

SURFACES VIA SPINORS AND SOLITON EQUATIONS

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ABSTRACT

This article surveys the Weierstrass representation of surfaces in the three- and four-dimensional spaces, with an emphasis on its relation to the Willmore functional. We also describe an application of this representation to constructing a new type of solutions to the Davey–Stewartson II equation. They have regular initial data, gain one-point singularities at certain moments of time, and extend to smooth solutions for the remaining times.

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1. THE WEIERSTRASS (SPINOR) REPRESENTATION OF SURFACES IN THE THREE-SPACE

The Weierstrass representation for minimal surfaces in the three-space is as follows: for any pair of holomorphic functions ψ_1 and $\bar{\psi}_2$ defined in a domain $U \subset \mathbb{C}$ in the complex plane, the formulae

$$\begin{aligned} x^1(P) &= \frac{i}{2} \int [(\psi_1^2 + \bar{\psi}_2^2)dz + (\bar{\psi}_1^2 + \psi_2^2)d\bar{z}] + x^1(P_0), \\ x^2(P) &= \frac{1}{2} \int [(-\psi_1^2 + \bar{\psi}_2^2)dz + (-\bar{\psi}_1^2 + \psi_2^2)d\bar{z}] + x^2(P_0), \\ x^3(P) &= \int [\psi_1\bar{\psi}_2dz + \bar{\psi}_1\psi_2d\bar{z}] + x^3(P_0) \end{aligned} \tag{1.1}$$

determine a minimal surface in \mathbb{R}^3 . Here we assume that U is simply-connected or the integrals over cycles in U vanish, and the integrals are taken along a path from a fixed point $P_0 \in U$ to P . Moreover, every minimal surface admits such a representation. Weierstrass used another data, namely $f = \bar{\psi}_2^2$ and $g = \frac{\psi_1}{\psi_2}$. However, for the generalization of this representation, it is worth to consider ψ_1 and ψ_2 and treat this pair as a solution of the Dirac equation

$$D\psi = 0, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.2}$$

for a two-dimensional Dirac operator of the form

$$D = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \quad U = \bar{U},$$

where a real-valued potential U vanishes for minimal surfaces. Now the Weierstrass representation generalizes as follows:

Theorem 1.1 ([16]). *For every solution ψ of (1.2), the formulae (1.1) define a surface in \mathbb{R}^3 for which z is a conformal parameter, the induced metric takes the form*

$$ds^2 = e^{2\alpha} dz d\bar{z}, \quad e^\alpha = |\psi_1|^2 + |\psi_2|^2,$$

and the potential U of the Dirac operator equals

$$U = \frac{He^\alpha}{2},$$

where H is the mean curvature.

Theorem 1.2 ([26]). *Every surface in \mathbb{R}^3 (with a fixed conformal parameter z on it) admits such a representation even globally. Therewith ψ is a section of a spinor bundle over the surface, the form $U^2 dx \wedge dy$ is globally defined and its integral over the surface is proportional to the Willmore functional*

$$\mathcal{W} = \int H^2 d\mu = 4 \int U^2 dx \wedge dy,$$

where $d\mu$ is the induced area form of the surface.

Hence, being considered for the Dirac operators with general real-valued potentials, the formulae (1.1) define the Weierstrass (spinor) representation of general surfaces in \mathbb{R}^3 .

Theorem 1.1 was derived from the similar formulae in the book by Eisenhart [9, PROBLEM 35.4] where instead of (1.2) the following condition is used:

$$L\psi_1 = L\bar{\psi}_2 = 0, \quad L = \partial\bar{\partial} - \frac{\partial \log U}{U}\bar{\partial} + U^2.$$

Here D naturally arises as the “square root” of the Schrödinger operator L . The representation based on the Dirac operator provides many more opportunities because its potential has no singularities and the operator has good spectral properties. In the advanced problems of his textbook, Eisenhart frequently proposed to prove results from various articles, and we cannot exclude that these formulae might be traced to some earlier publication. It appears that this local representation is equivalent to another one derived in [14], where the Dirac operator was not used either.

In [16] the Weierstrass representation was used for introducing the deformations of surfaces admitting such a representation. The operator D generates a hierarchy of solution equations of the form

$$\frac{\partial D}{\partial t_n} = [D, A_n] - B_n D,$$

where A_n and B_n are matrix differential operators such that the principal term of A_n takes the form

$$A_n = \begin{pmatrix} \partial^{2n+1} + \bar{\partial}^{2n+1} & 0 \\ 0 & \partial^{2n+1} + \bar{\partial}^{2n+1} \end{pmatrix} + \dots$$

This evolution preserves the zero energy level of D deforming the corresponding eigenfunctions

$$\frac{\partial \psi}{\partial t} + A\psi = 0 \tag{1.3}$$

and $D\psi_0 = 0$ for the initial data $\psi_0 = \psi|_{t=t_0}$, then $D\psi = 0$ for all $t \geq t_0$.

For $n = 1$, we have the modified Novikov–Veselov (mNV) equation [5]

$$U_t = \left(U_{zzz} + 3U_z V + \frac{3}{2} U V_z \right) + \left(U_{\bar{z}\bar{z}\bar{z}} + 3U_{\bar{z}} \bar{V} + \frac{3}{2} U \bar{V}_{\bar{z}} \right)$$

where

$$V_{\bar{z}} = (U^2)_{\bar{z}}.$$

In the case when $U|_{t=0}$ depends only on x , we have $U = U(x, t)$ and the mNV equation reduces to the modified Korteweg–de Vries equation $U_t = \frac{1}{4} U_{xxx} + 6U_x U^2$ (here $V = U^2$). In the same manner, the original Novikov–Veselov equation

$$U_t = U_{zzz} + U_{\bar{z}\bar{z}\bar{z}} + (VU)_z + (\bar{V}U)_{\bar{z}}, \quad V_{\bar{z}} = 3U_z$$

generalizes the Korteweg–de Vries equation.

