KÄHLER–RICCI FLOW ON FANO MANIFOLDS

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ABSTRACT

This is an expository paper. We will discuss some recent development in Kähler–Ricci flow on Fano manifolds.

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1. INTRODUCTION

Ricci flow was introduced by Hamilton in the early 1980s [27], and it preserves the Kählerian structure. The Kähler–Ricci (abbreciated as KR) flow is simply the Ricci flow restricted to Kähler metrics. If M is a Fano manifold, that is, a compact Kähler manifold with positive first Chern class $c_1(M) > 0$, we usually consider the following normalized KR-flow:

$$\frac{\partial \omega(t)}{\partial t} = -\operatorname{Ric}\big(\omega(t)\big) + \omega(t), \quad \omega(0) = \omega_0, \tag{1.1}$$

where ω_0 and $\omega(t)$ denote the Kähler forms of a given Kähler metric g_0 and the solutions of Ricci flow with initial metric g_0 in $2\pi c_1(M)$, respectively.[1] Then the flow preserves the Kähler class, i.e., $[\omega(t)] = 2\pi c_1(M)$ for all t. In particular, the flow preserves the volume of $\omega(t)$.

We may write solutions of (1.1) as

$$\omega(t) = \omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0$$

for some Kähler potential $\varphi = \varphi_t$. Let h be a Ricci potential of the background metric ω_0 such that

$$\operatorname{Ric}(\omega_0) - \omega_0 = \sqrt{-1}\partial\bar{\partial}h.$$

In 1985, Cao [11] first reduced (1.1) to solving a parabolic complex Monge–Ampère (MA) equation in the space of Kähler potentials as follows:

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega_0^n} + \varphi - h. \tag{1.2}$$

By using the maximum principle, he proved that (1.2) has a global solution ω_t for all $t \geq 0$. Thus the main interest in (1.1) is to study the limit behavior of $\omega(t)$, as well as φ_t of (1.2). In particular, if φ_t has a smooth limit, $\omega(t)$ will converge to a Kähler–Eintein (KE) metric. Hence, (1.1) also provides an approach to study KE-metrics on a Fano manifold. Compared to the continuity method used by Yau [68] and Aubin [4], Cao's argument also gives a variant proof via KR-flow of the existence of KE-metrics on a compact Kähler manifold with negative or trivial first Chern class.

In the one-dimensional case, i.e., $M = \mathbb{S}^2$, Hamilton proved the convergence of (1.1) to a round sphere under the assumption of positive curvature of ω_0 [28]. Later, Chow removed the Hamilton's condition [18, 19]. But both proofs depend on the uniformization theorem. An independent proof for the convergence of (1.1) on $\mathbb{S}^2$ was given by Chen–Lu–Tian [13]. As a consequence, they gave a proof of the uniformization theorem by using the Ricci flow.

Motivated by the Frankel conjecture, there are many influential works published for KR-flow on $\mathbb{C}P^n$ under the assumption of positive (or nonnegative) bisectional curvature, for instance, see [6, 16, 26, 41], among other references. In particular, Chen–Sun–Tian gave a proof of the Frankel conjecture by employing the Ricci flow [14].

1 For simplicity, we will denote a Kähler metric by its Kähler form thereafter.

Because there are some well-known obstructions for KE-manifolds (cf. [25, 40]), a Fano manifold may not admit a KE-metric, in general. Thus, the solutions φ_t of (1.2) may develop a singularity. It makes the investigation more complicated, when studying the limit behavior of the flow (1.1). In this paper, we will introduce some basic tools, as well as some recent developments of the KR-flow, including Perelman's fundamental estimates for KR-flow, the smooth convergence of KR-flow, the progress on Hamilton–Tian conjecture and the KR-flow on G-manifolds with singular limits.

2. KÄHLER–RICCI SOLITONS

A special class of solutions of (1.1) are related to KR-solitons. A KR-soliton on a Fano manifold M is a pair (X, ω), where X is a holomorphic vector field (HVF) on M and $\omega \in 2\pi c_1(M)$ is a Kähler metric on M such that

$$\mathrm{Ric}(\omega) - \omega = L_X(\omega), \tag{2.1}$$

where L_X denotes the Lie derivative along X. If $X = 0$, the KR-soliton becomes a KE-metric.

In 1985, Bando–Mabuchi proved the following uniqueness result for KE-metrics on a Fano manifold [7].

Theorem 2.1 (Bando–Mabuchi). *For any two KE-metrics ω and ω' on a Fano manifold M, there is a $\sigma \in \mathrm{Aut}(M)$ such that*

$$\omega' = \sigma^* \omega,$$

where $\mathrm{Aut}(M)$ is the group of holomorphism transformations of M.

Bando–Mabuchi's uniqueness theorem was generalized to KR-solitons by Tian and the author in 2000 [58, 59]: a KR-soliton on a compact complex manifold, if it exists, must be unique modulo $\mathrm{Aut}(M)$. Furthermore, X lies in the center of Lie algebra of a reductive part $\mathrm{Aut}_r(M)$ of $\mathrm{Aut}(M)$. We call such an X a soliton HVF, which is also unique modulo $\mathrm{Aut}(M)$. In fact, it is determined by the modified Futaki invariant, regardless of the existence of KR-solitons [59].

An important class of examples of KR-solitons were found in toric manifolds by Wang–Zhu in 2004. They solved (2.1) for a soliton HVF and torus-invariant metrics on a Fano toric manifold by using the technique of real MA-equations.

Let $\sigma_t = \exp\{t\,\mathrm{Re}(X)\}$ be a 1-PS in $\mathrm{Aut}(M)$. Then it is easy to see that the induced metrics by σ_t from a KR-soliton ω,

$$\omega(t) = \sigma_t^* \omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t, \tag{2.2}$$

are solutions of (1.1), as well as φ_t are solutions of (1.2). In particular, a KE-metric is a static solution of (1.1).

Note that φ_t in (2.2) is not uniformly bounded. Thus, we usually study the convergence of (1.1) or (1.2) in the sense of geometric metrics modulo holomorphism or diffeomorphism transformations; see [14, 16, 55, 56, 60, 61, 65, 66, 72, 73], etc.

3. PERELMAN'S ESTIMATES

There is a fundamental estimate for (1.1) established by Perelman in 2003 [43].

