MEAN CURVATURE AND VARIATIONAL THEORY

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ABSTRACT

In this article, we survey recent progress on the variational theory related to mean curvature. We will discuss the Morse theory of minimal hypersurfaces with an emphasis on the Multiplicity One Conjecture, generic spatial distributions of minimal hypersurfaces, variational theory for constant mean curvature (CMC) surfaces, and variational theory for minimal surfaces with free boundary.

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1. INTRODUCTION

In geometry, a large class of canonical objects are submanifolds which are stationary with respect to variations of length, area, or volume, possibly under various constraints. The condition of being stationary is tightly linked to a geometric quantity called the mean curvature function. The most notable examples are minimal surfaces, constant mean curvature surfaces, and more generally surfaces with prescribed mean curvature (PMC), where the mean curvature function respectively vanishes, is equal to a constant, or is prescribed by an ambient function. Such objects have been studied extensively by mathematicians for more than two centuries since the work of Lagrange on minimal surfaces in 1762, and various different methods have been developed, including but not limited to complex analysis, calculus of variations, partial differential equations, and geometric measure theory. In addition to their intrinsic beauty, profound applications of such canonical submanifolds have been found and led to solutions of many fundamental problems in other fields like topology, analysis, and physics. We refer to [15, 63, 67] for more discussions on historical backgrounds.

In the calculus of variations or variational theory, which we will focus on in this article, such surfaces are viewed as critical points of certain area- or volume-related functionals. In the past ten years, the variational-theoretic approach has enjoyed spectacular development, and deep new results have been proved on the existence of minimal, CMC, and PMC surfaces. In particular, the famous Yau's conjecture on the existence of infinitely many closed minimal surfaces was confirmed by combining the works of Marques–Neves [58] and Song [79], and general existence of closed CMC and PMC hypersurfaces was established by the author with Zhu [95, 96], and with Cheng [12]. Moreover, surprising new connections between these surfaces have been discovered, leading to the resolution of the Multiplicity One Conjecture for minimal hypersurfaces by the author [94]. In this article, we will provide a survey of these results, as well as some discussion of open problems along this direction.

1.1. Minimal surfaces

We start with a discussion of variational constructions of minimal surfaces in 3-dimensional spaces and, more generally, minimal hypersurfaces when the ambient space has dimension higher than 3. Minimal surfaces are mathematical models of soap films, where the surface tension tends to minimize the area. By the first variation formula of area, the mean curvature of such surfaces has to vanish. In general, minimal surfaces are defined as surfaces with vanishing mean curvature or, equivalently, stationary points of the area functional. The problem of finding area-minimizing surfaces with a given boundary in the 3-dimensional Euclidean space was raised by Lagrange, and later named after Joseph Plateau who systematically experimented with soap films in the 19th century. The Plateau's problem was solved independently by Douglas and Radó in 1930 using mapping methods. Since then, there have been various attempts to generalize this existence result to the case of higher-dimensional submanifolds and in Euclidean or Riemannian spaces of higher co-dimensions. In particular, this led to the development of geometric measure theory (GMT) by many outstanding mathematicians. By combining the works of Federer, Fleming, De Giorgi, Almgren, and Simons

[5,21,25,26,77], it is known that an area-minimizing current of codimension one is smoothly embedded outside a singular set of codimension 7. (We also refer to the work of De Lellis, particularly the survey [22], for the regularity of higher-codimensional area-minimizing currents.)

Besides the Plateau's problem, it is also natural to consider the existence of closed minimal surfaces in closed Riemannian manifolds. When the ambient space has rich topology, area-minimizing surfaces can be produced using either the mapping approach or geometric measure theory. For instance, when the ambient space M^n contains an incompressible surface $f : S_g \to M$ where S_g is a genus-g surface, Schoen–Yau [75] and Sacks–Uhlenbeck [72] proved the existence of an area-minimizing surface in its conjugacy class, by minimizing first the Dirichlet energy $E(f) = \int_{S_g} |\nabla f|^2$ and then within the Teichmüller space of conformal structures. In [62], Meeks–Simon–Yau proved the existence of embedded minimal surfaces by minimizing area within a nontrivial isotopy class in 3-manifolds. More generally, if there exists a nontrivial element $c \neq 0$ in the homology group $H_{n-1}(M^n, \mathbb{Z})$, by GMT there always exists an area-minimizing integral current $\Sigma \in c$, whose support is smoothly embedded outside a codimension 7 singular set.

The problem of finding closed minimal surfaces in general is more interesting and significantly harder. Inspired by earlier works on finding closed geodesics (one-dimensional minimal submanifolds) on 2-dimensional spheres [9,53], Almgren [2,3] initiated a program aiming at finding closed minimal submanifolds in closed Riemannian manifolds of any dimension and codimension. He designed a very general min–max theory applicable to families of integral cycles and showed the existence of a nontrivial stationary integral k-dimensional varifold in any closed M^n for $1 \leq k \leq n$. Later on, in a seminal work [68], Pitts further improved Almgren's theory and proved that the support of a min–max varifold is smoothly embedded in the codimension-one case ($k = n - 1$) when $3 \leq n \leq 6$, by using the famous curvature estimates for stable minimal hypersurfaces by Schoen–Simon–Yau [74]. Schoen–Simon [73] then extended the curvature estimates and hence obtained the regularity for codimension-one min–max varifolds in higher dimensions $n \geq 7$, allowing singular sets of codimension 7. Combining all the results above, the first general existence theorem is:

Theorem 1.1. *Every closed Riemannian manifold (M^n, g) of dimension $n \geq 3$ contains a nontrivial integral $(n-1)$-dimensional stationary varifold V whose support is a smoothly embedded minimal hypersurface outside a singular set of codimension 7. If $3 \leq n \leq 7$, the support of V is a smooth, closed, embedded, minimal hypersurface Σ.*

We also note that when M^n has nontrivial higher homotopy groups, Sacks–Uhlenbeck [72] produced branched, immersed, minimal 2-spheres by developing another min–max theory using perturbation arguments and classical Morse theory on Banach manifolds. Recently, in their proof of the finite-time extinction of the Ricci flow, Colding–Minicozzi [17] found a new proof of Sacks–Uhlenbeck's result by their harmonic replacements method. See also for the works of the author [92,93] and Rivière [69] for min–max constructions for higher genus minimal surfaces.

