GROUPS ACTING AT INFINITY

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ABSTRACT

An action of a group on a topological space is *rigid* if small perturbations to the action have little meaningful influence on its global dynamics. Many examples of rigid actions come from geometric considerations. This introductory survey describes the idea of "looking to infinity" as a source both of rigid examples and proofs of rigidity, starting with some early history then passing quickly to recent developments in topological rigidity of group actions. The examples considered include actions of hyperbolic manifold groups on the visual boundary of their universal cover, automorphism groups of surface groups, boundary actions of hyperbolic groups in the sense of Gromov, and group actions derived from Anosov flows on 3-manifolds.

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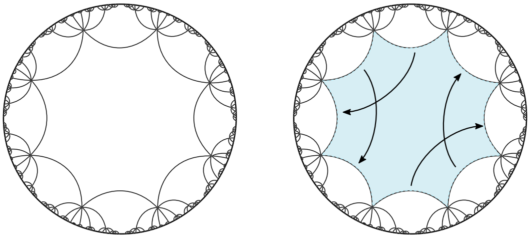

FIGURE 1

A regular tiling of the hyperbolic plane by octagons with geodesic sides. The symmetries of this tiling is a *hyperbolic group* with boundary S^1. Identifying sides of the octagonal domain by translations indicated on the figure gives a genus 2 surface with hyperbolic structure.

1. INTRODUCTION

The boundary at infinity. This survey concerns some recent developments in topological rigidity of group actions that come from "looking to infinity."

To start, let us take a short tour of a familiar object, the Poincaré ball model of the hyperbolic space. Recall that hyperbolic n-space, $\mathbb{H}^n$, is the unique complete, simply connected manifold of constant curvature -1. We can visualize it via the Poincaré ball model, the open unit ball in $\mathbb{R}^n$ equipped with the Riemannian metric $ds^2 = \frac{4\|d\mathbf{x}\|^2}{(1-\|\mathbf{x}\|^2)^2}$. In this model, infinite geodesics are the Euclidean lines and half-circles that meet the boundary of the ball orthogonally (see Figure 1 for an illustration when $n = 2$). Isometries of $\mathbb{H}^n$ preserve geodesics and totally geodesic half-spaces, from which one can deduce that they induce homeomorphisms of the sphere bounding the Poincaré ball. But what is this sphere of "points at infinity"?

Let us try to describe the boundary from the perspective of someone inside $\mathbb{H}^n$. Standing at a point in $\mathbb{H}^n$, your field of vision is a sphere, each line of sight a geodesic ray based at your eye. Each ray in this S^{n-1}-family tends to a unique point on the boundary sphere. To avoid privileging any one point as the eye of an observer, we might broaden our definition of the "sphere at infinity" as follows. In a geodesic metric space (X, d_X), define an equivalence relation on unit speed geodesic rays by declaring that two such rays, say α, $\gamma : [0, \infty) \to X$, are equivalent if there exists a constant D such that $d_X(\alpha(t), \gamma(t)) < D$ holds for all t. Applying this to $\mathbb{H}^n$, the equivalence classes of geodesic rays are in a one-to-one correspondence with points on the boundary sphere of the Poincaré ball, and also to visual directions based at a given point.

The beauty of this definition, beyond being independent of a basepoint, is that we did not need to model hyperbolic space as a ball in $\mathbb{R}^n$ to compactify it. Thus, it lends itself to other spaces with negative—or even coarse analogs of negative—curvature. Applying this definition to any proper geodesic metric space X that is *δ-hyperbolic* in the sense of Gromov (we will return to this later) gives an "ideal boundary" denoted $\partial_\infty X$, which is a compact space when equipped with the quotient topology induced from the compact-open topology on all geodesic rays. Informally, two boundary points are close if they can be represented by rays that stay a bounded distance from each other for a long time. This topology can be naturally extended to $X \cup \partial_\infty X$, compactifying X. The isometries of X preserve geodesics, and preserve the equivalence relation of staying a bounded distance apart, so extend to homeomorphisms of the boundary with this topology. This is the starting point for our story of groups acting on boundary spaces.

Rigidity. A group Γ acting on a space X is *locally rigid* if the only possible small deformations of the action are by trivial procedures—constructions that do not meaningfully change the global dynamics of the action. Of course, what “trivial procedure” and “meaningfully change” mean depends on the context, but a standard interpretation is that there should be either a surjective map, a self-homeomorphism, a diffeomorphism, or an isometry of X (semi)conjugating the perturbed action of Γ back to the original. That is, if $\rho : \Gamma \to \mathrm{Homeo}(X)$ is the original action, and ρ' a perturbation, then *rigid* means that there is a map $h : X \to X$ satisfying $h \circ \rho'(\gamma) = \rho(\gamma) \circ h$ for all $\gamma \in \Gamma$. If invertible, h is a genuine conjugacy; if merely surjective, it is called a *semiconjugacy*. One can strengthen this notion: *globally* rigid means that any deformation, no matter how big or small, is still (semi)conjugate to the original; and one can even do away with the idea of deforming continuously: *strong rigidity* typically means that any other action of the group is either semiconjugate to the original or essentially trivial.

The first major examples of local rigidity to attract significant attention were lattices in linear groups. A *lattice* in a group G is a discrete subgroup Γ such that G/Γ has finite volume, such as $\mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{SL}(n, \mathbb{R})$; it is *cocompact* if the quotient is compact. In 1960, as a means of showing the surprising fact that discrete, cocompact subgroups of $\mathrm{SL}(n, \mathbb{R})$ are conjugate to arithmetic groups, Selberg showed that the inclusion of such a lattice into $\mathrm{SL}(n, \mathbb{R})$ is locally rigid in the sense that any nearby representation is conjugate by an element of $\mathrm{SL}(n, \mathbb{R})$. This, together with concurrent work of Calabi, Weil, Calabi–Vesentini and others, was the birth of rigidity theory. At the time, the major techniques were algebraic (Selberg’s work hinged on traces of group elements) and differential geometric. But an important new idea of Selberg, picked up by Mostow, was that of *escape to infinity*, an idea that would prove extremely fruitful to others. Mostow writes:

> *“Upon analyzing Selberg’s proof of his rigidity theorem, the key relation shows its force as the elements [considered here in an abelian subgroup] go to infinity... It seemed to me desirable to exploit relations at infinity not only on abelian subfamilies, but among all elements of* Γ *near infinity.”* [35]

What does this mean? Mostow’s first success was with lattices in $\mathrm{Isom}(H^n) \cong O(n,1)/\pm 1$, so I will center our discussion here. Given isomorphic cocompact lattices Γ and Γ', one can build a map $f : \mathbb{H}^n \to \mathbb{H}^n$, equivariant with respect to their actions by isometries. Mostow’s idea was to ask whether this equivariant map of $\mathbb{H}^n$ extends to a map on the boundary at infinity (the answer is “yes, always”) and to study the *regularity* of the extension (a harder answer: “it is necessarily a conformal map, induced by conjugation by an isometry”). This led to the proof of

Theorem 1.1 (Mostow rigidity for hyperbolic manifolds [34]). Suppose M and N are compact, hyperbolic manifolds of dimension at least 3. If M and N are diffeomorphic, then they are isometric.

