$, a real orientation on (X, ω, ϕ) orients the real line bundle*

$$\Lambda_{\mathbb{R}}^{\text{top}}\big(T\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi\big) \otimes \mathfrak{f}^* \Lambda_{\mathbb{R}}^{\text{top}}(T\mathbb{R}\overline{\mathcal{M}}_{g,l}) \to \overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi.$$

Examples of real-orientable manifolds include $\mathbb{P}^{2n-1}$, X_5, many other projective complete intersections, and simply-connected real symplectic Calabi–Yau and real Kähler Calabi–Yau manifolds with spin fixed locus as described by the following propositions.

Proposition 2.5 ([19, PROPOSITION 1.2]). *Let (X, ω, ϕ) be a real symplectic manifold with $w_2(X^\phi) = 0$. If*

(1) *$H_1(X;\mathbb{Q}) = 0$ and $c_1(X) = 2(\mu - \phi^*\mu)$ for some $\mu \in H^2(X;\mathbb{Z})$ or*

(2) *X is compact Kähler, ϕ is antiholomorphic, and $\mathcal{K}_X = 2([D] + [\overline{\phi_* D}])$ for some divisor D on X,*

then (X, ω, ϕ) is a real-orientable symplectic manifold.

Corollary 2.6 ([19, COROLLARY 1.3]). *Let $n \in \mathbb{Z}^+$ and $\mathbf{a} \equiv (a_1, \ldots, a_{n-4}) \in (\mathbb{Z}^+)^{n-4}$ be such that*

$$a_1 + \cdots + a_{n-4} \equiv n \mod 4.$$

If $X_{n;\mathbf{a}} \subset \mathbb{P}^{n-1}$ is a complete intersection of multidegree $\mathbf{a}$ preserved by τ_n, then $(X_{n;\mathbf{a}}, \omega_{n;\mathbf{a}}, \tau_{n;\mathbf{a}})$ is a real-orientable symplectic manifold.

Proposition 2.7 ([19, PROPOSITION 1.4]). *Let $m, n \in \mathbb{Z}^+$, $k \in \mathbb{Z}^{\geq 0}$, and $\mathbf{a} \equiv (a_1, \ldots, a_k) \in (\mathbb{Z}^+)^k$.*

(1) *If $X_{n;\mathbf{a}} \subset \mathbb{P}^{n-1}$ is a complete intersection of multidegree $\mathbf{a}$ preserved by τ_n,*

$$a_1 + \cdots + a_k \equiv n \mod 2, \quad \text{and} \quad a_1^2 + \cdots + a_k^2 \equiv a_1 + \cdots + a_k \mod 4, \tag{2.6}$$

then $(X_{n;\mathbf{a}}, \omega_{n;\mathbf{a}}, \tau_{n;\mathbf{a}})$ is a real-orientable symplectic manifold.

(2) *If $X_{2m;\mathbf{a}} \subset \mathbb{P}^{2m-1}$ is a complete intersection of multidegree* **a** *preserved by η_{2m} and*

$$a_1 + \cdots + a_k \equiv 2m \mod 4,$$

then $(X_{2m;\mathbf{a}}, \omega_{2m;\mathbf{a}}, \eta_{2m;\mathbf{a}})$ is a real-orientable symplectic manifold.

2.3. Real Gromov–Witten theory

The moduli space $\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi$ is not smooth in general and its tangent bundle in Theorem 2.4 should be viewed in the usual moduli-theoretic (or virtual) sense. Since the (virtual) boundary of $\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi$ is empty, Theorem 2.4(1) implies that this moduli space carries a virtual fundamental class over $\mathbb{Q}$ (determined by the choice of orientation) and thus gives rise to real GW-invariants in arbitrary genus.

Theorem 2.8 ([18, THEOREM 1.4]). *Let (X, ω, ϕ) be a compact real-orientable $2n$-manifold with $n \notin 2\mathbb{Z}$, $g, l \in \mathbb{Z}^{\geq 0}$, $B \in H_2(X;\mathbb{Z})$, and $J \in \mathcal{J}_\omega^\phi$. Then a real orientation on (X, ω, ϕ) endows the moduli space $\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi$ with a virtual fundamental class and thus gives rise to genus g real GW-invariants of (X, ω, ϕ) that are independent of the choice of $J \in \mathcal{J}_\omega^\phi$.*

If $n \in 2\mathbb{Z}$ and $g + l \geq 2$, Theorem 2.4 implies that a real orientation on (X, ω, ϕ) induces an orientation on the real line bundle

$$\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi\big) \otimes \mathfrak{f}^*\big(\Lambda_{\mathbb{R}}^{\mathrm{top}}(T\mathbb{R}\overline{\mathcal{M}}_{g,l})\big) \to \overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi, \tag{2.7}$$

where $\mathfrak{f}$ is the forgetful morphism (2.4). This orientation can be used to construct GW invariants of (X, ω, ϕ) with classes twisted by the orientation system of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$.

