FLOER COHOMOLOGY, SINGULARITIES, AND BIRATIONAL GEOMETRY

MARK MCLEAN

ABSTRACT

We explain a few recent results concerning the application of various Floer theories to topics in algebraic geometry, including singularity theory and birational geometry. We will also state conjectures and open problems related to these results. We start out with a purely dynamical interpretation of the minimal discrepancy of an isolated singularity and explain how Floer theory fits into this story. Using similar ideas, we show how one can prove part of the cohomological McKay correspondence by computing a Floer cohomology group in two different ways. Finally, we illustrate how Hamiltonian Floer cohomology can be used to prove that birational Calabi–Yau manifolds have the same small quantum cohomology algebras, and we speculate how this might extend to orbifolds.

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1. INTRODUCTION

Objects in algebraic geometry provide a rich source of symplectic and contact manifolds. Smooth projective varieties $X \subset \mathbb{C}P^N$, for instance, have symplectic structures ω_X given by restricting the standard Fubini–Study form on $\mathbb{C}P^N$. Links of isolated singularities admit natural contact structures, given by a complex hyperplane distribution.

What can such structures tell us about the underlying algebraic variety? Typically, a large amount of data is lost when one forgets everything except the symplectic or contact structure. For instance, if we have a smooth family of projective varieties in $\mathbb{C}P^N$, a Moser argument tells us that they are all symplectomorphic. On the other hand, many properties are retained such as *uniruledness*, which is the property that a rational curve passes through every point. This was shown by Kollár and Ruan in [47, PROPOSITION G].

An important tool in symplectic geometry which can help us understand this question better is *Floer (co)homology*. In order to understand what Floer (co)homology is, it is best to first understand its finite-dimensional counterpart, namely *Morse homology*. Let us suppose that we have a generic Morse function f on a closed Riemannian manifold M. Such a function naturally decomposes M into cells, one for each critical point (Figure 1). Hence one can use f to compute the cellular homology of M. The underlying chain complex is generated as a $\mathbb{Z}$-module by critical points of f and the differential is a matrix with respect to this basis of critical points whose (p, q)-entry is the number (counted with sign) of gradient flowlines of f connecting p and q. The homology of this chain complex is called *Morse homology*.

Floer homology is an infinite-dimensional version of Morse homology. There are many different kinds of Floer homology groups. For instance, the infinite-dimensional version of the manifold M above could be the free loopspace $C^\infty(S^1, X)$ of a symplectic manifold (X, ω) with $\omega = d\theta$ satisfying an additional "convexity" condition at infinity. The infinite-dimensional version of the Morse function f could be an action functional

$$\mathcal{A} : C^\infty(S^1, X) \to \mathbb{R}, \quad \mathcal{A}(\gamma) := -\int_{S^1} \gamma^*\theta - \int_0^{2\pi} H\big(\vartheta, \gamma(e^{2\pi i \vartheta})\big) d\vartheta \qquad (1.1)$$

where $H : \mathbb{R}/\mathbb{Z} \times X \to \mathbb{R}$ is a time-dependent Hamiltonian, which also has a particular form near infinity. The generators of the chain complex for this Floer cohomology group as

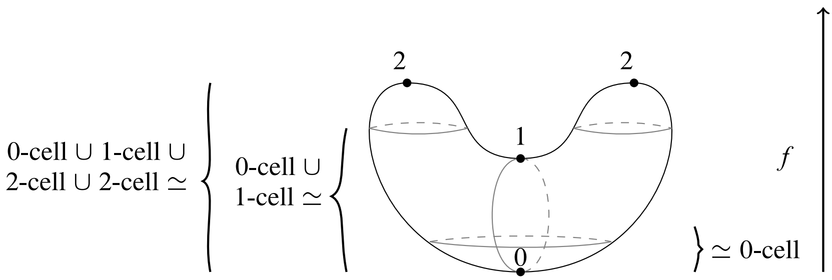

FIGURE 1
Cell decomposition from Morse function f.

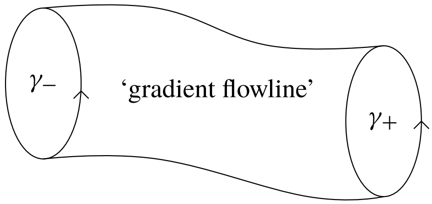

FIGURE 2
Floer differential.

a $\mathbb{Z}$-module are 1-periodic orbits of H, which are the critical points of $\mathcal{A}$. The differential is a matrix whose (γ_-, γ_+)-entry is the number of cylinders mapping to X connecting γ_- and γ_+ satisfying a certain PDE which represents the "gradient flowlines" of $\mathcal{A}$ (Figure 2). See [40] for a survey of some of these ideas.

In this article we will demonstrate how certain Floer (co)homology groups can be used to understand the following things:

(1) The minimal discrepancy of isolated singularities (Section 2);

(2) The Cohomological McKay correspondence (Section 3);

(3) Quantum cohomology of birational Calabi–Yau manifolds (Section 4).

2. MINIMAL DISCREPANCY OF ISOLATED SINGULARITIES

Let $A \subset \mathbb{C}^N$ be an irreducible affine variety of complex dimension n with at most one singularity at $0 \in \mathbb{C}^N$. In other words, $A \subset \mathbb{C}^N$ is cut out by a finite number of polynomial equations whose Jacobian matrix has constant rank along $A - 0$. The *link* of A at 0 is the manifold given by the intersection of A with the ε-sphere $S_\varepsilon := \{|z| = \varepsilon\} \subset \mathbb{C}^N$ where $\varepsilon > 0$ is any sufficiently small number. The link admits a contact structure $\xi_A := TL_A \cap iTL_A$ where $i : T\mathbb{C}^N \to T\mathbb{C}^N$ is multiplication by $i = \sqrt{-1}$ so long as $\varepsilon > 0$ is small enough.

Let us give two examples of such links. The first example is the smooth case, $A = \mathbb{C}^n$. Here the link $(L_{\mathbb{C}^n}, \xi_{\mathbb{C}^n})$ is contactomorphic to the $(2n-1)$-dimensional sphere S^{2n-1} with contact structure $\ker(\sum_{k=1}^n x_k dy_k - y_k dx_k)$ where $x_1 + iy_1, \ldots, x_n + iy_n$ are the standard complex coordinates on $\mathbb{C}^n$. The second example is the nondegenerate hypersurface singularity $A = \{\sum_{k=1}^{n+1} z_k^2 = 0\}$ inside $\mathbb{C}^{n+1}$ whose link is contactomorphic to the unit cotangent bundle of the n-sphere.

Suppose that $A' \subset \mathbb{C}^{N'}$ is another irreducible affine variety with at most one singularity at 0. If there are neighborhoods $U \subset A$, $U' \subset A'$ of 0 together with a homeomorphism $\phi : U \to U'$ sending $0 \in A$ to $0 \in A'$ so that ϕ is a biholomorphism from $U - 0$ to $U' - 0$, then the links (L_A, ξ_A) and $(L_{A'}, \xi_{A'})$ are contactomorphic ([56]).

Question 2.1. Conversely, suppose that (L_A, ξ_A) is contactomorphic to $(L_{A'}, \xi_{A'})$. What does A and A' have in common?

The following definition is inspired by a similar notion in Heegaard's thesis (see [22, PAGE 89]). We say that A is *topologically smooth at* 0 if L_A is diffeomorphic to a sphere. Mumford in [39] showed that if A is of dimension 2, normal, and topologically smooth then A is smooth at 0. Such a fact is false in higher dimensions. For instance, the three-dimensional singularity $\{x^2 + y^2 + z^2 + w^3 = 0\} \subset \mathbb{C}^4$ is not smooth, but it is normal and topologically smooth (see [7]). However, the link of such a singularity is not contactomorphic to the link of $\mathbb{C}^3$ (see [55] for a direct proof). Seidel in [50] conjectured the following:

Conjecture 2.2. *If A is normal and (L_A, ξ_A) is contactomorphic to $(L_{\mathbb{C}^n}, \xi_{\mathbb{C}^n})$ then A is smooth at* 0.

Theorem 2.3 ([34, COROLLAY 1.2]). *Conjecture 2.2 is true in dimension* 3.

