KÄHLER MANIFOLDS WITH CURVATURE BOUNDED BELOW

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ABSTRACT

This is a survey of certain Kähler manifolds with curvature bounded below. The topics include: (1) the uniformization conjecture of Yau, as well as its related problems; (2) compactification of certain Kähler manifolds of nonnegative curvature; and (3) Gromov–Hausdorff limits of Kähler manifolds.

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1. INTRODUCTION

The study of manifolds with a curvature lower bound has a long history in Riemannian geometry. For instance, we have the comparison theorems, Cheeger–Gromoll splitting theorem [5], Cheng–Yau gradient estimate [18], Li–Yau's heat kernel estimate [44], and many other important theorems. These theorems have been the basic tools to study manifolds with curvature lower bound. In 1981, Gromov proposed a fundamental notion, called the Gromov–Hausdorff convergence. Later, Cheeger, Colding, Tian, Naber [6–9, 19–22] developed an important theory studying the limit of under the Gromov–Hausdorff convergence.

In this survey, we shall consider the applications of Gromov–Hausdorff convergence theory to some problems in Kähler geometry. These include: (1) the uniformization conjecture of Yau, as well as its related problems; (2) compactification of certain noncompact Kähler manifolds of nonnegative curvature; and (3) the structure of Gromov–Hausdorff limits of Kähler manifolds.

2. YAU'S UNIFORMIZATION CONJECTURE AND ITS RELATED PROBLEMS

The classical uniformization theorem states that a simply connected Riemann surface is isomorphic to the Riemann sphere $\mathbb{CP}^1$, the Poincare disk $\mathbb{D}^2$, or the complex plane $\mathbb{C}$. A geometric consequence is that a complete orientable Riemannian surface of positive curvature is necessarily conformal to $\mathbb{CP}^1$ or $\mathbb{C}$. An orientable Riemannian surface can be regarded as a Kähler manifold of complex dimension 1. A natural question is to generalize such result to higher dimensional Kähler manifolds. The curvature we adopt here is the so-called (holomorphic) bisectional curvature.

Definition 2.1 ([42, 64]). On a Kähler manifold M^n, we say the bisectional curvature is greater than or equal to K ($BK \geq K$) if

$$\frac{R(X, \overline{X}, Y, \overline{Y})}{\|X\|^2\|Y\|^2 + |\langle X, \overline{Y}\rangle|^2} \geq K \tag{2.1}$$

for any two nonzero vectors $X, Y \in T^{1,0}M$.

Observe that the equality holds for complex space forms. The bisectional curvature lower bound condition is weaker than the sectional curvature lower bound, while stronger than the Ricci curvature lower bound.

So the question above can be refined as the classification of Kähler manifolds with positive bisectional curvature. In the compact case, the famous Frankel conjecture, solved by Mori [47] and Siu–Yau [57] independently, states that a compact Kähler manifold of positive bisectional curvature is biholomorphic to $\mathbb{CP}^n$ (in fact, Mori proved a stronger result). The noncompact analogue was proposed by Yau [66] in the 1970s; he asked whether or not a complete noncompact Kähler manifold with positive bisectional curvature is biholomorphic to a complex Euclidean space. For this, Yau further asked in [66] (see also [67, PAGE 117]) whether or not the ring of polynomial growth holomorphic functions is finitely generated, and whether or not the dimension of the spaces of holomorphic functions of polynomial growth is bounded from above by the dimension of the corresponding spaces of polynomials on $\mathbb{C}^n$.

On a complete Kähler manifold M, we say a holomorphic function $f \in \mathcal{O}_d(M)$ if there exists some $C > 0$ with $|f(x)| \leq C(1 + d(x, x_0))^d$ for all $x \in M$. Here x_0 is a fixed point on M. Let $\mathcal{O}_P(M) = \bigcup_{d>0} \mathcal{O}_d(M)$. If one wishes to prove the uniformization conjecture by considering $\mathcal{O}_P(M)$, it is important to know when $\mathcal{O}_P(M) \neq \mathbb{C}$. In [49], Ni proposed an interesting conjecture in this direction. Let us summarize the problems in the four conjectures below:

Conjecture 1. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then given any $d > 0$,* $\dim(\mathcal{O}_d(M)) \leq \dim(\mathcal{O}_d(\mathbb{C}^n))$.