Shortly afterwards, Margulis noticed that one needs only assume M and N have isomorphic fundamental groups.

Theorem 1.2 (Mostow rigidity, algebraic reformulation [30]). Let M and N be compact, hyperbolic manifolds of dimension at least 3. If $\pi_1(M) \cong \pi_1(N)$, then M and N are isometric. Equivalently, any two isomorphic cocompact lattices in $O(n,1)$ are conjugate provided $n \geq 3$.

To the reader meeting this theorem for the first time, I wish to emphasize the following dramatic consequence: for hyperbolic manifolds, *every metric invariant is actually a topological invariant*. Volume? Length of a shortest closed geodesic? Theorem 1.2 says these quantities are completely determined by the algebraic structure of the fundamental group of the manifold.

Mostow later extended his rigidity result to lattices in other semisimple Lie groups, the first step again being to construct boundary maps. There are many excellent surveys on these results and their influence in the work of Margulis, Zimmer, and many others; I recommend those of Fisher and Spatzier [12,36] as a starting point for the curious reader. Here we will take a different tack, skipping ahead to some very recent developments on rigidity of boundary actions. These will occur in settings where, for reasons of low dimension or low regularity, many of the dynamical techniques descended from Mostow and his contemporaries fall short, but the essence of the idea to look at the boundary prevails.

2. SURFACE GROUPS ACTING ON THE CIRCLE

Hyperbolic structures on surfaces. The attentive reader will have noticed the hypothesis "dimension at least 3" in Mostow's theorem. Indeed, this is necessary, as there is a continuum of nonisometric hyperbolic structures on any surface of genus $g \geq 2$, a fact that has been known (depending on how you count) at least since the time of Riemann or Poincaré. These correspond to the continuum of discrete, faithful representations, called *Fuchsian representations*, of the fundamental group of the surface into $\mathrm{Isom}(\mathbb{H}^2)$.

The way in which Mostow's original strategy fails in dimension 2 is quite subtle, having to do with the fact that there is no perfect analog of a quasiconformal map in dimension 1. Agard's survey "Mostow rigidity on the line" [1] contains a nice discussion of what goes right and wrong.[1] That said, for a fixed genus g, each boundary action corresponding to

1 There are many other proofs of Mostow rigidity, all falling short for surfaces in different ways. A proof using Gromov's simplicial volume relies on the fact that an ideal simplex of maximal volume in the hyperbolic n-space is *regular*—a meaningful distinction when $n \geq 3$, but in $\mathbb{H}^2$ all ideal triangles are isometric. Besson–Courtois–Gallot [5] have a strengthening of Mostow's theorem with a different endgame to the proof hinging on an inequality involving the determinant of an $n \times n$ derivative matrix, which simply fails for $n = 2$.

a hyperbolic structure on a genus g surface is conjugate by a *homeomorphism* of the circle.[2] This suggests that one might recover a notion of rigidity by weakening the regularity of the maps in question. Reducing regularity is actually quite a natural consideration: leaving the realm of hyperbolic structures and instead considering the boundary at infinity of the universal cover of the surface with an arbitrary Riemannian metric, or on a Cayley graph for the fundamental group of the surface, one retains only a *topological* circle with an action of the fundamental group by homeomorphisms. These actions are all conjugate to each other, and also all to any Fuchsian action, by a homeomorphism of the circle.

Rigidity at infinity for surfaces. This brings us to a beautiful theorem of Matsumoto on boundary rigidity for surfaces. To state it, I will first make precise the notion of rigidity described before. A group action of Γ on X is a homomorphism $\Gamma \to \mathrm{Homeo}(X)$. The *space of actions* $\mathrm{Hom}(\Gamma, \mathrm{Homeo}(X))$ is the set of all actions equipped with the compact-open topology: informally, two actions are close if, for every element γ in a large finite subset of Γ, the homeomorphisms given by the actions of γ are pointwise close on a large compact subset of X. *Global rigidity* is a statement about the homogeneity of a connected component of $\mathrm{Hom}(\Gamma, \mathrm{Homeo}(X))$; Matsumoto's version is as follows:

Theorem 2.1 (Matsumoto's rigidity [31]). Suppose that Σ is a surface of genus at least 2, and $\rho : \pi_1(\Sigma) \to \mathrm{Homeo}(S^1)$ lies in the same connected component as a Fuchsian representation ρ_0. Then there is a continuous, monotone, degree 1 map $h : S^1 \to S^1$ such that $h \circ \rho(\gamma) = \rho_0(\gamma) \circ h$ holds for all $\gamma \in \pi_1(\Sigma)$.

"Monotone" here means that h weakly preserves the cyclic ordering of points on S^1. A triple (x_1, x_2, x_3) of distinct points in S^1 has a *positive* or *negative* orientation depending on whether you proceed anticlockwise or clockwise around the circle as your read the points in order. To *weakly* preserve this order, a map sends each positively oriented triple either to a positively oriented triple or to a degenerate one where two or more points coincide. Thus, h may collapse an interval to a point, but does not double back on itself.

This weakening of the notion of conjugacy in the theorem statement is necessary, as it is easy to construct small, nonconjugate perturbations of almost any action of a countable group on the circle using a "blow up" trick. Enumerate your group Γ, fix a point x and some small $\varepsilon > 0$, and replace the image of x under the action of the nth element of Γ with a closed interval of length $\frac{\varepsilon}{2^n}$. The result is a circle whose circumference is increased by ε. Extend the original action of Γ over the inserted intervals by declaring that g takes the interval corresponding to $f(x)$ to that of $gf(x)$ by the unique affine map. This procedure gives an action by homeomorphisms, and the map $h : S^1 \to S^1$ collapsing each inserted interval to a point is a semiconjugacy.

Group cohomology. A remarkable theorem of Ghys [14] says that the equivalence relation on group actions generated by semiconjugacy, in the sense defined above using a mono-

2 This is again a subtle point: one can show that, if such a conjugating homeomorphism is differentiable with nonzero derivative at a single point, then it is real analytic, induced from an isometry of $\mathbb{H}^2$, and the surfaces are isometric.

tone maps, is actually a *cohomological phenomenon*. As is well known, the inclusion of the rotation subgroup $SO(2) \cong S^1$ into the group $\mathrm{Homeo}_+(S^1)$ of orientation-preserving homeomorphisms of the circle is a homotopy equivalence, thus the cohomology of the classifying space for $\mathrm{Homeo}_+(S^1)$ is generated by an *Euler class* in degree 2. It is less well known (following from a deep theorem of Thurston) that the cohomology of $\mathrm{Homeo}_+(S^1)$ as a *discrete group* is also generated by the Euler class: the map from the discrete $\mathrm{Homeo}_+(S^1)$ to $\mathrm{Homeo}_+(S^1)$ with the usual topology induces an isomorphism on cohomology.