The mNV deformation introduced in [16] is as follows: let a surface be induced by ψ via (1.1) and consider solutions U and ψ of the mNV equation and (1.3) with given initial data. Then for any moment of time, we have a spinor ψ that determines the

deformed surface. In fact, we have infinitely many deformations defined up to translations by $(x^1(P_0, t), x^2(P_0, t), x^3(P_0, t))$. This is some family of the mNV deformations of the surface.

Theorem 1.3 ([26]). *The mNV deformations evolve tori into tori and preserve their conformal classes and the values of the Willmore functional.*

Theorems 1.2 and 1.3 hint at the relation of this representation to the Willmore functional. Formulae (1.1) give immersions of the universal covers of surfaces and there are no compact minimal surfaces without boundary in \mathbb{R}^3 . Hence the infima for the Willmore functional for various conformal classes of closed surfaces show how much stress must be applied for converting an immersion of the universal cover into an immersion of a closed surface. In Section 2 we briefly expose how the Weierstrass representation was applied to studying the conformal geometry of surfaces.

In Section 4, in contrast to Section 2 where analysis was applied to geometry, we discuss the recent applications of geometry to analysis. We show how to construct exact solutions to the Davey–Stewartson II equation. Therewith, geometry of surfaces helps in finding a new scenario for creating singularities of solutions with regular initial data.

It would be interesting to apply the Weierstrass representation to other problems of the surface theory (bending, existence of umbilics, etc.). In particular, if some conjecture appears false, then methods of integrable systems can help in constructing an explicit counterexample (see, for instance, [1]).

2. SPECTRAL CHARACTERISTICS OF D AND CONFORMAL GEOMETRY OF SURFACES

The Willmore conjecture which states that the minimum of the Willmore functional among tori in \mathbb{R}^3 is attained at the Clifford torus was proved in [19] by means of the geometric measure theory and calculus of variations.

In the mid-1990s we proposed an approach to proving it using Theorem 1.3 and the integrable systems theory. This approach was not implemented, but we think it is worth to be briefly exposed here.

It was conjectured in [26] that

a nonstationary torus (with respect to the mNV flow and up to translations) cannot be a local minimum of the Willmore functional.

Otherwise, by Theorem 1.3, the minimum of the Willmore functional would contain an infinite family of tori invariant under the mNV flow and this would be very unlikely. By the general philosophy of integrable systems, the stationary solution to the mNV equation has the simplest possible spectral curve [27].

Since the flow preserves the conformal classes of tori, the same conjecture has to be valid for tori of every fixed conformal class.

For two-dimensional differential operators with periodic coefficients, the spectral curve (on the zero level energy) parameterizes its Floquet eigenfunctions [8]. In our case a Floquet eigenfunction ψ of the operator D with the eigenvalue (or the energy) E is a formal solution to the equation

$$D\psi = E\psi$$

which satisfies the periodicity conditions

$$\psi(q + \gamma_j) = e^{2\pi i(k, \gamma_j)} \psi(z, \bar{z}), \quad j = 1, 2,$$

where γ_1 and γ_2 generate the lattice of periods Λ of the potential U and $(k, \gamma) = k_1\gamma_1 + k_2\gamma_2$ is the inner product. The quantities $k_1, k_2 \in \mathbb{C}$ are called the quasimomenta of ψ and $\mu(\gamma_j) = e^{2\pi i(k, \gamma_j)}$ are Floquet multipliers. All possible triples (k_1, k_2, E) for which Floquet functions exist form an analytic subset $Q(U)$ in \mathbb{C}^3 , invariant under the dual lattice $\Lambda^* \subset \mathbb{R}^2 \subset \mathbb{C}^2$ acting on the quasimomenta. We proved that for the two-dimensional operators $\Delta + U$ and $\partial_y - \partial_x^2 + U$ in 1985. However, this paper was unpublished, although referred in [18] and was exposed in [30]. Now we define the spectral curve as the complex curve

$$\Gamma = (Q \cap \{E = 0\}) / \Lambda^*$$

and consider it up to biholomorphic equivalence, making the definition independent on the choice of a basis for Λ . The curve is an invariant of the mNV flow, it is naturally completed by a couple of points at infinity, which compactify it in the case of finite genus. The Floquet functions are glued into a meromorphic section over Γ . The above rough definition must be detailed for singular spectral curves. In general, the space of Floquet functions corresponding to a point from Γ is one-dimensional and the multiple points have to be normalized in such a manner that for the resulting curve Γ_ψ to every point there corresponds a one-dimensional space, there is a meromorphic section ψ of this bundle, and every Floquet function is a linear combination of sections at different points (see the definition of Γ_ψ in [30]). The spinor ψ generating a torus via (1.1) has the Floquet multipliers equal to ± 1 .

The spectral curve defined for D is a particular case of the general spectral curves which play a fundamental role in integrable systems. They are the first integrals of the system (that was first showed for the Korteweg–de Vries equation in [22]. The particular case of them are the spectral curves of constant mean curvature tori which are always of finite genus [13, 25]. In general, this spectral curve is of infinite genus. For finite genera cases, solutions to the integrable systems are expressed in terms of theta functions on spectral curves. In our case all Floquet functions are reconstructed from certain data related to Γ_ψ and the value of the Willmore functional is also determined by them [27]. We conjectured that

for tori in \mathbb{R}^3 , the curve Γ , i.e., the set of the multipliers $\mu(\gamma_j)$, is conformally invariant (as is the Willmore functional).

Since this is evident for translations and rotations, one was left to prove the same for the Möbius inversion, which was accomplished in [12].

