STABLE HOMOTOPY GROUPS OF SPHERES AND MOTIVIC HOMOTOPY THEORY

DANIEL C. ISAKSEN, GUOZHEN WANG, AND ZHOULI XU

Dedicated to Mark Mahowald

ABSTRACT

We consider the problem of computing the stable homotopy groups of spheres, including applications and history. We describe a new technique that yields streamlined computations through dimension 61 and gives new computations through dimension 90 with very few exceptions. We discuss questions and conjectures for further study, including a new approach to the computation of motivic stable homotopy groups over arbitrary base fields. We provide complete charts for the Adams spectral sequence through dimension 90.

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1. INTRODUCTION

The computation of stable homotopy groups of spheres is one of the most fundamental and important problems in topology. It has connections to many topics in topology, such as the cobordism theory of framed manifolds, the classification of smooth structures on spheres, obstruction theory, the theory of topological modular forms, algebraic K-theory, and equivariant homotopy theory.

Consider the set of homotopy classes of continuous based maps $S^{n+k} \to S^k$ between spheres of dimensions $n+k$ and k. This set admits a natural group structure. By the Freudenthal Suspension Theorem [12], this group only depends on k when $n > k+1$. This group is called the nth stable homotopy group of spheres, or the nth stable stem, and is denoted by π_n. If $n < 0$, then π_n is the zero group. Moreover, π_0 is isomorphic to the group of integers. Serre's finiteness theorem [40] tells us that π_n is a finite abelian group for $n > 0$.

Despite their simple definition, which was available 80 years ago, the stable homotopy groups are notoriously hard to compute. All known methods only give a complete answer through a range, and then reach an obstacle that can only be surmounted by the introduction of a new method. **Mahowald's Uncertainty Principles** attempt to quantify the inherent difficulty of the problem. Despite its difficulty, many mathematicians have made significant progress. We will briefly review the history and Mahowald's Uncertainty Principles in Section 3.

Recently, the authors have developed a new method [14] using motivic homotopy theory. Using this new method, we have already greatly improved our knowledge of stable stems [19,20], and we have ongoing computations into even higher dimensions. Our method is currently the most effective and less prone to human error, partly due to the fact that it relies more heavily on machine computation than previous methods.

The original purpose of motivic homotopy theory was to apply abstract homotopy theory to problems in number theory and algebraic geometry. In contrast, our work has reversed the information flow and applied motivic homotopy theory to discover new phenomena in classical topology.

2. SMOOTH STRUCTURES ON SPHERES

The work of Kervaire and Milnor [22] on the classification of smooth structures on spheres in dimensions at least 5 is an important example of an application of stable stem computations. Let Θ_n be the group of h-cobordism classes of homotopy n-spheres. This group classifies the differential structures on S^n for $n \geq 5$. Kervaire and Milnor [22] reduced the computation of the group Θ_n to the computation of the stable homotopy group π_n and the Kervaire invariant problem. The latter was resolved by Hill, Hopkins, and Ravenel [16] in all dimensions except for 126. In particular, Kervaire and Milnor observed that the spheres in dimensions 5, 6, and 12 have unique smooth structures.

We restate the following conjecture from [47], which is based on the current knowledge of stable stems and a problem proposed by Milnor [28].

Conjecture 2.1. *In dimensions greater than 4, the only spheres with unique smooth structures are S^5, S^6, S^{12}, S^{56}, and S^{61}.*

Uniqueness in dimension 56 is due to the first author [18], and uniqueness in dimension 61 is due to the second and third authors [47].

Conjecture 2.1 is equivalent to the claim that the group Θ_n is not of order 1 for dimensions greater than 61. This conjecture has been confirmed in all odd dimensions by the second and the third authors [47] based on the work of Hill, Hopkins, and Ravenel [16].

Theorem 2.2 ([47, COROLLARY 1.13]). *The only odd-dimensional spheres with unique smooth structures are S^1, S^3, S^5, and S^{61}.*

For even dimensions, Conjecture 2.1 has been confirmed for over half of all even dimensions by Behrens, Hill, Hopkins, Mahowald, and Quigley [6,7].