Lemma 3.1 (Perelman). *Let h_t be a Ricci potential of ω_t in* (1.1). *Then there are constants $c > 0$ and $C > 0$ depending only on the initial metric ω_0 such that the following are true:*

(1) $\operatorname{diam}(M, \omega_t) \le C$, $\operatorname{vol}(B_r(p), \omega_t) \ge c r^{2n}$;

(2) *For any $t \in (0, \infty)$, there is a constant c_t such that $h_t = -\dot{\phi}_t + c_t$ satisfies*

$$\|h_t\|_{C^0(M)} \le C, \quad \|\nabla h_t\|_{\omega_t} \le C, \quad \|\Delta h_t\|_{C^0(M)} \le C. \tag{3.1}$$

Perelamn's proof of Lemma 3.1 depends on the W-functional and the argument in proving noncollapsing of Ricci flow in the pioneering paper of his solution of the Poincaré conjecture [42]. A detailed proof of Lemma 3.1 can be found in a paper by Sesum–Tian [45]. We note that h_t is a Ricci potential of ω_t. Thus (3.1) means that the Ricci potential is uniformly bounded along the KR-flow (1.1) as well as the scalar curvature.

For Kähler metrics g in $2\pi c_1(M)$ on an n-dimensional Fano manifold M, Perelman's W-functional can be defined with a pair (g, f) by (cf. [62])

$$W(g, f) = (2\pi)^{-n} \int_M \big[R(g) + |\nabla f|^2 + f\big] e^{-f} \omega_g^n, \tag{3.2}$$

where f is a real smooth function normalized by

$$\int_M e^{-f} \omega_g^n = \int_M \omega_g^n = V. \tag{3.3}$$

Then Perelman's entropy $\lambda(g)$ is defined by

$$\lambda(g) = \inf_f \big\{ W(g, f) \mid (g, f) \text{ satisfies (3.3)} \big\}.$$

The number $\lambda(g)$ can be attained by some f (cf. [44]). In fact, such an f is a solution of the equation

$$2\triangle f + f - |Df|^2 + R = \lambda(g). \tag{3.4}$$

In particular, $f = \theta_X$ if $\omega_g = \omega_{KS}$, where θ_X is a potential of soliton HVF X associated to the KR-soliton ω_{KS} [61]. As a consequence, one can further prove that the minimizer of $W(g, \cdot)$ is uniquely associated to g near a KR-soliton (cf. [47]).

The first variation of $\lambda(\omega_g)$ with $\omega_g \in 2\pi c_1(M)$ has been computed [61, 62],

$$\delta\lambda(\omega_g) = -(2\pi)^{-n} \int_M \big\langle \operatorname{Ric}(g) - g + \operatorname{Hess} f, \delta g \big\rangle e^{-f} \omega_g^n.$$

Then it is easy to see that $\lambda(\omega_t)$ is monotonic along the flow (1.1). Thus the smooth limit of ω_t in Cheeger–Gromov topology should be a KR-soliton. In particular, if the curvature of ω_t is uniformly bounded, then by the regularity of Ricci flow [46]. together with the noncollapse property in Lemma 3.1(1), there exists a sequence of ω_t which converges smoothly to a KR-soliton in Cheeger–Gromov topology.

Since the scalar curvature of ω_t is uniformly bounded along (1.1) by Lemma 3.1(2), the monotonicity of the entropy $\lambda(\omega_t)$ implies a uniform log Sobolev inequality associated to ω_t. It was proved by Zhang that this log Sobolev inequality is equivalent to the following Sobolev inequality [70]:

$$\left(\int_M |\psi|^{\frac{2n}{n-1}} \omega_t^n\right)^{\frac{n-1}{n}} \leq C_s\left(\int_M |\nabla\psi|^2_{\omega_t} \omega_t^n + \int_M |\psi|^2_{\omega_t} \omega_t^n\right), \quad \forall \psi \in C^\infty(M), \tag{3.5}$$

where C_s is a uniform constant independent of t.

4. SMOOTH CONVERGENCE

As we know, the smooth limit of KR-flow (1.1) should be a KR-soliton. Thus it is natural to study the convergence of KR-flow on a Fano manifold which admits a KR-soliton. In 2002, Perelman first announced the convergence result on KE-manifolds in his distinguished paper [42] (see the last paragraph in the introduction part). In 2007, Tian and the author gave a proof of Perelman's result with discrete $\mathrm{Aut}(M)$ [60]. The proof is based on an inequality of Moser–Trudinger type established by Tian in his seminal work on KE-mertrics [51]. Then in 2013, we avoided using the Moser–Trudinger inequality and so gave a complete proof of Perelman's result for the convergence of KR-flow on KE-manifolds [61]. In the general case of KR-solitons, we have proved the following convergence result in [56].

Theorem 4.1. *Let (M, ω_{KS}) be a Fano manifold which admits a KR-soliton $\omega_{KS} \in 2\pi c_1(M)$ with respect to an HVF X. Let K_X be a compact 1-PS in* $\mathrm{Aut}(M)$ *generalized by* $\mathrm{Im}(X)$. *Then for any K_X-invariant initial metric $\omega_0 \in 2\pi c_1(M)$, the flow* (1.1) *converges to ω_{KS} exponentially modulo* $\mathrm{Aut}(M)$.

Proof of Theorem 4.1 is reduced to solving the following modified KR-flow equation:

$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}\big(\omega(t)\big) + \omega(t) + L_X\omega(t), \quad \omega(0) = \omega_0. \tag{4.1}$$

Analogous to (1.1), (4.1) is equivalent to

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega_0^n} + \varphi + X(\varphi) + \theta_X - h, \tag{4.2}$$

where θ_X is a potential of X associated to ω_0. We will deform the K_X-invariant initial metric ω_0 from ω_{KS} by a path ω^τ ($\tau \in [0,1]$), for example, $\omega^\tau = \tau\omega_0 + (1-\tau)\omega_{KS}$, to prove the convergence of φ_t in (4.2) for any initial ω^τ.

The first step is to prove the convergence of Kähler potentials φ_t in (4.2) for an initial metric ω_0 very close to ω_{KS}. This is related to the stability problem of KR-flow (4.1). We use a contradiction argument employing the fact of uniqueness of KR-solitons with the help of the regularity of (4.2) on $M \times [-1+t, 1+t]$ for any t. In a subsequent paper [73], the author actually proved that φ_t is convergent to a Kähler potential exponentially (without any holomorphism transformation). Moreover, the K_X-invariance condition for ω_0 can be removed. But we do not know whether the convergence is still with an exponential rate after holomorphism transformation, in general.