For the Clifford torus parameterized by x, y such that $0 \leq x, y \leq 2\pi$, the potential U of its Weierstrass representation is

$$U(x) = \frac{\sin x}{2\sqrt{2}(\sin x - \sqrt{2})}$$

and its spectral curve Γ_ψ is $\mathbb{C}P^1$ with two pairs of glued points.

For differential operators on surfaces of higher genera, the analog of Floquet–Bloch theory is unknown. It would be interesting to find it, if it exists, for the Dirac operator D .

For spheres, there are no analogs of the Floquet functions and the zero energy level of D just consists of the kernel $\text{Ker } D$.

We notice that there is an antiinvolution

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \xrightarrow{\sigma} \begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix}, \quad \sigma^2 = -1, \quad (2.1)$$

acting on $\text{Ker } D$. This implies that the dimension of the kernel over \mathbb{C} is always even.

We say that a sphere in \mathbb{R}^3 admits a spinor representation with a one-dimensional potential if after removing a certain pair of points we obtain the cylinder $\mathbb{R} \times S^1$ for which the potential of the representation depends on x only, i.e., $U = U(x)$. These are, for instance, spheres of revolution. By using the inverse scattering transform of one-dimensional Dirac operators on the line, we proved

Theorem 2.1 ([28]). *For spheres with a one-dimensional potential, we have*

$$\mathcal{W} = 4 \int U^2 dx \wedge dy \geq 4\pi N^2, \quad (2.2)$$

where $\dim_{\mathbb{C}} \text{Ker } D = 2N$, and the equalities are achieved at the soliton potentials

$$U_N(x) = \frac{N}{2 \cosh x}.$$

We call the spheres that correspond to these potentials soliton spheres, and it appears that they have very interesting geometrical properties [6]. In [28] we conjectured that *inequality (2.2) holds for all spheres.*

Soon after the preprint of [28] appeared, Friedrich showed that this conjecture implies the following statement:

Given an eigenvalue λ of the Dirac operator D on a two-dimensional spin-manifold homeomorphic to the two-sphere,

$$\lambda^2 \text{Area}(M) \geq \pi m^2(\lambda), \quad (2.3)$$

where $m(\lambda)$ is the multiplicity of λ .

For $m(\lambda) = 2$, inequality (2.3) was already proved by Bär [2].

The arguments by Friedrich were as follows. On a spin-manifold of dimension 2 with the metric $e^{2\alpha} dz d\bar{z}$, the Dirac operator (on the spin-manifold) takes the form

$$D = 2e^{-3\alpha/2} \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} e^{\alpha/2},$$

and the equation

$$D\varphi = \lambda\varphi$$

is rewritten as

$$\left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} - \frac{\lambda e^\alpha}{2} \right] \psi = 0,$$

where $\psi = e^{\alpha/2}\varphi$, and if λ is constant, then (2.2) implies (2.3). Moreover, if $\lambda = H$, then this is exactly the Dirac equation (1.2) (the sign of the mean curvature can be changed without any loss) and, since $e^\alpha = |\psi|^2$, we have $|\varphi| = 1$. Therefore the Weierstrass representation is rewritten in terms of solutions of the Dirac equation

$$D\varphi = H\varphi$$

of constant length, $|\varphi| = 1$ [11, THEOREM 13].

This embedding of the Weierstrass representation into the general framework of Dirac operators on spin-manifolds appears very fruitful: it led to its generalization, the spinorial representation of immersions of manifolds, which are not necessarily two-dimensional, into certain homogeneous spaces (see [3] and references therein).

The Weierstrass representation for surfaces in \mathbb{R}^3 was generalized for surfaces in three-dimensional Lie groups with left-invariant metrics in [4]. It helped establish some facts on constant mean curvature surfaces in these groups.

It would be interesting, at least as a test problem, to find a discretization of the Weierstrass representation by means of discrete complex analysis. In [36] that was done for the generalizations of the representation for time-like surfaces in $\mathbb{R}^{2,1}$, $\mathbb{R}^{3,1}$, and $\mathbb{R}^{2,2}$. But in these cases complex analysis is not involved because the principal term of the Dirac operator D has the form $\begin{pmatrix} 0 & \partial_\xi \\ \partial_\eta & 0 \end{pmatrix}$ where ξ and η are isotropic coordinates.

The conjectured inequality (2.2) was finally proved with its generalizations for surfaces of higher genera:

Theorem 2.2 ([10]). *For a closed oriented surface of genus g immersed into \mathbb{R}^3 via (1.1) and (1.2), we have*

$$\int U^2 dx \wedge dy \geq \begin{cases} \pi N^2, & \text{for } g = 0, \\ \begin{cases} \frac{\pi N^2}{4} & \text{for } N \text{ even,} \\ \frac{\pi(N^2-1)}{4} & \text{for } N \text{ odd,} \end{cases} & \text{for } g = 1, \\ \frac{\pi}{4g}(N^2 - g^2), & \text{for } g > 1, \end{cases}$$

where $\dim_{\mathbb{C}} \text{Ker } D = 2N$.

3. SURFACES IN THE FOUR-SPACE AND THE DAVEY–STEWARTSON EQUATION

Theorem 2.2 was derived from the Plücker formula in the quaternionic algebraic geometry [10].

The Weierstrass representation allows applying to surface theory other branches of mathematics. In Section 2 we discuss an approach based on the spectral theory of the Dirac operator. The quaternionic algebraic geometry applies algebro-geometrical methods by considering solutions of the Dirac equation as “holomorphic” sections of spinor bundles. It starts with treating the symmetry (2.1) as a multiplication by an imaginary unit j and considering $\text{Ker } D$ as a linear space over quaternions \mathbb{H} [24]. Therewith one may consider the Dirac operator of the more general form

$$D = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix} \quad (3.1)$$

whose kernel is also invariant under (2.1).