Motivated by the these results and the existence theory of closed geodesics, S. T. Yau formulated a famous conjecture in [90] asserting that every closed 3-manifold admits infinitely many distinct smooth, closed, immersed minimal surfaces. One peak of the recent developments on minimal hypersurface theory is the resolution of this conjecture by Marques–Neves [58] and Song [79]. Around the same time, a Morse theory for the area functional has been established [55, 57, 59, 94], and several striking results concerning the spatial distribution of these minimal hypersurfaces were proved [40, 60, 81]. All of these results were obtained by applying the Almgren–Pitts min–max theory to families of cycles of multiple parameters (deeply influenced by the solution of the Willmore Conjecture by Marques–Neves [56]). We will postpone detailed discussions to Sections 2 and 3.

A central difficulty in obtaining the aforementioned results is the a priori existence of integer multiplicity of the min–max varifolds. That is, the min–max varifolds may be represented by integer multiples of embedded minimal hypersurfaces. Therefore, applications of the min–max theory to higher-parameter families of integral cycles may just result in multiple covers of the minimal hypersurfaces associated with lower-parameter families. Motivated by classical Morse theory, Marques–Neves [59] formulated the Multiplicity One Conjecture:

Conjecture 1.2. *For smooth generic metrics on M^n when $3 \leq n \leq 7$, min–max varifolds are represented by multiplicity-one embedded minimal hypersurfaces.*

This conjecture was confirmed by the author in [94] using new ideas which were inspired by the investigation of the existence theory of CMC/PMC hypersurfaces [95, 96], to be discussed below. We will provide a sketch of proof of this conjecture in Section 2. Finally, we also note that the counterpart of the Multiplicity One Conjecture in the phase transition setting was proved by Chodosh–Mantoulidis [14] in 3 dimensions; see also [30, 35].

1.2. CMC and PMC surfaces

Surfaces with constant mean curvature (CMC) are mathematical models of soap bubbles. In the ideal situation, surface tension tends to minimize the surface area while the volume of enclosed air is fixed. Such surfaces must then be stationary points of the area subject to a volume constraint, and hence must have constant mean curvature by the first variation formula. CMC surfaces form a classical topic in differential geometry, and play an essential role in many areas, such as isoperimetric problems, interface theory for polymers, and general relativity. The classification of CMC surfaces in $\mathbb{R}^3$ and other homogeneous 3-manifolds has been a classical problem since the seminal work of Aleksandrov [1], and we refer to the survey paper [64] for this direction. In this article, we will focus on the existence theory of CMC surfaces in general manifolds.

We start with a brief and nonexhaustive review of several previous existence results that are closely related to our main results. The existence of CMC surfaces in $\mathbb{R}^3$ with prescribed mean curvature and Plateau boundary conditions was initiated by Heinz [38] and Hildebrandt [39]. The Rellich conjecture, which asserts the existence of at least two solutions to the CMC Plateau problem, was solved later by Brezis–Coron [11] and Struwe [83]. For the existence of closed CMC hypersurfaces, it is well known that the boundaries of isoperimetric regions are smoothly embedded CMC hypersurfaces (up to a singular set of codimension 7); see [4, 66]. By perturbation arguments, one can generate foliations by closed CMC hypersurfaces from a given nondegenerate closed minimal hypersurface, or near minimal submanifolds of strictly lower dimensions (see, for instance, the works of Ye, Mahmoudi–Mazzeo–Pacard [54, 91]). The gluing constructions pioneered by Kapouleas produced many important examples of complete or compact CMC surfaces in Euclidean spaces [10, 43]. In addition, there is the degree theory approach by Rosenberg–Smith [70]. However, these works left open the fundamental problem of finding closed hypersurfaces with arbitrary prescribed constant mean curvature in general manifolds.

In [95], Zhu and the author settled this problem by establishing the following general existence theory.

Theorem 1.3 ([95]). *Let M^n be a closed Riemannian manifold of dimension $3 \leq n \leq 7$. Given any $c \in \mathbb{R}$, there exists a nontrivial, smooth, closed, almost embedded hypersurface Σ of constant mean curvature c.*

Remark 1.4. A smooth almost embedded hypersurface is a smooth immersion where near any self-intersection point, the hypersurface decomposes into sheets which may touch but not cross. Such hypersurfaces are Alexandrov embedded.

We proved this result by establishing a min–max theory for the area functional with a volume term added, extending the Almgren–Pitts theory to the more general CMC setting. A sketch of proof will be provided in Section 4.

We note that no control on topology of the CMC hypersurfaces in Theorem 1.3 was known due to the use of integral currents as the total variation space. In contrast, we note that using a variant of the Almgren–Pitts theory, Simon–Smith [78] proved the existence of an embedded minimal 2-sphere in any Riemannian 3-sphere. Their work has been generalized to an arbitrary closed 3-manifold M by Colding–De Lellis [16] using sweepouts associated with Heegaard splittings, and the genus of the min–max surface is known to be bounded by the Heegaard genus of M [23, 44]. On the other hand, the min–max theory based on the harmonic mapping approach [17, 69, 72, 92, 93] naturally produces branched immersed minimal surfaces with controlled genus. With these contrasts in mind, it is tempting to search for closed CMC surfaces with both prescribed mean curvature and controlled genus (bounded by the Heegaard genus) in 3-manifolds. In particular, we note a conjecture by Rosenberg–Smith [70, PAGE 3]: "*for any $H \geq 0$ and any metric g on S^3 of positive sectional curvature, there exists an embedding of S^2 to S^3 of constant mean curvature H*". However, by the works of Torralbo [85] and Meeks–Mira–Pérez–Ros [65], it is known that in certain positively curved homogenous 3-spheres, there are mean curvature values for which the associated immersed CMC 2-spheres must have self-intersections. Motivated by the mapping approach for minimal surfaces, it is natural to modify *embedding* to *branched immersion* in the Rosenberg–Smith conjecture.

In [12], Cheng and the author solved this modified conjecture with a newly devised min–max theory using the mapping approach.

Theorem 1.5 ([12]). *Given a Riemannian 3-sphere (S^3, g) with nonnegative Ricci curvature, for every constant H, there exists a nontrivial branched immersed 2-sphere with constant mean curvature H.*

Remark 1.6. In [12], we also proved that a branched immersed H-CMC 2-sphere exists in (S^3, g) whenever $\mathrm{Ric}_g > -\frac{H^2}{2} g$, or for almost all H (with respect to the Lebesgue measure) without curvature assumptions on g.