Conjecture 2. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume M has positive bisectional curvature at one point p. Then the following three conditions are equivalent:*

(1) $\mathcal{O}_P(M) \neq \mathbb{C}$*;*

(2) *M has maximal volume growth;*

(3) *There exists a constant C independent of r so that $⨍_{B(p,r)} S \leq \frac{C}{r^2}$. Here S is the scalar curvature; $⨍$ means the average.*

Conjecture 3. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then the ring $\mathcal{O}_P(M)$ is finitely generated.*

Conjecture 4. *Let M^n be a complete noncompact Kähler manifold with positive bisectional curvature. Then M is biholomorphic to $\mathbb{C}^n$.*

Conjecture 1 was confirmed by Ni [49] with the assumption that M has maximal volume growth. Later, by using Ni's method, Chen–Fu–Le–Zhu [11] removed the extra condition. The key of Ni's method is a monotonicity formula for the heat flow on a Kähler manifold with nonnegative bisectional curvature. In [34], we discovered a logarithmic convexity result on Kähler manifolds with nonnegative holomorphic sectional curvature. This turns out to be very useful in dealing the problems above.

Definition 2.2. Let M be a complete Kähler manifold. We say that M satisfies the three circle theorem if, for any point $p \in M$, $r > 0$, any holomorphic function f on $B(p, r)$, $\log M_f(r)$ is a convex function of $\log r$. In other words, for $r_1 < r_2 < r_3$,

$$\log\left(\frac{r_3}{r_1}\right)\log M_f(r_2) \leq \log\left(\frac{r_3}{r_2}\right)\log M_f(r_1) + \log\left(\frac{r_2}{r_1}\right)\log M_f(r_3). \tag{2.2}$$

Here $M_f(r) = \sup|f(x)|$ for $x \in B(p, r)$.

Theorem 2.1. *Let M be a complete Kähler manifold. Then M satisfies the three circle theorem if and only if the holomorphic sectional curvature is nonnegative.*

The proof is a simple combination of Hessian comparison and a maximum principle argument.

Corollary 2.1. *Let M be a complete Kähler manifold with nonnegative holomorphic sectional curvature. If $f \in \mathcal{O}_d(M)$, then $\frac{M_f(r)}{r^d}$ is nonincreasing.*

Proof of the corollary. We need to show that

$$\frac{M_f(r_1)}{r_1^d} \geq \frac{M_f(r_2)}{r_2^d}$$

for $r_1 \leq r_2$. By rescaling, we may assume $r_1 = 1$. By the assumption, given any $\varepsilon > 0$, there exists a sequence $\lambda_j \to \infty$ such that

$$\log M_f(\lambda_j) \leq \log M_f(1) + (d + \varepsilon) \log \lambda_j.$$

If we take $r_3 = \lambda_j$ sufficiently large, then

$$M_f(r_2) \leq M_f(r_1) r_2^{d+\varepsilon}.$$

The corollary follows letting $\varepsilon \to 0$. ∎

Proof of Conjecture 1. Suppose the inequality fails for some $d > 0$. By linear algebra, at any point $p \in M$, there exists a nonzero holomorphic function $f \in \mathcal{O}_d(M)$ such that the vanishing order at p is at least $[d] + 1$ where $[d]$ is the greatest integer less than or equal to d. Therefore

$$\lim_{r \to 0^+} \frac{M_f(r)}{r^d} = 0.$$

Corollary 2.1 says that $\frac{M_f(r)}{r^d}$ is nonincreasing. Thus $f \equiv 0$. This is a contradiction. ∎

Now we come to Conjecture 2. Ni and Tam have made important contributions. For example, they proved that (1) is equivalent to (3). In **[35, 36]**, we were able to prove the equivalence of (1) and (2).

Theorem 2.2. *Let (M^n, g) be a complete Kähler manifold with nonnegative bisectional curvature. Assume the universal cover does not split as a product and there exists a nonconstant holomorphic function of polynomial growth on M, then M has maximal volume growth.*

Theorem 2.3. *Let (M^n, g) be a complete Kähler manifold with nonnegative bisectional curvature and maximal volume growth. Then there exists a nonconstant holomorphic function of polynomial growth on M.*

Let us sketch the proof of Theorem 2.2. By a result of Ni and Tam, we have that $\dim(\mathcal{O}_d(M)) \geq cd^n$, where c is independent of n. Assume M does not have maximal volume growth. We can look at a tangent cone of M at infinity. According to Cheeger–Colding, the tangent cone has Hausdorff dimension at most $2n - 1$. Then we pick a regular point on the tangent cone. The tangent cone at that regular point would be $\mathbb{R}^k$, where $1 \leq k \leq 2n - 1$. By the three circle theorem, we were able to pass these polynomial growth holomorphic functions to that Euclidean space without changing the growth rate and the linear independence. If $k < n$, then we obtain a contradiction, by counting the dimension of harmonic functions. However, if $k \geq n+1$, then this will not work. In this situation, we managed to find some partial complex structure on $\mathbb{R}^k$. This gave extra restriction, which sufficed for the proof.