By the first step, we see that there is $\tau_0 \leq 1$ such that the flow (4.1) is convergent for the initial metric ω^τ with any $\tau < \tau_0$. It remains to prove that the convergence still holds for ω^{τ_0}. We can also use a contradiction argument. A key estimate is to show that the energy level $L(\omega^{\tau_0})$ of the flow for the initial metric ω^{τ_0} satisfies

$$L(\omega^{\tau_0}) = \lim_{t\to\infty} \lambda(\omega_t^{\tau_o}) = \lambda(\omega_{KS}). \tag{4.3}$$

In fact, we prove

Proposition 4.2. *Suppose that M is a Fano manifold which admits a KR-soliton (ω_{KS}, X). Then for any K_X-invariant initial metric ω of KR-flow (1.1), it holds that*

$$L(\omega) = (2\pi)^{-n}\big(nV - N_X\big(c_1(M)\big)\big), \tag{4.4}$$

where

$$N_X\big(c_1(M)\big) = \int_M \theta_X(\omega) e^{\theta_X(\omega)} \omega^n \tag{4.5}$$

is a holomorphic invariant for HVFs, which is independent of $\omega \in 2\pi c_1(M)$, and $\theta_X(\omega)$ is a potential of X associated to ω with a normalization

$$\int_M e^{\theta_X(\omega)} \omega^n = \int_M \omega^n = V. \tag{4.6}$$

The proof of Proposition 4.2 depends on an estimate of the asymptotic behavior of minimizer f_t in (3.4) for metric ω_t of the flow (1.1) via the Sobolev inequality (3.5) [56, LEMMA 3.3, PROPOSITION 4.2].

Remark 4.3. (1) For a K_X-invariant initial metric ω_0, φ_t in (4.2) is convergent to a Kähler potential exponentially as in the first step (without any holomorphism transformation).

(2) In case of Fano toric manifolds, the author proved the convergence of (1.2) for a torus-invariant initial metric ω_0 after torus transformations by using the technique of the real MA-equation [72]. As a consequence, the result gives an alternative proof of Wang–Zhu's theorem [67] for the existence of KR-solitons on toric manifolds via the Ricci flow.

5. H-INVARIANT

The invariant $N_X(c_1(M))$ in (4.5) can be defined for any $Y \in \eta_r(M)$ as follows (cf. [56]):

$$H(Y) = F_Y(Y) + N_Y\big(c_1(M)\big), \tag{5.1}$$

where

$$F_Y(Y) = \int_M Y(h_\omega - \theta_Y(\omega)) e^{\theta_Y(\omega)} \omega^n$$

and

$$N_Y\big(c_1(M)\big) = \int_M \theta_Y(\omega) e^{\theta_Y(\omega)} \omega^n.$$

Here ω is chosen as a K-invariant metric in $2\pi c_1(M)$ and $\mathrm{Im}(Y)$ generates a compact 1-PS in $\mathrm{Aut}_r(M)$ so that $\theta_Y(\omega)$ is a real potential of HVF Y. Note that $F_X(\cdot)$ is just the modified Futaki-invariant and so $F_X(X) = 0$ [59]. Thus,

$$H(X) = N_X\big(c_1(M)\big).$$

Moreover, $H(Y)$ can be also written as (cf. [56, (5.3)])

$$H(Y) = \int_M \theta_Y(\omega) e^{h_\omega} \omega^n, \tag{5.2}$$

where the Ricci potential h_ω is normalized by

$$\int_M e^{h_\omega} \omega^n = \int_M \omega^n = V$$

and is $\theta_Y(\omega)$ given in (4.6). Thus if we do not care about the normalization of $\theta_Y(\omega)$,

$$H(Y) = \int_M \theta_Y(\omega) e^{h_\omega} \omega^n - V \ln\left[\frac{1}{V}\int_M e^{\theta_Y(\omega)} \omega^n\right], \quad \forall Y \in \eta_r(M). \tag{5.3}$$

In the above formula, $(-H(\cdot))$ is usually called an H-invariant in the literature [9, 22, 29, 30], which can be defined for any special degeneration induced by C^*-actions on M as the generalized Futaki-invariant of Ding–Tian [23].

By calculating the H-invariant for the special degeneration arising from the KR-flow (1.1) with the help of Hamilton–Tian conjecture [51] (also see the next section), Dervan–Székelyhidi recently proved (4.4) for any $\omega \in 2\pi c_1(M)$ [22]. As a consequence, they removed the assumption for a K_X-invariant initial metric ω_0 in Theorem 4.1 as follows.

Theorem 5.1 (Dervan–Székelyhidi). *Let (M, ω_{KS}) be a Fano manifold which admits a KR-soliton ω_{KS}. Then for any initial metric $\omega_0 \in 2\pi c_1(M)$, the flow* (1.1) *converges smoothly to ω_{KS} in the sense of Kähler potentials modulo* $\mathrm{Aut}(M)$.

There are other applications of H-invariant to the uniqueness of limit of KR-flow and the optimal degeneration of Fano manifolds; we refer the reader to recent papers [9, 29, 35, 65].

Remark 5.2. As we see, Dervan–Székelyhid's proof of Theorem 5.1 depends on the Hamilton–Tian conjecture. It would be interesting to give a direct proof without using the conjecture as done for Theorem 4.1 in [56].

6. A NEW APPROACH TO THE HAMILTON–TIAN CONJECTURE

In [51], Tian proposed the following conjecture (a folklore conjecture of Hamilton–Tian (HT-conjecture) [5, 17, 42, 57]):

Any sequence of $(M, \omega(t))$ contains a subsequence converging to a length space $(M_\infty, \omega_\infty)$ in the Gromov–Hausdorff topology and $(M_\infty, \omega_\infty)$ is a smooth KR-soliton outside a closed subset S, called the singular set, of codimension at least 4. *Moreover, this subsequence of $(M, \omega(t))$ converges locally to the regular part of $(M_\infty, \omega_\infty)$ in the Cheeger–Gromov topology.*

The HT-conjecture asserts the existence of a singular limit of (1.1) with local regularity. The above Theorem 4.1 (and Theorem 5.1) confirms the conjecture for the Fano manifold admitting a KR-soliton. In this case, the convergence of flow ω_t is smooth in the sense of Kähler potentials, in particular, the curvature of ω_t is uniformly bounded. However, it has been found in some special Fano manifolds with large symmetric group action that the curvature of ω_t cannot stay uniformly bounded [37] (also see the next section). In other words, there are examples of Fano manifolds on which the KR-flow develops singularities of type II. Thus, in general, there is no smooth limit of a KR-flow.