3. HISTORY AND MAHOWALD'S UNCERTAINTY PRINCIPLES

We review the history of computing stable stems. See [46] for a survey of classical methods and also Section 2 of [47].

After the geometric computation of the first three stems, Serre [39] computed π_n for $n \leq 8$ using the cohomology of Eilenberg–MacLane spaces and the Serre spectral sequence. Using the EHP sequence and higher compositions such as Toda brackets, Toda [41] computed a large range of unstable homotopy groups of spheres and obtained π_n for $n \leq 19$.

Since π_n is finite abelian, it can be reconstructed from its p-primary components for each prime p. History has demonstrated the effectiveness of this approach. The standard approach to computing stable stems at each prime is to use Adams-type spectral sequences that converge from algebra to homotopy. To identify the algebraic E_2-pages, one needs auxiliary algebraic spectral sequences that converge from simpler algebra to more complicated algebra. For any spectral sequence, difficulties arise in computing differentials and in solving extension problems. Typically, a variety of complementary methods are required to compute a spectral sequence. One method may compute some types of differentials and extension problems efficiently but leave other types unanswered. To obtain complete computations, one must be *eclectic*, applying and combining different methodologies. Even so, combining all known methods, there are eventually some problems that cannot currently be solved.

In fact, we have the following principle, first named by Ravenel [15].

The First Mahowald Uncertainty Principle. Any spectral sequence converging to the homotopy groups of spheres with an E_2-page that can be named using homological algebra will be infinitely far from the actual answer.

The first principle essentially says that the computation of stable stems is not an algebraic problem—there are infinitely many nonzero differentials that must be resolved in such a spectral sequence. Based on experience of learning from Mark Mahowald, the third author [48] named the second principle:

The Second Mahowald Uncertainty Principle. Any method that computes nontrivial differentials in such a spectral sequence will leave infinitely many differentials undecided.

At odd primes, the state-of-the-art is computed by Ravenel in [36] with the Adams–Novikov spectral sequence [33] and the chromatic spectral sequence, which are based on complex cobordism and formal groups. As the prime grows, so does the range of computation, since the spectral sequences become sparser. For example, for $p = 3$ and $p = 5$, we have complete knowledge up to around 100 and 1000 stems, respectively [36]. These ranges are both approximately equal to $p^3(2p - 2)$.

At the prime 2, the classical Adams spectral sequence [1] is still the most efficient method. May [26] constructed the May spectral sequence at all primes, which converges to the E_2-page of the Adams spectral sequence, and he computed π_n for $n \leq 28$. Using higher structure such as the interactions between Massey products and Toda brackets, Mahowald (with Barratt, Bruner, Jones, and Tangora) [3, 4, 8, 25] computed π_n for $n \leq 47$. We also mention [23, 24], which take an entirely different approach. However, the computations in [23, 24] are now known to contain several errors.

More recently, in 2014, the first author [18] gave a thorough accounting of the Adams spectral sequence up to dimension 59 with the exception of only one differential, but then he reached an obstacle as predicted by Mahowald's Uncertainty Principles. The new idea was to compare classical computations with the motivic Adams spectral sequence. The exception was later proved by the third author [21] based on the first author's computations.

In 2016, with tremendous efforts, the second and third authors [47] bypassed the above obstacle by computing two more stable stems using the $\mathbb{R}\mathrm{P}^\infty$-method. In particular, the second and third authors proved that π_{61} is the zero group; Theorem 2.2 is a consequence. The $\mathbb{R}\mathrm{P}^\infty$-method is useful for finding specific, particularly difficult, Adams differentials and is not designed to study all differentials systematically.

A major breakthrough occurred in 2017 and the next few years. A new method [14] allowed the authors [19, 20] to recompute most Adams differentials up to dimension 61 very easily and to extend computations to dimension 90 with only a few exceptions. For example, the hardest differential $d_3(D_3) = B_3$ proved in [47] is now an immediate consequence of this new method; it comes immediately from the output of a computer program. Our new method is discussed in the next section.

Further computations into higher dimensions are still ongoing. We have not yet reached an insurmountable obstacle that will require a new method to resolve.