Given an action $\rho : \Gamma \to \mathrm{Homeo}_+(S^1)$ of a group by orientation-preserving homeomorphisms of the circle, one can pull back the discrete, integral Euler class to an element of $H^2(\Gamma; \mathbb{Z})$. Typically, remembering only the Euler class and forgetting the action results in a great loss of information. For instance, if Γ is a surface group, then $H^2(\Gamma; \mathbb{Z}) \cong \mathbb{Z}$, but there are uncountably many distinct actions of Γ on the circle, even up to semiconjugacy. However, the Euler class is a *bounded* cocycle in the sense of Gromov, and the second *bounded cohomology* of a surface group is infinite dimensional. Ghys showed, remarkably, that this pullback of the Euler class in bounded cohomology determines the action up to semiconjugay. The rigidity statement of Matsumoto quoted above is, in fact, a consequence of a stronger theorem, in which he shows that there is a unique, maximal (meaning pairing maximally with the fundamental class of the surface) bounded second cohomology class, corresponding to the semiconjugacy class of a Fuchsian representation.

Rigidity of geometric actions. The Fuchsian surface groups acting on the circle in Matsumoto's rigidity theorem are lattices in $\mathrm{PSL}(2, \mathbb{R})$, the group of orientation-preserving isometries of $\mathbb{H}^2$. With this in mind, we make the following definition:

Definition 2.2. An action of a group Γ on a manifold X is *geometric* if it is obtained by the embedding of Γ as a cocompact lattice in a connected Lie group G acting transitively on X.

This definition is modeled after the idea of a geometry from Klein's *Erlangen program*: a connected Lie group acting transitively on a space with compact point stabilizers. When X is the circle, we can easily list all geometric actions of groups. The connected Lie groups acting transitively on S^1 are the rotation group SO(2), the projective linear group $\mathrm{PSL}(2, \mathbb{R})$ acting on $\mathbb{R}P^1 = S^1$, and finite cyclic covers (i.e., central extensions by finite cyclic groups) of $\mathrm{PSL}(2, \mathbb{R})$, which act naturally on finite covers of S^1—conveniently, also topological circles (see [15]). Any cocompact lattice in one of these groups is, up to finite index, the fundamental group of a surface of genus g for some $g \geq 2$, and these are all obtained by lifting Fuchsian surface groups to a finite cyclic cover. Provided that the degree of the cover divides the Euler characteristic of the surface, such a lift exists and gives a lattice in the corresponding cyclic extension of $\mathrm{PSL}(2, \mathbb{R})$.

Matsumoto's theorem gives global rigidity for geometric examples where the ambient Lie group is $\mathrm{PSL}(2, \mathbb{R})$. With Maxime Wolff, we showed this for all geometric surface groups:

Theorem 2.3 (Rigid $\Leftrightarrow$ geometric on the circle [24,29]). An action of a surface group on S^1 is globally rigid if and only if it is geometric.

The direction *geometric implies rigid* consists in a careful study of geometric representations using the Poincaré rotation number and techniques of Calegari and Walker; a detailed expository account is given in [25] and an alternative proof given later by Matsumoto, using a Markov partition and Maskit combination theorem style of argument, in [32]. The statement that *rigid implies geometric* is much more difficult: handed an action of $\pi_1(\Sigma)$ on the circle, with no knowledge about it except that you cannot deform it, you need to reconstruct the ambient Lie group and lattice surface group.

The proof in [29] is rather technical, but, under a major simplifying assumption one can produce a much easier proof in which the strategy of "reconstruction" is evident. This is carried out in the relative short paper [27]. The simplifying assumption eliminates the possibility that $\rho(\pi_1(\Sigma))$ lies in one of the nontrivial covers of $\mathrm{PSL}(2,\mathbb{R})$. As a consequence of this assumption, one can deform the representation so that some simple closed curve a on the surface has its action $\rho(a)$ conjugate to the boundary action of a hyperbolic isometry of $\mathrm{PSL}(2,\mathbb{R})$. This means that the action of $\rho(a)$ on the circle has exactly two fixed points, one attracting and one repelling.

Having found one such simple closed curve, we find many, and then start the reconstruction: we show that the arrangement of attracting and repelling points of the action of these simple closed curves on the circle agrees with the intersection pattern of lifts of geodesic representatives of these curves on the surface. Under a Fuchsian representation, the axes of translation of elements are exactly such lifts of curves, and this allows us to build the desired semiconjugacy to the Fuchsian class. Figure 1 gives a representative visual: the geodesics shown there are all represented by simple closed curves; it is the cyclic order of their endpoints that we recover.

Open questions. For any real linear algebraic group G and finitely presented group $\Gamma = \langle S \mid R \rangle$, the space of representations $\mathrm{Hom}(\Gamma, G)$ is a real algebraic variety, a subspace of $G^{|S|}$ cut out by the finitely many relations from R. Thus $\mathrm{Hom}(\Gamma, G)$ has finitely many connected components. Goldman [17, THEOREM A] gives a precise count in the case where Γ is the fundamental group of a closed surface, and G the k-fold covering group of $\mathrm{PSL}(2,\mathbb{R})$. Geometric representations exist precisely when k divides $2g-2$; if k is larger than $2g-2$, then there is only a single connected component: every representation is deformable to the trivial representation. Using this, one can show that for a given genus g, *up to deformation, there are only finitely many representations of $\pi_1(\Sigma_g)$ into a Lie subgroup of S^1*. By contrast, we do not know:

Question 2.4. Does the space $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{Homeo}_+(S^1))$ have finitely many connected components?

Question 2.5. Can every action of a surface group on the circle be deformed into one with image in a Lie group?

In fact, the situation is even messier than this. "Deformation" suggests movement along a *path* of representations, while my definition of rigidity referred to homogeneity of *connected components*. Alas, we do not know if path components and connected components of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{Homeo}_+(S^1))$ agree.

At the time of framing Definition 2.2, I was very enthusiastic about pursuing the theme "rigid $\leftrightarrow$ geometric" to see to what extent it might play out in settings beyond group actions on the circle. In retrospect, that seems too optimistic: geometric examples are excellent candidates for rigidity, but I do not think they are precisely all the rigid examples of actions in a general setting. This is not to deter anyone in their attempt to prove such a theorem—perhaps for the right modification of the definition of geometric (maximal dimension Lie group might be a good start), and the right definition of rigid, such a statement is true.

Our next topic should provide evidence for continued optimism, as we look at the first obvious generalization of surface groups acting on the circle—the boundary actions of fundamental groups of hyperbolic manifolds of higher dimension. In fact, one does not need $\mathbb{H}^n$ here (and so we will bid a temporary farewell to lattices) but only a Riemannian metric of negative curvature.