For each $i = 1, \ldots, l$, let

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi \to X, \quad \big[u, (z_1^+, z_1^-), \ldots, (z_l^+, z_l^-)\big] \to u(z_i^+),$$

be the evaluation at the first point in the ith pair of conjugate points. For $\mu_1, \ldots, \mu_l \in H^*(X)$, the numbers

$$\langle \mu_1, \ldots, \mu_l \rangle_{g,B}^\phi \equiv \int_{[\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi]} \mathrm{ev}_1^*\mu_1 \cdots \mathrm{ev}_l^*\mu_l \in \mathbb{Q}$$

are virtual counts of real J-holomorphic curves in X passing through generic cycle representatives for the Poincaré duals of $\mu_1, \ldots, \mu_l$, i.e., real GW invariants of (X, ω, ϕ) with conjugate pairs of insertions. They are independent of the choices of cycles representatives and of J.

Moreover, for each $i = 1, \ldots, l$, let

$$\psi_i \in H^2\big(\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi; \mathbb{Q}\big)$$

be the Chern class of the universal cotangent line bundle for the marked point z_i^+. For $a_1, \ldots, a_l \in \mathbb{Z}^{\geq 0}$ and $\mu_1, \ldots, \mu_l \in H^*(X;\mathbb{Q})$, let

$$\big\langle \tau_{a_1}(\mu_1), \ldots, \tau_{a_l}(\mu_l)\big\rangle_{g,B}^\phi = \int_{[\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi]^{\mathrm{vir}}} \psi_1^{a_1}\big(\mathrm{ev}_1^*\mu_1\big) \cdots \psi_l^{a_l}\big(\mathrm{ev}_l^*\mu_l\big) \tag{2.8}$$

be the associated real descendant GW invariant. This number is again independent of the choices of cycle representatives and of $J \in \mathcal{J}_\omega^\phi$.

Given the existence of a full descendant theory, there are many natural questions that arise and that are not well understood at the moment. In particular, they are related to finding structures governing the invariants. One expects to find a real cohomological field theory behind them and a Givental–Teleman-type classification result would be very valuable for reconstruction results, mirror symmetry, and connections to Dubrovin–Zhang-type integrable hierarchies. Further connections to integrable systems that parallel those established in the classical case for KdV, KP, and Toda [25, 31, 32, 40] are also expected.

3. STRUCTURAL RESULTS

Here we consider the real Gromov–Witten theory of real 3-folds which are the total space of real bundles over curves with an antisymplectic involution. The motivation for considering 3-folds of this type comes from the virtual contribution to the real GW invariants of a real elementary curve in a compact real Calabi–Yau 3-fold, sometimes referred to as multiple-covers contribution, and the real Gopakumar–Vafa conjecture [37] expressing the connected real Gromov–Witten invariants in terms of integer invariants.

The invariants associated with this setup are called local real Gromov–Witten (RGW) invariants and are discussed in Section 3.1. They give rise to a semisimple 2D Klein TQFT defined on an extension of the category of unorientable surfaces. This structure allows us to completely solve the theory by providing a closed formula for the local RGW invariants in terms of representation-theoretic data, extending earlier results of Bryan and Pandharipande [9]. The local version of the real Gopakumar–Vafa formula is obtained as a consequence of the structural results. Furthermore, in the case of the resolved conifold, we find that the partition function of the RGW invariants agrees with that of the SO/Sp Chern–Simons theory [3].

3.1. Local real Gromov–Witten invariants

Let (Σ, c) be a symmetric Riemann surface and $L \to \Sigma$ a holomorphic line bundle. Then the total space of

$$L \oplus c^* \bar{L} \to \Sigma, \quad c_{tw}(z; u, v) = \big(c(z); v, u\big) \tag{3.1}$$

is a real manifold with an antiholomorphic involution c_{tw}. An $U(1)$-action on the line bundle $L \to \Sigma$ induces an action on the 3-fold (3.1) compatible with the real structure. We define local (relative) RGW invariants associated to the real 3-fold (3.1) as pairings between the $U(1)$-equivariant Euler class of the index bundle $\operatorname{Ind} \bar{\partial}_L$ (regarded as an element in K-theory) and the virtual fundamental class of the (relative) real moduli space $\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)$ discussed below.

Definition 3.1. Let (Σ, c) be a marked symmetric surface, with r pairs of conjugate marked points $(x_1^+, c(x_1^+)), \ldots, (x_r^+, c(x_r^+))$, and $\vec{\lambda} = (\lambda^1, \ldots, \lambda^r)$ be a collection of r partitions of d. Denote by

$$\overline{\mathcal{M}}_{d,\chi}^{\bullet,c}(\Sigma)_{\lambda^1,\ldots,\lambda^r} \tag{3.2}$$

the relative real moduli space of degree d stable real maps $f : (C, \sigma) \to (\Sigma, c)$ such that

- f has ramification pattern λ^i over x_i^+ (and thus also over $x_i^- = c(x_i^+)$), for all $i = 1, \dots, r$;
- the domain C is possibly disconnected and has total Euler characteristic χ;
- f is nontrivial on each connected component of C.