A hypersurface Σ^{n-1} in M^n has prescribed mean curvature (PMC) by some function $h : M \to \mathbb{R}$ if its mean curvature is everywhere identical to the value of h. PMC hypersurfaces are natural generalizations of CMC hypersurfaces and are models for capillary surfaces; see [27, §1.6]. The local existence theory for PMC hypersurfaces is quite well understood in the case of Plateau boundary conditions and in the graphical case; see [96] for references. On the other hand, the global theory or the existence for closed PMC hypersurfaces had been largely open except for constant prescription functions. The global existence problem for closed PMC surfaces in closed three manifolds is a conjecture of Yau in the 1980s (by personal communication, see also [90, PROBLEM 59] for a version of his conjecture in $\mathbb{R}^3$).

In [96], Zhu and the author extended our CMC min–max theory developed in [95] to nonconstant prescription functions. In particular, we solved the existence problem for closed PMC hypersurfaces for a generic class of smooth prescription functions.

Theorem 1.7 ([96]). *Let M^n be a closed Riemannian manifold of dimension $3 \le n \le 7$. There is an open dense set (in the smooth topology) $\mathcal{S} \subset C^\infty(M)$ of prescription functions h for which there exists a nontrivial, smooth, closed, almost embedded hypersurface Σ of prescribed mean curvature h. That is, $H_\Sigma = h|_\Sigma$.*

In both Theorems 1.3 and 1.7, the min–max theory was devised only for one-parameter families. These results were later generalized to multiparameter min–max constructions together with Morse index upper bounds by the author in [94]. The PMC min–max Theorem 1.7 and its generalizations in [94] had played an essential role in the proof of the Multiplicity One Conjecture by the author in [94].

Finally, we also note the phase transition approach to the PMC existence problem by Bellettini–Wickramasekera [8] for nonnegative Lipschitz prescribing functions.

2. VARIATIONAL THEORY FOR AREA AND THE MULTIPLICITY ONE CONJECTURE

In this part, we introduce the recently developed Morse theory for the area functional and a sketch of proof of the Multiplicity One Conjecture. For simplicity, in what follows, we will denote by n the dimension of a hypersurface Σ^n, and by $(n+1)$ the dimension of the ambient manifold M^{n+1}.

The principle behind Morse theory is to relate the topology of a given total space to all the critical points of a functional defined therein in a generic scenario. We choose the total space for the area functional to be the space of mod-2 n-cycles, denoted by $\mathcal{Z}_n(M, \mathbb{Z}_2)$, which can roughly be regarded as the boundaries of open sets with finite n-dimensional Hausdorff measure. In [2], Almgren calculated all the homotopy groups of $\mathcal{Z}_n(M, \mathbb{Z}_2)$, and proved the following:

Theorem 2.1. *$\mathcal{Z}_n(M, \mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$.*

Here $\mathbb{RP}^\infty$ denotes the infinite-dimensional real projective space. This fact implies that the $\mathbb{Z}_2$-cohomological ring of $\mathcal{Z}_n(M, \mathbb{Z}_2)$ is a polynomial ring whose generator we denote by $\bar{\lambda}$. That is, $\mathcal{H}^*(\mathcal{Z}_n(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\bar{\lambda}]$. Motivated by the topological structures, Gromov [31,32], Guth [36], and Marques–Neves [58] introduced the notion of the volume spectrum for the area functional in $\mathcal{Z}_n(M, \mathbb{Z}_2)$ as a nonlinear version of the Laplacian spectrum. Below, we let X be any finite-dimensional parameter space, for instance, a cubical complex.

Definition 2.2 (Volume spectrum). Given $k \in \mathbb{N}$, a continuous map $\Phi : X \to \mathcal{Z}_n(M, \mathbb{Z}_2)$ is called a *k-sweepout* if $\Phi^*(\bar{\lambda}^k) \neq 0$ in $H^k(X, \mathbb{Z}_2)$. The *kth volume spectrum*, or the *k-width*, is just the min–max value

$$\omega_k(M) = \inf_{\Phi:k\text{-sweepout}} \sup_{x \in \mathrm{dmn}(\Phi)} \mathrm{Area}\big(\Phi(x)\big),$$

where $\mathrm{dmn}(\Phi)$ stands for the domain of Φ.

It was proved that the sequence $\{\omega_k(M)\}$ grows sublinearly at the rate of $k^{\frac{1}{n+1}}$ as $k \to \infty$ [31,32,36,58]. Moreover, the sequence satisfies a Weyl Law.

Theorem 2.3 (Liokumovich–Marques–Neves [52]). *There exists a universal constant $a(n) > 0$ such that for any compact Riemannian manifold M^{n+1},*

$$\lim_{k\to\infty} \omega_k(M) k^{-\frac{1}{n+1}} = a(n)\,\mathrm{Vol}(M)^{\frac{n}{n+1}}.$$

Note that the Almgren–Pitts min–max theory works for families of cycles within a homotopy class, while the definition of the volume spectrum concerns all families via the cohomological condition. To link them together, Marques–Neves systematically studied the Morse index for minimal hypersurfaces produced by the Almgren–Pitts theory [57]. In particular, they proved the following version of the min–max theorem.

Theorem 2.4. *Let M^{n+1} be a closed Riemannian manifold with $3 \leq n+1 \leq 7$. For each $k \in \mathbb{N}$, there exists a disjoint collection of connected, closed, smoothly embedded minimal hypersurfaces $\{\Sigma_i^k : i = 1, \ldots, l_k\}$ with integer multiplicities $\{m_i^k : i = 1, \ldots, l_k\} \subset \mathbb{N}$ such that*

$$\omega_k(M) = \sum_{i=1}^{l_k} m_i^k \cdot \mathrm{Area}\big(\Sigma_i^k\big) \quad \text{and} \quad \sum_{i=1}^{l_k} \mathrm{Ind}\big(\Sigma_i^k\big) \leq k.$$

Here $\text{Ind}(\Sigma)$ *stands for the Morse index of* Σ*, which is the number of negative eigenvalues of the second variation of area.*

The possible existence of multiplicities greater than 1 formed a major obstacle in applications of the Almgren–Pitts theory since the 1980s. In addition to the possible repeated occurrence of minimal hypersurfaces when applying Theorem 2.4 to $\{\omega_k\}_{k\in\mathbb{N}}$, min–max varifolds with higher multiplicities cannot fit into the program of Marques–Neves [59] to obtain Morse index lower bounds; (see also [55]). The following famous conjecture was formulated by Marques–Neves [59].