4. MOTIVIC HOMOTOPY THEORY AND ALGEBRAICITY OF THE COFIBER OF τ

Morel and Voevodsky [30, 31] developed motivic homotopy theory in the mid-1990s in order to import homotopical techniques into algebraic geometry. This program found great success in Voevodsky's resolutions of the Milnor Conjecture [42] and the Bloch–Kato Conjecture [43].

There is a cellular subcategory of the motivic stable homotopy category that is generated by two types of spheres: the simplicial sphere $S^{1,0}$, and the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 - 0$, denoted by $S^{1,1}$. After p-completion, there is a stable map $\tau : S^{0,-1} \to S^{0,0}$ over $\mathbb{C}$ that induces a nonzero map on mod p motivic homology. We denote by $S^{0,0}/\tau$ the cofiber of τ.

One may view the p-completed $\mathbb{C}$-motivic stable homotopy category as a deformation of the p-completed classical stable homotopy category, with this element τ as a parameter. Dugger and the first author [11] identified the generic fiber "$\tau = 1$" with the p-completed classical stable homotopy category. Gheorghe and the second and third authors [14] identified the special fiber "$\tau = 0$" with a purely algebraic category.

Theorem 4.1 ([14]). *At each prime p, there is an equivalence*

$$S^{0,0}/\tau\text{-}\mathbf{Mod} \simeq \mathcal{D}(\mathrm{BP}_{2*}\mathrm{BP}\text{-}\mathbf{Comod})$$

of stable ∞-categories, equipped with t-structures, between the category of cellular module spectra over $S^{0,0}/\tau$ and Hovey's [17] derived category of $\mathrm{BP}_{2*}\mathrm{BP}$-comodules.

The right-hand side is also known as the derived category of quasicoherent sheaves on the moduli stack of formal groups over $\mathbb{Z}_p$-algebras, which is foundational to chromatic homotopy theory [29, 35].

The first author [18] observed that the homotopy groups of $S^{0,0}/\tau$ are isomorphic to the classical Adams–Novikov E_2-page $\mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$. In 2017, the second author [45] made a computer program that computes the algebraic Novikov spectral sequence, which converges to the Adams–Novikov E_2-page, in a large range. The computer data aligned with the motivic Adams spectral sequence for $S^{0,0}/\tau$ obtained by the first author. This discovery motivated the following theorem by Gheorghe and the second and third authors [14], which is crucial to the computation of classical and $\mathbb{C}$-motivic stable homotopy groups of spheres.

Theorem 4.2 ([14]). *The tri-graded motivic Adams spectral sequence for $S^{0,0}/\tau$ is isomorphic to the algebraic Novikov spectral sequence for BP_{2*} [27, 33]:*

$$\begin{tikzcd}
\mathrm{Ext}^{a,2w-s+a,w}_{A^{\mathrm{mot}}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \arrow[r, "\cong"] \arrow[d, Rightarrow, "\textbf{Motivic Adams SS}"'] & \mathrm{Ext}^{s,2w}_{\mathrm{BP}_{2*}\mathrm{BP}/I}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) \arrow[d, Rightarrow, "\textbf{Algebraic Novikov SS}"'] \\
\pi_{2w-s,w}(S^{0,0}/\tau) \arrow[r, "\cong"] & \mathrm{Ext}^{s,2w}_{\mathrm{BP}_{2*}\mathrm{BP}}(\mathrm{BP}_{2*}, \mathrm{BP}_{2*}).
\end{tikzcd}$$

Here A^{mot} is the motivic Steenrod algebra, and $\mathbb{F}_p[\tau]$ is the mod p motivic cohomology of $S^{0,0}$.

There is a Betti realization functor $\mathbf{Re}$ from the motivic stable homotopy category over $\mathbb{C}$ to the classical stable homotopy category, which extends the functor that sends a complex algebraic variety to its $\mathbb{C}$-points. We have $\mathbf{Re}(S^{n,w}) \simeq S^n$ and $\mathbf{Re}(\tau) = 1$.