3. MANIFOLD GROUPS ACTING ON BOUNDARY SPHERES

We described in the introduction how to compactify certain spaces by equivalence classes of geodesic rays. When the space in question is the universal cover of a closed hyperbolic or negatively curved surface, Matusmoto's Theorem 2.1 says that the induced boundary action of the fundamental group is rigid. While his proof does not apply for manifolds of higher dimension (there is no bounded Euler class and no cyclic order of points at infinity for groups acting on higher dimensional spheres), Bowden and I recently proved an analogous local result in generality:

Theorem 3.1 ([6]). Let M be a compact, orientable n-manifold with negative curvature, and $\rho_0 : \pi_1(M) \to \mathrm{Homeo}(S^{n-1})$ the boundary action. There exists a neighborhood U of ρ_0 in $\mathrm{Hom}(\pi_1(M), \mathrm{Homeo}(S^{n-1}))$ consisting of representations which are *topological factors* of ρ_0 in the sense of classical dynamics: for each representation $\rho \in U$, there is a continuous, surjective map $h : S^{n-1} \to S^{n-1}$ satisfying $h \circ \rho(\gamma) = \rho_0(\gamma) \circ h$ for all $\gamma \in \pi_1(M)$.

The key idea of the proof is to encode the dynamics of the action ρ_0 and a perturbation ρ in *foliated spaces*, and promote *topologically stable properties* (such as transversality) to the desired dynamical stability. We describe a few details of the strategy now.

From group actions to foliated spaces. Given an arbitrary manifold M and an action ρ of $\pi_1(M)$ by homeomorphisms on a space F, the associated *foliated F-bundle over M* is the quotient of $\tilde{M} \times F$ by the diagonal action of $\pi_1(M)$ via deck transformations on $\tilde{M}$ and via ρ on F. This quotient space is an F-bundle over M, and since the action is diagonal, the "horizontal foliation" of $\tilde{M} \times F$ by leaves of the form $\tilde{M} \times \{*\}$ descends to a topological foliation on this bundle, topologically transverse to the fibers. If M and F have smooth structures (which we will always take to be the case) and the action is by diffeomorphisms, then the result is a smooth bundle with smooth foliation, transverse to the fibers. The idea to

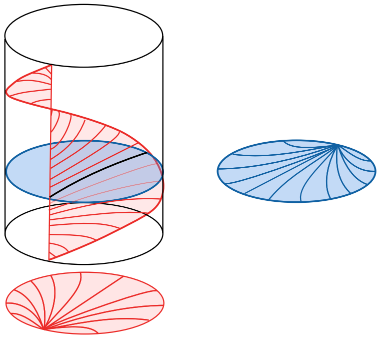

FIGURE 2
$UT(\mathbb{H}^2)$ is trivialized as $\mathbb{H}^2 \times S^1$, where the point on S^1 (height) is given by the endpoint at infinity of a geodesic ray with given tangent vector. Leaves of the weak-stable foliation are horizontal, weak-unstable leaves intersect them transversely along geodesics.

encode actions or representations in foliated spaces is not new—for instance, as one example of a use quite close to our theme, Goldman's 1980 thesis [16] presents the idea of a geometric structure as a section of a foliated bundle (following Kulkarni, Sullivan, and Thurston), using this perspective to understand representations of surface groups into $\mathrm{PSL}(2, \mathbb{R})$.

We are interested in the special case where M and ρ are as in Theorem 3.1. When $\rho = \rho_0$ is the action on the boundary at infinity, the associated foliated sphere bundle over M is isomorphic to the unit tangent bundle of M. This gives additional structure which we exploit in the proof.

Foliations on the unit tangent bundle. The unit tangent bundle $UT(M)$ of a negatively curved manifold M has a natural foliation $\mathcal{F}$ transverse to its sphere fibers. We can describe this by looking at the unit tangent bundle of the universal cover $\tilde{M}$ and using the boundary at infinity. For each point $z \in \partial_\infty \tilde{M}$, let $L_z \subset UT\tilde{M}$ be the set of all unit tangent vectors to geodesic rays that represent z. The leaf L_z is homeomorphic to $\tilde{M}$ (there is exactly one tangent vector in a given direction at each point), so the sets L_z partition $UT\tilde{M}$ into n-dimensional hyperplanes. Following classical work of Anosov, these sets are actually smooth, embedded submanifolds. The action of $\pi_1(M)$ on $UT\tilde{M}$ sends leaves to leaves, so this foliation descends to one on $UT(M)$, and the bundle isomorphism between the foliated sphere bundle associated to the boundary action can be chosen to naturally identify this foliation on $UT(M)$, called the *weak-stable foliation* with the "horizontal" foliation on the foliated bundle. See Figure 2 left for an illustration when $\tilde{M} = \mathbb{H}^2$. Of course, one could equally well make the opposite choice of considering the set of tangent vectors v that *emmanate* from a common point at infinity. This gives the *weak-unstable* foliation, which is also transverse to the fibers, and transverse to leaves of the weak-stable foliation.

Maps between bundles. The proof of Theorem 3.1 starts with the construction of a particularly well behaved map between the foliated bundles associated with nearby actions, as illustrarted in Figure 3. Given a perturbation ρ of the standard boundary action ρ_0, one

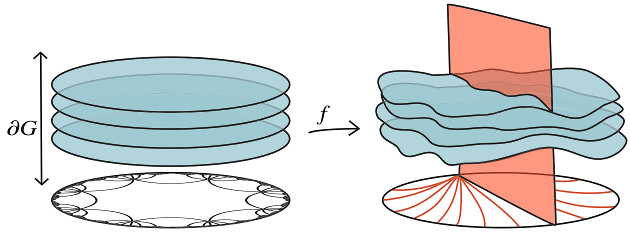

FIGURE 3

An equivariant map locally close to the identity on $\tilde{M} \times \partial G$ captures a perturbation of the action of $G = \pi_1(M)$ on ∂G. The images of horizontal leaves $\tilde{M} \times \{x\}$ intersect the leaves of the stable foliation of geodesic flow (in red) along *quasigeodesics* allowing us to use large-scale metric stability to prove dynamical stability. For general groups, one needs a more complicated space and a substitute for the stable foliation.

builds, by hand, a $\pi_1(M)$-equivariant map $\tilde{M} \times S^{n-1} \to UT\tilde{M}$ where on the left we have an action via ρ on the sphere factor, and on the right the standard action on the unit tangent bundle. Although ρ does not act by diffeomorphisms (only homeomorphisms), by sacrificing injectivity we can design this equivariant map to send each horizontal leaf $\tilde{M} \times \{p\}$ to a C^1-embedded submanifold of $UT\tilde{M}$ that stays C^1-close to leaves of the weak-stable foliation over large compact sets (i.e., its tangent distribution is uniformly close to the weak-stable distribution). This means that the images of horizontal leaves remain *transverse* to the leaves of the weak-*unstable* foliation on $UT\tilde{M}$. We show that any two such leaves that intersect do so along a path that is uniformly close to a geodesic in $UT\tilde{M}$.

Now the boundary at infinity makes another appearance. Using some deep results on the dynamics of the action of $\pi_1(M)$ on its boundary (including the remarkable *convergence group property*, which we will, alas, not have space to discuss here), we show that the near-geodesics on the image of each "horizontal" leaf $\tilde{M} \times \{p\}$ cut out by the weak-stable leaves all share a common endpoint at infinity, depending only on the parameter p. This association of such a point p to this common endpoint at infinity gives a map $S^{n-1} \to S^{n-1}$. And this map, it turns out, gives the desired semiconjugacy of the actions.