Conjecture 2.5 (Multiplicity One Conjecture). *For a bumpy metric on* M^{n+1}, $3 \le n+1 \le 7$, *there exists a collection* $\{\Sigma_i^k\}$ *as in Theorem* 2.4 *such that every component* Σ_i^k *is two-sided and of multiplicity one.*

Remark 2.6. A hypersurface is *two-sided* if its normal bundle is trivial. A Riemannian metric is *bumpy* if every closed immersed minimal hypersurface is a nondegenerate critical point of the area functional. White proved that the set of bumpy metrics is generic in the sense of Baire [88, 89].

This conjecture was confirmed by the author in [94].

Theorem 2.7. *Conjecture* 2.5 *is true.*

Theorem 2.7, together with the program on Morse index lower bounds developed by Marques–Neves [59], implies that for bumpy metrics, there exists a closed minimal hypersurface of Morse index k and area $\omega_k(M)$ for each $k \in \mathbb{N}$. The above works together established a satisfactory global Morse theory for the area functional. Recently, Marques–Montezuma–Neves proved Morse inequalities for the area functional [55], and hence established a local Morse theory as well.

By the convergence theorems for minimal hypersurfaces of Sharp [76], the same conclusions in Theorem 2.7 hold true for metrics with a positive Ricci curvature, as well as the following results concerning the multiplicity and Morse index of min–max minimal hypersurfaces for general metrics.

Theorem 2.8 ([94]). *In Theorem* 2.4*, every component* Σ_j^k *which is not weakly stable is two-sided with* $m_j^k = 1$*; and* $\sum_{\Sigma_j^k:\text{ two-sided}} \text{Ind}(\Sigma_j^k) \le k$.

Remark 2.9. A closed minimal hypersurface Σ is weakly stable if the second variation of area at Σ is nonnegative definite with a nontrivial kernel. The results in Theorem 2.8 have been partially generalized to dimensions $n+1 > 7$ by Li [50].

Sketch of proof of Theorem 2.7. The key idea of our proof in [94] is to approximate the area functional by the weighted $\mathcal{A}^h$-functional used in the PMC min–max theory [96]. Here $\mathcal{A}^h$ is defined for Caccioppoli sets Ω by $\mathcal{A}^h(\Omega) = \text{Area}(\partial\Omega) - \int_\Omega h\,dM$, where $h \in C^\infty(M)$. A smooth critical point of $\mathcal{A}^h$ is a Caccioppoli set Ω whose boundary is a smooth hypersurface $\Sigma = \partial\Omega$ and has mean curvature (with respect to the outward unit normal) given by h

restricted to Σ. There are two crucial parts in the proof. First, we show that, given a bumpy metric, the volume spectrum $\omega_k(M)$ can be realized by the area of some minimal hypersurfaces coming from relative min–max constructions using sweepouts of boundaries. Next, we observe that, still assuming bumpiness, if one approximates Area by a sequence $\{\mathcal{A}^{\varepsilon_k h}\}_{k\in\mathbb{N}}$ where $\varepsilon_k \to 0$, and if $h : M \to \mathbb{R}$ is carefully chosen, then the limit min–max minimal hypersurfaces (of min–max PMC hypersurfaces associated with $\mathcal{A}^{\varepsilon_k h}$) are all two-sided and have multiplicity one.

Part 1. Given a bumpy metric, for each $k \in \mathbb{N}$, by [57] there exists a free homotopy class Π of maps $\Phi : X \to \mathcal{Z}_n(M, \mathbb{Z}_2)$, where X is a fixed k-dimensional parameter space such that the min–max value $\mathbf{L} = \inf_{\Phi\in\Pi} \max_{x\in X} \text{Area}(\Phi(x))$ equals $\omega_k(M)$. Choose $\Phi_0 \in \Pi$ so that $\max_{x\in X} \text{Area}(\Phi_0(x))$ is very close to $\mathbf{L}$. Since the space of Caccioppoli sets $\mathcal{C}(M)$ forms a double cover of $\mathcal{Z}_n(M, \mathbb{Z}_2)$ via the boundary map $\partial : \Omega \to [\partial\Omega]$ (see [59]), we can lift Φ_0 to $\widetilde{\Phi}_0 : \tilde{X} \to \mathcal{C}(M)$, where $\pi : \tilde{X} \to X$ is also a double cover. Next let Y be the subset of $x \in X$ where $\Phi_0(x)$ is ε-close to the set $\mathcal{S}$ of closed embedded minimal hypersurfaces Σ with Area $\leq \mathbf{L} + 1$ and Ind $\leq k$, and let $Z = \overline{X \setminus Y}$. As $\mathcal{S}$ is a finite set by [76], Y is topologically trivial, and hence $\tilde{Y} = \pi^{-1}(Y)$ is a disjoint union of two homeomorphic copies of Y, that is, $\tilde{Y} = Y^+ \bigsqcup Y^-$ with $Y \simeq Y^+ \simeq Y^-$. On the other hand, since no element in $\Phi_0(Z)$ is close to being regular, we can deform $\Phi_0|_Z$ based on Pitts's combinatorial argument [68, 4.10], so that

$$\max_{x\in Z} \text{Area}\big(\Phi_0(x)\big) < \mathbf{L}. \tag{2.1}$$

Now consider the $(\tilde{X}, \tilde{Z})$-relative homotopy class of maps generated by $\widetilde{\Phi}_0$: $\widetilde{\Pi} = \{\Psi : \tilde{X} \to \mathcal{C}(M) : \Psi|_{\tilde{Z}} = \widetilde{\Phi}_0|_{\tilde{Z}}\}$.

Lemma 2.10 ([94, LEMMA 5.8]). *The min–max value $\tilde{\mathbf{L}}$ of $\widetilde{\Pi}$ satisfies*

$$\tilde{\mathbf{L}} := \inf_{\Psi\in\widetilde{\Pi}} \max_{x\in\tilde{X}} \text{Area}\big(\partial\Psi(x)\big) \geq \mathbf{L} = \omega_k(M).$$

Hence by (2.1)*, we have the nontriviality condition* $\tilde{\mathbf{L}} > \max_{x\in Z} \text{Area}(\Phi_0(x))$.