The naturality of Adams spectral sequences then gives us a zigzag diagram

$$\begin{tikzcd}
\mathrm{Ext}_A^{s,t}(\mathbb{F}_p,\mathbb{F}_p) \arrow[d, Rightarrow, "\text{Adams SS}"] & \mathrm{Ext}_{A^{\mathrm{mot}}}^{s,t,w}(\mathbb{F}_p[\tau],\mathbb{F}_p[\tau]) \arrow[l, "\mathrm{Re}"'] \arrow[r] \arrow[d, Rightarrow, "\text{Motivic Adams SS}"] & \mathrm{Ext}_{A^{\mathrm{mot}}}^{s,t,w}(\mathbb{F}_p[\tau],\mathbb{F}_p) \arrow[d, Rightarrow, "\text{Motivic Adams SS}"] \\
\pi_{t-s}S^0 & \pi_{t-s,w}S^{0,0} \arrow[l, "\mathrm{Re}"'] \arrow[r] & \pi_{t-s,w}S^{0,0}/\tau
\end{tikzcd}$$

of spectral sequences. Here the left map is given by the Betti realization functor, and the right map is induced by the quotient map $S^{0,0} \to S^{0,0}/\tau$. This diagram of spectral sequences is very powerful. The differentials on the right side are purely algebraic by Theorem 4.2 and can be obtained by the output of a computer program!

In fact, this method obtains all differentials up to the 45-stem with essentially only one exception. Consistent with the Second Mahowald Uncertainty Principle, more and more exceptions occur in higher dimensions. See Appendix A of [14] for more details.

In practice, our method can be summarized in the following steps:

(1) Compute the $\mathbb{C}$-motivic Adams E_2-page with a machine in a large range.

(2) Compute the algebraic Novikov spectral sequence with a machine in a large range, including all differentials and multiplicative structure, and use Theorem 4.2 to identify it with the motivic Adams spectral sequence for $S^{0,0}/\tau$.

(3) Use the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \to S^{0,0}/\tau \to S^{1,-1}$$

and naturality of Adams spectral sequences to pull back and push forward Adams differentials for $S^{0,0}/\tau$ to Adams differentials for the motivic sphere.

(4) Apply a variety of ad hoc arguments to deduce additional Adams differentials for the motivic sphere.

(5) Invert τ to obtain the classical Adams spectral sequence and the classical stable homotopy groups.

The machine-generated data that we use in steps (1) and (2) are available at [44].

5. RESULTS AND ADAMS CHARTS

Our computational results of the classical Adams spectral sequence are best summarized in charts, which we include at the end of this article. The charts are displayed in pieces so that they fit onto individual pages. For tables that describe the stable homotopy groups π_n for $n \leq 90$, see [19, 20].

The first eight charts (Figures 1–8) represent the Adams E_2-page. The dimension is on the horizontal axis, and Adams filtration is on the vertical axis. Each dot represents a copy of $\mathbb{F}_2$. Dark gray vertical lines and lines of slope 1 and 1/3 represent multiplications by h_0, h_1, and h_2, respectively. Light gray lines of slope $-r$ represent Adams d_r differentials.

Nearly all of the differentials through dimension 90 have been computed. The only exceptions are that $d_9(x_{85,6} + h_0^3 c_3)$ or $d_{10}(h_1 f_2)$ might equal $M\Delta h_1 d_0$ in the 84-stem.

The last three charts (Figures 9–11) represent the Adams E_∞-page. The dark gray lines represent a multiplicative structure that is inherited from the E_2-page. Light gray lines represent a multiplicative structure that is hidden by the Adams filtration. Beyond the 70-stem, there remain some unresolved ν extensions that are not shown on the chart. Beyond the 80-stem, there remain unresolved 2 extensions and η extensions that are not shown.

The 2-primary part of π_n can be read from this chart. The vertical column in dimension n represents the associated graded object of the Adams filtration of π_n. The presence of k dots in the nth column means that π_n has order 2^k.

The vertical lines determine the group structure of π_n. Each vertical line represents a nontrivial extension of abelian groups. Therefore, a sequence of m dots connected by vertical lines represents a copy of $\mathbb{Z}/2^m$ inside of π_n. For example, the 2-primary part of π_{23} is $\mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/16$.

In stems beyond 30, a regular pattern emerges along the top of the E_∞-page that is distinct from the much more complicated and irregular pattern below. This regular pattern represents the v_1-periodic part. We omit this pattern starting from the high 40s.