4. COARSE HYPERBOLICITY: FROM SPACES TO GROUPS

Following Gromov, a geodesic metric space X is called *δ-hyperbolic* (for some $\delta \geq 0$) if every geodesic triangle T in X has the property that each side of T lies in the metric δ-neighborhood of the union of the other two sides. For example, any tree is 0-hyperbolic, and it is a pleasant exercise in hyperbolic trigonometry to show that $\mathbb{H}^n$, with its metric of constant curvature -1, is δ-hyperbolic for the constant $\delta = \ln(1 + \sqrt{2})$.

This definition also works for groups: a finitely generated group is *hyperbolic* if its Cayley graph is δ-hyperbolic for some δ. While the constant δ depends on the generating set, the notion *hyperbolic for some δ* does not. Indeed, "δ-hyperbolic for some δ" is a metric invariant up to *quasiisometry*, the relation shared between Cayley graphs with different generating sets. This concept is the center of Gromov's highly influential essay *Hyperbolic groups*, and the setting in which one can compactify a space by a boundary at infinity with the method given in the introduction—although Gromov attributes this idea as "essentially due to Mostow and Margulis" due to its appearance in the theorems we quoted earlier [18, 0.3B].

A finitely generated group acts naturally on its Cayley graph by auotmorphisms, hence by isometries when edges are taken to have unit length. Thus, it induces an action by homeomorphisms on the boundary. One can therefore ask:

Question 4.1. Let Γ be a finitely generated hyperbolic group. Is the action of Γ on its boundary (locally) rigid, up to semiconjugacy?

Rather than repeatedly writing "locally rigid up to semiconjugacy," we borrow terminology from classical dynamics, traditionally applied to actions of $\mathbb{Z}$, but just as valid for group actions. An action ρ_0 of Γ on a space X is *topologically stable* if, for any sufficiently nearby action ρ, there is a surjective, continuous map h satisfying $h \circ \rho(\gamma) = \rho_0(\gamma) \circ h$ for all $\gamma \in \Gamma$. Typically, one requires h to depend continuously on the perturbation, being close to the identity if ρ is sufficiently close to ρ_0. Thus, the statement of Theorem 3.1 is simply an assertion of topological stability for manifold fundamental groups.

Boundaries of groups. While the examples of boundaries at infinity we have looked at so far have been spheres, the topology of the boundary of a group is typically quite complicated, both locally and globally (see [20]). In some cases, the topology is so complicated that a positive answer to Question 4.1 follows from the structure of the boundary itself. Kapovich and Kleiner [22] have examples of groups where the only automorphisms of their boundary come from left multiplication. From this one can easily deduce rigidity: the action of the group on its boundary is an isolated point in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}(\partial_\infty \Gamma))$. For free groups, whose boundary is a Cantor set, one can also use the "ping-pong" dynamics of the action to prove local rigidity using a relatively hands-on argument.

At the other end of the spectrum are groups with sphere boundary. In contrast to Kapovich–Kleiner's boundaries, homeomorphisms of the sphere are very easy to perturb, each having an infinite-dimensional family of deformations, making Question 4.1 particularly interesting. It is in this context that Manning and I recently proved an analog to Theorem 3.1:

Theorem 4.2 (Rigidity for sphere boundary actions [26]). For any Gromov hyperbolic group Γ with sphere boundary, the natural action of Γ on $\partial_\infty \Gamma$ is topologically stable.

In proving this, we remove all the differential topological machinery (such as transversality and the regularity of weak-stable foliations) from the proof of Theorem 3.1, replacing it with *coarse metric* machinery. The fundamental starting point is the stability of *quasigeodesics*.

Quasigeodesic stability. A *quasigeodesic* is a map γ from $\mathbb{R}$ into a metric space (X, d_X) such that, for some constants K, C, the bounds

$$\frac{1}{K} d_X\big(\gamma(t), \gamma(s)\big) - C \leq |t - s| \leq K d_X\big(\gamma(t), \gamma(s)\big) + C$$

hold for all $t, s \in \mathbb{R}$. More generally, a (K, C) *quasiisometric embedding* of a metric space (Y, d_Y) into (X, d_X) is a map $\gamma : Y \to X$ that satisfies the above bounds for all points t, s in Y, with $|t - s|$ replaced by the distance $d_Y(t, s)$. Such a map is called a *quasiisometry* if it has the additional property of being *coarsely surjective*, meaning that each point of Y lies a uniformly bounded distance away from some point in the image.

The idea of quasigeodesic stability comes from work of H. M. Morse in the early 1920s. In [33], he considers the following question: suppose we take a closed surface Σ of genus $g \geq 2$ equipped with an arbitrary Riemannian metric, and a homeomorphism $f : \Sigma \to \Sigma_{\text{hyp}}$ identifying it with some fixed hyperbolic genus g surface Σ_{hyp}. What do length-minimizing geodesic paths on $\widetilde{\Sigma}$ look like under the lifted map $\tilde{f} : \widetilde{\Sigma} \to \widetilde{\Sigma}_{\text{hyp}}$? Do they share properties with genuine hyperbolic geodesics, such as tending in each direction to a unique point on the boundary of the disc in the Poincaré model?

Morse's map $\tilde{f}$ is an example of a quasiisometry, and the image of a length-minimizing geodesic under $\tilde{f}$ is a quasigeodesic. In answering the question above, Morse proves what is now known as the *Morse lemma* on quasigeodesic stability. In its modern, more general form, this lemma states:

Lemma 4.3 (Morse lemma). There exists a constant $B = B(K, C, \delta)$ such that for any δ-hyperbolic metric space X, every (K, C) quasigeodesic $\gamma : \mathbb{R} \to X$ lies in the B-neighborhood of a unique geodesic, and hence every quasigeodesic ray defines a unique point on $\partial_\infty(X)$.

In addition, quasigeodesics satisfy a *local-to-global* principle: a map which is a (K, C) quasigeodesic embedding when restricted to all sufficiently long segments is, in fact, globally a quasigeodesic. This allows us to pursue the broad strategy used in proving Theorem 3.1 in this coarse setting. We first translate a perturbation of an action into a nice map between foliated metric spaces, then show that images of leaves intersect leaves in (a preferred section of) the target foliated space along quasigeodesics. Of course, having no smooth manifold or universal cover on hand makes the strategy nontrivial to even set up, and much harder to execute!

Related results and open questions. Much existing work on the dynamics of groups acting on their boundaries relies on *local expansivity*: for each point of the boundary, there is an element of the group that contracts this neighborhood a uniform amount (thus, one has uniform *expansion* under the inverse, hence the name).[3] Sullivan [37] used this property to demonstrate a structural stability result for Kleinian groups acting on the boundary sphere of $\mathbb{H}^3$: a C^1-small perturbation of such an action has an invariant set on which the action is conjugate

3 There are many related definitions of expansivity, here I am roughly following Sullivan.

to the original action of the group on its limit set. This was recently improved and generalized by Kapovich–Kim–Lee [21] to prove structural stability for a much broader class of expansive actions, including boundary actions of hyperbolic groups, under perturbations which preserve a generalized expansivity property. Lipschitz-small perturbations are one example to which their theory applies, however, general continuous perturbations do not preserve local expansivity and so are not covered by this strategy. Sullivan's method involves a dynamical "coding" of points by sequences of group elements, suggesting a connection to classical symbolic dynamics. This is no coincidence, and we now know a number of rigidity results in this direction (see [7]). However, the following general problem remains open:

Question 4.4. Is the action of every hyperbolic group Γ on its boundary topologically stable? What techniques apply to intermediate cases between the boundary sphere case and Kapovich–Kleiner's rigid examples? Which examples exhibit a rigidity property stronger than topological stability, and what phenomena are responsible for this behavior?