In fact, we proved a stronger result, which we will see has some connections with birational geometry. First of all, we need some definitions. Let (C, ξ) be a cooriented contact manifold. A 1-form $\alpha \in \Omega^1(C)$ is *compatible* with ξ if $\ker(\alpha) = \xi$ and α respects the coorientation of ξ. The restriction $d\alpha|_\xi$ is a symplectic structure on ξ. Therefore since the natural inclusion map from the unitary group to the linear symplectomorphism group is a homotopy equivalence, we get that the structure group of ξ naturally lifts to the unitary group and hence we can define its first Chern class $c_1(\xi)$. We say that our singularity $0 \in A$ is *numerically $\mathbb{Q}$-Gorenstein* if $c_1(\xi_A)$ vanishes in $H^2(L_A; \mathbb{Q})$.

We will now define the minimal discrepancy of such a singularity. This is an important invariant in the minimal model program (see [51]). Let X be a complex n-manifold with boundary and suppose that the natural map $H^2(X, \partial X; \mathbb{Q}) \to H^2(X; \mathbb{Q})$ is injective. Suppose also that $c_1(TX|_{\partial X})$ vanishes inside $H^2(\partial X; \mathbb{Q})$. Then, we can define the *relative first Chern class* $c_1(X, \partial X) \in H^2(X, \partial X; \mathbb{Q})$ as follows. Consider the long exact sequence

$$H^1(\partial X; \mathbb{Q}) \xrightarrow{0} H^2(X; \partial X; \mathbb{Q}) \to H^2(X; \mathbb{Q}) \xrightarrow{\beta} H^2(\partial X; \mathbb{Q}). \tag{2.1}$$

The vanishing condition on $c_1(TX|_{\partial X})$ implies that $\beta(c_1(X)) = 0$ and so $c_1(X)$ lifts uniquely to a class $c_1(X, \partial X; \mathbb{Q}) \in H^2(X; \partial X; \mathbb{Q})$ which we call the *relative first Chern class of X*.

Now let $\pi : \tilde{A} \to A$ be a *resolution* of A at 0. In other words, a proper morphism from a smooth variety $\tilde{A}$ which is an isomorphism onto its image away from $0 \in A$ so that $\pi^{-1}(0)$ is a union of transversally intersecting connected complex hypersurfaces $E := \bigcup_{i \in S} E_i$. The hypersurfaces $(E_i)_{i \in S}$ are the *irreducible exceptional divisors* of our resolution. Resolutions always exist according to Hironaka [23]. If A is smooth at 0 then we require $\tilde{A} \neq A$. Let $B_\varepsilon \subset \mathbb{C}^N$ be the closed ε-ball. Then $\tilde{A}_\varepsilon := \pi^{-1}(B_\varepsilon)$ deformation retracts onto E for ε small enough and so $H^2(\tilde{A}_\varepsilon; \mathbb{Q})$ is generated freely by the Poincaré duals $\mathrm{PD}(E_i)$, $i \in S$ of $(E_i)_{i \in S}$. Such a fact combined with the negativity lemma can be used to show that the natural map $H^2(\tilde{A}_\varepsilon; \partial\tilde{A}_\varepsilon; \mathbb{Q}) \to H^2(\tilde{A}_\varepsilon; \mathbb{Q})$ is injective (see [34, LEMMA 3.2]). Also $\xi_A \oplus \mathbb{C}$ is isomorphic to $TA|_{L_A}$ and so $c_1(\xi_A) = c_1(TA|_{L_A})$.

Now suppose that our singularity is numerically $\mathbb{Q}$-Gorenstein. Then by the discussion above, $\tilde{A}_\varepsilon$ has a relative first Chern class which is a sum $\sum_{i \in S} a_i \mathrm{PD}(E_i)$ for some unique rational numbers $(a_i)_{i \in S}$. The *discrepancy* of E_i is defined to be a_i for each $i \in S$. We define the *minimal discrepancy* $\mathrm{md}(A) \in \mathbb{Q}$ of $0 \in A$ to be $a := \min_{i \in S} a_i$ if $a \geq -1$ and $-\infty$ otherwise.

We will give a dynamical interpretation of the minimal discrepancy using ξ_A. The *Reeb vector field* associated to a contact form α compatible with ξ_A is the unique vector field R_α in the kernel of $d\alpha$ satisfying $\alpha(R_\alpha) = 1$. The dynamics of the flow of such a vector field can change drastically depending on the choice of contact form compatible with ξ_A. A *Reeb orbit of* α of period $L > 0$ is a periodic flowline $\gamma : \mathbb{R}/L\mathbb{Z} \to L_A$ of this vector field. If (L_A, ξ_A) is $\mathbb{Q}$-Gorenstein and satisfies $H^1(L_A; \mathbb{Q}) = 0$, one can associate an index to this orbit γ called the *Conley–Zehnder index* $\mathrm{CZ}(\gamma) \in \mathbb{Q}$ (see [34, DEFINITION 4.2]). Very roughly, this index "counts" the number of times the Reeb flow "wraps" around γ. We define the *lower SFT index* to be

$$\mathrm{CZ}(\gamma) - \frac{1}{2} \dim \ker(D\phi_L|_{(\xi|_{\gamma(0)})} - \mathrm{id}) + (n-3), \tag{2.2}$$

where $\phi_t : L_A \to L_A$, $t \in \mathbb{R}$ is the flow of R_α. We define the *minimal SFT index* of α to be $\mathrm{mi}(\alpha) := \inf_\gamma \mathrm{lSFT}(\gamma)$ where the infimum is taken over all Reeb orbits γ of α. We define the *highest minimal SFT index* of (L_A, ξ_A) to be $\mathrm{hmi}(L_A, \xi_A) := \sup_\alpha \mathrm{mi}(\alpha)$ where the supremum is taken over all contact forms α compatible with ξ_A. By construction, this is an invariant of (L_A, ξ_A) up to coorientation preserving contactomorphism.

Theorem 2.4 ([34, THEOREM 1.1]). *Let* $0 \in A$ *be normal and numerically* $\mathbb{Q}$*-Gorenstein. Suppose* $H^1(L_A; \mathbb{Q}) = 0$. *Then*

- *if* $\mathrm{md}(A, 0) \geq 0$ *then* $\mathrm{hmi}(L_A, \xi_A) = 2\mathrm{md}(A, 0)$, *and*
- *if* $\mathrm{md}(A, 0) < 0$ *then* $\mathrm{hmi}(L_A, \xi_A) < 0$.

Seidel's conjecture follows immediately from Theorem 2.4 above and the conjecture below due to the fact that $\mathrm{md}(\mathbb{C}^n, 0) = n - 1$.

Conjecture 2.5 ([52, CONJECTURE 2]). *Suppose* A *is normal and numerically* $\mathbb{Q}$*-Gorenstein with* $\mathrm{md}(A, 0) = n - 1$ *then* A *is smooth at* 0.

This conjecture is true when $n = 3$ by [45, MAIN THEOREM (I)] combined with minimal discrepancy calculations from [33] and [25], as well as [5, COROLLARY 5.17]. Therefore we have a proof of Theorem 2.3.

There are two parts to the proof of Theorem 2.4. The first part gives an upper bound of $\mathrm{md}(A, 0)$, and the second part gives a lower bound. It is easier to prove the upper bound since one only needs to find an explicit contact form α compatible with ξ_A satisfying $\mathrm{md}(A, 0) \leq \mathrm{mi}(\alpha)$. In order to construct such a contact form, one starts with a resolution $\pi : \tilde{A} \to A$ as above. Since $\pi^{-1}(0)$ is a transverse intersection of complex hypersurfaces, one can deform the link $\pi^{-1}(S_\varepsilon)$ through contact hypersurfaces so that it is compatible with these hypersurfaces in some sense. The periodic orbits of the corresponding Reeb flow

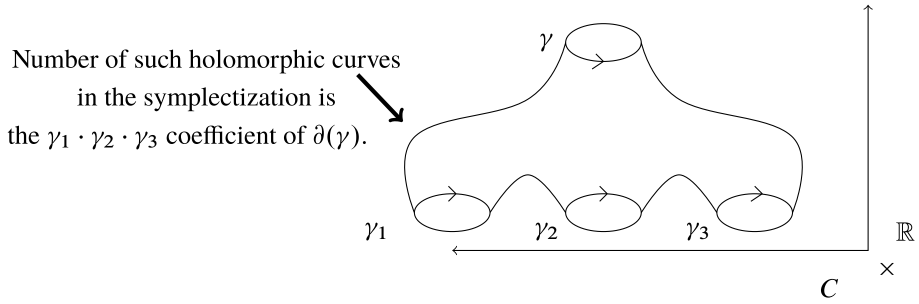

FIGURE 3
Full contact homology differential.