The moduli space $\overline{\mathcal{M}}_{d,\chi}^{\bullet,c}(\Sigma)_{\lambda^1,\dots,\lambda^r}$ has virtual dimension b, where

$$b = d\chi(\Sigma) - \chi - 2\delta(\vec{\lambda}) \quad \text{and} \quad \delta(\vec{\lambda}) = \sum_{i=1}^{r} \big(d - \ell(\lambda^i)\big). \tag{3.3}$$

Here $\ell(\lambda^i)$ is the length of the partition λ^i, i.e., the cardinality of $f^{-1}(x_i^+)$.

These real moduli spaces are orientable, but a priori the local RGW invariants depend on the choice of real orientation (cf. Definition 2.3) and on the topological type of the real structure c on Σ. We show in [15] that there is a canonical choice of orientation for the local RGW invariants, compatible with the splitting formula (3.21), and, moreover, that they do not depend on the real structure c. We therefore omit these choices from the notation below.

If $L \to \Sigma$ is a holomorphic bundle, the operator $\bar{\partial}_L$ determines a family of complex operators over the moduli spaces of maps to Σ; the fiber at a stable map $f : C \to \Sigma$ is the pullback operator $\bar{\partial}_{f^*L}$. Denote by Ind $\bar{\partial}_L$ the index bundle associated to this family of operators, regarded as an element in K-theory.

Let $\bar{\partial}_{(L\oplus c^*\bar{L}, c_{tw})}$ denote the restriction of $\bar{\partial}_{L\oplus c^*\bar{L}}$ to the invariant part of its domain and target, cf. [18, SECTION 4.3]. Via the projection onto the first factor, the kernel and cokernel of $\bar{\partial}_{(L\oplus c^*\bar{L}, c_{tw})}$ are canonically identified with the kernel and cokernel of $\bar{\partial}_L$ and

$$\operatorname{Ind} \bar{\partial}_{(L\oplus c^*\bar{L}, c_{tw})} \cong \operatorname{Ind} \bar{\partial}_L. \tag{3.4}$$

The right-hand side carries a natural complex structure, which pulls back to one on the left-hand side. An $U(1)$-action on L induces one on $(L \oplus c^*\bar{L}, c_{tw})$, compatible with the real structure. In turn, these induce $U(1)$-actions on Ind $\bar{\partial}_L$ and Ind $\bar{\partial}_{(L\oplus c^*\bar{L}, c_{tw})}$ and the isomorphism (3.4) identifies their equivariant Euler classes.

Definition 3.2. Assume (Σ, c) is a symmetric surface with r pairs of marked points. Let $L \to \Sigma$ be a holomorphic line bundle and $\vec{\lambda} = (\lambda^1, \dots, \lambda^r)$ a collection of r partitions of d. The local real relative GW invariants associated with the real 3-fold $(L \oplus c^*\bar{L}, c_{tw}) \to (\Sigma, c)$ are the equivariant pairings

$$RZ_{d,\chi}^c(\Sigma, L)_{\vec{\lambda}} = \int_{[\overline{\mathcal{M}}_{d,\chi}^{c,\bullet}(\Sigma)_{\vec{\lambda}}]^{\text{vir}}} e_{U(1)}(-\operatorname{Ind} \bar{\partial}_L). \tag{3.5}$$

We further consider the shifted generating function

$$\operatorname{RGW}_d(\Sigma, L)_{\vec{\lambda}} = \sum_{\chi} u^{d(\frac{\chi(\Sigma)}{2} + c_1(L)[\Sigma]) - \frac{\chi}{2} - \delta(\vec{\lambda})} RZ_{d,\chi}^c(\Sigma, L)_{\vec{\lambda}} \in \mathbb{Q}(t)\big((u)\big), \tag{3.6}$$

where $\delta(\vec{\lambda})$ is as in (3.3). It takes values in the localized equivariant cohomology ring of $U(1)$ generated by t.

3.2. TQFT and Klein TQFT

Let $2\mathbf{Cob}$ be the usual (oriented, closed) 2-dimensional cobordism category. It is the symmetric monoidal category with objects given by compact oriented 1-manifolds (without boundary) and morphisms given by (diffeomorphism classes of) oriented cobordisms. A 2-dimensional topological quantum field theory (2D TQFT) with values in a commutative ring R is a symmetric monoidal functor

$$F : 2\mathbf{Cob} \to R\,\mathrm{mod},$$

where $R\,\mathrm{mod}$ is the category of R-modules. This is equivalent to a commutative Frobenius algebra over R; the product and coproduct correspond to the pair of pants while the unit and counit to the cap and cup, respectively. In [9], Bryan and Pandharipande enlarge the category $2\mathbf{Cob}$ to a category $2\mathbf{Cob}^{L_1,L_2}$ with the same objects, but with morphisms decorated by a pair of complex line bundles (L_1, L_2) trivialized over the boundary; the Euler numbers (k_1, k_2) of these bundles determine the level of the theory. Restricting the morphisms to $k_1 = k_2 = 0$ defines an embedding

$$2\mathbf{Cob} \subset 2\mathbf{Cob}^{L_1,L_2}.$$

Bryan and Pandharipande use the local GW invariants to define a symmetric monoidal functor

$$\mathbf{GW} : 2\mathbf{Cob}^{L_1,L_2} \to R\,\mathrm{mod} \tag{3.7}$$

on this larger category. The functor (3.7) extends the classical 2D TQFT that appeared in the work of Dijkgraaf–Witten [12] and Freed–Quinn [14], and whose Frobenius algebra is the center $\mathbb{Q}[S_d]^{S_d}$ of the group algebra of the symmetric group S_d. It is used to completely solve the local Gromov–Witten theory.