As mentioned before, group boundaries can be topologically complex. One way the sphere plays an essential role in the proof of Theorem 4.2 is that its homeomorphism group is locally contractible, and there is no obvious substitute for this property in other settings. An interesting first case to attack might be the Menger curve, this being the boundary of a *random* group in the standard density model.

We note also that it is unknown which topological spaces occur as boundaries of groups. Thus, an approach to Question 4.4 either has to be restricted to families of understood examples, or avoid explicitly describing the boundary altogether.

5. AUTOMORPHISM GROUPS ACTING AT INFINITY

An automorphism of a finitely generated group Γ defines a quasiisometry of the Cayley graph of Γ, and therefore extends to a homeomorphism of $\partial_\infty \Gamma$. The inner automorphism defined by conjugation by γ is a bounded distance (with bound given by the word length of γ) from the map induced by left-multiplication by γ, so $\mathrm{Inn}(\Gamma) \cong \Gamma$ agrees with the actions we have already discussed and considering the action of $\mathrm{Aut}(\Gamma)$ is a natural next step.

Enlarging $\Gamma \cong \mathrm{Inn}(\Gamma)$ to $\mathrm{Aut}(\Gamma)$ is most interesting when the outer automorphism group of Γ is large. Many hyperbolic groups have trivial or finite outer automorphism group, the most basic case where $\mathrm{Out}(\Gamma)$ is infinite is when Γ is the fundamental group of a surface.[4] This case is particularly interesting to low dimensional topologists due to its relationship with mapping class groups.

Mapping class groups. A *mapping class* is an equivalence class of homeomorphism up to isotopy. Let $\mathrm{MCG}_{\pm}(\Sigma) := \pi_0(\mathrm{Homeo}(\Sigma))$ denote the group of all mapping classes of

4 Following the work of Paulin and Rips, Levitt showed that one-ended hyperbolic groups have infinite outer automorphism group if and only if they split, as an HNN extension or an amalgam of groups with finite center, over a virtually cyclic subgroup with infinite center, so in some sense resemble the surface groups we will discuss.

a surface Σ. One may also consider the subgroup of homeomorphisms fixing a basepoint, in which case $\mathrm{MCG}_{\pm}(\Sigma, x)$ denotes the group of homeomorphisms fixing x up to isotopy preserving x. The subscript $\pm$ here indicates that we consider both orientation preserving and reversing homeomorphisms, the mapping class groups denoted $\mathrm{MCG}(\Sigma, x)$ and $\mathrm{MCG}(\Sigma)$, respectively, are the index two subgroups of orientation-preserving elements.

For a surface Σ (which we continue to assume is of genus at least two, so that its fundamental group is hyperbolic), the *Dehn–Nielsen–Baer theorem* is the statement that the exact sequence

$$\mathrm{Inn}\big(\pi_1(\Sigma)\big) \to \mathrm{Aut}\big(\pi_1(\Sigma)\big) \to \mathrm{Out}\big(\pi_1(\Sigma)\big)$$

is isomorphic, term by term, to the Birman exact sequence

$$\pi_1(\Sigma) \to \mathrm{MCG}_{\pm}(\Sigma, x) \to \mathrm{MCG}_{\pm}(\Sigma).$$

This isomorphism has a particularly nice geometric description, using boundary actions and the identification of $\widetilde{\Sigma}$ with $\mathbb{H}^2$ coming from a choice of hyperbolic structure on Σ. Choose a lift $\tilde{x}$ of the point x on Σ, so each $f \in \mathrm{Homeo}(\Sigma)$ fixing x has a unique lift $\tilde{f}$ to $\mathbb{H}^2$ that fixes $\tilde{x}$. Since Σ is compact and f is continuous, this lift is a quasiisometry of $\mathbb{H}^2$ so induces a continuous map on the boundary circle. If f and g represent the same element of $\mathrm{MCG}_{\pm}(\Sigma, x)$, lifting an isotopy preserving $\tilde{x}$ will move all points on $\widetilde{\Sigma}$ a uniformly bounded distance, so will not change boundary homeomorphism. Thus, considering the action of lifts on the boundary gives a well-defined map from $\mathrm{MCG}_{\pm}(\Sigma, x)$ to homeomorphisms of S^1, agreeing with the action of $\mathrm{Aut}(\pi_1(\Sigma))$ under the identification above.

In his problem list on mapping class groups [8, QUESTION 6.2], Farb asks whether these actions are rigid:

Question 5.1 (Farb). Is every faithful action of $\mathrm{MCG}(\Sigma, x)$ on S^1 by homeomorphisms necessarily semiconjugate to the standard action on the boundary?

Question 5.1 asks for a much stronger form of rigidity than exhibited by the action of the fundamental group $\pi_1(\Sigma)$. There are many distinct semiconjugacy classes of faithful actions of $\pi_1(\Sigma)$ on the circle; in fact, one can even take these to have image in $\mathrm{PSL}(2, \mathbb{R})$ (see, e.g., [23] for a detailed discussion and references). Despite this, Farb's question is actually quite reasonable because of *torsion*. The mapping class group of a surface group contains many finite-order elements, and any action of a finite cyclic group on the circle is conjugate to an action by rotations. Thus, the presence of torsion is suggestive, though no guarantee, of rigidity.

Unexpectedly, something even stronger than what Farbs asks for is true—the hypothesis "faithful" is not needed, but only *nontrivial*.

Theorem 5.2 (Mapping class rigidity [28]). For any surface Σ of genus at least 2, every nontrivial action of $\mathrm{MCG}(\Sigma, x)$ on the circle is (up to choice of orientation) semiconjugate to the standard boundary action.

The proof, as expected, makes use of torsion, but in a perhaps unexpected way: we study the action of *orbifold fundamental groups* that contain $\pi_1(\Sigma)$. The definition of Euler number for surface group actions on S^1 that we introduced in Section 2 can be extended to actions of orbifold fundamental groups in a natural way so that it is multiplicative under covers, like Euler characteristic. We use geometric and topological arguments (relying on torsion) to show that nontrivial actions of $\mathrm{MCG}(\Sigma, x)$ have a maximal Euler number, and use Matsumoto's theorem to prove rigidity.

Since torsion plays a critical role in this argument, we do not know if a similar result holds for all finite-index subgroups of $\mathrm{MCG}(\Sigma, x)$. It would be interesting to see another approach to mapping class rigidity, relying more on group structure and less on the geometry and topology of the surface, perhaps towards a general theory for rigidity of actions of automorphism groups of other hyperbolic groups.