For that we identify \mathbb{C}^2 with \mathbb{H} as follows:

$$(z_1, z_2) \rightarrow z_1 + jz_2 = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

and consider the two matrix operators

$$\bar{\partial} = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \quad jU = j \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix} = \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix},$$

where $j \in \mathbb{H}$ is the imaginary unit for which we have $j^2 = -1$, $zj = j\bar{z}$, and $\bar{\partial}j = j\partial$. Then the Dirac equation $D\psi = 0$ takes the form

$$(\bar{\partial} + jU)(\psi_1 + j\psi_2) = (\bar{\partial}\psi_1 - \bar{U}\psi_2) + j(\partial\psi_2 + U\psi_1) = 0.$$

Since ψ_1 and $\bar{\psi}_2$ are sections of the same bundle E , we rewrite the Dirac equation as

$$(\bar{\partial} + jU)(\psi_1 + \bar{\psi}_2j) = 0$$

and treat $E \oplus E$ as a quaternionic line bundle whose sections are of the form $\psi_1 + \bar{\psi}_2j$. The symmetry (2.1) induces some quaternion linear endomorphism J of E such that $J^2 = -1$, $\psi_1 + \bar{\psi}_2j \rightarrow (\psi_1 + \bar{\psi}_2j)j = -\bar{\psi}_2 + \psi_1j$, and this J defines for any quaternion fiber a canonical splitting into $\mathbb{C} \oplus \mathbb{C}$ (in our case this is a splitting into ψ_1 and $\bar{\psi}_2$) and such a bundle is called a “complex quaternionic line bundle.” The kernel of $D = \bar{\partial} + jU$ is invariant under the right-side multiplications by constant quaternions and hence is a linear space over \mathbb{H} .

The “quaternionic” analog of the classical the Plücker formula established in [10] implies (2.2) and (2.3).

By using the analogy with complex algebraic geometry, other interesting results were obtained, in particular on Bäcklund transformations and special classes of surfaces. Moreover, this approach offers another opportunity: in its framework the Weierstrass representation was also extended to surfaces in \mathbb{R}^4 and therewith \mathbb{R}^4 was naturally identified with \mathbb{H} . In the coordinate language, the representation was written down in [17] and is as follows.

Let D be of the form (3.1) and introduce the formally conjugate operator

$$D^\vee = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \bar{U} & 0 \\ 0 & U \end{pmatrix}.$$

Theorem 3.1 ([17]). *If ψ and φ satisfy the equations*

$$D\psi = 0, \quad D^\vee\varphi = 0. \quad (3.2)$$

then the formulae

$$\begin{aligned} x^k(P) &= x^k(P_0) + \int (x_z^k dz + \bar{x}_z^k d\bar{z}), \quad k = 1, 2, 3, 4, \\ x_z^1 &= \frac{i}{2}(\bar{\varphi}_2\bar{\psi}_2 + \varphi_1\psi_1), \quad x_z^2 = \frac{1}{2}(\bar{\varphi}_2\bar{\psi}_2 - \varphi_1\psi_1), \\ x_z^3 &= \frac{1}{2}(\bar{\varphi}_2\psi_1 + \varphi_1\bar{\psi}_2), \quad x_z^4 = \frac{i}{2}(\bar{\varphi}_2\psi_1 - \varphi_1\bar{\psi}_2), \end{aligned} \quad (3.3)$$

define the surface in \mathbb{R}^4 for which the induced metric is given by $e^{2\alpha} dz d\bar{z} = (|\psi_1|^2 + |\psi_2|^2)(|\varphi_1|^2 + |\varphi_2|^2) dz d\bar{z}$ and $|U| = \frac{|\mathbf{H}|e^\alpha}{2}$ with \mathbf{H} being the mean curvature vector.

For $U = \bar{U}$ and $\psi = \varphi$, this representation reduces to (1.1).

The converse is also true but there is a difference with surfaces in \mathbb{R}^3 for which a choice of a parameter z defines ψ uniquely up to multiplication by ± 1 .

Theorem 3.2 ([29]). *Every oriented surface (with a given conformal parameter) has representation (3.3). The spinors ψ and φ are defined up to the gauge transformations*

$$\psi_1 \rightarrow e^h \psi_1, \quad \psi_2 \rightarrow e^{\bar{h}} \psi_2, \quad \varphi_1 \rightarrow e^{-h} \varphi_1, \quad \varphi_2 \rightarrow e^{-\bar{h}} \varphi_2, \quad U \rightarrow e^{\bar{h}-h} U,$$

where h is holomorphic. For every torus, the potential U may be taken doubly periodic.

Let us explain the appearance of these gauge transformations and, at the same time, why the dimensions 3 and 4 are distinguished by the existence of such spinor representations.

The Grassmannian $\tilde{G}_{n,2}$ of oriented two-planes in \mathbb{R}^n is diffeomorphic to the quadric Q :

$$z_1^2 + \dots + z_n^2 = 0, \quad (z_1 : \dots : z_n) \in Q_n \subset \mathbb{C}P^{n-1}.$$

To every oriented plane with an positively oriented orthonormal basis $e_1 = (x_1, \dots, x_n)$, $e_2 = (y_1, \dots, y_n)$ there corresponds the point $(z_1 : \dots : z_n)$, $z_k = x_k + iy_k$, $k = 1, \dots, n$, of this quadric. Given a surface $(X^1(z, \bar{z}), \dots, X^n(z, \bar{z}))$ in \mathbb{R}^n with a conformal parameter z , we define the Gauss map as

$$z \rightarrow \left(\frac{\partial X^1}{\partial z} : \dots : \frac{\partial X^n}{\partial z} \right) \in Q_n.$$