The existence of polynomial growth holomorphic functions is also very interesting. Usually, this is done by finding plurisubharmonic functions of logarithmic growth. Here we took a different approach. The idea is to look at a tangent cone of M at infinity. Essentially, we managed to establish a complex analytic structure on a tangent cone. Then by pulling back those functions to M, we obtained holomorphic functions on a larger and larger domain. Then the three circle theorem ensured that we can take a subsequence and obtain a nontrivial polynomial growth holomorphic function. The tools we used were the Gromov–Hausdorff convergence theory, Hörmander L^2-estimate of $\bar{\partial}$ [30], heat flow technique by Ni and Tam [52, 53], and the three circle theorem.

Now we come to Conjecture 3. In [45], Mok proved the following:

Theorem 2.4 (Mok). *Let M^n be a complete noncompact Kähler manifold with positive bisectional curvature such that for some fixed point $p \in M$,*

(1) *Scalar curvature $\leq \frac{C_0}{d(p,x)^2}$ for some $C_0 > 0$;*

(2) $\operatorname{vol}(B(p,r)) \geq C_1 r^{2n}$ *for some $C_1 > 0$.*

Then M^n is biholomorphic to an affine algebraic variety.

In Mok's proof, the biholomorphism was given by holomorphic functions of polynomial growth. Therefore, $\mathcal{O}_P(M)$ is finitely generated. In the general case, it was proved by Ni [49] that the transcendental dimension of $\mathcal{O}_P(M)$ over $\mathbb{C}$ is at most n. In [36], we confirmed Conjecture 2:

Theorem 2.5. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then the ring $\mathcal{O}_P(M)$ is finitely generated.*

During the course of the proof, we obtained a partial result for Conjecture 4:

Theorem 2.6. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume M is of maximal volume growth, then M is biholomorphic to an affine algebraic variety.*

Our idea is based on the resolution of Conjecture 2. Assume M admits a nontrivial polynomial growth holomorphic function. If we cheat a little bit, say the universal cover does not split, then M is necessarily of maximal volume growth. The main point is to prove that there exist finitely many polynomial growth holomorphic functions $(f_1, \ldots, f_k)$ such that the mapping is proper. Again this heavily uses the convergence theory and the three circle theorem.

Now we come to Conjecture 4. So far, this conjecture is still open. However, there has been much important progress due to various authors. In earlier works, Mok–Siu–Yau [48] and Mok [45] considered embedding by using holomorphic functions of polynomial growth. Later, with the Kähler–Ricci flow (Chern–Ricci flow), results were improved significantly. See, for example, [23–25,29,31,50,55,56] for related works. In [38], we obtained

Theorem 2.7. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume M has maximal volume growth, then M is biholomorphic to $\mathbb{C}^n$. In fact, we can find n polynomial growth holomorphic functions $f_1, \dots, f_n$ which serve as the biholomorphism.*

Remark 2.1. Note that Theorem 2.7 was also proved in [43] by M. C. Li and L. F. Tam. The proof is different from ours.

Let f be a polynomial growth holomorphic function on M. We define the degree of f be the infimum of $d > 0$ such that $f \in \mathcal{O}_d(M)$.

Corollary 2.2. *Under the same assumption as above, if f is a nonconstant polynomial growth holomorphic function on M with minimal degree, then $df \neq 0$ at any point.*

Now let us explain the basic strategy to Theorem 2.7. We follow [26,27,36,37] closely. Recall under the same assumption of Theorem 2.7, it was proved in [36] that the manifold is biholomorphic to an affine algebraic variety. How to prove that the affine variety is in fact $\mathbb{C}^n$? If $n = 2$, Ramanujam's result says that an algebraic surface homeomorphic to $\mathbb{R}^4$ is necessarily isomorphic to $\mathbb{C}^2$. Unfortunately, there is no such criterion in higher dimensions. Moreover, the argument in [36] does not provide information about the topology of the manifold.

Consider a tangent cone V of M at infinity. That is, there exists $r_i \to \infty$ so that the sequence $(M_i, p_i, d_i) = (M, p, \frac{d}{r_i})$ converges to V in the pointed Gromov–Hausdorff sense. Cheeger–Colding theory asserts that V is a metric cone. Let r be the distance to the vertex. Then the vector field $-r\frac{\partial}{\partial r}$ retracts V to the vertex. A key new idea is to solve $\bar{\partial}$ equation on the holomorphic tangent bundle. More precisely, we constructed holomorphic vector fields Z_i on $B(p_i, 1)$ so that in a natural sense, $Re Z_i$ converges to $-r\frac{\partial}{\partial r}$. By using some complex-analytic techniques, we managed to prove that the flow generated by $Re Z_i$ contracts a domain containing $B(p_i, \frac{1}{2})$ to a point. Since $B(p_i, \frac{1}{2})$ exhausts M, we see M is, in fact, exhausted by topological balls. Then by Stalling's result, the manifold is diffeomorphic to $\mathbb{R}^{2n}$. As we see before, if $n = 2$, the manifold is biholomorphic to $\mathbb{C}^2$.