The Gromov–Hausdorff convergence part in the HT-conjecture follows from Perelman's noncollapse result and Zhang's upper volume estimate [71]. There is significant progress on this conjecture, first by Tian and Zhang in dimension less than 4 [57], then by Chen–Wang [17] and Bamler [5] in higher dimensions. In fact, by using the tools of geometric measure theory, Bamler proved a generalized version of the conjecture for a Ricci flow with uniformly bounded scalar curvature. His result can be also regarded as a version of Ricci flow for Cheeger–Colding compactness theorem [12] for Riemannian metrics with a bounded Ricci curvature.

In this section, we discuss an alternative proof of the HT-conjecture in a joint work with Wang [66]. Precisely, we prove

Theorem 6.1. *For any sequence of (M, ω_t) of (1.1), there is a subsequence $t_i \to \infty$ and a $\mathbb{Q}$-Fano variety $\tilde{M}_\infty$ with klt singularities such that ω_{t_i} is locally C^∞-convergent to a KR-soliton ω_∞ on $\mathrm{Reg}(\tilde{M}_\infty)$ in the Cheeger–Gromov topology. Moreover, ω_∞ can be extended to a singular KR-soliton on $\tilde{M}_\infty$ with a L^∞-bounded Kähler potential ψ_∞ and the completion of $(\mathrm{Reg}(\tilde{M}_\infty), \omega_\infty)$ is isometric to the global limit $(M_\infty, \omega'_\infty)$ of ω_{t_i} in the Gromov–Hausdorff topology. In addition, if ω_∞ is a singular KE-metrics, ψ_∞ is continuous and M_∞ is homeomorphic to $\tilde{M}_\infty$ which has Hausdorff codimension of singularities of $(M_\infty, \omega'_\infty)$ equal to at least 4.*

Compared to the proofs by blowing-up arguments in the two long papers [17] and [5], our proof of Theorem 6.1 is purely analytic by using the technique of the complex MA-equation. In Theorem 6.1, we also obtain a structure of $\mathbb{Q}$-Fano variety with klt singularities for the Gromov–Hausdorff limit in the HT-conjecture.

The proof of Theorem 6.1 is based on a recent result of Liu–Székelyhidi on Tian's partical C^0-estimate for polarized Kähler metrics with Ricci curvature bounded below [39]. In a paper of Zhang [69], it has been observed that Liu–Székelyhidi's result can be applied to prove a partial C^0-estimate for a sequence of Kähler metrics raised from the flow ω_t of (1.1). We note that the HT-conjecture also implies a partial C^0-estimate for the flow (1.1) (cf. [15, 57]). Thus we actually prove that the HT-conjecture and the partial C^0-estimate for the KR-flow are equivalent.

Let (M, L, ω) be a polarized manifold such that ω is a Kähler metric in $2\pi c_1(L)$. Choose a Hermitian metric h on L such that $\mathrm{R}(h) = \omega$. Then for any positive integer l, we have an L^2-inner product on $H^0(M, L^l, \omega)$,

$$(s_1, s_2) = \int_M \langle s_1, s_2\rangle_{h^{\otimes l}}\omega^n, \quad \forall s_1, s_2 \in H^0(M, L^l, \omega). \tag{6.1}$$

Thus for any orthonormal basis $\{s^\alpha\}$ $(0 \le \alpha \le N = N(l))$ of $H^0(M, L^l, \omega)$, we define the Bergman kernel by (cf. [48])

$$\rho_l(M, \omega)(x) = \sum_{i=0}^{N} |s^\alpha|^2_{h^{\otimes l}}(x), \quad \forall x \in M, \tag{6.2}$$

which is independent of choice of the basis $\{s^\alpha\}$.

The following fundamental result was proved by Liu–Székelyhidi [39].

Lemma 6.2 (Liu–Székelyhidi). *Given $n, D, v > 0$, there is a positive integer l and a real number $b > 0$ with the following property: Suppose that (M, L, ω) is a polarized Kähler manifold with $\omega \in 2\pi c_1(L)$ such that*

$$\mathrm{Ric}(\omega) \ge -\omega, \quad \mathrm{vol}(M, \omega) \ge v, \quad \mathrm{diam}(M, \omega) \le D. \tag{6.3}$$

Then for any $x \in M$, one has

$$\rho_l(M, \omega)(x) \ge b. \tag{6.4}$$

An inequality like (6.4) was called a partial C^0-estimate by Tian [49,50,52,53], which plays a critical role in his proof of YTD-conjecture [54]. The upper bound of $\rho_l(M, \omega)$ can be also obtained by using the standard Moser iteration (for example, see [31, Lemma 3.2]).

By (6.4), we can write ω as a metric with bounded Kähler potential using the Fubini–Study metric as the background metric. In fact, if the orthonormal basis $\{s^\alpha\}$ $(0 \le \alpha \le N)$ defines an embedding Φ, then we have

$$\omega = \Phi^*\left(\frac{1}{l}\omega_{FS}\right) - \frac{1}{l}\sqrt{-1}\partial\bar\partial \log \rho_l(M, \omega).$$

By the gradient estimate for s^α (cf. [24,49]) and the lower bound (6.4) for $\rho_l(M, \omega)$, it holds that

$$\Phi^*\left(\frac{1}{l}\omega_{FS}\right) \le C(n, D, v)\omega. \tag{6.5}$$

This is because

$$\Phi^*(\omega_{FS}) = \sqrt{-1}\frac{\sum_{\alpha=0}^N \langle \nabla s^\alpha, \nabla s^\alpha\rangle}{\rho_l(M, \omega)} - \sqrt{-1}\frac{(\sum_{\alpha=0}^N \langle \nabla s^\alpha, s^\alpha\rangle)(\sum_{\alpha=0}^N \langle s^\alpha, \nabla s^\alpha\rangle)}{\rho_l^2(M, \omega)}.$$