Proof. If the conclusion were false, then since

$$\max_{x\in\tilde{Z}} \text{Area}(\partial\widetilde{\Phi}_0(x)) = \max_{x\in Z} \text{Area}(\Phi_0(x)) < \mathbf{L},$$

one could deform $\widetilde{\Phi}_0$ on $\tilde{Y}$ so that the maximum area is less than $\mathbf{L}$. However, as Y^+ and Y^- are disjoint, the deformations on Y^+ (or on Y^-) can be passed to the quotient to give deformations of $\Phi_0|_Y$ in $\mathcal{Z}_n(M, \mathbb{Z}_2)$. As all the maps are fixed on Z, we then obtain deformations of Φ_0 after which the maximum area is less than $\mathbf{L}$, which is a contradiction. ∎

Part 2: The main conclusion follows from the result below.

Theorem 2.11 ([94, THEOREM 4.1]). *In the above notation, if g is bumpy, $\tilde{\mathbf{L}}$ can be realized as the area of a multiplicity one, closed, embedded, two-sided, minimal hypersurface.*

To derive Theorem 2.7, first note that by the choice of Φ_0, we know that $\tilde{\mathbf{L}}$ is very close to $\mathbf{L}$. By the bumpiness of g, the values of $\tilde{\mathbf{L}}$ should stabilize to $\mathbf{L}$ when they are close enough.

Proof of Theorem 2.11. To simplify notions, we will drop all the tildes in this part. Given a smooth function $h : M \to \mathbb{R}$ and $\varepsilon > 0$, we can approximate $\mathbf{L}$ by the min–max values for the $\mathcal{A}^{\varepsilon h}$-functional, $\mathbf{L}^{\varepsilon h} = \inf_{\Psi \in \Pi} \max_{x \in X} \mathcal{A}^{\varepsilon h}(\partial \Psi(x))$, that is, $\mathbf{L}^{\varepsilon h} \to \mathbf{L}$ as $\varepsilon \to 0$. Note that we require $\Psi|_Z = \Phi_0|_Z$ for all $\Psi \in \Pi$. By the fact $\mathbf{L} > \max_{x \in Z} \operatorname{Area}(\partial \Phi_0(x))$, and that the term $\varepsilon \int_\Omega h dM$ in $\mathcal{A}^{\varepsilon h}(\Omega)$ is uniformly small, we have, for ε small enough,

$$\mathbf{L}^{\varepsilon h} > \max_{x \in Z} \mathcal{A}^{\varepsilon h}\big(\Psi(x)\big). \tag{2.2}$$

For a generic choice of h, applying the multiparameter PMC min–max theory [94, THEOREM 1.7] (based on the one-parameter version in [96]), we obtain some $\Omega_\varepsilon \in \mathcal{C}(M)$ such that: (1) $\Sigma_\varepsilon = \partial \Omega_\varepsilon$ is an almost embedded hypersurface; (2) the mean curvature (with respect to outward unit normal) $H_{\Sigma_\varepsilon} = \varepsilon h|_{\Sigma_\varepsilon}$; (3) $\mathcal{A}^{\varepsilon h}(\Omega_\varepsilon) = \mathbf{L}^{\varepsilon h}$; and (4) the Morse index (with respect to $\mathcal{A}^{\varepsilon h}$) $\operatorname{Ind}(\Sigma_\varepsilon) \le k$.

Letting $\varepsilon \to 0$, by (2)–(4) and [94, THEOREM 2.6], up to taking a subsequence, Σ_ε converge locally smoothly away from a finite set $\mathcal{W}$ to a closed embedded minimal hypersurface Σ_0 (assumed to be connected without loss of generality) with an integer multiplicity $m \in \mathbb{N}$. Therefore, $\mathbf{L} = m \operatorname{Area}(\Sigma_0)$, and it suffices to prove that Σ_0 is two-sided (which we skip here) and $m = 1$.

The convergence implies that Σ_ε locally decomposes as an m-sheeted graph over $\Sigma_0 \setminus \mathcal{W}$, with graphing functions: $u_\varepsilon^1 \le u_\varepsilon^2 \le \cdots \le u_\varepsilon^m$. And by (1), the outward unit normal of Ω_ε will alternate orientations along these sheets. The proof proceeds depending on whether m is odd or even.

Claim 1. *If $m \ge 3$ is odd, then Σ is degenerate, hence a contradiction.*

Proof. Since m is odd, the top and bottom sheets have the same orientation, so by subtracting the PMC equations of the two sheets, we have $L(u_\varepsilon^m - u_\varepsilon^1) + o(u_\varepsilon^m - u_\varepsilon^1) = \varepsilon(h(x, u_\varepsilon^m) - h(x, u_\varepsilon^1)) = o(u_\varepsilon^m - u_\varepsilon^1)$, where L is the Jacobi operator associated with the second variation of Σ. After renormalizations, the height differences $u_\varepsilon^m - u_\varepsilon^1$ will converge subsequentially to a positive Jacobi field of $\Sigma \setminus \mathcal{W}$, which extends to Σ by a standard trick. ■

Claim 2. *If m is even, there exists a solution of $L\varphi = 2h|_{\Sigma_0}$ which does not change sign.*

Proof. Now the top and bottom sheets have opposite orientations. Thus $L(u_\varepsilon^m - u_\varepsilon^1) + o(u_\varepsilon^m - u_\varepsilon^1) = \pm\varepsilon(h(x, u_\varepsilon^1) + h(x, u_\varepsilon^m))$. Using the renormalization procedure again and noting that $u_\varepsilon^m - u_\varepsilon^1 > 0$, we get either a positive Jacobi field (which cannot happen) or a positive function φ satisfying $L\varphi = 2h|_{\Sigma_0}$ or $L\varphi = -2h|_{\Sigma_0}$. ■

The following key lemma says that Claim 2 cannot hold for a suitably chosen h. Hence the proof of Theorem 2.11 is complete.