"wrap" around the divisors $(E_i)_{i\in S}$. One can explicitly compute all of their Conley–Zehnder indices, giving our result (see [34, THEOREM 5.23]).

In the paper [34], we used pseudoholomorphic curve techniques to give the lower bound for $\mathrm{md}(A, 0)$ (see [34, SECTIONS 6,7]). However, this lower bound conjecturally can also be proven using a Floer homology group, called *full contact homology*. We will give a brief sketch of this idea in the case where $\mathrm{md}(A, 0) \geq 0$.

Very roughly, *full contact homology* $\mathrm{CH}_*(C, \xi)$ of a $(2n-1)$-contact manifold (C, ξ) is defined in the following way (see [2,14,24,42]). The chain complex is the free supercommutative algebra over $\mathbb{Q}$ generated by Reeb orbits of a generic compatible contact form λ and graded by Conley–Zehnder index plus $(n-3)$. We now put an appropriate translation-invariant almost-complex structure on the symplectization $(\mathbb{R} \times C, d(e^t\lambda))$ of (C, ξ). The differential is the unique $\mathbb{Q}$-linear differential on this algebra satisfying the Leibniz rule and whose $(\gamma, \prod_{i=1}^k \gamma_i)$ coefficient is a count of genus-zero holomorphic curves in $\mathbb{R} \times C$ up to translation "limiting" to the corresponding Reeb orbits γ, $(\gamma_i)_{i=1}^k$ of λ (Figure 3). Full contact homology does not depend on the choice of a compatible contact form.

We have the following conjectural spectral sequence computing $\mathrm{CH}_*(L_A, \xi_A)$. To set up this spectral sequence, we need some preliminary definitions. For each $I \subset S$, let $E_I := \bigcap_{i\in I} E_i$ where $(E_i)_{i\in S}$ are the irreducible exceptional divisors of our resolution as above. Define $E_I^o := E_I - \bigcup_{I \subsetneq I'} E_{I'}$ and let NE_I^o be its normal bundle in $\tilde{A}$ for each $I \subset S$. For each tuple $(k_i)_{i\in I}$ of integers, there is a $U(1)$ action on NE_I^o preserving the fibers so that $\beta \in U(1)$ sends a point $(x_i)_{i\in I} \in NE_I^o = \bigoplus_{i\in S-I}((TE_{I-i}|_{E_I^o})/TE_I^o)$ to $(\beta^{k_i} x_i)_{i\in I}$. Let $NE_I^{/(k_i)_{i\in I}}$ be the quotient of $NE_I^o - E_I^o$ by this action. Suppose our resolution $\tilde{A}$ admits a Kähler form ω with an integral lift. Then one can construct a line bundle with curvature a positive multiple of $-2\pi i\omega$ together with a meromorphic section s so that the divisor associated to s is equal to $-\sum_{i\in S} w_i E_i$ for some positive integers $(w_i)_{i\in S}$.

Conjecture 2.6. *Define*

$$A_{p,q} \equiv \bigoplus_{\{(k_i)\in\mathbb{N}_{\geq 0}^S : \sum_i k_i w_i = p\}} H_{p+q-2\sum_i k_i a_i}\big(NE_{I_{(k_i)_{i\in I}}}^{/(k_i)}; \mathbb{Q}\big) \tag{2.3}$$

where $I_{(k_i)} \equiv \{i \in S : k_i \neq 0\}$. *Then there is a spectral sequence converging to* $\mathrm{CH}_*(L_A, \xi_A)$ *with* E^1 *page equal to the free supercommutative algebra generated by the bigraded vector space* $A_{*,*}$, *i.e.*,

$$E^1_{*,*} = \bigoplus_{n \geq 0} \mathrm{Sym}^n_{\mathbb{Q}}(A_{*,*}). \tag{2.4}$$

This spectral sequence is very similar to those in [19] and [35], and we expect the method of proof for that above to be similar in spirit.

Now let us continue with the proof of the lower bound for $\mathrm{md}(A, 0)$ in the case where $\mathrm{md}(A, 0) \geq 0$. Consider the smallest value of p for which the entry $E^1_{p,2\mathrm{md}(A,0)-p}$ in the spectral sequence above is nonzero. Then for degree reasons, this entry cannot be killed. Hence full contact homology is nonzero in degree $2\mathrm{md}(A, 0)$. This implies that there is a Reeb orbit of lower SFT index $2\mathrm{md}(A, 0)$ for any generic contact form compatible with ξ_A and hence $\mathrm{hmi}(L_A, \xi_A) \leq 2\mathrm{md}(A, 0)$, giving us our lower bound.

It would be interesting to know if there are other properties of the singularity $0 \in A$ captured by full contact homology. Full contact homology is typically very hard to compute since one has to compute the differential by solving a PDE with asymptotic boundary conditions. However, the following definition and conjecture might be of help.

Definition 2.7. Let $\mathbb{D}_z(\delta) \subset \mathbb{C}$ be the closed disk of radius $\delta > 0$ centered at $z \in \mathbb{C}$. Define $\mathbb{D}(\delta) := \mathbb{D}_0(\delta)$ and $\mathbb{D} := \mathbb{D}(1)$. Define the *short arc space* $\mathrm{Arc}^o(A)$ to be the space of holomorphic maps $u : \mathbb{D} \to A$ whose boundary is disjoint from 0 equipped with the C^∞ topology coming from the embedding in $A \subset \mathbb{C}^N$ (see [27, DEFINITION 2(2)]). Let $\mathrm{Arc}^*(A)$ be the disjoint union $\bigsqcup_{m \in \mathbb{N}_{\geq 0}} (\mathrm{Arc}^o(A))^m$. For each $w \in \mathbb{R}$, we define $\mathrm{Arc}^*_{\leq w}(A)$ to be the subspace of those tuples $(u_i)_{i=1}^m$ of arcs for which the sum of the degrees of $u_i^*(\sum_{j \in S} w_j E_j)$, $i = 1, \dots, m$ is $\leq w$, as well as the case $m = 0$.

For each $l \in \mathbb{N}$, we define $\widetilde{\mathrm{Jet}}^l(A)$ to be the set of l-jets of holomorphic maps $\phi : \mathbb{D}(\delta) \to A$, $\delta > 0$ satisfying $u^{-1}(0) = 0$. We let $\overline{\mathrm{Jet}}^l(A) := \widetilde{\mathrm{Jet}}^l(A)/S^1$ where the S^1 action rotates the arcs. For each $l \in \mathbb{N}$, define $S\overline{\mathrm{Jet}}^{l!}(A) := \{\emptyset\} \sqcup \bigsqcup_{j=1}^l (\overline{\mathrm{Jet}}^{l!/j}(A))^j / S_j$ where S_j is the permutation group on j elements. For $w \in \mathbb{R}$ and $l > w$, we define the map $\pi_{l,w} : \mathrm{Arc}^*_{\leq w}(A) \to S\overline{\mathrm{Jet}}^{l!}(A)$ as follows: Let $(u_i)_{i=1}^m$ be an element of $\mathrm{Arc}^*(A)$ and for each $j = 1, \dots, m$ let $u_i^{-1}(0) = \{z_1^i, \dots, z_{j_i}^i\} \subset \mathbb{D}$, $j_i \in \mathbb{N}$. Let $\delta > 0$ be very small. Then the collection of arcs $\phi|_{\mathbb{D}_{z_k^j}(\delta)}$, $k = 1, \dots, j_i$, $i = 1, \dots, m$ defines an element of $S\overline{\mathrm{Jet}}^{l!}(A)$. If $m = 0$ then this corresponds to having no arcs, and we map this to $\{\emptyset\}$.