As in [69], we modify metric ω_t to η_t so that (6.3) is satisfied by solving the following MA-equation:

$$(\eta_t)^n = (\omega_t + \sqrt{-1}\partial\bar\partial\kappa_t)^n = e^{h_t}\omega_t^n, \quad \sup_M \kappa_t = 0, \tag{6.6}$$

where h_t is a uniformly bounded Ricci potential of ω_t chosen as in Lemma 3.1. By Yau's solution to Calabi's problem [68], (6.6) can be solved, and by Moser iteration (cf. [58]) we have

$$|\kappa_t| \le C\left(\|h_t\|_{C^0(M)}, \omega_0\right) \le A. \tag{6.7}$$

By (6.7), each orthonormal basis of $H^0(M, K_M^{-l}, \omega_t)$ is comparable to any one of $H^0(M, K_M^{-l}, \eta_t)$. Thus by Lemma 6.2, we prove [66, PROPOSITION 2.7],

Proposition 6.3. *Given any sequence ω_{t_i} $(i \to \infty)$ of flow ω_t, there is a subsequence of ω_{t_i}, which is still denoted as ω_{t_i}, such that (6.4) holds for ω_{t_i} and the embedding of M by an orthonormal basis $\{s_{t_i}^\alpha\}$ of $H^0(M, K_M^{-l}, \omega_{t_i})$ in $\mathbb{C}P^N$ converges to a normal variety $\tilde{M}_\infty$.*

Denoting the embedding of $\{s_{t_i}^\alpha\}$ by Φ_i, we have $\tilde{M}_i = \Phi_i(M)$ converging to a normal variety $\tilde{M}_\infty$ by Proposition 6.3. Thus

$$(\Phi_i^{-1})^* \omega_{t_i} = \frac{1}{l}\omega_{FS} + \sqrt{-1}\partial\bar{\partial}\varphi_i, \tag{6.8}$$

where $\phi_i = -\frac{1}{l}(\Phi_i^{-1})^*(\log \rho_l(M, \omega_i))$ satisfies

$$|\varphi_i| \le C. \tag{6.9}$$

Moreover, by the gradient estimate for $s_{t_i}^\alpha$ [31, LEMMA 3.1],

$$\left\| \nabla s_{t_i}^\alpha \right\|_{\omega_{t_i}} \le C_s\big(\|h_{t_i}\|_{C^0(M)}, C_s, n\big) l^{\frac{n}{2}+1}, \tag{6.10}$$

where C_s is the Sobolev constant in (3.5). Thus as in (6.5), we get

$$\frac{1}{l}\omega_{FS}|_{\tilde{M}_i} \le C(\Phi_i^{-1})^* \omega_{t_i}. \tag{6.11}$$

By (6.9) and (6.11), we can derive a local $C^{k,\alpha}$-estimate for ϕ_i via the parabolic equation (1.2). In fact, we may choose exhausting open sets $\Omega_\gamma \subset \tilde{M}_\infty$. Then by Proposition 6.3, there are diffeomorphisms $\Psi_\gamma^i : \Omega_\gamma \to \tilde{M}_i$ such that the curvature of $\omega_{FS}|_{\tilde{\Omega}_\gamma^i}$ is C^k-uniformly bounded independently of i, where $\tilde{\Omega}_\gamma^i = \Psi_\gamma^i(\Omega_\gamma)$. For simplicity, we let $\tilde{\omega}_i = \frac{1}{l}\omega_{FS}|_{\tilde{M}_i}$.

The following key estimate was obtained in [66, LEMMA 3.1].

Lemma 6.4. *There exist constants A, C_γ, A_γ such that*

$$|\varphi_i| \le A \quad \text{in } \tilde{M}_i, \tag{6.12}$$

$$C_\gamma^{-1}\tilde{\omega}_i \le (\Phi_i^{-1})^*\omega_{t_i} \le C_\gamma \tilde{\omega}_i \quad \text{in } \tilde{\Omega}_\gamma^i, \tag{6.13}$$

$$\|\varphi_i\|_{C^{k,\alpha}(\tilde{\Omega}_\gamma^i)} \le A_\gamma. \tag{6.14}$$

The estimates (6.12)–(6.14) in Lemma 6.4 can be extended to Kähler potentials ϕ_i^s of metrics ω_{t_i+s} associated to the background $\hat{\omega}_i$, where $s \in [-1, 1]$ and ϕ_i^s satisfies

$$(\Phi_i^{-1})^*\omega_{t_i} = \hat{\omega}_i + \sqrt{-1}\partial\bar{\partial}\varphi_i^s.$$

By Lemma 6.4, we see that ω_{t_i} (by taking a subsequence) is locally C^∞-convergent to a KR-soliton ω_∞ on $\mathrm{Reg}(\tilde{M}_\infty)$ in the Cheeger–Gromov topology, which can be extended to a singular KR-soliton on $\tilde{M}_\infty$ with a L^∞-bounded Kähler potential ψ_∞ in the sense of full MA-measure [8]. In the case of KE-metrics ω_∞, one can further show that the local limit of η_{t_i} on $\mathrm{Reg}(\tilde{M}_\infty)$ associated to ω_{t_i} in (6.6) is just ω_∞. Thus in this case, we can actually prove that the Gromov–Hausdorff limit $(M_\infty, \omega_\infty')$ is homeomorphic to $\tilde{M}_\infty$ and the Hausdorff codimension of singularities of $(M_\infty, \omega'_\infty)$ is at least 4. The $\mathbb{Q}$-Fano structure of $\tilde{M}_\infty$ with klt singularities can be proved as in [8,31,54].

Remark 6.5. The uniqueness of $\mathbb{Q}$-Fano structure of $\tilde{M}_\infty$ in Theorem 6.1 is independent both of the sequence and initial metric $\omega_0 \in 2\pi c_1(M)$. We refer the reader to recent papers [15,29,65].