A different extension of $2\mathbf{Cob}$ is obtained by allowing unoriented and possibly unorientable surfaces as cobordisms; see [2,4]. We refer to this category as $2\mathbf{KCob}$, where $\mathbf{K}$ stands for Klein (surface). The objects are closed unoriented 1-manifolds and the morphisms are diffeomorphism classes of *unoriented* (and possibly unorientable) cobordisms. An equivalent point of view is to consider the orientation double covers of both the objects and the morphisms: the objects are then closed oriented 1-manifolds with an orientation-reversing involution (deck transformation) exchanging the sheets of the cover and the morphisms are compact oriented 2-dimensional manifolds with a *fixed-point free* orientation-reversing involution extending the one on the boundary. Such 2-dimensional manifolds are called symmetric surfaces, and we denote this category by $2\mathbf{SymCob}$. Moreover,

$$2\mathbf{KCob} \equiv 2\mathbf{SymCob},$$

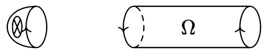
where the identification is obtained by passing to the orientation double cover in one direction and taking the quotient by the involution in the other. Working from the perspective of $2\mathbf{SymCob}$ allows us to construct an extension $2\mathbf{SymCob}^L$ of this category related to that of [9] and completely solve the local real Gromov–Witten theory.

The category 2**Cob** can be regarded as a subcategory of 2**KCob** with the same objects, but fewer morphisms

$$2\mathbf{Cob} \subset 2\mathbf{KCob}.$$

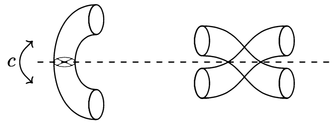
Note that even if a cobordism in 2**KCob** is orientable, there may not be way to orient it in a way compatible with the boundary identifications.

The generators of 2**Cob** ⊂ 2**KCob** are the usual cap, cup, tube, twist, and pair of pants cobordisms, and the corresponding elements of 2**SymCob** are their orientation double covers. The category 2**KCob** has two extra generators, the cross-cap (a Möbius band) and the involution



(3.8)

respectively. In 2**SymCob** these correspond to their orientation double covers



(3.9)

Note that in 2**SymCob** the involution swaps the two outgoing circles – this distinguishes it from the tube which acts as the identity.

The extra generators satisfy certain relations in 2**KCob** (see [4, pp. 1840–1841]). For example, moving a puncture once around the Möbius band changes the orientation of the puncture, cf. Figure 2; equivalently, the involution acts trivially on the product of the cross-cap with another element, cf. (3.13). Another relation comes from decomposing the product of two cross-caps as in Figure 3, cf. (3.14).

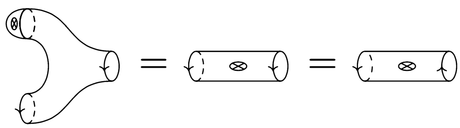


FIGURE 2
Relation in 2**KCob**: involution acts trivially on products with a cross-cap.

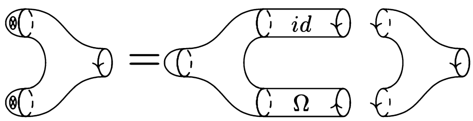


FIGURE 3
Relation in 2**KCob**: decomposing the punctured Klein bottle.

3.2.1. Semisimple Klein TQFT

Definition 3.3. A (closed) 2D Klein TQFT is a symmetric monoidal functor

$$F : 2\mathbf{KCob} \to R\,\mathrm{mod}\,. \tag{3.10}$$

When (3.10) is regarded as a morphism on $2\mathbf{SymCob} \equiv 2\mathbf{KCob}$ via the orientation double cover construction, we denote it by

$$\tilde{F} : 2\mathbf{SymCob} \to R\,\mathrm{mod}\,. \tag{3.11}$$

In fact, cf. [4, PROPOSITION 1.11], a (closed) 2D Klein TQFT is equivalent to a commutative Frobenius algebra $H = F(S^1)$ together with two extra structures:

(a) an involutive (anti)automorphism Ω of the Frobenius algebra H, denoted $x \mapsto x^*$. This means

$$(x^*)^* = x, \quad (xy)^* = y^*x^* \quad \text{and} \quad \langle x^*, y^*\rangle = \langle x, y\rangle \quad \text{for all } x, y \in H. \tag{3.12}$$

(b) an element $U \in H$ such that

$$(aU)^* = aU \quad \text{for all } a \in H \quad \text{and} \tag{3.13}$$

$$U^2 = m(id \otimes \Omega)\big(\Delta(1)\big) = \sum \alpha_i \beta_i^*. \tag{3.14}$$

Here the coproduct is $\Delta(1) = \sum \alpha_i \otimes \beta_i$. The involution Ω and the element U correspond to the cobordisms (3.8). The relations (b) correspond to Figures 2 and 3.