It is straightforward to derive that the image of the Gauss map lies in the quadric from the conformality of z . For $n = 3$, the quadric Q_3 is diffeomorphic to \mathbb{C}^1 and its rational parameterization is

$$z_1 = \frac{i}{2}(a^2 - b^2), \quad z_2 = \frac{1}{2}(b^2 - a^2), \quad z_3 = ab, \quad (a : b) \in \mathbb{C}P^1,$$

and the spinor ψ is reconstructed from the Gauss map as $\psi_1 = a, \bar{\psi}_2 = b$. For $n = 4$, we have the diffeomorphic Segre mapping

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \rightarrow Q_4$$

of the form $z_1 = \frac{i}{2}(a_1b_1 + a_2b_2), z_2 = \frac{1}{2}(a_2b_2 - a_1b_1), z_3 = \frac{1}{2}(a_1b_2 - a_2b_1), z_4 = \frac{i}{2}(a_2b_1 - a_1b_2), (a_1 : a_2) \in \mathbb{C}P^1, (b_1 : b_2) \in \mathbb{C}P^1$, the spinors take the form $\varphi = (a_1, \bar{a}_2), \psi = (b_1, \bar{b}_2)$ and are reconstructed up to the gauge transformations. Since they have to satisfy (3.2), h has to be holomorphic. For $n > 4$, the quadrics Q_n have no such rational parameterizations.

The operators D and D^\vee enter the representation of the Davey–Stewartson (DS) equations via compatibility of linear systems. That led to introducing the DS deformations of surfaces, the four-dimensional analog of the mNV deformations [17].

We consider one of such deformations for which we proved that it transforms tori into tori and preserves the Willmore functional $4 \int |U|^2 dx \wedge dy$ [29]. It has the form

$$U_t = i(U_{zz} + U_{\bar{z}\bar{z}} + (V + \bar{V})U), \quad V_{\bar{z}} = 2(|U|^2)_z \tag{3.4}$$

and is the compatibility condition for the linear problems

$$D\psi = 0, \quad \partial_t \psi = A\psi$$

where

$$A = i \begin{pmatrix} -\partial^2 - V & \bar{U}\bar{\partial} - \bar{U}_{\bar{z}} \\ U\partial - U_z & \bar{\partial}^2 + \bar{V} \end{pmatrix}.$$

It is also the compatibility condition for the system

$$D^\vee \varphi = 0, \quad \varphi_t = A^\vee \varphi,$$

where

$$A^\vee = -i \begin{pmatrix} -\partial^2 - V & U\bar{\partial} - U_{\bar{z}} \\ \bar{U}\partial - \bar{U}_z & \bar{\partial}^2 + \bar{V} \end{pmatrix}.$$

This equation is called the Davey–Stewartson II (DSII) equation.

The evolution of ψ and φ gives us a deformation of the Gauss map of surfaces (3.3) which are at every moment of time defined up to a translation depending on the temporal variable.

4. THE MOUTARD TRANSFORMATION FOR THE DAVEY–STEWARTSON II EQUATION AND ITS APPLICATIONS

The Moutard transformation was introduced in 1876 in projective differential geometry for the equation

$$f_{xy} + Uf = 0.$$

Given a solution f_0 of this equation, the transformation constructs another equation of this form with a different potential \tilde{U} such that to every solution of the first equation there corresponds a solution of the new one and this is done by an explicit analytical formula. One of the

problems to which the transformation was applied is an explicit construction of an immersion of the hyperbolic plane into \mathbb{R}^3 which, by Hilbert's theorem, appeared to be impossible. Later, the one-dimensional version, the Darboux transformation, was constructed and has found many important applications in mathematical physics.

Recently, the version for the elliptic equation $f_{z\bar{z}} + Uf = 0$ was applied, for instance, to constructing in terms of explicit analytical formulae

- (1) blowing up solutions of the Novikov–Veselov equation with regular and fast decaying initial data [34],
- (2) two-dimensional von Neumann–Wigner potentials with multiple positive eigenvalues [21].

We recall that a potential of the Schrödinger operator on \mathbb{R}^n is called von Neumann–Wigner if it has a positive eigenvalue.

Here we construct a Moutard-type transformation for (3.2) and extend it to a transformation of solutions of the DSII equation.

Extend spinors ψ and φ to \mathbb{H} -valued functions, i.e.,

$$\Psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \varphi_1 & -\bar{\varphi}_2 \\ \varphi_2 & \bar{\varphi}_1 \end{pmatrix}$$

and put

$$\omega(\Phi, \Psi) = -\frac{i}{2}(\Phi^\top \sigma_3 \Psi + \Phi^\top \Psi)dz - \frac{i}{2}(\Phi^\top \sigma_3 \Psi - \Phi^\top \Psi)d\bar{z},$$

where $X \rightarrow X^\top$ is the conjugation of X , and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix. If Ψ and Φ satisfy the Dirac equations (3.2) then $\omega(\Phi, \Psi)$ and $\omega(\Psi, \Phi)$ are closed forms. Denote, for brevity, $\Gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The \mathbb{H} -valued function

$$\begin{aligned} S(\Phi, \Psi)(z, \bar{z}) &= \Gamma \int \omega(\Phi, \Psi) \\ &= \int \left[i \begin{pmatrix} \psi_1 \bar{\varphi}_2 & -\bar{\psi}_2 \bar{\varphi}_2 \\ \psi_1 \varphi_1 & -\bar{\psi}_2 \varphi_1 \end{pmatrix} dz + i \begin{pmatrix} \psi_2 \bar{\varphi}_1 & \bar{\psi}_1 \bar{\varphi}_1 \\ -\psi_2 \varphi_2 & -\bar{\psi}_1 \varphi_2 \end{pmatrix} d\bar{z} \right] \\ &= \int d \begin{pmatrix} ix^3 + x^4 & -x^1 - ix^2 \\ x^1 - ix^2 & -ix^3 + x^4 \end{pmatrix} \end{aligned}$$

defines a surface in $\mathbb{R}^4 = \mathbb{H}$ with z as the conformal parameter (3.3). Hence we identify S with a surface in \mathbb{R}^4 .