Lemma 2.12. *For a suitably chosen h, the solutions of $L\varphi = 2h|_\Sigma$ on a closed embedded minimal hypersurface Σ with* Area $\le C$ *and* Ind $\le k$ *must change sign.*

Proof. By [76], the set of minimal hypersurfaces with Area $\le C$ and Ind $\le k$ is finite, which we denote by $\{\Sigma_1, \Sigma_2, \dots, \Sigma_N\}$. Take pairwise disjoint neighborhoods $U_j^\pm \subset \Sigma_j$ and a smooth function f defined on $\bigcup U_j^\pm$ with compact support such that (1) $f|_{U_j^+}$ is nonnegative and is positive at some point; (2) $f|_{U_j^-}$ is nonpositive and is negative at some point. Next extend Lf to some $h_0 \in C^\infty(M)$ and take a generic h as close to h_0 as we want. Then any solution φ of $L\varphi = 2h|_{\Sigma_j}$ would be close to $2f$ for each Σ_j, and hence must change sign. ■

3. GENERIC DENSENESS, EQUIDISTRIBUTION, AND SCARRING

En route to the proof of Yau's conjecture and the establishment of a Morse theory for the area functional, we have also observed several striking results on the spatial distribution of closed minimal hypersurfaces for generic smooth metrics, which we introduce in this part. There is an intimate analog between closed minimal hypersurfaces and L^2-density of Laplace eigenfunctions regarding their spatial distributions. For instance, both exhibit equidistribution and scarring phenomena. We refer to [81] for a survey of this analogy.

Using the Weyl Law for the volume spectrum (Theorem 2.3), Irie–Marques–Neves [40] obtained a very surprising generic density result for closed minimal hypersurfaces, and hence settled Yau's conjecture in generic case. (See [49] by Li for the generalization to higher dimensions.)

Theorem 3.1 (Irie–Marques–Neves [40]). *Let M^{n+1} be a closed manifold with $3 \le n+1 \le 7$. Then for a C^∞-generic Riemannian metric, the union of all closed, smoothly embedded minimal hypersurfaces is dense in M.*

This result was later quantified by Marques–Neves–Song [60] to prove the following generic equidistribution result for closed minimal hypersurfaces.

Theorem 3.2 (Marques–Neves–Song [60]). *Let M^{n+1} be a closed manifold with $3 \le n+1 \le 7$. Then for a C^∞-generic Riemannian metric, there exists a sequence of closed, smoothly embedded minimal hypersurfaces $\{\Sigma_j\}_{j\in\mathbb{N}}$ that is equidistributed in M. That is, $\forall f \in C^\infty(M)$,*

$$\lim_{q\to\infty} \frac{1}{\sum_{j=1}^q \mathrm{Area}(\Sigma_j)} \sum_{j=1}^q \int_{\Sigma_j} f\, d\Sigma_j = \frac{1}{\mathrm{Vol}(M,g)} \int_M f\, dM.$$

The key idea behind these results is that, after bumping up the metric in a neighborhood U of a point $p \in M$ (for instance, by conformal changes), the min–max theory necessarily yields a closed minimal hypersurface passing through U according to the Weyl Law.

In his proof of Yau's conjecture for general nongeneric metrics, Song [79] introduced a localized version of the volume spectrum $\{\widetilde{\omega}_k\}_{k\in\mathbb{N}}$, called the *cylindrical volume spectrum* and defined as the volume spectrum of certain noncompact manifolds with Lipschitz metrics obtained by gluing infinite cylinders to compact manifolds with stable minimal boundaries. In contrast to the sublinear growth of the standard volume spectrum, $\{\widetilde{\omega}_k\}_{k\in\mathbb{N}}$ grows linearly by [79]. By extending the ideas in [60] to the cylindrical volume spectrum, Song and the author [81] obtained a generic scarring result. Namely, we showed that generically there exist closed embedded minimal hypersurfaces with large area and Morse index, which accumulate surrounding any stable minimal hypersurface in a quantitative way. Such a phenomenon is called *scarring*.

Theorem 3.3 (Generic scarring, [81]). *Let M^{n+1} be a closed manifold with $3 \le n+1 \le 7$. For a C^∞-generic metric, we have: for any connected, closed, embedded, 2-sided, minimal hypersurface S in M which is stable, there is a sequence $\{\Sigma_k\}$ of closed, embedded, minimal hypersurfaces, such that*

$$(1)\ \Sigma_k \cap S = \emptyset; \quad (2)\ \lim_{k\to\infty} \|\Sigma_k\| = \infty; \quad (3)\ \lim_{k\to\infty} \mathrm{Ind}(\Sigma_k)\|\Sigma_k\|^{-1} = \|S\|^{-1};$$

$$(4)\ \mathbf{F}\left(\frac{[S]}{\|S\|}, \frac{[\Sigma_k]}{\|\Sigma_k\|}\right) \le 1/\log\big(\|\Sigma_k\|\big).$$

Here $[\Sigma]$ is the varifold associated to Σ, $\|\Sigma\|$ is its area, and $\mathbf{F}$ is the varifold distance.

In dimension $n+1=3$, we also explored the 3-manifold topology to find stable minimal surfaces and showed that generic scarring happens for all closed 3-manifolds but the spherical quotients.

4. MIN–MAX THEORY FOR CMC SURFACES

In this part, we present a sketch of the proof of the CMC existence Theorem 1.3, focusing for simplicity on the one-parameter min–max construction. The proof of the PMC Theorem 1.7, which we omit here, shared several key ideas with the CMC case, with additional challenges including the correct choice of prescribing functions [96, PROPOSITION 0.2], and a more complicated gluing scheme.

Sketch of proof of Theorem 1.3. Fixing a closed manifold (M^{n+1}, g) and a number $c > 0$, a given Morse function $f : M \to [0,1]$ generates a continuous map $\Phi_0 : [0,1] \to \mathcal{C}(M)$ by $\Phi_0(x) = \{f(p) < x\}$ with $\Phi_0(0) = \emptyset$ and $\Phi_0(1) = [M]$. The min–max value of $\mathcal{A}^c$ associated with the relative homotopy class $\Pi = \{\Phi : [0,1] \to \mathcal{C}(M), \Phi|_{\{0,1\}} = \Phi_0|_{\{0,1\}}\}$ is

$$\mathbf{L}^c = \inf_{\Phi\in\Pi} \max_{x\in[0,1]} \mathcal{A}^c\big(\Phi(x)\big),$$

where

$$\mathcal{A}^c(\Omega) = \mathrm{Area}(\partial\Omega) - c\,\mathrm{Vol}_g(\Omega).$$

Using the isoperimetric inequalities for small volumes, we have

Theorem 4.1 ([95, THEOREM 3.9]). $\mathbf{L}^c > 0$.