Definition 3.4. A semisimple Klein TQFT is a Klein TQFT whose associated Frobenius algebra is semisimple.

A semisimple TQFT is determined by the structure constants $\{\lambda_\rho\}$, i.e., the coefficients of the comultiplication $\Delta(v_\rho) = \lambda_\rho v_\rho \otimes v_\rho$ in the idempotent basis $\{v_\rho\}$. Moreover,

Proposition 3.5 ([15, PROPOSITION 7.4]). *Assume (3.10) is a semisimple KTQFT with idempotent basis $\{v_\rho\}$ and structure constants $\{\lambda_\rho\}$, and assume that the ground ring R has no zero divisors. Then*

(i) *Ω defines an involution on the idempotent basis $\Omega(v_\rho) = v_{\rho^*}$.*

(ii) *If $U = \sum_\rho U_\rho v_\rho$ then $U_\rho^2 = \lambda_\rho$ if $\rho = \rho^*$, and $U_\rho = 0$ if $\rho \neq \rho^*$.*

Assume Σ is a closed symmetric surface, considered as a morphism in $2\mathbf{SymCob}$ from the ground ring to the ground ring.

Corollary 3.6 ([15, COROLLARY 7.5]). *With the notation of Proposition 3.5, the morphism (3.11) is given by:*

$$\tilde{F}(\Sigma) = \sum_{\rho=\rho^*} U_\rho^{g-1}, \quad \textit{when } \Sigma \textit{ is a connected genus } g \textit{ surface, and}$$

$$\tilde{F}(\Sigma \sqcup \overline{\Sigma}) = \sum_\rho \lambda_\rho^{g-1}, \quad \textit{when } \Sigma \sqcup \overline{\Sigma} \textit{ is a } g\textit{-doublet.}$$

3.2.2. The category $2\mathbf{SymCob}^L$

Consider the category $2\mathbf{SymCob}^L$ whose objects are disjoint unions of copies of $\mathcal{S} = (S^1 \sqcup \overline{S^1}, \varepsilon)$, where ε swaps the two components, and morphisms correspond to isomorphism classes relative boundary of decorated cobordisms $W = (\Sigma, c, L)$, where Σ is an oriented cobordism with a fixed-point free orientation-reversing involution c, extending ε, and L is a complex line bundle over Σ, trivialized along the boundary of Σ.

The level 0 theory corresponds to a trivial bundle L, and defines embeddings

$$2\mathbf{Cob} \subset 2\mathbf{KCob} \equiv 2\mathbf{SymCob} \subset 2\mathbf{SymCob}^L. \tag{3.15}$$

The doubling procedure defines an embedding

$$2\mathbf{Cob}^{L_1,L_2} \subset 2\mathbf{SymCob}^L, \quad (\Sigma, L_1, L_2) \mapsto (\Sigma \sqcup \overline{\Sigma}, L_1 \sqcup \bar{L}_2). \tag{3.16}$$

The category $2\mathbf{Cob}^{L_1,L_2}$ has 4 extra generators, the level $(\pm 1, 0)$, $(0, \pm 1)$-caps, besides those of $2\mathbf{Cob}$, cf. [9, SECTION 4.3]. Similarly, the generators of the category $2\mathbf{SymCob}^L$ are those of $2\mathbf{SymCob}$ together with the images of the $(\pm 1, 0)$, $(0, \pm 1)$-caps under (3.16).

Proposition 3.7 ([15, PROPOSITION 7.6]). *A symmetric monoidal functor*

$$F : 2\mathbf{SymCob}^L \to R\,\mathrm{mod} \tag{3.17}$$

is uniquely determined by the level 0 *theory and the images* η *and* $\bar{\eta}$ *of the level* $(-1, 0)$ *and* $(0, -1)$*-caps.*

If the restriction of (3.17) to the level 0 theory defines a semisimple KTQFT with idempotent basis $\{v_\rho\}$ let

$$\eta = \sum_\rho \eta_\rho v_\rho \quad \text{and} \quad \bar{\eta} = \sum_\rho \bar{\eta}_\rho v_\rho. \tag{3.18}$$

As in Corollary 3.6, then the value of F on a closed connected genus g symmetric surface Σ at level $k = c_1(L)[\Sigma]$ is equal to

$$F(\Sigma|L) = \sum_{\rho=\rho^*} U_\rho^{g-1} \eta_\rho^{-k}. \tag{3.19}$$

The value of F on a g-doublet $\Sigma \sqcup \overline{\Sigma}$ with a line bundle $L_1 \sqcup L_2$ is similarly equal to

$$F(\Sigma \sqcup \overline{\Sigma}\,|L_1, L_2) = \sum_\rho \lambda_\rho^{g-1} \eta_\rho^{-k_1} \bar{\eta}_\rho^{-k_2},$$

where $k_1 = c_1(L_1)[\Sigma]$ and $k_2 = c_1(L_2)[\overline{\Sigma}]$.