Recall that the domain of Z_i exhausts M. However, it seems difficult to glue these Z_i together. A technical reason is that the unique zero point of Z_i might diverge to infinity.

There are two possible ways to get around this difficulty. One is to prove that the tangent cone V is complex-analytically smooth. Eventually, by using results in algebraic geometry, we can prove that if $n \leq 3$, then V is complex-analytically smooth. Then it is relatively easily to prove that M is biholomorphic to $\mathbb{C}^n$. Unfortunately, the algebro-geometric method fails for higher dimensions.

Another approach is to construct a nice *global* holomorphic vector field on M. This is how we prove Theorem 2.7. A key point is to study a linear space Z consisting of holomorphic vector fields on M so that the action (derivative) on any polynomial growth holomorphic function preserves the degree. It turns out that Z has finite dimension. Arguing by contradiction, we managed to prove that in Z there exists a global holomorphic vector field which contracts M to a point. This gives us the desired biholomorphism from M to $\mathbb{C}^n$. A detailed analysis also gives "canonical" holomorphic coordinate on M.

Finally, let us mention that there have been important progress of Chen–Zhu on the uniformization conjecture without assuming maximal volume growth condition. The reader is referred to [25] for the details.

3. COMPACTIFICATION OF CERTAIN KÄHLER MANIFOLDS OF NONNEGATIVE CURVATURE

In this section, we extend some techniques in [34–37] to study the compactification of certain complete Kähler manifolds with nonnegative Ricci curvature.

In [39], we proved the following

Theorem 3.1. *Let (M^n, g) $(n \geq 2)$ be a complete noncompact Kähler manifold with nonnegative Ricci curvature and maximal volume growth. Fix a point $p \in M$ and set $r(x) := d_g(x, p)$, where d_g denotes the distance with respect to g. Then*

- (I) *M is biholomorphic to a Zariski open set of a Moishezon manifold if, for some $\varepsilon > 0$, the bisectional curvature $BK \geq -\frac{C}{r^{2+\varepsilon}}$. In fact, on M, the ring of polynomial growth holomorphic functions is finitely generated;*
- (II) *If $BK \geq -\frac{C}{r^2}$ and M has a unique tangent cone at infinity, then M is biholomorphic to a Zariski open set of a Moishezon manifold;*
- (III) *M is quasiprojective if the Ricci curvature is positive and $|\mathrm{Rm}| \leq \frac{C}{r^2}$.*

Part (II) states that one can relax the decay assumption on BK in part (I) by assuming the existence of a unique tangent cone at infinity in order to reach the same conclusion. It is desirable to remove the uniqueness of the tangent cone in part (II).

Part (I) is a generalization of Theorem 2.6. Part (II) is connected with many previous results. For instance, Bando–Kasue–Nakajima [2], examples of Tian–Yau [60, 61], Tian [59], and more recent results of Conlon and Hein [12–14]. Part (III) generalizes a theorem of Mok [46, MAIN THEOREM]. There, the same result was obtained under an additional assumption that $\int_M \mathrm{Ric}^n < +\infty$. As a consequence of Theorem 3.1, part (II), we obtain

Corollary 3.1. *Let M be a complete noncompact Ricci-flat Kähler manifold with maximal volume growth. Assume the curvature has quadratic decay. Then M is a crepant resolution of a normal affine algebraic variety. Furthermore, there exists a two-step degeneration from that affine variety to the unique metric tangent cone of M at infinity.*

For a Ricci-flat Riemannian manifold with maximal volume growth, having quadratic curvature decay is equivalent to one (and hence all [15]) tangent cone having a smooth link. Indeed, by [15], the metric on M converges to that on the unique tangent cone at a logarithmic rate. Corollary 3.1 mirrors the result of [27] for tangent cones at a point. In that case, the two-step degeneration should comprise a degeneration of the normal affine variety to the "weighted tangent cone" followed by a $\mathbb{C}^*$-equivariant degeneration of the weighted tangent cone to the tangent cone. Intuitively, the weighted tangent cone is obtained by taking the quotient of the ring of polynomial growth holomorphic functions by the homogeneous ideal generated by weighted homogeneous polynomial growth holomorphic functions so that the restriction to the normal affine variety has lower degree (compare the local weighted tangent cone [27, pp. 354]). For Ricci-flat Kähler manifolds with maximal volume growth and quadratic curvature decay, the weighted tangent cone is expected to be distinct from the tangent cone when the metric converges logarithmically to that on the tangent cone, and is expected to coincide with the tangent cone when the metric converges at a polynomial rate to that on the tangent cone. Note that no metric information is contained in the weighted tangent cone.