For each $w \in \mathbb{R}$, $l > w$, we define $S\mathrm{Jet}^{l!}_{\leq w}(A)$ to be the image of $\pi_{l,w}$ equipped with the finest topology making $\pi_{l,w}$ continuous (this can be different from the usual jet space topology, [27, EXAMPLE 4]). For each $w \in \mathbb{R}$, there is a natural integration map

$$H^*_c\big(S\mathrm{Jet}^{l!}_{\leq w}(A)\big) \to H_c^{*-n(l!-k!)}\big(S\mathrm{Jet}^{k!}_{\leq w}(A)\big) \tag{2.5}$$

for each $k \leq l$ sufficiently large. Define $H^*_c(S\mathrm{Jet}(A))$ to be

$$\varinjlim_{w \in \mathbb{R}} \varprojlim_{l} H_c^{*+n(l!+1)}\big(S\mathrm{Jet}^{l!}(A)\big).$$

Conjecture 2.8. *We have a natural isomorphism* $\mathrm{CH}_*(L_A, \xi_A) \cong H^*_c(S\mathrm{Jet}(A))$.

The parameter w should, very roughly, correspond to the natural action filtration on full contact homology. Note that for many examples these groups can be nontrivial in both positive and negative degrees. The following conjecture provides evidence for Conjecture 2.8.

Conjecture 2.9. *The same spectral sequence from Conjecture 2.6 computes* $H^*_c(S\mathrm{Jet}(A))$.

We hope that the same methods from [8] can be used to prove Conjecture 2.9. The filtration associated to this spectral sequence should come from the parameter w above. In order to prove Conjecture 2.8, one needs to write down an enhanced "PSS" map (see [43]) and show that it respects both spectral sequences (Conjectures 2.6 and 2.9) (or, more precisely, the action filtration and the filtration coming from w). A simpler version of this map is described later on in the next section.

3. COHOMOLOGICAL MCKAY CORRESPONDENCE

Quotient singularities $\mathbb{C}^n/G$, where $G \subset \mathrm{SU}(n)$ is a finite group, are natural examples of singularities to study from a Floer-theoretic perspective. One reason for this is that they are homogeneous, and this ensures that the link has a compatible contact 1-form with nice Reeb dynamics. In this section, we will show how Floer theory can shed light on the *cohomological McKay correspondence* [44].

Definition 3.1. A *crepant resolution* of $\mathbb{C}^n/G$ is a resolution $\pi : Y \to \mathbb{C}^n/G$ satisfying $c_1(Y) = 0$.

Let us consider the following open problem.

Conjecture 3.2 (Cohomological McKay correspondence over $\mathbb{K}$, [46, CONJECTURE 1.1]). *Let* $\mathbb{K}$ *be a field. There is a natural basis of* $H^*(Y;\mathbb{K})$ *consisting of irreducible representations of* G*. In particular, its dimension is the number of conjugacy classes* $|\mathrm{Conj}(G)|$ *of* G.

By blowing up an existing resolution, one can construct new resolutions of the same singularity whose cohomology has arbitrarily large rank. However, such resolutions are typically not crepant. In dimension 3 it was shown that crepant resolutions always exist (see [6, THEOREM 1.2]). However, in dimension 4 there are examples which do not admit any crepant resolution (see [11, EXAMPLE 2.28]). Batyrev [4] showed that when $\mathbb{K} = \mathbb{Q}$, the rank of $H^*(Y;\mathbb{K})$ is the number of conjugacy classes of G. However, he did not give a natural basis for this group.

Theorem 3.3 ([37, THEOREMS 1.4 AND 1.5]). *Suppose that* G *acts freely away from* $0 \in \mathbb{C}^n$ *and suppose* $\mathbb{K}$ *is a field whose characteristic does not divide* $|G|$*. Let* Y *be a quasiprojective crepant resolution. Then there is a Floer cohomology group* $\mathrm{SH}^*_+(Y;\Lambda_\mathbb{K})$ *defined over a field* $\Lambda_\mathbb{K}$ *of the same characteristic as* $\mathbb{K}$ *satisfying the following properties:*

(1) $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$ *is naturally isomorphic to* $H^*(Y;\Lambda_{\mathbb{K}})$ *(up to a shift in degree) and*

(2) $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$ *has rank equal to* $|\mathrm{Conj}(G)|$.

Corollary 3.4. *Let* $\mathbb{K}$ *be a field whose characteristic does not divide* $|G|$*. If* G *acts freely away from* 0 *then the rank of* $H^*(Y;\mathbb{K})$ *is equal to the number of conjugacy classes of* G*.*

The field $\Lambda_{\mathbb{K}}$ is called the *Novikov field* over $\mathbb{K}$ and is defined as the power series ring

$$\Lambda_{\mathbb{K}} = \left\{ \sum_{i\in\mathbb{N}} a_i t^{r_i} \,\middle|\, a_i \in \mathbb{K},\ r_i \in \mathbb{R},\ \forall\, i \in \mathbb{N},\ r_i \to \infty \text{ as } i \to \infty \right\}. \tag{3.1}$$

Let us now explain how the Floer group $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$, called *positive symplectic cohomology*, is constructed. In order to do this, we need to define *Hamiltonian Floer cohomology* first. Let $H = (H_t)_{t\in[0,1]}$ be a generic time-dependent Hamiltonian on a symplectic manifold (X,ω) and let us assume $c_1(X) = 0$. The chain complex for Hamiltonian Floer cohomology $\mathrm{HF}^*(H;\Lambda_{\mathbb{K}})$ is freely generated over $\Lambda_{\mathbb{K}}$ by 1-periodic orbits $\gamma : \mathbb{R}/\mathbb{Z} \to X$ of H. This is graded by a version of the Conley–Zehnder index. The differential is a matrix with respect to this basis of 1-periodic orbits whose (γ_-,γ_+) entry is a count of cylinders $u : \mathbb{R}\times\mathbb{R}/\mathbb{Z} \to X$ joining γ_- and γ_+ satisfying a Cauchy–Riemann-like PDE $\partial_s u + J_t(\partial_t u + X_{H_t}) = 0$ where $(J_t)_{t\in\mathbb{R}/\mathbb{Z}}$ is a family of almost complex structures on X (these are called *Floer trajectories*). The count is weighted by energy, which is a particular integral over this cylinder, and this is why we need the Novikov ring $\Lambda_{\mathbb{K}}$. If we did not do this, then the count might be infinite. Hamiltonian Floer cohomology was originally developed by Floer in [15]. The book [1] provides a very good introduction to Hamiltonian Floer cohomology.

Symplectic cohomology is a Hamiltonian Floer cohomology group that is usually defined for noncompact symplectic manifolds satisfying certain convexity properties at infinity. Very roughly, these are Hamiltonian Floer groups associated to Hamiltonians that tend to infinity very rapidly as one travels to infinity in the symplectic manifold. The fact that the symplectic manifold is noncompact can create problems such as infinite counts or the differential not squaring to zero since Floer trajectories can escape to infinity. However, in nice cases, one can define symplectic cohomology. There are many different versions of symplectic cohomology (e.g., [9, 10, 16, 21, 57–59]). Two good surveys of symplectic cohomology are contained in [40] and [49].

Now let us define $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$ for our crepant resolution Y. Since Y is crepant, we have that π is an isomorphism onto its image away from the exceptional locus $\pi^{-1}(0)$, and so we have a natural identification $Y - \pi^{-1}(0) = (\mathbb{C}^n - 0)/G$. Since Y is quasiprojective, we can put a natural symplectic structure ω_Y on Y which coincides with the standard linear symplectic structure on $\mathbb{C}^n/G$ away from a small neighborhood of $\pi^{-1}(0)$ (see [37, SECTION 2.5]). We now let H be a Hamiltonian on Y which is equal to $|z|^4$ near infinity (or some other rapidly increasing function of $|z|$). Then we define $\mathrm{SH}^*(Y;\Lambda_{\mathbb{K}}) := \mathrm{HF}^*(H;\Lambda_{\mathbb{K}})$. This group is a symplectomorphism invariant of (Y,ω_Y). There is a natural map $H^*(Y;\Lambda_{\mathbb{K}}) \to \mathrm{SH}^*(Y;\Lambda_{\mathbb{K}})$ and the cone of the corresponding chain map is called the *positive symplectic* *cohomology* $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$, which is the key Floer group from Theorem 3.3. Strictly speaking, the Hamiltonian H cannot be *exactly* $|z|^4$ near infinity since it needs to be generic, and so in reality it is a very small generic perturbation of such a function near infinity. A standard argument ensures that the definition does not depend on the specific choice of Hamiltonian H.