Let us define the \mathbb{H} -valued function

$$K(\Phi, \Psi) = \Psi S^{-1}(\Phi, \Psi) \Gamma \Phi^\top \Gamma^{-1} = \begin{pmatrix} i\bar{W} & a \\ -\bar{a} & -iW \end{pmatrix}. \quad (4.1)$$

The following theorem gives a Moutard-type transformation for D .

Theorem 4.1 ([20]). *Given Ψ_0 and Φ_0 , the solutions of (3.2), for every pair Ψ and Φ of solutions of the same equations, the \mathbb{H} -valued functions*

$$\tilde{\Psi} = \Psi - \Psi_0 S^{-1}(\Phi_0, \Psi_0) S(\Phi_0, \Psi), \quad \tilde{\Phi} = \Phi - \Phi_0 S^{-1}(\Psi_0, \Phi_0) S(\Psi_0, \Phi)$$

satisfy the Dirac equations

$$\tilde{D}\tilde{\Psi} = 0, \quad \tilde{D}^\vee\tilde{\Phi} = 0$$

for the Dirac operators with the potential

$$\tilde{U} = U + W, \tag{4.2}$$

where W is defined by (4.1) for $K(\Phi_0, \Psi_0)$. Here $S(\Psi_0, \Phi_0)$ is normalized by the condition

$$\Gamma S^{-1}(\Phi_0, \Psi_0)\Gamma = (S^{-1}(\Psi_0, \Phi_0))^\top.$$

The potential \tilde{U} is the potential of the Weierstrass representation of the surface S^{-1} with z being a conformal parameter. The surface S^{-1} is obtained from S by composition of the inversion centered at the origin and the reflection $(x_1, x_2, x_3, x_4) \rightarrow (-x_1, -x_2, -x_3, x_4)$.

For $U = \tilde{U}$ and $\Psi = \Phi$, this transformation reduces to the transformation of Dirac operators with real-valued potentials given in [35] in different form. In [32] it was related to the Weierstrass representation of surfaces in \mathbb{R}^3 by proving that it corresponds to the Möbius inversion $S \rightarrow S^{-1}$. This gives another proof of the conformal invariance of the Floquet multipliers by explicitly describing the transformations of the Floquet functions. Theorem 4.1 implies its analog for tori in \mathbb{R}^4 . However, in this case the curve Γ_ψ is not preserved by the Möbius inversions. For instance, for the Clifford torus in the unit sphere $S^3 \subset \mathbb{R}^4$, the spectral curve Γ_ψ of its Möbius inversion centered at some point is $\mathbb{C}P^1$ except for the case when the surface lies in a plane, in which case it is $\mathbb{C}P^1$ with a pair of double points [20].

Let us replace $K(\Phi, \Psi)$ in (4.1) with

$$S(\Phi, \Psi)(z, \bar{z}, t) = \Gamma \int \omega(\Phi, \Psi) + \Gamma \int \omega_1(\Phi, \Psi),$$

where

$$\begin{aligned} \omega_1(\Phi, \Psi) = & \left(\left[\Phi_z^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \Phi_{\bar{z}}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \Psi \\ & - \Phi^\top \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi_z + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Psi_{\bar{z}} \right] \right) dt. \end{aligned}$$

We have

Theorem 4.2 ([33]). *If U solves the Davey–Stewartson II equation (3.4) and Ψ and Φ satisfy the equations $D\Psi = 0$, $\Psi_t = A\Psi$, $D^\vee\Phi = 0$, and $\Phi_t = A^\vee\Phi$, then the Moutard transformation (4.2) of U gives the solution \tilde{U} of the DSII equation*

$$\tilde{U}_t = i(\tilde{U}_{zz} + \tilde{U}_{\bar{z}\bar{z}} + 2(\tilde{V} + \bar{\tilde{V}})\tilde{U}), \quad \tilde{V}_{\bar{z}} = (|\tilde{U}|^2)_z$$

with

$$\tilde{V} = V + 2ia_z$$

where a is given by (4.1).

The geometrical meaning of this transformation is as follows: for every fixed t , the spinors Ψ and Φ determine some surface $S(t)$ in \mathbb{R}^4 and U is the potential of such a representation. The surfaces $S(t)$ evolve via the DSII equation. We invert every such surface and obtain the t -parameter family of surfaces $\tilde{S}(t) = S^{-1}(t)$ which evolve via the DSII equation. Starting with a family of smooth surfaces and the corresponding smooth potentials U , we may construct singular solutions of the DSII equation: when $S(t)$ passes through the origin, the function \tilde{U} loses continuity or regularity because the origin is mapped into the infinity by the inversion.

One of the simplest applications of Theorem 4.2 consists in constructing exact solutions from holomorphic functions. In this case we start from the trivial solution $U = V = 0$ for which Ψ and Φ are defined by holomorphic data. For instance, we have

Theorem 4.3 ([33]). *Let $f(z, t)$ be a function which is holomorphic in z and satisfies the equation*

$$\frac{\partial f}{\partial t} = i \frac{\partial^2 f}{\partial z^2}.$$

Then

$$U = \frac{i(zf' - f)}{|z|^2 + |f|^2}, \quad V = 2ia_z,$$

where

$$a = -\frac{i(\bar{z} + f')\bar{f}}{|z|^2 + |f|^2},$$

satisfy the Davey–Stewartson II equation.