Note that $\max\{\mathcal{A}^c(\emptyset), \mathcal{A}^c(M)\} = 0$. This directly implies that $\mathbf{L}^c$ can be realized by some nontrivial weak limit. In the multiparameter cases, one needs to assume that $\mathbf{L}^c$ is strictly greater than the values assumed on the relative boundary; see (2.2).

For an arbitrary critical sequence $\{\Phi_i\} \subset \Pi$, that is, if $\max_{x\in[0,1]} \mathcal{A}^c(\Phi_i(x)) \to \mathbf{L}^c$, we define the *critical set* as the collection of all varifold limits:

$$\mathbf{C}(\{\Phi_i\}) = \Big\{V : V = \lim_{i_j\to\infty} \big|\partial\Phi_{i_j}(x)\big| \text{ as varifolds, where } \mathcal{A}^c\big(\Phi_{i_j}(x_j)\big) = \mathbf{L}^c\Big\}.$$

By a tightening argument adapted from that of Almgren–Pitts, we can homotopically deform $\{\Phi_i\}$ to a new critical sequence **[95, §4]**, denoted still by $\{\Phi_i\}$ by abuse of notation, such that

Lemma 4.2. *Every element in* $\mathbf{C}(\{\Phi_i\})$ *has c-bounded first variation.*

Note that this is the first key novel idea in comparison with the minimal case **[68]**: the $\mathcal{A}^c$-functional is not defined for general varifolds, so we cannot show that every element in $\mathbf{C}(\{\Phi_i\})$ is a stationary point of $\mathcal{A}^c$ as in **[68]**. Nevertheless, having c-bounded first variation provides enough control on elements of $\mathbf{C}(\{\Phi_i\})$ to proceed.

We can then adapt the Almgren–Pitts combinatorial argument to show that at least one element $V \in \mathbf{C}(\{\Phi_i\})$ satisfies an "*almost-minimizing*" property. Heuristically, V is almost-minimizing in an open set $U \subset M$ if it can be approximated by the boundaries of a sequence $\{\Omega_i\} \subset \mathcal{C}(M)$ such that, if we deform Ω_i in U without increasing the $\mathcal{A}^c$-value by δ_i in the process, then we are not allowed to decrease the $\mathcal{A}^c$-value by ε_i at the end. Here $\delta_i, \varepsilon_i \to 0$ as $i \to \infty$. That is, writing the deformation as $\{\Omega_i^t\}_{t\in[0,1]}$,

$$\mathcal{A}^c\big(\Omega_i^t\big) \le \mathcal{A}^c(\Omega_i) + \delta_i, \forall t \in [0,1] \Rightarrow \mathcal{A}^c\big(\Omega_i^1\big) \ge \mathcal{A}^c(\Omega_i) - \varepsilon_i.$$

Using this property, we can construct replacements V^* of V inside any $K \subset\subset U$, satisfying:

Proposition 4.3 (**[95, Proposition 5.8]**). (1) *V^* is the same as V outside K;* (2) $-c\,\mathrm{Vol}(K) \le \|V^*\| - \|V\| \le c\,\mathrm{Vol}(K)$*;* (3) *$V^*$ is the limit of boundaries $\partial\Omega_i^*$ which locally minimize $\mathcal{A}^c$ in the interior of K.*

Note that (2) and (3) form two main differences of the CMC case compared to the minimal case **[68]**. In (2), the mass may change due to the volume term in $\mathcal{A}^c$, but luckily the errors for mass change converge to zero in any blowup procedure. Moreover, in (3) we gain more regularity. In fact, $\partial\Omega_i^*$ are stable CMC hypersurfaces, and hence form a compact family in the smooth topology by curvature estimates, and the limit can still be represented as a boundary due to the one-sided maximum principle satisfied by CMC **[95, Lemma 2.7]**. That is, two embedded CMC hypersurfaces which do not cross each other and have opposite orientations must either be disjoint or touch on at most a codimension-one subset. We point out that in the minimal case, the replacement V^* is smoothly embedded inside K, but may have integer multiplicities. This phenomenon forms the key mechanism for separating minimal sheets in the PMC approximation used in **[94]**.

To obtain the regularity of V, we showed, heuristically, that V coincides with V^*. One key step is to prove that two such replacements V^* and V^{**} glue together as a smooth almost embedded CMC hypersurface along a particularly chosen interface. This amounts to showing that the unit normal vectors match modulo standard regularity theory of elliptic PDEs. Since the CMC equation is not homogeneous, we need to make sure that the orientations of V^* and V^{**} match at a gluing point. Fortunately, this can be justified using the boundary structures. Another challenge is to glue near a self-touching point of V^* and V^{**}. We observed that a blowup of V^* satisfies all the requirements for the existence of good replacements in the minimal case, and hence must be an embedded minimal hypersurface. This fact, together with our particular gluing configurations, implies that all blowups appearing in the gluing procedure are planes. The matching of normal vectors then follows in a standard way. ■

5. MINIMAL SURFACES WITH FREE BOUNDARY AND APPLICATIONS

All the aforementioned results have their counterparts in compact manifolds M with smooth boundary ∂M. The variational problem in $(M, \partial M)$ concerns submanifolds $\Sigma \subset M$ with boundary $\partial\Sigma$ (possibly empty) constrained to lie in ∂M, that is, $\partial\Sigma \subset \partial M$. Critical points of the area functional for this type of variational problems are minimal submanifolds Σ with boundary $\partial\Sigma$ meeting ∂M orthogonally, usually called *minimal submanifolds with free boundary*. Other than earlier works of Gergonne in 1816 and H. A. Schwarz in 1890, Courant was the first mathematician who studied systematically the free boundary problems for minimal surfaces; see [19, CHAPTER VI]. We refer to [48] for a brief historical account of this topic. The study of free boundary minimal surfaces was recently revived by the seminal work of Fraser–Schoen [29], where they revealed a deep connection between extremal Steklov eigenvalue problems and free boundary minimal surface theory in the unit ball. We refer to [47] for a nice survey on this connection and on various constructions of examples in the unit ball.