Let us explain very roughly how to prove parts (1) and (2) of Theorem 3.3. Let us start with part (1), which states that $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}}) \cong H^*(Y;\Lambda_{\mathbb{K}})$. By the definitions above, it is sufficient to show $\mathrm{SH}^*(Y;\Lambda_{\mathbb{K}}) = 0$. Consider the natural $U(1)$-action on $\mathbb{C}^n$ given by sending a vector $z \in \mathbb{C}^n$ to $e^{2\pi i\vartheta} z$ for each $\vartheta \in \mathbb{R}/\mathbb{Z}$. Such an action lifts to a $U(1)$-action on Y (see [37, LEMMA 3.4]). For appropriate ω_Y, one can show that this $U(1)$-action is the flow of a Hamiltonian $K : Y \to \mathbb{R}$. Now the key point is that we can deform H in a compact region of Y so that it is equal to a large multiple of K near the exceptional locus and is a rapidly increasing function of $|z|$ away from this locus, with high derivatives. This forces the Conley–Zehnder indices of all the orbits to be very large, since the linearized flow near each orbit "spins" extremely fast. Hence $\mathrm{SH}^*(Y;\Lambda_{\mathbb{K}})$ vanishes since the chain complex can be made to vanish at any given degree.

The proof of part (2) of Theorem 3.3 is a spectral sequence argument. One first deforms H in a compact region of Y so that it is C^2-small near the exceptional locus and is a generic perturbation of a function of $|z|$ elsewhere. The generators of $\mathrm{HF}^*(H;\Lambda_{\mathbb{K}})$ away from the exceptional locus correspond to Reeb orbits of an appropriate contact form on the link S^{2n-1}_ε/G of $\mathbb{C}^n/G$ where S^{2n-1}_ε is the sphere of radius $\varepsilon > 0$. Since our link S^{2n-1}_ε is simply connected, we have have a natural bijection $\pi_0(\mathcal{L}(S^{2n-1}/G)) \cong \mathrm{Conj}(G)$ where $\mathcal{L}(S^{2n-1})/G$ is the free loop space of our link. Hence the chain complex computing $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$ splits as a direct sum of groups indexed by conjugacy classes of G. However, the Floer differential might not respect this direct sum structure.

The contact form on our link S^{2n-1}_ε/G is the radial one $\alpha = \frac{1}{2}\sum_{i=1}^n r_i^2\, d\vartheta_i$ where (r_i,ϑ_i), $i = 1,\dots,n$ are polar coordinates on each factor of $\mathbb{C}^n$. The Reeb flow of this contact form is the same as the flow of the $U(1)$-action on S^{2n-1}_ε/G up to scaling. Hence one can compute the generators of the chain complex for $\mathrm{SH}^*_+(Y)$ using this $U(1)$-action.

The orbits come in families associated to the eigenspaces of each matrix element $g \in G \subset \mathrm{SU}(n)$ and the cohomology of these families give us the E_1 page of a spectral sequence computing $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$. Now, instead of computing $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$, one must first compute a variant $\mathrm{SH}^*_{S^1,+}(Y;\Lambda_{\mathbb{K}})$ since in this case the spectral sequence degenerates. One can then show that $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$ has rank $|\mathrm{Conj}(G)|$. This ends the sketch of the proof of Theorem 3.3.

The proof of part (1) of Theorem 3.3 naturally identifies $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$ with $H^*(Y;\Lambda_{\mathbb{K}})$. However, the proof of part (2) does not produce a similar natural identification.

Open Problem 3.5. Does the natural grading by conjugacy classes of G of the chain complex computing $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$ also get respected by the differential?

If the answer to this problem is yes, then we get a natural basis of $H^*(Y;\Lambda_{\mathbb{K}})$ by elements of $\mathrm{Conj}(G)$. Another issue is that we require that the characteristic of $\mathbb{K}$ does not divide $|G|$. We do not know how to show that the spectral sequence computing $\mathrm{SH}^*_{S^1,+}(Y;\Lambda_{\mathbb{K}})$ degenerates when the characteristic of $\mathbb{K}$ divides $|G|$.

Open Problem 3.6. What does the spectral sequence look like for $\mathrm{SH}^*_{S^1,+}(Y;\Lambda_{\mathbb{K}})$ when the characteristic $\mathbb{K}$ divides $|G|$?

Our work [37] was partly inspired by [27, SECTION 6], which tries to understand the cohomological McKay correspondence using arc spaces. Recall in Definition 2.7 that, for an isolated singularity A, we defined the *short arc space* $\mathrm{Arc}^o(A)$. Consider the subspace $\mathrm{ShArc}(A) \subset \mathrm{Arc}^o(A)$ of those arcs $u : \mathbb{D} \to \mathbb{C}^n/G$ satisfying $u^{-1}(0) = \{0\}$. Kollár and Némethi in [27, COROLLARY 32] show that the "irreducible components" of $\mathrm{ShArc}(\mathbb{C}^n/G)$ are in natural 1–1 correspondence with $\mathrm{Conj}(G)$. This correspondence is given by the boundary of each short arc $u : \mathbb{D} \to \mathbb{C}^n/G$, viewed as an element of

$$\pi_0\big(\mathcal{L}(\mathbb{C}^n - 0)/G\big) = \pi_0\big(\mathcal{L}\big(S^{2n-1}_\varepsilon/G\big)\big) = \mathrm{Conj}(G). \tag{3.2}$$

One way of connecting $\mathrm{ShArc}(\mathbb{C}^n/G)$ with $\mathrm{SH}^*(Y;\Lambda_{\mathbb{K}})$ might be through the *PSS map* (see [43]). The *PSS map* is a natural map from $\mathrm{SH}^*_+(Y;\Lambda_{\mathbb{K}})$ to $H_*(\mathrm{ShArc}(\mathbb{C}^n/G);\Lambda_{\mathbb{K}})$ given by sending an orbit γ to a "cycle" swept out by the moduli space of maps $u : \mathbb{C} \to Y$ so that $u(re^{2\pi i\vartheta})$ converges to $\gamma(\vartheta)$ as $r \to \infty$ and where the "cycle" is swept out by 0 (Figure 4).

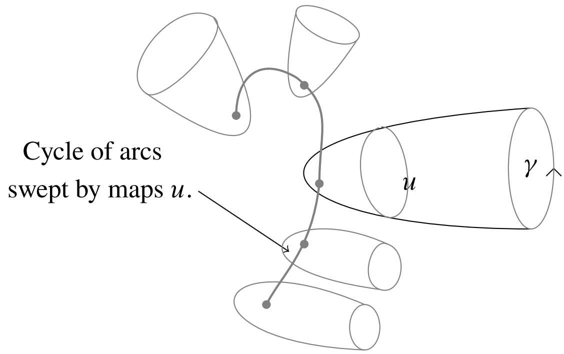

FIGURE 4
Arc space PSS map.

The Floer-theoretic methods used to prove Theorem 3.3 work very well if G acts freely away from 0.

Open Problem 3.7. Is there a way of using the Floer-theoretic methods above to deal with the case where G does not necessarily act freely away from 0?

The ideas of Section 4 below might be of use when we are dealing with this problem (see Open Problem 4.7 below).

4. QUANTUM COHOMOLOGY OF BIRATIONAL CALABI–YAU MANIFOLDS

Recall that two algebraic varieties are *birational* to each other if they have isomorphic dense Zariski-open subsets. The *minimal model program* in algebraic geometry, very roughly, is concerned with finding the "smallest" varieties in their birational equivalence class (minimal models). These minimal models are not necessarily unique. Calabi–Yau manifolds are examples of such minimal models. Therefore it is very natural to ask what properties birational Calabi–Yau manifolds have in common. For our purposes, we will say that a *Calabi–Yau manifold* is a smooth projective variety with trivial first Chern class.