6. ANOSOV FLOWS ON 3-MANIFOLDS

We conclude this survey by describing a different method of producing actions at infinity, this one coming from the orbit space of an Anosov flows on a 3-manifold. A flow ϕ_t on a Riemannian manifold M is called *Anosov* if there is a ϕ_t-invariant global splitting of the tangent bundle TM as a direct sum

$$TM = X \oplus E^{ss} \oplus E^{uu},$$

where X is the direction of the flow, E^{ss} is the "stable distribution" consisting of vectors whose length is uniformly contracted by the flow, and E^{uu} is the "unstable distribution" consisting of vectors that are uniformly expanded (or, more precisely, uniformly contracted when the direction of the flow is reversed). *Contracted* has a specific meaning: there are positive constants c and $\lambda > 0$ such that the length of the pushforward of a tangent vector $(\phi_t)_*(v)$ under the time t map of the flow is bounded above by $ce^{-\lambda t}||v||$ for all $t > 0$. On a compact manifold, one may always find a Riemannian metric on M *adapted* to the flow for which one may take the multiplicative constant $c = 1$.

It is a classical result that the two distributions $E^s := X \oplus E^{ss}$ and $E^u := X \oplus E^{uu}$ are integrable, meaning they are everywhere tangent to a foliation. We will restrict our attention in this section to 3-manifolds, thus E^{ss}, E^{uu}, and X are all one-dimensional, and the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ tangent to $X \oplus E^{ss}$ and $X \oplus E^{uu}$ are 2-dimensional transverse foliations that meet along the orbits of the flow.

The two most basic examples of Anosov flows in dimension 3, and indeed the starting points for the construction of all other known examples, are suspensions of linear maps on tori and geodesic flows on surfaces.

Example 6.1 (Linear Anosov maps on tori). Consider a transformation $A \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{trace}(A) > 2$, or equivalently, two distinct, real eigenvalues of norm not equal to 1. Since A preserves the integer lattice in $\mathbb{R}^2$, it descends to a self-diffeomorphism $\bar{A}$ of the square torus $T^2 := \mathbb{R}^2/\mathbb{Z}^2$. Let M be the *mapping torus* of $\bar{A}$, the quotient of $T^2 \times \mathbb{R}$ under the relation $(x, 0) \sim (\bar{A}^n(x), n)$ for $n \in \mathbb{Z}$. The straight line flow $(x, s) \mapsto (x, s + t)$ on $T^2 \times \mathbb{R}$ descends to a flow ϕ_t on M. Each eigendirection of A defines a 1-dimensional $\bar{A}$-invariant line field on M, one of which is uniformly contracted by ϕ_t flow and one uniformly expanded, giving the desired Anosov property.

The reason Anosov flows are such interesting examples in dynamics is that they simultaneously exhibit global stability and local chaos. *Local chaos* means that nearby points have vastly different trajectories. This is already apparent in Example 6.1, for instance, the origin on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is fixed by $\bar{A}$ so is a periodic orbit of ϕ_t, but arbitrarily nearby points have infinite trajectories that intersect the torus $T^2 \times \{0\}$ along a dense set. Moreover, this interspersing of periodic and infinite trajectories happens everywhere: since the induced action of elements of $\mathrm{SL}(2, \mathbb{Z})$ preserve the finite set of points of the form $\{(p/q, r/q) : 0 \leq p, r < q\}$ (for any fixed q) on the fundamental domain $[0, 1]^2$ for $\mathbb{R}^2/\mathbb{Z}^2$, any such point eventually returns to itself under iterates of $\bar{A}^n$, giving a closed orbit for ϕ_t.

By contrast, the global picture of the flow is overall stable. Replacing A by a nearby nonlinear diffeomorphism of T^2 and doing the same construction gives a new flow on the same topological space, which turns out always to be *conjugate* to the flow just discussed. In fact, here one can even replace A by any map with the same action on homology of T^2 **[13, 19]**.

The second building block for Anosov flows is geodesic flow in negative curvature. The simplest such examples come from hyperbolic surfaces.

Example 6.2 (Geodesic flow). Let Σ be a surface equipped with a metric of constant curvature -1, and let $M = UT(\Sigma)$ be its unit tangent bundle. *Geodesic flow* is the map that, at time t, sends a unit tangent vector $v \in UT(\Sigma)$ to the vector tangent to the line $\{\exp(sv) : s \in \mathbb{R}\}$ at the point $\exp(tv)$. One can compute explicitly using the identification of $UT(\Sigma)$ with $\mathrm{PSL}(2, \mathbb{R})$ that this flow is Anosov. The weak-stable and weak-unstable foliations from the flow (lifted to $\widetilde{\Sigma}$) are exactly those shown in Figure 2. Leaves of $\mathcal{F}^s$ consist of tangent vectors to geodesics with a common forward endpoint, and $\mathcal{F}^u$ those with a common negative endpoint at infinity.

Classifying flows. Having given two families of examples, we now embark on the ambitious program to understand all Anosov flows in dimension 3. For this, one must answer questions of:

- *Existence.* Which 3-manifolds support an Anosov flow? What techniques can be used to construct families of examples?
- *Abundance.* If M^3 supports one Anosov flow, can it support many dynamically distinct ones?
- *Classification.* What invariants can be used to distinguish distinct flows?

The existence problem has a long history, but is still not completely solved. The work of Palmeira and Verjovsky from the 1970s shows that if M supports an Anosov flow, then $\tilde{M}$ is homeomorphic to $\mathbb{R}^3$, thus M must be irreducible. An irreducible 3-manifold admits a decomposition along tori into geometric pieces, so the next question to ask is which kinds of pieces M may have. Margulis [2, APPENDIX] showed that $\pi_1(M)$ is large in the sense that it has *exponential growth*: for any fixed generating set, the number of reduced words of length r in the group grows exponentially in r. This rules out, for instance, geometric manifolds with a Euclidean structure. There are a number of constructions—Dehn surgery and other gluing techniques—that produce examples on geometric manifolds with exponential growth, or manifolds with a nontrivial torus decomposition, and a few other known special constraints, but we still lack a complete picture.

The approach to classification that I wish to discuss here is the purely topological one, namely, classifying flows *up to orbit equivalence*. Two flows ϕ_t and ψ_t on a manifold M are called *orbit equivalent* if there is a self-homeomorphism of M taking orbits of ϕ_t to orbits of ψ_t, in other words, the 1-dimensional foliations on M by flowlines are homeomorphic.

The remainder of this survey is devoted to describing a very recent rigidity result (Theorem 6.3 and its generalization) saying that Anosov flows can be distinguished up to this equivalence by the set of homotopy classes of loops represented by closed orbits. The proof of this result comes, again, from looking to a boundary at infinity. This time, the boundary is that of the *orbit space* of the flow.