3.3. Splitting formulas

Let (Σ_0, c_0) be a nodal symmetric surface with a pair of conjugate nodes and r pairs of conjugate marked points. It has a normalization $(\widetilde{\Sigma}, \tilde{c})$ which has $r + 2$ pairs of conjugate marked points. Similarly, (Σ_0, c_0) has a family of smooth deformations $(\mathcal{F}, c_{\mathcal{F}}) = \bigcup_s (\Sigma_s, c_s)$, simultaneously smoothing out the conjugate nodes using complex conjugate gluing parameters. The generic fiber (Σ_s, c_s) of the family is a symmetric surface with r

pairs of conjugate marked points, and a pair of "splitting circles" (disjoint vanishing cycles) swapped by the involution; as the gluing parameters converge to 0, these circles pinch to produce the two complex conjugate nodes of Σ_0; see Figure 4. Any complex line bundle L over Σ_s can be deformed to the nodal surface and then lifted to its normalization to give a line bundle $\tilde{L}$ over $\widetilde{\Sigma}$.

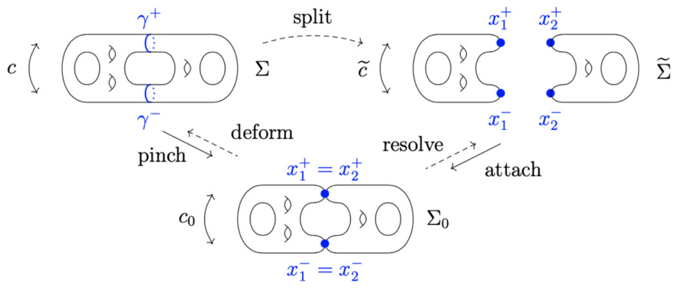


FIGURE 4
Splitting Σ along a pair of conjugate circles (γ^+, γ^-).

In order to state the splitting theorem in a more compact form, we define the raising of the indices by the formula

$$\mathrm{RGW}(\Sigma, L)^{\nu^1\ldots\nu^s}_{\mu^1\ldots\mu^r} = \mathrm{RGW}(\Sigma, L)_{\mu^1\ldots\mu^r,\nu^1\ldots\nu^s}\left(\prod_{i=1}^{s}\zeta(\nu^i)t^{2\ell(\nu^i)}\right), \tag{3.20}$$

where $\zeta(\lambda) = \prod m_k! k^{m_k}$ for a partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$.

Theorem 3.8 (RGW splitting theorem, [16]). *Assume (Σ, c) is a marked symmetric surface with r pairs of conjugate points and L is a complex line bundle over Σ. Let $(\widetilde{\Sigma}, \tilde{c})$ denote the symmetric surface obtained as described above from (Σ, c) by splitting it along two conjugate circles, and let $\tilde{L}$ be the corresponding line bundle over $\widetilde{\Sigma}$.*

Then for any collection $\vec{\mu} = (\mu^1, \ldots, \mu^r)$ of r partitions of d, the RGW invariants (3.6) satisfy

$$\mathrm{RGW}_d(\Sigma, L)_{\vec{\mu}} = \sum_{\lambda \vdash d} \mathrm{RGW}_d(\widetilde{\Sigma}, \tilde{L})^{\lambda}_{\vec{\mu},\lambda}. \tag{3.21}$$

This result is used to show that the local RGW theory gives rise to (an extension of) a KTQFT; it corresponds to compatibility of cobordism decompositions.

3.4. The RGW Klein TQFT

In this section we use the local RGW invariants (3.6) to define an extension of a Klein TQFT, i.e., a functor **RGW** from the category $2\mathbf{SymCob}^L$ described in Section 3.2.2. This extends the Bryan–Pandharipande TQFT constructed from the GW theory for the anti-diagonal action.

Let $R = \mathbb{C}(t)((u))$ be the ring of Laurent series in u whose coefficients are rational functions of t and d be a positive integer. Denote by $\mathcal{S} = (S^1 \sqcup \overline{S^1}, \varepsilon)$ the disjoint union of two copies of a circle with opposite orientations and with the involution ε swapping them. To the object $\mathcal{S}$ we associate

$$\mathbf{RGW}_d(\mathcal{S}) = H = \bigoplus_{\alpha \vdash d} R e_\alpha, \tag{3.22}$$

the free module with basis $\{e_\alpha\}_{\alpha \vdash d}$ indexed by partitions α of d. Let

$$\mathbf{RGW}_d(\mathcal{S} \sqcup \cdots \sqcup \mathcal{S}) = H \otimes \cdots \otimes H.$$

To each cobordism $W = (\Sigma, c, L)$ in $2\mathbf{SymCob}^L$ from n copies of $\mathcal{S}$ to m copies of $\mathcal{S}$, associate the R-module homomorphism

$$\mathbf{RGW}_d(W) : H^{\otimes n} \to H^{\otimes m} \tag{3.23}$$

defined by

$$e_{\lambda^1} \otimes \cdots \otimes e_{\lambda^n} \mapsto \sum_{\mu^i \vdash d} \mathrm{RGW}_d(\Sigma_W | L_W)^{\mu^1..\mu^m}_{\lambda^1..\lambda^n}\ e_{\mu^1} \otimes \cdots \otimes e_{\mu^m}.$$

Here Σ_W is a closed marked symmetric Riemann surface whose topological type is that of Σ after removing small disks around the pairs of marked points and $L_W \to \Sigma_W$ is a holomorphic line bundle whose first Chern class corresponds to the Euler class of $L \to \Sigma$.