7. KR-FLOW ON G-MANIFOLDS

In this section, we discuss the KR-flow on a Fano G-manifold, which develops singularities of type II [37]. In this joint paper, Li, Tian, and the author prove

Theorem 7.1. *Let G be a complex reductive Lie group and (M, J) be a Fano G-manifold. Suppose that (M, J) does not admit any KR-soliton. Then any solution of KR-flow on (M, J) with an initial metric $\omega_0 \in 2\pi c_1(M, J)$ is of type II.*

Here by a G-manifold we mean a *(biequivariant) compactification of G* which admits a holomorphic $G \times G$-action and has an open and dense orbit isomorphic to G as a $G \times G$-homogeneous space [1–3]. Clearly, toric manifolds form a special class of G-manifolds with G being the torus group.

A criterion theorem for the existence of KE-metrics on Fano G-manifolds was established by Delcroix [20] several years ago.

Theorem 7.2 (Delcroix). *Let M be a Fano G-manifold with associated moment polytope P. Let P_+ be the positive part of P defined by a positive roots' system $\Phi_+ = \{\alpha\}$ of G. Then M admits a KE-metric if and only if the barycenter of P_+ with respect to Φ_+ satisfies*

$$\mathrm{bar}(P_+) \in 2\rho + \Xi, \tag{7.1}$$

where Ξ is the relative interior of the cone generated by Φ_+ and $\rho = \frac{1}{2}\sum_{\alpha \in \Phi_+} \alpha$.

Delcroix's proof obtains a prior C^0-estimate for a class of real MA-equations on the positive cone $\mathfrak{a}_+ \subset \mathfrak{a} = \mathbb{R}^r$ (r is the rank of G, i.e., the dimension of a maximal torus in G) defined by Φ_+ as done for toric Fano manifolds in [67]. Later, Li, Zhou, and the author gave another proof of Delcroix's theorem by verifying the properness of K-energy and also generalized the theorem to the case of KR-solitons [38].

By Theorem 7.2, it is possible to classify all Fano G-manifolds which admit a KE-metric or KR-solitons. For example, for the rank $r = 2$, there are two $SO_4(\mathbb{C})$-manifolds and one $Sp_4(\mathbb{C})$-manifold which cannot admit any KR-solitons, see [20,21,37,74]. Thus Theorem 7.1 provides a class of examples of Ricci flow with singularities of type II on Fano manifolds.

By the HT-conjecture and the uniqueness result in [29,65], we may assume that the initial metric ω_0 in Theorem 7.1 is $K \times K$-invariant. Here K is a maximal compact subgroup of G. The proof of Theorem 7.1 includes two main steps by using a contradiction argument under the assumption of uniformly bounded curvature: first, proving that the

Cheeger–Gromov limit (M_∞, J_∞) of any sequence of $K \times K$-invariant metrics on (M, J) is still a G-manifold; second, showing that the complex structure J_∞ of limit will not jump from J. It is useful to mention that the above two steps work for any sequence of $K \times K$-invariant metrics in $2\pi c_1(M, J)$ and can be also generalized to any sequence of $K \times K$-invariant metrics with uniformly bounded curvature on a polarized G-manifold. For example, as an application one can establish an analogue of Theorem 7.1 for Calabi's flows [10] with singularities of type II on polarized G-manifolds.

7.1. A direct proof of Theorem 7.1 for a sequence of ω_t

In the following, we present a more direct proof of Theorem 7.1 from a paper jointly with Tian [63]. We turn to prove

Theorem 7.3. *Let ω_i $(= \omega_{t_i}, t_i \to \infty)$ be a sequence of (M, ω_t, J) in the flow of (1.1) with a $K \times K$-invariant initial metric $\omega_0 \in 2\pi c_1(M, J)$. Suppose that the curvature of ω_i is uniformly bounded. Then ω_i converges to a KR-soliton in the sense of Kähler potentials on (M, J). In particular, (M, J) admits a KR-soliton.*

We need to recall some notation and facts proved in [37]. Let $\{E_1, \dots, E_n\}$ be a basis of the Lie algebra $\mathfrak{g}$. Then the right (left) action of G induces a space of $\mathrm{span}\{e_1, \dots, e_n\}$ of HVFs with $\mathrm{Im}(e_a) \in \mathfrak{k}$ on M, where $\mathfrak{k}$ is the Lie algebra of K. By the partial C^0-estimate as in Section 6, there is a sequence of Kodaira embeddings $\Phi_i : M \to \mathbb{C}P^N$ induced by an orthonormal basis $\{s_\alpha^i\}$ in $H^0(K_M^{-m}, \omega_i)$ such that the image $\Phi_i(M) = \hat{M}_i$ converges to the image $\Phi_\infty(M_\infty) = \hat{M}_\infty$ in the topology of complex submanifolds, where $\Phi_\infty : M_\infty \to \mathbb{C}P^N$ is the Kodaira embedding induced by an orthonormal basis $\{s_\alpha^\infty\}$ in $H^0(K_{M_\infty}^{-m}, \omega_\infty)$.

Let $\mathrm{span}\{\hat{e}_1^i, \dots, \hat{e}_n^i\}$ be a space of HVFs on $\hat{M}_i$ induced by Φ_i. It has been proved in [37, (4.6)] that for each a, $(\hat{e}_a^i, \hat{\omega}_i)$ converges to an HVF $(\hat{e}_\alpha^\infty, \hat{\omega}_\infty)$ on $\hat{M}_\infty$ in the sense of [37, DEFINITION 3.1], where $\hat{\omega}_i = \frac{1}{m}\omega_{FS}|_{\hat{M}_i}$ and $\hat{\omega}_\infty = \frac{1}{m}\omega_{FS}|_{\hat{M}_\infty}$. Moreover, the basis $\{\hat{e}_1^\infty, \dots, \hat{e}_n^\infty\}$ induces an effective G-action on $\hat{M}_\infty$.