A conjecture of Yau [67] states that if a complete Ricci-flat Kähler manifold has finite topological type, then it can be compactified complex analytically. Corollary 2.1 supports this conjecture, at least in this very special setting. Another conjecture of Yau [68, Question 71] states that complete noncompact Kähler manifolds with positive Ricci curvature are biholomorphic to a Zariski open set of a compact Kähler manifold. Part (III) of Theorem 2.2 supports this conjecture.

Now we introduce the strategy of the proof of Theorem 2.2. In parts (I) and (II), we consider polynomial growth holomorphic functions. The three circle theorem was replaced by Donaldson–Sun's three circles theorem [27, Proposition 3.7]. Also note, in the setting of [36], polynomial growth holomorphic functions separate points and tangents. This is no longer true in parts (I) and (II), due to the possibility of compact subvarieties. The two step degeneration in Corollary 3.1 follows from the argument in [17, 27].

The statement of part (III) is very similar to parts (I) and (II). However, the argument is very different. We essentially follow the argument of Mok [46]. The strategy is to consider plurianticanonical sections with polynomial growth. The key new result is a uniform multiplicity estimate for plurianticanonical sections. which provides the dimension estimate for polynomial growth plurianticanonical sections, without the extra assumption $\int_M \mathrm{Ric}^n < +\infty$ (compare [46, Theorem 2.2]).

4. GROMOV–HAUSDORFF LIMITS OF KÄHLER MANIFOLDS

Recall the seminar work of Cheeger [4]:

Theorem 4.1 (Cheeger, 1970). *Given $K, v, d, n > 0$, consider a class of compact Riemannian n-manifolds with $|\sec| \leq K$, $\operatorname{diam} < d$, $\operatorname{vol} > v$. Then such class is precompact in $C^{1,\alpha}$-topology. In other words, given a suquence of manifolds in this class, there exists a sub-*

sequence convergent in Cheeger–Gromov sense to a smooth manifold M (metric is $C^{1,\alpha}$). As a corollary, this class contains only finite diffeomorphism types.

Later, Anderson [1] generalized Cheeger's work in the Ricci curvature setting:

Theorem 4.2 (Anderson, 1990). *Given $C, i, d, n > 0$, consider a class of compact Riemannian n-manifolds with $|\mathrm{Ric}| \leq C$, $\mathrm{inj} > i$, $\mathrm{diam} < d$. Then the previous theorem holds for such class.*

Remark 4.1. 1. Anderson's theorem satisfies the noncollapsing condition; 2. One cannot replace the injectivity radius bound by noncollapsing condition as in Cheeger's theorem. Otherwise, the limit may not be smooth.

In order to obtain precompactness when the limit is not smooth, one has to consider weaker convergence. An important notion is Gromov–Hausdorff distance. This defines a distance between two compact metric spaces. We say a sequence of metric spaces converge in the Gromov–Hausdorff sense, if the Gromov–Hausdorff distance is approaching zero.

Theorem 4.3 (Gromov, [28]). *Given $C, d, n > 0$, consider a class of compact Riemannian n-manifolds with $\mathrm{Ric} \geq -C$, $\mathrm{diam} < d$. Then this class is precompact in the Gromov–Hausdorff sense (note that the Gromov–Hausdorff limit may be far from smooth).*

For noncompact manifolds, one can consider a manifold with a base pointed. Then the notion of pointed-Gromov–Hausdorff convergence makes sense, i.e., first consider the Gromov–Hausdorff convergence in a geodesic ball of fixed radius, then let the radius go to infinity (diagonal sequence).

A basic problem in metric differential geometry is to study the regularity of the Gromov–Hausdorff limit of manifolds with Ricci curvature lower bound (noncollapsed). There are fundamental contributions by Cheeger, Colding, Tian, and Naber. Given a limit space X and a point $p \in X$, we can consider a blow up of X at p. A blow up limit is called a tangent cone at p (note that the tangent cone at p need not be unique).

Definition 4.1. A point $p \in X$ is called regular if a tangent cone is isometric to a Euclidean space $\mathbb{R}^m$. A point is singular, it is not regular.