Theorem 3.9 ([15, THEOREM 8.1]). *The assignment* (3.23) *defines a symmetric monoidal functor*

$$\mathbf{RGW}_d : 2\mathbf{SymCob}^L \to R\,\mathrm{mod}\,. \tag{3.24}$$

Its restriction to $2\mathbf{KCob}$ *under* (3.15) *is a Klein TQFT, while its restriction to* $2\mathbf{Cob}^{L_1,L_2}$ *under* (3.16) *is*

$$\mathbf{RGW}_d(\Sigma \sqcup \overline{\Sigma} | L_1 \sqcup \bar{L}_2)(u,t) = (-1)^{dk_2} \mathbf{GW}_d(\Sigma | L_1, L_2)(iu, it). \tag{3.25}$$

Here k_i *is the total degree of* L_i *and* $\mathbf{GW}_d$ *is the TQFT* (3.7) *considered by Bryan–Pandharipande (for the antidiagonal action).*

The KTQFT determined by the level 0 local RGW invariants is semisimple, cf [15]. It corresponds in fact to *signed* counts of degree d real Hurwitz covers. The idempotent basis is indexed by irreducible representations of the symmetric group S_d and $\Omega(v_\rho) = v_{\rho'}$ where ρ' is the conjugate representation. In order to calculate the coefficients of U in the idempotent basis, we introduced in [15] the signed Frobenius–Schur indicator (SFS). The SFS takes values $0, \pm 1$ on irreducible real representations, unlike the standard FS indicator which is $+1$ on them. The SFS is 0 if and only if the representation is *not* self-conjugate and the sign of a self-conjugate representation is given as a function of its characters. While these considerations are valid for real representations of any finite group, in the case of the symmetric group we find a simpler expression for the latter function using the Weyl formula for B_n. In particular, for an irreducible self-conjugate representation ρ of S_d,

$$\mathrm{SFS}(\rho) = (-1)^{(d - r(\rho))/2},$$

where $r(\rho)$ is the rank of ρ, i.e., the length of the main diagonal of the Young diagram associated to ρ. This is precisely the sign that appears in the partition function of the SO/Sp Chern–Simons theory **[3, (6.1)]**; in the case of the resolved conifold, Theorems 3.12 and 3.13 below recover the partition function **[3, (6.3)]** and the free energy **[3, (3.2)]**, respectively.

The idempotent basis for the theory is given by

$$v_\rho = \frac{\dim \rho}{d!} \sum_\alpha (-t)^{\ell(\alpha)-d} \chi_\rho(\alpha) e_\alpha, \tag{3.26}$$

indexed by the irreducible representations ρ of S_d. We then have the following results.

Lemma 3.10 (**[15, Lemma 9.2]**). *In the idempotent basis $\{v_\rho\}$, the structure constants $\{\lambda_\rho\}$ and the coefficients $\{\eta_\rho\}$, $\{\bar\eta_\rho\}$ of the $(-1,0)$ and $(0,-1)$-caps are given by*

$$\lambda_\rho = t^{2d}\left(\frac{d!}{\dim \rho}\right)^2, \quad \eta_\rho = t^d Q^{c_\rho/2}\left(\frac{\operatorname{dimh}_Q \rho}{\dim \rho}\right), \quad \bar\eta_\rho = t^d Q^{-c_\rho/2}\left(\frac{\operatorname{dimh}_Q \rho}{\dim \rho}\right). \tag{3.27}$$

Here $Q = e^u$, c_ρ is the total content of the Young diagram associated to ρ, and

$$\operatorname{dimh}_Q \rho = d! \prod_{\square\in\rho}\left(2\sinh\frac{h(\square)u}{2}\right)^{-1} = d! \prod_{\square\in\rho}\left(Q^{\frac{h(\square)}{2}} - Q^{-\frac{h(\square)}{2}}\right)^{-1}, \tag{3.28}$$

where $h(\square)$ denotes the hooklength of the square $\square$ in the Young diagram associated to ρ.

Proposition 3.11 (**[15, Corollary 9.7]**). *In the idempotent basis, the level 0 cross-cap U is given by*

$$U = \sum_{\substack{\rho\vdash d\\ \rho=\rho'}} (-1)^{(d-r(\rho))/2}\, t^d \frac{d!}{\dim \rho} v_\rho, \tag{3.29}$$

where $r(\rho)$ is the length of the main diagonal of the Young diagram of ρ.

Combining these results with the results of Section 3.2 we obtain a closed expression for the local RGW theory of the 3-folds (3.1) in terms of representation theoretic data. The Calabi–Yau case is given in the following theorem.

Theorem 3.12 (**[15, Lemma 9.14]** (Local CY)). *Let Σ be a connected genus g symmetric surface and $L \to \Sigma$ a holomorphic line bundle with Chern number $g-1$. Then the generating function of the degree d local RGW invariants is equal to*

$$\mathrm{RGW}_d(\Sigma, L) = \sum_{\rho=\rho'}\left((-1)^{\frac{d-r(\rho)}{2}} \prod_{\square\in\rho} 2\sinh\frac{h(\square)u}{2}\right)^{g-1}.$$

Here the sum is over all self-conjugate partitions ρ of d, the product is over all boxes $\square$ in the Young diagram of ρ, $h(\square)$ is the hooklength of $\square$, and $r(\rho)$ is the length of the main diagonal of the Young diagram of ρ.