Let $T^{\mathbb{C}}$ be a torus subgroup of G acting on $\hat{M}_\infty$ with a basis $\{X_1, \dots, X_r\}$ of $\mathfrak{a} = J\mathfrak{k} \cap \mathfrak{t}^{\mathbb{C}}$. Then it can be regarded as a subgroup of the maximal torus group $\tilde{T}^{\mathbb{C}}$ in $\mathbb{C}P^N$. Let $\tilde{W}_1, \dots, \tilde{W}_{N+1}$ be the $N+1$ hyperplanes in $\mathbb{C}P^N$ where $\tilde{T}^{\mathbb{C}}$ does not act freely. Thus for any induced HVF $\tilde{X}$ of $X \in \mathfrak{t}^{\mathbb{C}}$ on $\hat{M}_\infty$, one has

$$\{\hat{x} \in \hat{M}_\infty \mid \tilde{X}(\hat{x}) = 0\} \subset \bigcup_\alpha \tilde{W}_\alpha. \tag{7.2}$$

Let $\mathcal{O}$ denote an open dense G-orbit in M. Since M has finitely many $G \times G$-orbits [1, 2], there are basis points $x_\delta \in M \setminus \mathcal{O}$, $\delta = 1, \dots, k$, such that

$$M = \mathcal{O} \bigcup_\delta (G \times G)x_\delta. \tag{7.3}$$

Note that the closure of each $G \times G$-orbit $(G \times G)x_\delta$ is a smooth algebraic variety whose dimension is less than n. Then, up to a subsequence, the closure of $\Phi_i((G \times G)x_\delta)$ converges to an algebraic limit in $\mathbb{C}P^N$. As a consequence, $\Phi_i(M\backslash\mathcal{O})$ has an algebraic limit $D\hat{M}_\infty$ in $\hat{M}_\infty \subset \mathbb{C}P^N$.

Let $\hat{\mathcal{O}}_\infty = \hat{M}_\infty \setminus D\hat{M}_\infty$ be an open set in $\hat{M}_\infty$. We set

$$\hat{\mathcal{O}}^0_\infty = \hat{\mathcal{O}}_\infty \setminus \left(\bigcup_\alpha \tilde{W}_\alpha\right) \quad \text{and} \quad \mathcal{O}^0_\infty = \Phi^{-1}_\infty(\hat{\mathcal{O}}^0_\infty) \subset M_\infty. \tag{7.4}$$

Note that

$$e^\infty_a = (\Phi^{-1}_\infty)_* \hat{e}^\infty_a, \quad a = 1, \dots, n.$$

Thus

$$\bar{X}(x_\infty) \neq 0, \quad \forall x_\infty \in \mathcal{O}^0_\infty, \quad \bar{X} \in \operatorname{span}\{e^\infty_1, \dots, e^\infty_r\}. \tag{7.5}$$

Fix a point $\hat{x}_\infty \in \hat{\mathcal{O}}_\infty$. We choose $\hat{x}_i \in \hat{M} \to \hat{x}_\infty$ and let $x_i = \Phi_i^{-1}(\hat{x}_i) \in \mathcal{O}_\infty$. Then $x_i \to x_\infty \in M_\infty$ in the Gromov–Hausdorff topology. Let $x_i' = (x_i^1, \dots, x_i^r) \in \mathfrak{a}$ be the real part of local partial coordinates z_i in [37, SECTION 2.1]. Without loss of generality, we may assume $x_i' \in \mathfrak{a}_+$ since the metric ω_i is $K \times K$-invariant. By the argument in the proof of [37, LEMMA 4.4], we have actually proved the following key lemma.

Lemma 7.4. *Suppose that the center of $\mathfrak{g}$ is zero. Then there is an absolute constant A such that*

$$\left|x_i'\right|^2 = \left|x_i^1\right|^2 + \cdots + \left|x_i^r\right|^2 \leq A. \tag{7.6}$$

Proof of Theorem 7.3. Let ψ, ψ^i be Weyl-invariant convex functions on $\mathfrak{a}$ associated to the background metric $\omega_0 = \sqrt{-1}\partial\bar{\partial}\psi$ and $K \times K$-invariant metrics ω_i, respectively. It suffices to prove that

$$|\varphi_i| = \left|\psi^i - \psi\right| \leq C \tag{7.7}$$

for some absolute constant C [56].

We first consider Case 1: G is semisimple. Then by Lemma 7.4, (7.6) holds. Thus, by [37, (4.18)], there is a small Euclidean ball B_ε in $\mathfrak{a}$ such that

$$\det(\psi^i_{ab})(x') \geq \delta_0, \quad \forall x' \in B_\varepsilon \cap \mathfrak{a}_+. \tag{7.8}$$

Moreover, as in the proof of [37, (4.23)], there is an open set $B' \subset B_\varepsilon \cap \mathfrak{a}_+$ such that for any $\alpha \in \Phi_+$, we have

$$\left\langle \alpha, \nabla\psi^i(x')\right\rangle \geq c_0, \quad \forall x' \in B'. \tag{7.9}$$

Thus, by the metric matrix (2.3) in [37, LEMMA 2.1], we get

$$a_0\omega \leq \omega_i \leq \frac{1}{a_0}\omega \quad \text{in } \Delta'_\varepsilon, \tag{7.10}$$

where a_0 is a small absolute constant and

$$\Delta'_\varepsilon = \left\{z \in \Delta_\varepsilon \mid x'_z \in B'\right\} \tag{7.11}$$

is an open set of $\Delta_\varepsilon = \{z = (z^1, \dots, z^n) \mid |z^l - x_i^l| < \varepsilon\}$.

We claim that there are a point $z_0^i \in \Delta'_\varepsilon$ and an absolute constant C_1 such that

$$\varphi_i(z_0^i) \geq \sup_M \varphi_i - C_1 \quad \text{and} \quad -\varphi_i(z_0^i) \geq \sup_M(-\varphi_i) - C_1. \tag{7.12}$$

By the Green formula, we have

$$\sup_M \varphi_i \leq \frac{1}{V}\int_M \varphi_i \omega_i^n + C_2. \tag{7.13}$$

On the other hand, by the Sobolev inequality (3.5) and the Green function estimate [7], there is an absolute constant A_1 such that the Green function $G_t(\cdot,\cdot)$ associated to ω_t satisfies

$$\int_M G_t(x,\cdot)\omega_t^n = 0 \quad \text{and} \quad G_t(x,\cdot) \geq A_1.$$

Thus as in (7.13), we also have

$$\sup_M(-\varphi_i) \leq \frac{1}{V}\int_M (-\varphi_i)\omega_i^n + C_3, \tag{7.14}$$

where C_3 is an absolute constant.