Theorem 4.4 (Cheeger–Colding, [6–9]). *Given $n, v > 0$, let (X, o) be the pointed-Gromov–Hausdorff limit of a sequence of n-manifolds (M_i, p_i) with $\mathrm{Ric} \geq -(n-1)$ and $\mathrm{vol}(B(p_i, 1)) > v$. Then*

(1) *X is metric length space of Hausdorff dimension n. The Hausdorff measure is equal to the limit of volume element on M_i.*

(2) *The singular set has Hausdorff codimension at least 2 (sharp).*

(3) *Any tangent cone is a metric cone.*

Note the following: (1) regular points are not so "regular." For example, consider a doubled disk. Then all points are regular. Near the boundary of the disk, the metric is only Lipschitz. It is a conjecture that near a regular point on X, the metric is bi-Lipschitz to a Euclidean ball. Currently, the best known regularity is bi-Hölder; (2) Regular set is not necessarily open. In other words, the singular set need not be closed. In fact, it could be dense. This already appears in the real two-dimensional case (the singular set in this case is countable). In higher dimensions, Li–Naber [19] constructed a limit space so that the singular set is given by a fat Cantor set. In other words, the topology of singular set could be very complicated.

In the above, we considered the Gromov–Hausdorff limit of Riemannian manifolds with Ricci curvature lower bound and noncollapsed volume. What if these manifolds are all Kähler? Can we get extra results? Observe all two-dimensional Riemannian manifolds (oriented) are Kähler. So we cannot expect too much from the extra Kähler assumption. For simplicity, let us call the Gromov–Hausdorff limit of Kähler manifolds with Ricci curvature lower bound Kähler–Ricci limit space. The following is the first important result in this direction:

Theorem 4.5 (Cheeger–Colding–Tian, [10]). *Let X be a noncollapsed Kähler–Ricci limit space. Then any tangent cone splits even dimensional Euclidean factor. In other words, the splitting lines must come in pairs.*

Below is a breakthrough result of Donaldson–Sun [26] and Tian [62] (for simplicity, we only listed a part of their result)

Theorem 4.6. *Let X be the Gromov–Hausdorff limit of a sequence of polarized Kähler manifolds M_i with $|\mathrm{Ric}| \leq C$ and $\mathrm{diam} < d$, $\mathrm{vol} > v$. Then X is homeomorphic to a normal projective variety.*

Such result is a key to the existence of Kähler–Einstein metrics on Fano manifolds. In a joint work with G. Szekelyhidi [40], we generalized the result above to the case when the Ricci curvature has a lower bound:

Theorem 4.7. *Let X be the Gromov–Hausdorff limit of a sequence of polarized Kähler manifolds M_i with $\mathrm{Ric} \geq -1$ and $\mathrm{diam} < d$, $\mathrm{vol} > v$. Then X is homeomorphic to a normal projective variety.*

The argument follows [26, 62], and a key step is Tian's partial C^0-estimate [63]. During the proof, we also need a recent deep result of Cheeger–Jiang–Naber [19] on the Minkowski content of the singular set.

A basic technical ingredient in Theorem 4.7 is a result on the existence of holomorphic charts in balls that are Gromov–Hausdorf-close to the Euclidean ball. This is an extension of Proposition 1.3 of [36], where the bisectional curvature lower bound was assumed.

Theorem 4.8. *There exists $\varepsilon > 0$, depending on the dimension n with the following property. Suppose that $B(p, \varepsilon^{-1})$ is a relatively compact ball in a (not necessarily complete) Kähler manifold (M^n, p, ω), satisfying $\mathrm{Ric}(\omega) > -\varepsilon\omega$, and*

$$d_{GH}\big(B(p, \varepsilon^{-1}), B_{\mathbb{C}^n}(0, \varepsilon^{-1})\big) < \varepsilon.$$

Then there is a holomorphic chart $F : B(p,1) \to \mathbb{C}^n$ which is a $\Psi(\varepsilon|n)$-Gromov–Hausdorff approximation to its image. In addition, on $B(p,1)$ we can write $\omega = i\partial\bar{\partial}\phi$ with $|\phi - r^2| < \Psi(\varepsilon|n)$, where r is the distance from p.

We give two applications of Theorem 4.8. The first shows that under Gromov–Hausdorff convergence to a smooth Riemannian manifold, the scalar curvature functions converge as measures. Here we state a simple corollary of this.

Corollary 4.1. *Given any $\varepsilon > 0$, there is a $\delta > 0$ depending on ε, n satisfying the following. Let $B(p,1)$ be a relatively compact unit ball in a Kähler manifold (M,ω) satisfying* $\mathrm{Ric} > -1$, *and $d_{GH}(B(p,1), B_{\mathbb{C}^n}(0,1)) < \delta$. Then $|\int_{B(p,\frac{1}{2})} S| < \varepsilon$, where S is the scalar curvature of ω.*

The other application is the following, which was proved previously under the assumption of non-negative bisectional curvature.