Batyrev showed in [3] that any two birational Calabi–Yau manifolds have the same Betti numbers. In fact, by using ideas in [12,28] combined with [20], or by [61, COROLLARY 1.6], they have the same integral cohomology groups. However, the methods used do not produce an explicit isomorphism between these groups. Also the cup product structures might not agree (see [17, EXAMPLE 7.7]).

There is a deformed version of the cup product called the quantum cup product. Let us define this. We will fix a field $\mathbb{K}$ and a Calabi–Yau manifold X with a Kähler form ω admitting an integral lift. We define the Novikov ring

$$\Lambda_{\mathbb{K}}^{\omega} = \left\{ \sum_{i \in \mathbb{N}} a_i t^{\beta_i} \;\middle|\; a_i \in \mathbb{K},\ \beta_i \in H_2(X;\mathbb{Z}),\ \omega(\beta_i) \to \infty \text{ as } i \to \infty \right\}. \tag{4.1}$$

Let $A, B, C \in H^*(X;\mathbb{K})$ be cohomology classes whose degrees sum up to $2n$, where n is the complex dimension of X, and let $a, b, c \in C_*(X;\mathbb{K})$ be cycles representing the corresponding Poincaré duals of A, B, C. For each $\beta \in H_2(X;\mathbb{Z})$, we define the *Gromov–Witten invariant* $\mathrm{GW}_{0,3}^{X,\beta}(A, B, C) \in \mathbb{Z}$ to be the "count" of holomorphic maps $u : \mathbb{P}^1 \to X$ representing β so that $u(0)$ maps to a, $u(1)$ maps to b, and $u(\infty)$ maps to c. Technically, in order for this count to make sense, one needs to perturb the complex structure on X to a generic domain-dependent family of almost complex structures and count these curves with sign. Now let $A_1, \dots, A_k \in H^*(X;\mathbb{K})$ be a basis of homogeneous elements and let $\hat{A}_1, \dots, \hat{A}_k \in H^*(X;\mathbb{K})$ be the dual elements with respect to the pairing $(\eta, \nu) \to \int_X \eta \cup \nu$. We define *small quantum cohomology* to be the unique $\Lambda_{\mathbb{K}}^{\omega}$-algebra $\mathrm{QH}^*(X;\Lambda_{\mathbb{K}}^{\omega})$ which is isomorphic as a graded $\Lambda_{\mathbb{K}}^{\omega}$-module to $H^*(X;\Lambda_{\mathbb{K}}^{\omega})$ and whose product $\star_X$ satisfies

$$A_i \star_X A_j = \sum_{\beta \in H_2(X;\mathbb{Z})} \sum_{l=1}^{k} \mathrm{GW}_{0,3}^{X,\beta}(A_i, A_j, A_l) \hat{A}_l t^{\beta}. \tag{4.2}$$

One should think of this product as the cup product which has additional "correction" terms coming from counts of nonconstant holomorphic maps (Figure 5). For instance, if there were no nonconstant genus zero holomorphic maps (e.g., when X is an abelian variety) then this would be equal to the cup product.

Big quantum cohomology is also a deformation of the cup product which is more general than small quantum cohomology. Its definition involves counts of genus-zero curves passing through arbitrarily many cycles.

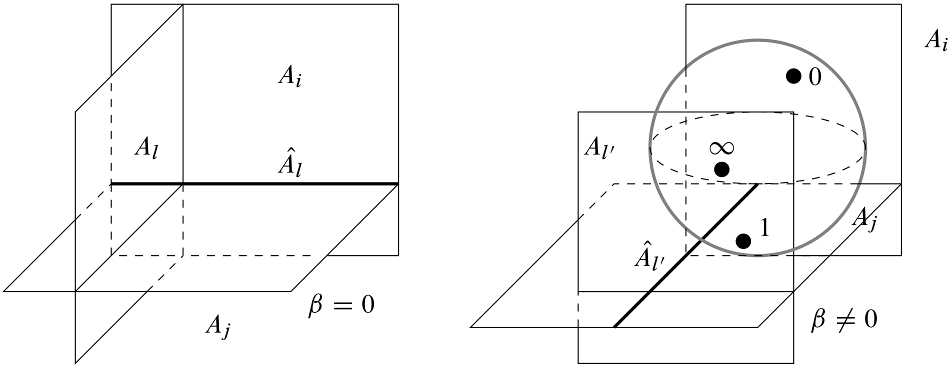

FIGURE 5
Terms in small quantum product. Cycles Poincaré dual to their respective cohomology classes are illustrated.

Conjecture 4.1 (Morrison [38] and Ruan [48]). *Any two birational Calabi–Yau manifolds have isomorphic (small or big) quantum cohomology rings up to analytic continuation.*

This conjecture was proven in dimension 3 in [32]. It was shown in [29–31] that if both Calabi–Yau manifolds are related by a sequence of birational transformations called *ordinary flops* then the conjecture above is true for big quantum cohomology (and hence also for small quantum cohomology). Wang in [60, SECTION 4.3, CONJECTURE IV] conjectured that all such Calabi–Yau manifolds, after deformation, are related by these operations, and so this would imply Conjecture 4.1. The method of proof in the papers [29–32] above is given by degenerating the Calabi–Yau manifold in a particular way and looking at Gromov–Witten invariants on this degeneration. We will describe a completely different approach to Conjecture 4.1 above using Floer theory, and in particular a modified version of symplectic cohomology.

Let X and $\check{X}$ be birational Calabi–Yau manifolds and let ω and $\check{\omega}$ be Kähler forms on X and $\check{X}$, respectively, admitting integral lifts. We get two Novikov rings $\Lambda_{\mathbb{K}}^{\omega}$ and $\Lambda_{\mathbb{K}}^{\check{\omega}}$ defined as in equation (4.1). By [26, LEMMA 4.2], there are natural identifications $H_2(X;\mathbb{Z}) \cong H_2(\hat{X};\mathbb{Z})$, due to the fact that the region in which the birational transform is not an isomorphism has complex codimension ≥ 2. Hence from now on, we will not distinguish between these groups, and so we can define the intersection of both Novikov rings $\Lambda_{\mathbb{K}}^{\omega,\check{\omega}} := \Lambda_{\mathbb{K}}^{\omega} \cap \Lambda_{\mathbb{K}}^{\check{\omega}}$. More explicitly,

$$\Lambda_{\mathbb{K}}^{\omega,\check{\omega}} = \left\{ \sum_{i \in \mathbb{N}} a_i t^{\beta_i} \,\middle|\, a_i \in \mathbb{K},\ \beta_i \in H_2(X;\mathbb{Z}),\ \min\big(\omega(\beta_i), \check{\omega}(\beta_i)\big) \to \infty \text{ as } i \to \infty \right\}. \quad (4.3)$$

The following theorem essentially proves Conjecture 4.1 for small quantum cohomology algebras.

Theorem 4.2 ([36, THEOREM 1.2]). *There exists a graded $\Lambda_{\mathbb{K}}^{\omega,\check{\omega}}$-algebra Z together with algebra isomorphisms*

$$Z \otimes_{\Lambda_{\mathbb{K}}^{\omega,\check{\omega}}} \Lambda_{\mathbb{K}}^{\omega} \cong \mathrm{QH}^*\big(X;\Lambda_{\mathbb{K}}^{\omega}\big), \quad Z \otimes_{\Lambda_{\mathbb{K}}^{\omega,\check{\omega}}} \Lambda_{\mathbb{K}}^{\check{\omega}} \cong \mathrm{QH}^*\big(\check{X};\Lambda_{\mathbb{K}}^{\check{\omega}}\big). \tag{4.4}$$

The downside of this theorem is that the algebra Z is unknown in general, as is the isomorphisms in (4.4).