3.5. Real Gopakumar–Vafa formula

The local RGW invariants correspond to possibly disconnected counts. As usual they can be expressed in terms of more basic invariants. In the real GW theory, these basic counts come in two flavors, $\mathrm{CRGW}_d(\Sigma, L)$ and $\mathrm{DRGW}_d(\Sigma, L)$, corresponding to maps from connected real domains and respectively from doublet domains, i.e., domains consisting of two copies of a connected surface with opposite complex structures and the real structure exchanging the two copies. In fact,

$$1 + \sum_{d=1}^{\infty} \mathrm{RGW}_d(\Sigma, L)q^d = \exp\left(\sum_{d=1}^{\infty} \mathrm{CRGW}_d(\Sigma, L)q^d + \sum_{d=1}^{\infty} \mathrm{DRGW}_{2d}(\Sigma, L)q^{2d}\right).$$

Furthermore, the doublet invariants are related to half of the complex GW invariants whenever the target Σ is connected,

$$\mathrm{DRGW}_{2d}(\Sigma, L)(u,t) = (-1)^{d(k+1-g)} \frac{1}{2} \mathrm{GW}_d^{\mathrm{conn}}(g|k,k)(iu, it),$$

where g is the genus of Σ, $k = c_1(L)[\Sigma]$ is the degree of L, and $\mathrm{GW}_d^{\mathrm{conn}}(g|k,k)$ are the connected invariants defined in [9] for the anti-diagonal action.

As a consequence of the structure result provided by Theorem 3.12, we obtain the local real Gopakumar–Vafa formula (cf. [37, SECTION 5]). The local GV conjecture in the classical setting, proved in [22, PROPOSITION 3.4], states that the connected GW invariants defined in [9] have the following structure:

$$\sum_{d=1}^{\infty} \mathrm{GW}_d^{\mathrm{conn}}(g|g-1, g-1)(u)q^d = \sum_{d=1}^{\infty} \sum_h n_{d,h}^{\mathbb{C}}(g) \sum_{k=1}^{\infty} \frac{1}{k}\left(2\sin\left(\frac{ku}{2}\right)\right)^{2h-2} q^{kd}, \tag{3.30}$$

where the coefficients $n_{d,h}^{\mathbb{C}}(g)$, called the local BPS states, satisfy (i) $n_{d,h}^{\mathbb{C}}(g) \in \mathbb{Z}$ and (ii) for each d, $n_{d,h}^{\mathbb{C}}(g) = 0$ for large h.

In the real setting, the local real GV formula takes the following form.

Theorem 3.13 ([15, THEOREM 10.1] (Local real GV formula)). *Fix a genus g symmetric surface Σ and consider the local real Calabi–Yau 3-fold $(L \oplus c^*\bar{L}, c_{tw}) \to \Sigma$. Then the generating function for the connected local RGW invariants has the following structure:*

$$\sum_{d=1}^{\infty} \mathrm{CRGW}_d(\Sigma|L)(u)q^d = \sum_{d=1}^{\infty}\sum_{h=0}^{\infty} n_{d,h}^{\mathbb{R}}(g) \sum_{\substack{k \text{ odd} \\ k>0}} \frac{1}{k}\left(2\sinh\left(\frac{ku}{2}\right)\right)^{h-1} q^{kd}, \tag{3.31}$$

where the coefficients $n_{d,h}^{\mathbb{R}}(g)$ satisfy (i) (integrality) $n_{d,h}^{\mathbb{R}}(g) \in \mathbb{Z}$, (ii) (finiteness) for each d, $n_{d,h}^{\mathbb{R}}(g) = 0$ for large h, and (iii) (parity) $n_{d,h}^{\mathbb{R}}(g) = n_{d,h}^{\mathbb{C}}(g) \mod 2$. Moreover,

(a) *for $g = 0$, $n_{d,h}^{\mathbb{R}}(0) = 1$ when $d = 1$ and $h = 0$ and vanish otherwise.*

(b) *for $g = 1$, $n_{d,h}^{\mathbb{R}}(1) = (-1)^{d-1}$ when $h = 1$ and vanish otherwise.*

(c) *for any $g \geq 0$, $n_{1,h}^{\mathbb{R}}(g) = 1$ when $h = g$ and vanish otherwise.*

The $g = 0$ case of Theorems 3.12, 3.13 give the real Gromov–Witten invariants of the resolved conifold and coincide with the SO/Sp Chern–Simons theory on S^3. This is an instance of the real analogue of the large N-duality [20, 28]. Developing a mathematical theory of the real topological vertex [1, 26] would allow establishing this correspondence for any toric real Calabi–Yau 3-fold. Furthermore, a relation between Kauffman polynomials and real GW invariants is also expected based on this duality and it would be very interesting to investigate the potential implications of such a relation.

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