In this part, we will focus on the codimension-one case, namely *minimal hypersurfaces with free boundary*, abbreviated as FBMHs, in general compact manifolds. We refer to [28,51] for higher-codimensional cases. Parallel to the proof of Yau's conjecture and the development of Morse theory in closed manifolds, we have also witnessed fruitful results in the free boundary setting. In his original proposal, Almgren [2] already included compact manifolds with boundary $(M, \partial M)$. He considered the space of relative cycles $\mathcal{Z}_{\text{rel}}(M, \partial M, G)$, where $G = \mathbb{Z}$ or $\mathbb{Z}_2$. (Those are integral currents or flat chains in M with boundary supported on ∂M.) The min–max procedure was expected to produce smoothly embedded FBMHs in dimensions between 3 and 7. The works by Grüter, Jost [33,41] in the 1980s and by De Lellis–Ramic [24] recently confirmed this regularity result with an additional convexity assumption on ∂M. Without assuming any boundary convexity, Li [46] first attempted this problem in dimension 3, and a general existence and regularity result was later completely established by Li and the author [48] in dimensions between 3 and 7, hence finishing the first step of Almgren's program in the free boundary setting.

A subtle difficulty present in the nonconvex boundary case is the possible touching of the interior of an FBMH with ∂M, usually called the touching phenomenon. In [48], we proved that the min–max varifold is smoothly embedded even if it has nontrivial support on

∂M. That is, the min–max FBMH may touch ∂M along an arbitrary set. This has been further developed by Guang, Li, Wang, and the author [34] to obtain Morse index upper bounds, as well as a generic density result. Very recently, Wang [86] solved the free boundary version of Yau's conjecture based on ideas in [79], thereby proving the existence of infinitely many FBMHs in an arbitrary compact $(M, \partial M)$. With Sun and Wang, we [84] proved the free boundary version of the Multiplicity One Conjecture based on [94]. As an essential tool, we also established the free boundary min–max theory for CMC/PMC hypersurfaces in [84].

Variational theory in ambient manifolds with boundary has potential applications to constructing minimal hypersurfaces in noncompact or singular spaces. The idea is to exhaust those spaces by compact domains with smooth boundary and then take the limit of the free boundary FBMHs constructed therein. In [84], we found one such application to Gaussian spaces, and constructed minimal surfaces in such spaces with arbitrarily large Gaussian area; see also [45]. Note that those minimal surfaces are self-shrinking solutions of the mean curvature flow; see [18]. We hope to see more applications of this idea in the future, for instance, in compact spaces with singularities.

6. FURTHER DISCUSSIONS

We have seen many celebrated results of the min–max theory in closed manifolds with bumpy metrics or with metrics of positive Ricci curvature. However, there still remain many interesting open problems for general metrics, besides those that can be proved by approximations. The solution of Yau's conjecture by Song [79] is almost the only general result about an arbitrary metric in this field. Since contradiction arguments were used in [79] (see also the refined proof in [80] using the notion of saddle point minimal hypersurfaces), it is tempting to find a direct construction of infinitely many closed minimal hypersurfaces by variational methods. This would require a better understanding of multiplicities for nonbumpy metrics. In particular, it would be interesting to know for a general metric whether there exist infinitely many $k \in \mathbb{N}$ such that the min–max minimal hypersurfaces associated with ω_k have multiplicity one. On the other hand, in an ongoing work joint with Wang [87], we exhibit the first nontrivial examples of nonbumpy metrics on $\mathbb{S}^3$ under which the min–max varifolds associated with the second width ω_2 must have multiplicity two. We conjecture that for any $k \in \mathbb{N}$, there exist nonbumpy metrics under which the k-width must be realized by minimal hypersurfaces with higher multiplicity. Upon finishing this survey, we learned that in an ongoing work [82], Stevens and Sun proved a nice dichotomy result for a closed manifold with an arbitrary metric, that is, there exist either closed minimal hypersurfaces with arbitrarily large area, or uncountably many closed minimal hypersurfaces. It would be interesting to know when the second situation happens. It is also natural to ask to what extent equidistribution and scarring of closed minimal hypersurfaces hold for general metrics, where one may first search for a sequence of closed minimal hypersurfaces whose average measures converge to a limit measure with positive density everywhere. Enlightened by the Quantum Unique Ergodicity Conjecture [71], it would be desirable to show that for generic metrics the sequence of min–max minimal hypersurfaces associated with the volume spectrum individually equidistribute (or even just to find a sequence of closed minimal hypersurfaces which individually equidistribute); see [60]. Another interesting question is whether the generic scarring phenomenon can occur surrounding unstable (for instance, index-one) minimal hypersurfaces in general; see [81]. Finally, it would be very interesting to know to what extent the volume spectrum reflects the ambient geometry.

Compared to minimal (hyper)surfaces, the existence theory of CMC (hyper)surfaces, particularly for multiple solutions, is still largely open. For instance, it would be very interesting to know whether the Simon–Smith [78] min–max constructions work for the CMC setting with small prescribed mean curvatures in an arbitrary 3-sphere. If so, the multiplicity-one result for the Simon–Smith min–max could be proved using the ideas in [94], and this will shed light on another famous conjecture of Yau which asserts the existence of four distinct minimal spheres [90, PROBLEM 89]; see also [37, 42]. The existence of multiple closed CMC hypersurfaces with prescribed mean curvature is a very interesting and natural problem (compare with Yau's conjecture on minimal hypersurfaces). Recently, there was some nice progress on this problem presented by Dey [20] and Mazurowski [61] based on [95]. Motivated by a well-known conjecture of Arnold [7, 1981-9] which asserts the existence of at least two distinct closed curves of any prescribed constant geodesic curvature on an arbitrary Riemannian 2-sphere (see [13,97] for more discussions), it is tempting to conjecture that every closed manifold (M^{n+1}, g), $3 \leq n+1 \leq 7$, contains at least two distinct closed CMC hypersurfaces with mean curvature c for any $c > 0$. On a related note, it would be interesting to extend the Rellich conjecture mentioned earlier to higher-dimensional Euclidean spaces using the theory in [95]. Also, since the Euclidean spaces contain closed embedded CMC hypersurfaces of any prescribed curvature, it is natural to conjecture that any asymptotically flat manifold of low dimension contains at least one closed CMC hypersurface for any prescribed curvature. (Note that there is an extensive literature on stable CMC hypersurfaces in those spaces which we do not go into here.) Finally, we note that the equation satisfied by marginally outer-trapped surfaces (MOTS) in general relativity is also of prescribed mean curvature type; see [6]. Even though the MOTS equation is not variational, it is still an interesting question whether one can construct them using a min–max scheme.

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