4.1. Example

We will now illustrate Theorem 4.2 with an example (see [38, SECTION 7.3]). Suppose that X and $\check{X}$ are connected Calabi–Yau 3-folds and that there exists a disjoint union of connected genus 0 curves $C_1,\ldots,C_k$ in X and $\check{C}_1,\ldots,\check{C}_k$ in $\check{X}$ together with a class $\Gamma \in H_2(X;\mathbb{Z})$ so that

- $[C_j] = \Gamma \in H_2(X;\mathbb{Z})$ and $[\check{C}_j] = -\Gamma \in H_2(\check{X};\mathbb{Z})$ for each j and all connected genus-zero curves mapping to X or $\check{X}$, representing a multiple of Γ, have image equal to one of these curves,
- the normal bundle of C_j and $\check{C}_j$ is $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ for each j, and
- X and $\check{X}$ are related by an Atiyah flop along all of these curves.

Very roughly, an *Atiyah flop* along C_j removes C_j and glues it back in with the two $\mathcal{O}(-1)$ factors of its normal bundle swapped. One can think of an Atiyah flop as a kind of 0-surgery along the "knot" C_j. Since $H_2(X;\mathbb{Z})$ is naturally identified with $H_2(\check{X};\mathbb{Z})$, we have by Poincaré duality a natural identification $H^k(X;\mathbb{Z}) = H^k(\check{X};\mathbb{Z})$ where k is even. Hence from now on we will identify these cohomology groups. Let $\hat{A}_0,\ldots,\hat{A}_l \in H^4(X;\mathbb{Q})$ be a basis so that $\hat{A}_0$ is Poincaré dual to Γ and let $A_0,\ldots,A_l \in H^2(X;\mathbb{Q})$ be the dual basis with respect to the pairing $(\alpha,\beta) \to \int_X \alpha \cup \beta$. The algebra Z from Theorem 4.2 is equal to $H^*(X;\Lambda_{\mathbb{Q}}^{\omega,\check{\omega}})$ as a $\Lambda^{\omega,\check{\omega}}$-module, and its product $\star_Z$ is the unique $\Lambda^{\omega,\check{\omega}}$-bilinear map satisfying

$$A_i \star_Z A_j = A_i \cup_X A_j + k\delta_{0i}\delta_{0j}\hat{A}_0 t^{\Gamma} + \sum_{k=0}^{l}\sum_{\beta\notin\mathbb{Z}\Gamma} \mathrm{GW}_{0,3}^{X,\beta}(A_i,A_j,A_k)\hat{A}_k t^{\beta} \tag{4.5}$$

for each $i,j \in \{0,\ldots,l\}$. By replacing the class A_0 in (4.5) with $\frac{1}{1-t^{\Gamma}}A_0$ and $-\frac{t^{-\Gamma}}{1-t^{-\Gamma}}A_0$, and the class $\hat{A}_0$ with $(1-t^{\Gamma})\hat{A}_0$ and $-\frac{1-t^{-\Gamma}}{t^{-\Gamma}}\hat{A}_0$, we get the respective isomorphisms (4.4).

4.2. Symplectic cohomology of compact subsets

The main tool in the proof of Theorem 4.2 is a version of symplectic cohomology, which is very similar to definitions of symplectic cohomology in [21, 57, 58].

Definition 4.3. Let (M,ω) be a closed symplectic manifold and let $K \subset M$ be a compact subset. Then we define *symplectic cohomology of $K \subset M$* to be

$$\mathrm{SH}^*(K\subset M) := \varinjlim_{a} \varprojlim_{b} \varinjlim_{H|_K<0} \mathrm{HF}^*_{[a,b]}(H) \tag{4.6}$$

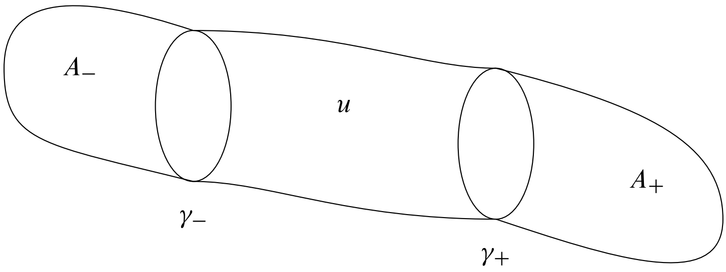

FIGURE 6
Capped Floer trajectory.

where $\mathrm{HF}^*_{[a,b]}(H)$ is a Hamiltonian Floer cohomology group which is defined in the same way as in Section 3, with a few differences:

(1) The chain complex is freely generated over $\mathbb{K}$ by pairs (γ, A), called *capped orbits of action in* $[a,b]$, where $\gamma : \mathbb{R}/\mathbb{Z} \to M$ is a 1-periodic orbit of H and $A \in H_2(M,\gamma;\mathbb{Z})$ is a homology cycle with boundary γ (called a *capping*);

(2) The *action* of (γ, A) is $-\int_A \omega - \int_{\mathbb{R}/\mathbb{Z}} H(\gamma(t))dt$;

(3) The differential only counts cylinders u connecting (γ_-, A_-) and (γ_+, A_+), so that when one caps off each end of u by A_- and A_+, respectively, one gets a null-homologous sphere (Figure 6).

Also the limits are taken with respect to the ordering $\leq$.

The group $\mathrm{SH}^*(K \subset M)$ naturally has a $\Lambda^\omega_{\mathbb{K}}$-module structure induced by the natural $H_2(X;\mathbb{Z})$-action on capped orbits given by adding these classes to the cappings A. It also has a natural "pair of pants" product (see **[43]**) making it a $\Lambda^\omega_{\mathbb{K}}$ algebra. The maps $\mathrm{HF}^*(H_1) \to \mathrm{HF}^*(H_2)$ with $H_1 \leq H_2$ in equation (4.6) above are defined by counting cylinders in a similar way to the differential. As demonstrated in **[57]**, one should really take the direct and inverse limits in equation (4.6) at the chain level in some appropriate homotopy-theoretic sense before taking homology, but we will not do this here for simplicity.

Symplectic cohomology seems to be quite useful when K is a *Liouville subdomain* of (M,ω). A *Liouville subdomain* is a codimension-0 submanifold $K \subset M$ satisfying $\omega|_K = d\theta$ for some 1-form θ with the property that the ω-dual X_θ of θ points outwards along ∂K. One should think of the last condition as a "convexity" condition. Symplectic cohomology satisfies the following properties:

(1) If $c_1(M) = 0$ and K is a Liouville subdomain satisfying a certain "index boundedness" property then $\mathrm{SH}^*(K \subset M)$ only depends on the isotopy class of K. In other words, if we have a smooth family of index-bounded Liouville domains

then the corresponding symplectic cohomology groups are naturally isomorphic.

(2) If $K - M$ is *stably displaceable* then $\mathrm{SH}^*(K \subset M) = \mathrm{SH}^*(M \subset M)$ (a set $P \subset M$ is *stably displaceable* if $\phi(P \times S^1) \cap P \times S^1 = \emptyset$ for some Hamiltonian symplectomorphism ϕ of $M \times T^*S^1$).

(3) $\mathrm{SH}^*(M \subset M) \cong \mathrm{QH}^*(M; \Lambda_{\mathbb{K}}^{\omega})$.

(4) If $c_1(M) = 0$ and K is a Liouville domain satisfying this "index boundedness" property then $\mathrm{SH}^*(K \subset M)$ can be computed using Hamiltonians that are constant outside a neighborhood of K and where these constant orbits do not contribute the chain complex.

4.3. Idea of proof

Here we will give an idea of the proof of Theorem 4.2. Let $\phi : X \dashrightarrow \check{X}$ be our birational isomorphism between Calabi–Yau manifolds X and $\check{X}$, and let ω and $\check{\omega}$ be Kähler forms on X and $\check{X}$, respectively, which admit integral lifts. Choose Zariski-dense affine subvarieties $A \subset X$ and $\check{A} \subset \check{X}$ so that ϕ maps A isomorphically to $\check{A}$. We can modify $\check{\omega}$ so that $\omega|_Q = \phi^*(\check{\omega})|_Q$ for an arbitrarily large compact subset Q of A.