Proposition 4.1. *There exists $\varepsilon > 0$ depending on n, so that if M^n is a complete noncompact Kähler manifold with* $\mathrm{Ric} \geq 0$ *and $\lim_{r\to\infty} r^{-2n}\mathrm{vol}(B(p,r)) \geq \omega_{2n} - \varepsilon$, then M is biholomophic to $\mathbb{C}^n$. Here ω_{2n} is the volume of the Euclidean unit ball.*

Remark 4.2. In the Riemannian setting, Perelman [54] first showed that such a manifold must be contractible. Cheeger–Colding [7] proved such manifold is diffeomorphic to the Euclidean space.

Let us briefly mention the strategy to Theorem 4.8. First, we conformally scale the metric away from a compact domain, in order to make the metric complete (note the metric is no longer Kähler). Thanks to a result of Cavalletti–Mondino [16], our assumptions imply that the almost Euclidean isoperimetric inequality holds in some smaller balls. As in [29], there exists a Ricci flow solution $h(t)$ for a definite time $t \in [0,T]$, satisfying $|\mathrm{Rm}| \leq A/t$ for $t \in (0,T]$. In short, after a fixed time, the metric becomes almost Euclidean is smooth sense. The problem is that, the new metric is not Kähler (not even compatible with the complex structure). Now the observation is that the complex structure J is almost compatible with the new metric. Therefore, by the approach of Newlander–Nirenberg theorem [51], we can find a fixed size holomorphic chart near the center.

Now we glue such chart to a a domain of $\mathbb{CP}^n$. In this way, we were able to run the genuine Kähler–Ricci flow. Thanks to a result of Tian–Zhang [65], the flow has a definite existence time. After a short time, the metric has good regularity near p. The potential estimate follows by integrating along the time line. Finally, the desired holomorphic chart followed by solving another $\bar{\partial}$-problem.

Let us now study the structure of the metric singular set in the Kähler setting. Assume in addition that the N_i^m is a sequence of polarized Kähler manifolds. Then, as we saw above, the limit Y is naturally identified with a projective variety. When the metrics along the sequence are Kähler–Einstein, Donaldson–Sun [26] showed that the metric singular set of Y is the same as the complex analytic singular set of the corresponding projective variety.

Let $\mathcal{R}$ stand for the metric regular set. For small $\varepsilon > 0$, denote by $\mathcal{R}_\varepsilon$ the set of points p so that $\omega_m - \lim_{r\to 0} \frac{\mathrm{vol}(B(p,r))}{r^m} < \varepsilon$. Here ω_m is the volume of the unit ball in $\mathbb{R}^m$. Then $\mathcal{R} = \bigcap_{\varepsilon>0} \mathcal{R}_\varepsilon$. Note that $\mathcal{R}_\varepsilon$ is an open set, while in general $\mathcal{R}$ may not be open. Now we state the theorem

Theorem 4.9. *Let (X, d) be a Gromov–Hausdorff limit as in Theorem 4.7. Then for any $\varepsilon > 0$, $X \setminus \mathcal{R}_\varepsilon$ is contained in a finite union of analytic subvarieties of X. Furthermore, the singular set $X \setminus \mathcal{R}$ is equal to a countable union of subvarieties.*

Remark 4.3. Since the singular set could be dense, the countable union in Theorem 4.9 cannot be replaced by a finite union.

Remark 4.4. In view of Li–Naber's example **[19]**, this result shows that the behavior of singularities in the Kähler case is in sharp contrast with the Riemannian case (see also Theorem 4.10 without polarization). On the one hand, the metric singularities in the Kähler case might seem flexible, since one can perturb Kähler potentials locally. On the other hand, analytic sets are very rigid, and so in particular Theorem 4.9 implies the following: if we perturb the Kähler metric inside a holomorphic chart and assume that the geometric conditions are preserved, then the metric singular set can change by at most a countable set of points.

Let us explain why Theorem 4.9 should be true. The key is the following characterization of the metric singular set:

Proposition 4.2. *A point p is regular in the metric sense if and only if it is complex-analytically regular and the Lelong number for* Ric *vanishes at p.*

Here Ric is the positive $(1, 1)$-current on the limit space, regarded as the limit of Ricci form on manifolds. With this in hand, the argument goes as follows: If a point is metric singular, then either it is holomorphic singular point, or the Lelong number of Ric is positive. In the first case, it belongs to a complex-analytic set; in the second case, according to a theorem of Siu **[58]**, this set is given by a countable union of analytic sets.