$\mathbb{R}$-covered flows and orbit spaces. A first topological invariant to distinguish flows comes from the global transverse structure $\mathcal{F}^s$. Lifting $\mathcal{F}^s$ to a foliation $\tilde{\mathcal{F}}^s$ on $\tilde{M}$ gives a foliation of $\mathbb{R}^3$ by planes. Collapsing each plane to a point produces a 1-dimensional manifold called the *leaf space* of $\tilde{\mathcal{F}}^s$. The leaf space is either non-Hausdorff, or homeomorphic to $\mathbb{R}$; in the latter case we say the flow is $\mathbb{R}$*-covered*. This terminology does not privilege $\mathcal{F}^s$, by [3, 10] the leaf space of $\tilde{\mathcal{F}}^s$ is Hausdorff if and only if that of the lifted unstable foliation $\tilde{\mathcal{F}}^u$ is as well. These two cases ($\mathbb{R}$ covered and non-Hausdorff leaf space) lend themselves to different techniques for classification. We discuss the $\mathbb{R}$ covered case first.

While being $\mathbb{R}$-covered may seem like a restrictive hypothesis, there are, in fact, many diverse examples. Both Examples 6.1 and 6.2 are $\mathbb{R}$-covered, and many more can be produced by modifying these manifolds using Dehn surgery. A given manifold may support *arbitrarily many* inequivalent $\mathbb{R}$-covered flows even if geometric; Bowden and Mann [6] give constructions of such on closed hyperbolic manifolds.

Our understanding of $\mathbb{R}$-covered flows is due largely to the work of Barbot and Fenley, starting with the work in [3, 10]. They consider the *orbit space* $\mathcal{O}$ of the flow, the quotient of $\tilde{M}$ obtained by collapsing each flowline to a point. Any $\mathbb{R}$-covered flow that is *not* obtained from a hyperbolic toral automorphism such as in Example 6.1 has the remarkable property that its orbit space is homeomorphic to an infinite diagonal strip in the plane, as shown on the left of Figure 4, with $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ being the vertical and horizontal foliations. Such flows are called *skew*. Since the topology of the foliations on this orbit space does not distinguish flows, any classification theorem must rest on a new algebraic or topological invariant. In recent work, Barthelmé and I show that the *spectrum of periodic orbits* (the free homotopy classes of loops represented by periodic orbits of the flow) does the job:

Theorem 6.3 (Spectral rigidity for flows [4]). Suppose ϕ_t and ψ_t are $\mathbb{R}$-covered Anosov flows on a compact 3-manifold M. The conjugacy classes in $\pi_1(M)$ represented by the free

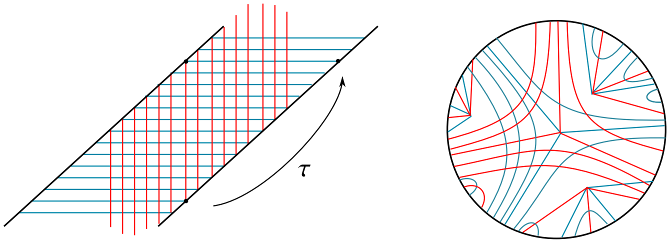

FIGURE 4
The orbit space of a skew flow (left) and a schematic of that of a pseudo-Anosov flow (right). Nonseparated stable and unstable leaves "meeting at infinity" define a shift τ commuting with the action of $\pi_1(M)$ on the skew picture.

homotopy classes of closed orbits for ϕ_t and ψ_t agree if and only if the flows are orbit equivalent via a map isotopic to the identity.

The action on $\mathcal{O}$ at infinity. The interesting case in Theorem 6.3 is for skew flows, as the mapping torus of a linear Anosov map admits only the obvious suspension flow and its inverse. To solve the problem for skew flows, we look to the boundary at infinity—the compactification of $\mathcal{O}$ by the lines in the diagonal strip model. The action of $\pi_1(M)$ on $\tilde{M}$ descends to $\mathcal{O}$ and extends to an action by homeomorphisms on each boundary line, commuting with a translation of the line that comes from the structure of the lifted foliations. The dynamics of individual elements acting on the line at infinity are also well understood. Up to passing to the index-two subgroup of elements preserving orientation, it is an example of what we call a *hyperbolic-like action*.

Definition 6.4 ([4]). An action of a group G on the line is *hyperbolic-like* if it commutes with the translation $x \mapsto x + 1$, and every nontrivial element either acts freely, or has precisely two fixed points in each unit interval, one attracting and one repelling.

In the tradition of classical theorems of Hölder and Solodov, which promote information about the dynamics of individual homeomorphisms of the line to a global conclusion about the structure of a group action, we prove a general result on hyperbolic-like actions on the line.

Theorem 6.5 (Hyperbolic actions are determined by fixed spectra [4]). Given two faithful, minimal, hyperbolic-like actions of a group G on $\mathbb{R}$, if the sets of elements acting with fixed points for each action agree, then the two actions are conjugate by a homeomorphism.

The strategy of the proof for Theorem 6.5 is to recover the linear order of fixed points of elements (and hence reconstruct a dense subset of the line) from the algebraic data of the set of elements with fixed points. This is not far in spirit from the "reconstruction" strategies described in Section 2. Theorem 6.5 is really the heart of the proof of Theorem 6.3, what remains is to promote the conjugacy of actions at infinity to an honest orbit equivalence, a technique already used by Barbot.

Non-ℝ-covered flows and pseudo-Anosov flows. In the case where the leaf spaces of $\tilde{F}^s$ and $\tilde{F}^u$ are non-Hausdorff, one can leverage the topology of these foliations to get information about the flow, a perspective fruitfully exploited by Fenley in [11]. It turns out that the same family of techniques also applies to a strictly broader class of *pseudo-Anosov* flows—topological flows with expanding/contracting behavior as in the Anosov case, but where $\mathcal{F}^s$ and $\mathcal{F}^u$ are allowed to *branch* in a specified way along a discrete set of periodic orbits.

The orbit space of such a flow is a topological plane with two transverse, 1-dimensional, possibly singular foliations, as cartooned in Figure 4 (right). In [9], Fenley gives a natural construction of a compactification of the orbit space of any pseudo-Anosov flow by a boundary circle so that the compactified space is homeomorphic to a disk and the natural action of the fundamental group of the manifold by homeomorphisms of $\mathcal{O}$ extends to the boundary. In the work in preparation with Barthelmé and Frankel, we use this boundary circle and the induced action of $\pi_1(M)$ to prove spectral rigidity for all transitive, non-ℝ-covered Anosov and pseudo-Anosov flows on compact 3-manifolds. Combined with Theorem 6.3, this gives a full spectral rigidity result in the Anosov and pseudo-Anosov setting: provided the flow is transitive, if the conjugacy classes in $\pi_1(M)$ represented by the free homotopy classes of closed orbits for two flows ϕ_t and ψ_t agree, then the flows are orbit equivalent via a map isotopic to the identity.

Although this work gives one answer to the classification problem, many open questions remain, especially regarding existence and abundance. Of particular interest to me is the interplay between geometry of a manifold and topology of the leaf spaces of such flows, hyperbolic manifolds being a particularly interesting example. Which hyperbolic 3-manifolds admit Anosov flows? Does the complexity of the manifold bound the number of distinct flows it may admit? May a hyperbolic manifold admit infinitely many inequivalent Anosov flows?

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