Geometrically, we have the deformation of graphs $w = f(z, t)$ which are minimal surfaces in $\mathbb{R}^4 = \mathbb{C}^2$. Whenever $f(z, t)$ vanishes at $z = 0$, the graph passes through the origin and the solution \tilde{U} loses continuity or regularity. Hence the Weierstrass representation visualizes the creation of singularity and gives a method for finding such solutions.

We already applied this idea to constructing a solution with a one-point singularity for the modified Novikov–Veselov equation by using the Enneper surface [31]. However, in contrast to the mNV equation, the DSII has an important physical meaning.

In the variables

$$X = 2y, \quad Y = 2x.$$

the Davey–Stewartson II equation takes the form known in mathematical physics, namely

$$\begin{aligned} iU_t - U_{XX} + U_{YY} &= -4|U|^2U + 8\varphi_X U, \\ \Delta\varphi &= \frac{\partial^2\varphi}{\partial X^2} + \frac{\partial^2\varphi}{\partial Y^2} = \frac{\partial}{\partial X}|U|^2, \end{aligned} \tag{4.3}$$

where $\text{Re } V = 2|U|^2 - 4\varphi_X$, $\varphi_X = \frac{\partial\varphi}{\partial X}$ [7]. This version of the DSII equation is called focusing.

Ozawa constructed a blow-up solution to (4.3) with the initial data

$$U(X, Y, 0) = \frac{e^{-ib(4a)^{-1}(X^2 - Y^2)}}{a(1 + ((X/a)^2 + (Y/a)^2)/2)}$$

and showed that, for constants a and b such that $ab < 0$, we have

$$\|U\|^2 \rightarrow 2\pi \cdot \delta \quad \text{as } t \rightarrow T = -a/b$$

in \mathcal{S}' where $\|U\|^2 = \int_{\mathbb{R}^2} |U|^2 dx dy$ is the squared L_2 -norm of U and δ is the Dirac distribution centered at the origin [23]. We remark that $\|U\|^2 = 2\pi$ and the solution extends to $T > -a/b$ and gains regularity. In [15] it is conjectured for this equation that the blow-up in all cases is self-similar and the time-dependent scaling is as in the Ozawa solution. This conjecture is based on numerical results.

Let us consider the simplest examples of the solutions given by Theorem 4.3. We denote by c a constant which may take arbitrary complex values, and by r we denote $|z|$, $z \in \mathbb{C}$.

Next we consider

$$(1) \quad f = z^2 + 2it + c,$$

$$\begin{aligned} U &= \frac{i(z^2 - 2it - c)}{|z|^2 + |z^2 + 2it + c|^2}, \\ V &= \frac{4(\bar{z}^2 - 2it + \bar{c})}{|z|^2 + |z^2 + 2it + c|^2} - \frac{2(2z(\bar{z}^2 - 2it + \bar{c}) + \bar{z})^2}{(|z|^2 + |z^2 + 2it + c|^2)^2}, \end{aligned} \quad (4.4)$$

and $|U| = O(\frac{1}{r^2})$ as $r \rightarrow \infty$. If c is not purely imaginary, then the solution is always smooth. If $c = it\tau$, $\tau \in \mathbb{R}$, then for $t = -\frac{\tau}{2}$, U has singularity at $z = 0$ of the type

$$U \sim ie^{2i\phi} \quad \text{as } r \rightarrow 0, \text{ where } z = re^{i\phi}.$$

We remark that $U \in L_2(\mathbb{R}^2)$ for all t and c . Since a small variation of c removes singularities, they are unstable.

$$(2) \quad f = z^4 + 12itz^2 - 12t^2 + c,$$

$$U = \frac{i(3z^4 + 12itz^2 + 12t^2 - c)}{|z|^2 + |z^4 + 12itz^2 - 12t^2 + c|^2}. \quad (4.5)$$

This solution becomes singular for $c = 12t^2$ which is possible if and only if c is real-valued and positive. In this case it has singularities $U \sim -12te^{2i\phi}$ at $z = 0$ for $t = \pm\sqrt{c/12}$.

The solution to the mNV equation constructed in [31] is real-valued and regular except for the time T_{sing} when it has singularity at the origin of the form $U \sim -\cos 2\phi$.

We remark that $\|U\|^2$ is the first integral of the system. For (4.4), it is always equal to 2π except for the time T_{sing} when the solution becomes singular. For $t = T_{\text{sing}}$, it is equal to π . Analogously, for (4.5) it is equal to 4π for t such that U is nonsingular, and is equal to 3π for $t = T_{\text{sing}}$. The multiplicity of the value of this functional to π in both cases is explained by that the surfaces \tilde{S} are immersed Willmore spheres (with singularities for singular moments of time).

By taking polynomials of higher degree for f , we can construct such singular solutions for which the regular initial data have any polynomial decay.

Are there another physically relevant wave equations that admit solutions with such singularities?