Let us also sketch the argument in Proposition 4.2. If a point is metric regular, then by Theorem 4.8, we see it is complex-analytically regular. So without loss of generality, we may assume that the point is complex-analytically regular. We are left to show that a point is metric regular iff the Lelong number of Ric vanishes. Note the Lelong number for $2\pi\mathrm{Ric}$ at p is given by

$$\lim\inf_{x\to p} \frac{\log |dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n|^2(x)}{\log |z(x)|},$$

where $z_1, \ldots, z_n$ is a holomorphic chart near p. If the point is metric regular, then by the estimate of Cheeger–Colding, we can show that the value above vanishes. Now assume the point is metric singular, the key is the following claim:

Claim 4.1. Assume p is not a regular point in the metric sense. Then there exist $\varepsilon > 0$ and $r_0 > 0$ so that for all $r < r_0$, if nonzero holomorphic functions $f_1, \ldots, f_n$ on $B(p, 4r)$ satisfy

$f_j(p) = 0$ and $\int_{B(p,r)} f_j \bar{f_k} = 0$ for $j \neq k$, there exists $1 \leq l \leq n$ so that

$$\frac{⨍_{B(p,2r)} |f_l|^2}{⨍_{B(p,r)} |f_l|^2} \geq 2^{2+10n\varepsilon}.$$

Given such claim, we can show that for all small r, on $B(p,r)$,

$$|dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n| \leq 2Cr^{\varepsilon}.$$

Then we obtained the desired lower bound of Lelong number by using the following

Claim 4.2. Let p be a complex-analytically regular point on X. Let $(z_1, \ldots, z_n)$ be a holomorphic chart near p. Assume $z_j(p) = 0$ for all j. Then there exists $\alpha = \alpha(n, v, d) > 0$, $C > 0$, $c > 0$ so that $cr(q)^{\alpha} \leq |z(q)| \leq Cr(q)$ for all q sufficiently close to p. Here r is the distance function to p.

Let us also mention a parallel result under the bisectional curvature lower bound.

Theorem 4.10. *Let (X, p) be the pointed Gromov–Hausdorff limit of complete Kähler manifolds (M_i^n, p_i) with bisectional curvature lower bound -1 and $\mathrm{vol}(B(p_i, 1)) \geq v > 0$. Then X is homeomorphic to a normal complex-analytic space. The metric singular set $X \setminus \mathcal{R}$ is exactly given by a countable union of complex-analytic sets, and for any $\varepsilon > 0$, each compact subset of $X \setminus \mathcal{R}_{\varepsilon}$ is contained in a finite union of subvarieties.*

Remark 4.5. B. Wilking and R. Bamler [3], M. Lee and L. Tam [32] proved that the limit space is, in fact, complex-analytically smooth. Also J. Lott [33] has developed some theory on the limit of Kähler manifolds with bisectional curvature lower bound.

Now we come to tangent cones of noncollapsed Kähler–Ricci limit space. Recall that, according to Cheeger–Colding, such tangent cones must be metric cones. Also according to Cheeger–Colding–Tian, such tangent cones must split off even dimensional lines. Jointly with G. Szekelyhidi [41], we proved the following:

Theorem 4.11. *Every tangent cone of Z is homeomorphic to a normal affine algebraic variety such that under a suitable embedding into $\mathbf{C}^N$, the homothetic action on the tangent cone extends to a linear torus action.*

Theorem 4.11 was shown previously by Donaldson–Sun [27] under the assumptions that the ω_i are curvature forms of line bundles $L_i \to M_i$, and $|\mathrm{Ric}(\omega_i)| < 1$. An important application of their result is that in their setting the holomorphic spectrum of the tangent cones is rigid, which in turn they used to show the uniqueness of tangent cones. While we are not able to show uniqueness, our result does imply the rigidity of the holomorphic spectrum under two-sided Ricci curvature bounds, even when the (M_i, ω_i) are not polarized.

More precisely, recall that for a Kähler cone $C(Y)$, possibly with singularities, the holomorphic spectrum is defined by

$$\mathcal{S} = \{\deg(f) : f \text{ is homogeneous and holomorphic on } C(Y)\} \subset \mathbf{R}.$$

We then have

Corollary 4.2. *Suppose that we have two sided bounds* $|\mathrm{Ric}(\omega_i)| < 1$ *along the sequence above. Then for any* $q \in Z$ *the holomorphic spectrum of every tangent cone at* q *is the same.*

As in [27], the rigidity of the holomorphic spectrum follows from the fact that the space of tangent cones at each point is connected, and the holomorphic spectrum consists of algebraic numbers. Note that these results hold in particular for tangent cones at infinity of Calabi–Yau manifolds with Euclidean volume growth.

Corollary 4.3. *Let* M *be a complete noncompact Ricci flat Kähler manifold of maximal volume growth. Then the asymptotic volume ratio is an algebraic number.*

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