For any small number $\delta \ll 1$, we let $M_\delta = \frac{C_2 V}{\delta}$ and $M'_\delta = \frac{C_3 V}{\delta}$. Set

$$E_\delta = \left\{z \in \Delta'_\varepsilon \mid \varphi_i(z) \leq \sup_M \varphi_i - M_\delta\right\}$$

and

$$E_\delta^i = \left\{z \in \Delta'_\varepsilon \mid \big(-\varphi_i(z)\big) \leq \sup_M(-\varphi_i) - M'_\delta\right\}.$$

Then, by (7.13) and (7.14), it is easy to see that

$$\operatorname{meas}_\omega(E_\delta) \leq \delta \quad \text{and} \quad \operatorname{mess}_{\omega_i}(E_\delta^i) \leq \delta.$$

By (7.10), it follows that

$$\operatorname{meas}_\omega(E_\delta),\ \operatorname{meas}_\omega(E_\delta^i) \to 0 \quad \text{as } \delta \to 0.$$

Note that $\operatorname{meas}_\omega(\Delta'_\varepsilon)$ is strictly positive. Hence (7.12) must be true.

By (7.12), we get

$$\operatorname{osc}_M \varphi_i \leq \sup_M \varphi_i + \sup_M(-\varphi_i) + C \leq C_4.$$

Recall that $\omega_s = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_s$ satisfies the following complex MA-equation:

$$\frac{\partial \varphi_s}{\partial s} = \log\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_s)^n}{\omega_0^n} + \varphi_s - h, \quad s \in [-1+t_i, t_i+1]. \tag{7.15}$$

Then by Lemma 3.1, we may assume that φ_s satisfies (cf. [60])

$$|\varphi_s| \leq C \quad \text{and} \quad \left|\frac{\partial \varphi_s}{\partial s}\right| \leq C, \quad \forall s \in [-1+t_i, t_i+1].$$

Thus by the regularity, we get

$$\|\varphi_i\|_{C^{k,\alpha}} \leq C. \tag{7.16}$$

As a consequence, ω_i converges to a KR-soliton on (M, J) [42,60].

Next we deal with the general Case 2: The center $\mathfrak{g}_c$ of $\mathfrak{g}$ is not zero. Let $\mathfrak{a}_c \subseteq \mathfrak{a}$ be the real part of $\mathfrak{g}_c$. By the above case 1 and the argument in the proof of [37, LEMMA 4.4] (Case 1 on page 14), we may assume that $|x_i'| \to \infty$ as $i \to \infty$ and

$$\langle \alpha, x_i' \rangle \leq A, \quad \forall \alpha \in \Phi_+, \tag{7.17}$$

for some absolute constant A, where Φ_+ is a positive roots' system. Write x_i' as

$$x_i' = x_i^0 + x_i'',$$

where $x_i^0 \in \mathfrak{a}_c$. Then

$$\langle \alpha, x_i' \rangle = \langle \alpha, x_i'' \rangle \leq A. \tag{7.18}$$

Let

$$\tilde{\psi}^i(x) = \psi^i(x + x_i^0).$$

Then $\tilde{\psi}^i(x)$ is still a Weyl-invariant convex function on $\mathfrak{a}$ and it still satisfies equation (7.15). Moreover, by (7.18), x_i'' is uniformly bounded if $\mathfrak{a}_c \subsetneq \mathfrak{a}$. Thus we can argue as in Case 1 for $\tilde{\psi}^i$ such that (7.16) holds, while the ε-cube Δ_ε of dimension n centered at x_i in (7.11) is replaced by

$$\tilde{\Delta}_\varepsilon = \{ z = (z^1, \dots, z^n) \mid |z^l - \tilde{x}_i^l| < \varepsilon \}$$

in the local coordinates $\{z^l\}_{l=1,\dots,n}$, where $x_i'' = (\tilde{x}_i^l, \dots, \tilde{x}_i^l)$. As a consequence,

$$\omega_i = \tilde{\omega}_i = \sqrt{-1}\partial\bar{\partial}\tilde{\psi}^i$$

converges to a KR-soliton in the sense of Kähler potentials on (M, J). The proof is complete. ■

7.2. Examples by Li–Li

By Theorems 7.1 and 6.1 for the HT-conjecture, the limit of KR-flow should be a singular KR-soliton on a $\mathbb{Q}$-Fano variety $\tilde{M}_\infty$ if a Fano G-manifold M does not admit a KR-soliton. It is interesting to study the degenerate structure of $\tilde{M}_\infty$ from M. Recently, Y. Li and Z. Li classified $\tilde{M}_\infty$ for the case of $G = \mathrm{SO}_4(\mathbb{C})$ in [35].

It is known that there are three possible Fano compactifications of $\mathrm{SO}_4(\mathbb{C})$ of dimension 6 (cf. [21]). By Theorem 7.2, one of the compactifications admits a KE-metric and the other two cannot admit any KE-metric (cf. [74]). Since $\mathrm{SO}_4(\mathbb{C})$ is semisimple, the latter two also cannot admit any KR-soliton. Thus by Theorem 7.1, the limit $\tilde{M}_\infty$ of a flow on these two manifolds should be a singular Q-Fano variety.

By studying the minimizer of H-invariant (see Section 5) for $G \times G$-equivariant special degenerations on a Fano G-manifold, Y. Li and Z. Li proved that the minimizer can be attained by a $G \times G$-equivalent special degeneration with a center fiber of $G \times G$-spherical variety [35]. In the case of $G = \mathrm{SO}_4(\mathbb{C})$, they further showed that the two $G \times G$-spherical varieties corresponding to the two non-KE-manifolds above are both relatively modified K-polystable. By Han–Li's uniqueness result for the minimizers of H-invariant [29] and the fact that the Perelman's entropy (see Section 3) attains the maximum along the KR-flow [15, 22, 65], these two spherical varieties should be limits $\tilde{M}_\infty$ of KR-flows.

Remark 7.5. (1) Examples in [36] show that the singular limit $\tilde{M}_\infty$ of KR-flow on a G-manifold cannot be a $\mathbb{Q}$-Fano variety of G-compactification, in general. We expect it is a $G \times G$-spherical variety as in the above case of $G = \mathrm{SO}_4(\mathbb{C})$.

(2) To the best of the author's knowledge, there is no known example of Fano manifold M with discrete $\mathrm{Aut}(M)$ on which the solution of a KR-flow is of type II. In fact, we do not know whether there is a Fano manifold with discrete $\mathrm{Aut}(M)$ on which the limit of a KR-flow is a singular KE-metric. In the latter, M must be K-semistable (cf. [34, 65]).

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