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REFERENCES

- [1] U. Abresch, Constant mean curvature tori in terms of elliptic functions. *J. Reine Angew. Math.* **374** (1987), 169–192.
- [2] C. Bär, Lower eigenvalue estimates for Dirac operators. *Math. Ann.* **293** (1992), no. 1, 39–46.
- [3] P. Bayard, Spinorial representation of submanifolds in $SL(n, \mathbb{C})/SU(n)$. *Adv. Appl. Clifford Algebr.* **29** (2019), 51, 38 pp.
- [4] D. A. Berdinskii and I. A. Taimanov, Surfaces in three-dimensional Lie groups. *Sib. Math. J.* **46** (2005), no. 6, 1005–1019.
- [5] L. V. Bogdanov, Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg–de Vries equation. *Theoret. and Math. Phys.* **70** (1987), no. 2, 309–314.
- [6] C. Bohle and P. Peters, Soliton spheres. *Trans. Amer. Math. Soc.* **363** (2011), no. 10, 5419–5463.
- [7] A. Davey and K. Stewartson, On three-dimensional packets of surface waves. *Proc. R. Soc. Lond. A* **338** (1974), 101–110.
- [8] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, The Schrödinger equation in a periodic field and Riemann surfaces. *Sov. Math., Dokl.* **17** (1976), 947–951.
- [9] L. P. Eisenhart, *A treatise on the differential geometry of curves and surfaces*. Allyn and Bacon, Boston, 1909.
- [10] D. Ferus, K. Leschke, F. Pedit, and U. Pinkall, Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2-tori. *Invent. Math.* **146** (2001), no. 3, 505–593.
- [11] T. Friedrich, On the spinor representation of surfaces in Euclidean 3-space. *J. Geom. Phys.* **28** (1998), no. 1–2, 143–157.
- [12] P. G. Grinevich and M. U. Schmidt, Conformal invariant functionals of immersions of tori into \mathbb{R}^3 . *J. Geom. Phys.* **26** (1997), no. 1–2, 51–78.
- [13] N. J. Hitchin, Harmonic maps from a 2-torus to the 3-sphere. *J. Differential Geom.* **31** (1990), no. 3, 627–710.
- [14] K. Kenmotsu, Weierstrass formula for surfaces of prescribed mean curvature. *Math. Ann.* **245** (1979), no. 2, 89–99.
- [15] C. Klein and N. Stoilov, Numerical study of blow-up mechanisms for Davey–Stewartson II systems. *Stud. Appl. Math.* **141** (2018), no. 1, 89–112.

- [16] B. G. Konopelchenko, Induced surfaces and their integrable dynamics. *Stud. Appl. Math.* **96** (1996), no. 1, 9–51.
- [17] B. G. Konopelchenko, Weierstrass representations for surfaces in 4D spaces and their integrable deformations via DS hierarchy. *Ann. Global Anal. Geom.* **18** (2000), no. 1, 61–74.
- [18] I. M. Krichever, Spectral theory of two-dimensional periodic operators and its applications. *Russian Math. Surveys* **44** (1989), no. 2, 145–225.
- [19] F. C. Marques and A. Neves, Min–max theory and the Willmore conjecture. *Ann. of Math. (2)* **179** (2014), no. 2, 683–782.
- [20] R. M. Matuev and I. A. Taimanov, The Moutard transformation of two-dimensional Dirac operators and the conformal geometry of surfaces in four-space. *Math. Notes* **100** (2016), no. 5–6, 835–846.
- [21] R. G. Novikov, I. A. Taimanov, and S. P. Tsarev, Two-dimensional von Neumann–Wigner potentials with a multiple positive eigenvalue. *Funct. Anal. Appl.* **48** (2014), no. 4, 295–297.
- [22] S. P. Novikov, A periodic problem for the Korteweg–de Vries equation. I. *Funct. Anal. Appl.* **8** (1974), no. 3, 236–246.
- [23] T. Ozawa, Exact blow-up solutions to the Cauchy problem for the Davey–Stewartson systems. *Proc. R. Soc. Lond. Ser. A* **436** (1992), no. 1897, 345–349.
- [24] F. Pedit and U. Pinkall, Quaternionic analysis on Riemann surfaces and differential geometry. Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). *Doc. Math. Extra Vol. II* (1998), 389–400.
- [25] U. Pinkall and I. Sterling, On the classification of constant mean curvature tori. *Ann. of Math. (2)* **130** (1989), no. 2, 407–451.
- [26] I. A. Taimanov, Modified Novikov–Veselov equation and differential geometry of surfaces. In *Solitons, geometry, and topology: on the crossroad*, pp. 133–151, Amer. Math. Soc. Transl. Ser. 2 179, Adv. Math. Sci. 33, Amer. Math. Soc., Providence, RI, 1997.
- [27] I. A. Taimanov, The Weierstrass representation of closed surfaces in \mathbb{R}^3 . *Funct. Anal. Appl.* **32** (1998), no. 4, 258–267.
- [28] I. A. Taimanov, The Weierstrass representation of spheres in \mathbb{R}^3 , the Willmore numbers, and soliton spheres. *Proc. Steklov Inst. Math.* **225** (1999), 322–343.
- [29] I. A. Taimanov, Surfaces in the four-space and the Davey–Stewartson equations. *J. Geom. Phys.* **56** (2006), no. 8, 1235–1256.
- [30] I. A. Taimanov, The two-dimensional Dirac operator and the theory of surfaces. *Russian Math. Surveys* **61** (2006), no. 1, 79–159.
- [31] I. A. Taimanov, Blowing up solutions of the modified Novikov–Veselov equation and minimal surfaces. *Theoret. and Math. Phys.* **182** (2015), no. 2, 173–181.
- [32] I. A. Taimanov, The Moutard transformation of two-dimensional Dirac operators and the Möbius geometry. *Math. Notes* **97** (2015), no. 5–6, 124–135.
- [33] I. A. Taimanov, The Moutard transformation for the Davey–Stewartson II equation and its geometrical meaning. *Math. Notes* **110** (2021), no. 5–6, 754–766.

- [34] I. A. Taimanov and S. P. Tsarev, Two-dimensional rational solitons and their blow-up via the Moutard transformation. *Theoret. and Math. Phys.* **157** (2008), no. 2, 1525–1541.
- [35] D. Yu, Q. P. Liu, and S. Wang, Darboux transformation for the modified Veselov–Novikov equation. *J. of Physics A* **35** (2001), no. 16, 3779–3785.
- [36] D. V. Zakharov, The Weierstrass representation of discrete isotropic surfaces in $\mathbb{R}^{2,1}$, $\mathbb{R}^{3,1}$, and $\mathbb{R}^{2,2}$. *Funct. Anal. Appl.* **45** (2011), no. 1, 25–32.

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