Now one of the key observations is that codimension ≥ 1 subvarieties of Kähler manifolds are stably displaceable (see [36, SECTION 6.3]). Combining this with the fact that $c_1(X) = c_1(\check{X}) = 0$, one can choose $\check{\omega}$ very carefully so that there exists a smooth family D_t, $t \in [0,1]$ of Liouville subdomains in $(\check{A}, \check{\omega})$ satisfying an index boundedness property so that $\phi^{-1}(D_0) \subset Q$, $X - \phi^{-1}(D_0)$ is stably displaceable in (X, ω) and $\check{X} - D_1$ is stably displaceable in $(\check{X}, \check{\omega})$. Strictly speaking, such a family of Liouville subdomains $(D_t)_{t \in [0,1]}$ is not constructed in the paper [36], and something slightly more complicated is done instead (see [36, SECTION 7]). However, we will assume $(D_t)_{t \in [0,1]}$ exists for simplicity.

We will now explain how to construct the $\Lambda_{\mathbb{K}}^{\omega,\check{\omega}}$-algebra Z in the statement of Theorem 4.2. By property (4), we can find chain complexes computing $\mathrm{SH}^*(X - \phi^{-1}(D_0) \subset X)$ and $\mathrm{SH}^*(\check{X} - D_0)$ involving Hamiltonians which are constant outside a small neighborhood of $\phi^{-1}(D_0)$ (resp. D_0) and such that the constant orbits do not contribute to the chain complex. Now, the regions $V_X \subset X$, $V_{\check{X}} \subset X$ where ϕ and its inverse are ill-defined are of complex codimension at least 2 and hence, by a genericity argument, one can ensure that the Floer trajectories map to the domain or image of ϕ only (Figure 7). This means that we can define these Hamiltonian Floer groups over $\Lambda_{\mathbb{K}}^{\omega,\check{\omega}}$ giving us a new "symplectic cohomology" group Z associated to $\phi^{-1}(D_0) \subset X$. As a result, we can show

$$Z \otimes_{\Lambda_{\mathbb{K}}^{\omega,\check{\omega}}} \Lambda_{\mathbb{K}}^{\omega} \cong \mathrm{SH}^*\big(X - \phi^{-1}(D_0) \subset X\big), \quad Z \otimes_{\Lambda_{\mathbb{K}}^{\omega,\check{\omega}}} \Lambda_{\mathbb{K}}^{\check{\omega}} \cong \mathrm{SH}^*(\check{X} - D_0 \subset \check{X}). \tag{4.7}$$

The following two equations also hold:

$$\mathrm{SH}^*\big(X - \phi^{-1}(D_0)\big) \overset{(2)}{=} \mathrm{SH}^*(X \subset X) \overset{(3)}{=} \mathrm{QH}^*\big(X; \Lambda_{\mathbb{K}}^{\check{\omega}}\big), \tag{4.8}$$

$$\mathrm{SH}^*(\check{X} - D_0 \subset \check{X}) \overset{(1)}{=} \mathrm{SH}^*(\check{X} - D_1 \subset X) \overset{(2)}{=} \mathrm{SH}^*(\check{X} \subset \check{X}) \overset{(3)}{=} \mathrm{QH}^*\big(\check{X}; \Lambda_{\mathbb{K}}^{\omega}\big). \tag{4.9}$$

Our result now follows from equations (4.7)–(4.9).

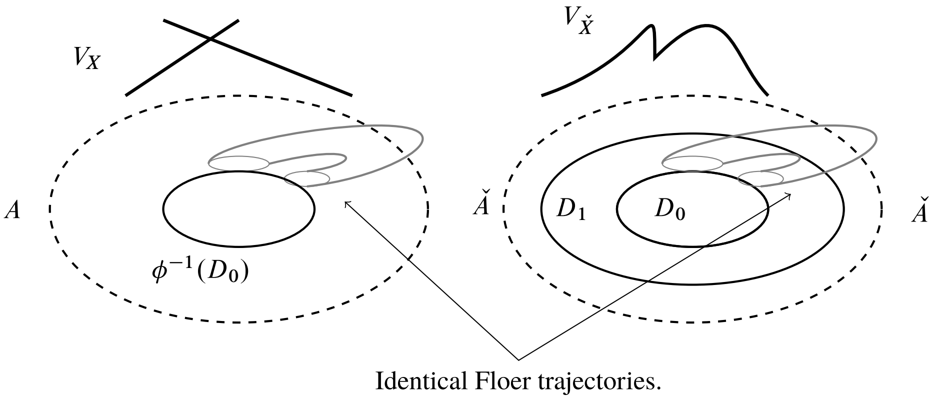

FIGURE 7
Floer trajectories avoiding V_X and $V_{\hat{X}}$.

4.4. Further directions

One of the problems with Theorem 4.2 is that the isomorphisms are not explicit. Let $\Delta \subset X \times \check{X}$ be the closure of the graph of the birational isomorphism ϕ. Then we have a push–pull map

$$\Psi_\Delta : H^*(X;\mathbb{K}) \to H^*(\check{X};\mathbb{K}), \quad \Psi_\Delta(\alpha) := \mathrm{PD}\big((\mathrm{pr}_{\check{X}})_*\big(\Delta \cap \mathrm{pr}_X^* \alpha\big)\big) \tag{4.10}$$

where pr_X and $\mathrm{pr}_{\check{X}}$ are the natural projection maps from $X \times \check{X}$ to X and $\check{X}$, respectively, and PD is Poincaré duality.

Conjecture 4.4 ([60, SECTION 4.3, CONJECTURE I]). *If $\mathbb{K} = \mathbb{Q}$, then we can identify the quantum cohomology groups of X and $\check{X}$ using the equivalence Ψ_Δ.*

Since the regions V_X and $V_{\check{X}}$ where ϕ and its inverse are ill-defined have complex codimension 2, it should be possible to show that the above conjecture is true if we restrict ourselves to the subalgebra of $H^*(X;\Lambda_X^\omega)$ and $H^*(\check{X};\Lambda_{\check{X}}^\omega)$ generated by elements of degree 0, 1, 2 and $2n-2$, $2n-1$, and $2n$. Motivated by the fact that symplectic cohomology could, in principle, be computed by relative Gromov–Witten invariants ([49, REMARK 8.3], [13, 19]), it would be interesting to investigate (over any field $\mathbb{K}$) whether the equivalences (4.4) can be realized in some way by counts of curves in $X \times \check{X}$. This leads us to the following very difficult open problem:

Open Problem 4.5. Can one produce a purely algebraic proof of Theorem 4.2 using relative Gromov–Witten invariants motivated by themes in [13] or [19]?

Trying to understand what is going on in dimension 3 could be of use here. There should be a version of symplectic cohomology of $M \subset M$ which is defined using *bulk deformed* Hamiltonian Floer cohomology (see [18, 54]). This is naturally isomorphic to big quantum cohomology. However, the methods of Section 4.3 do not work using this bulk deformed version of Hamiltonian Floer cohomology due to the fact that the definition involves both cycles and orbits. There should be a version of Hamiltonian Floer cohomology which only uses orbits and Riemann surfaces satisfying the perturbed Floer equation joining them so that the associated symplectic cohomology group of $M \subset M$ is isomorphic to big quantum cohomology (see [41]). However, such a construction requires an additional choice of a "trivialization of a circle action."

Open Problem 4.6. Can one use the techniques above to prove that birational Calabi–Yau manifolds have the "same" big quantum cohomology groups (maybe, up to some additional choices).

The article [53] gave a potential definition for Hamiltonian Floer cohomology in the setting of orbifolds. This leads to the following open problem:

Open Problem 4.7. Suppose that X and $\check{X}$ are birational Calabi–Yau orbifolds. Can one use the techniques in the previous section to relate the quantum cohomology of X and $\check{X}$.

An example of such a birational transform is a crepant resolution as in Section 3. This problem has an additional serious difficulty which is that the birational transform might be ill-defined on a codimension 1 region. This means that the analogue of equation (4.7) does not hold. However, there might be additional genus Gromov–Witten invariants counting curves mapping to the locus where ϕ and ϕ^{-1} are ill-defined which might correct for this.

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MARK MCLEAN

Stony Brook Mathematics Department, Stony Brook University, 100 Nicolls Rd, Stony Brook, NY 11794, USA, markmclean@math.stonybrook.edu