6. DEFORMATIONS OF STABLE HOMOTOPY THEORY

One interpretation of Theorem 4.1 is that the $\mathbb{C}$-motivic cellular stable homotopy category is a deformation of classical stable homotopy category. Although our work is heavily motivated by motivic homotopy theory, it is not logically dependent because of purely topological constructions [13, 34] of this cellular subcategory.

There are other deformations of classical stable homotopy theory that are also computationally useful, such as $\mathrm{H}\mathbb{F}_2$-synthetic stable homotopy theory [9]. Beyond the 90-stem, $\mathrm{H}\mathbb{F}_2$-synthetic stable homotopy theory has provided additional information to our method, and can be viewed as one more tool in the "ad hoc" step (4) of Section 4.

Lately, we have begun to study $\mathrm{H}\mathbb{F}_2$-synthetic $\mathbb{C}$-motivic stable homotopy theory. This can be viewed as a deformation of a deformation of classical stable homotopy theory. On the other hand, one could also perform the deformations in the other order by considering BP-synthetic, $\mathrm{H}\mathbb{F}_2$-synthetic stable homotopy theory. We believe that these double deformations are equivalent, and we propose the name "bimotivic homotopy theory" for this triply-graded stable homotopy theory.

7. THE CHOW t-STRUCTURE

Over an arbitrary based field k, the story is more complicated than just a deformation—it becomes the Postnikov–Whitehead tower associated to the Chow t-structure.

In [2], Bachmann, Kong, and the second and third authors defined the Chow t-structure on the motivic stable homotopy category $\mathrm{SH}(k)$ over any base field k. Its non-negative part $\mathrm{SH}(k)_{c\geq 0}$ is generated by Thom spectra $\mathrm{Th}(\xi)$ associated to K-theory points $\xi \in K(X)$ on smooth and proper schemes X. We implicitly invert the exponential characteristic of k and denote by $E \mapsto E_{c=i}$ the truncations with respect to the Chow t-structure.

Theorem 7.1 ([2]). *Let $E \in \mathrm{SH}(k)$. Then there is a canonical isomorphism*

$$\pi_{2w-s,w} E_{c=i} \cong \mathrm{Ext}^{s,2w}_{\mathrm{MU}_{2*}\mathrm{MU}}(\mathrm{MU}_{2*}, \mathrm{MGL}_{2*+i,*}E).$$

Here MGL is the algebraic cobordism spectrum. Theorem 7.1 generalizes the isomorphism on the abutments in Theorem 4.2 over $\mathbb{C}$ to an arbitrary base field and is an integral statement.

Moreover, the heart of the Chow t-structure $\mathrm{SH}(k)^{c\heartsuit}$ can be described as a category of enriched presheaves (see, e.g., [37, SECTION 3.5]) over the category of pure MGL-motives $\mathrm{PM}_{\mathrm{MGL}}(k)$ [2, DEFINITION 1.4].

Theorem 7.2 ([2]). *The Chow heart $\mathrm{SH}(k)^{c\heartsuit}$ is equivalent to the category of enriched presheaves on $\mathrm{PM}_{\mathrm{MGL}}(k)$ with values in $\mathrm{MU}_{2*}\mathrm{MU}$-comodules.*

Restricting to the subcategory of cellular objects, the Chow heart can be identified as the abelian category of $\mathrm{MU}_{2*}\mathrm{MU}$-comodules. The category of cellular objects over $(S^{0,0})_{c=0}$ is equivalent to Hovey's [17] derived category of $\mathrm{MU}_{2*}\mathrm{MU}$-comodules.

Theorem 7.3 ([2]). *There are equivalences of stable ∞-categories*

$$\mathrm{SH}(k)^{\mathrm{cell},c\heartsuit} \simeq \mathrm{MU}_{2*}\mathrm{MU}\text{-}\mathbf{Comod},$$
$$(S^{0,0})_{c=0}\text{-}\mathbf{Mod}^{\mathrm{cell}} \simeq \mathcal{D}(\mathrm{MU}_{2*}\mathrm{MU}\text{-}\mathbf{Comod}).$$

Theorem 7.3 allows us to identify the motivic Adams spectral sequence of $(S^{0,0})_{c=i}$ as an algebraic Novikov spectral sequence, which can be computed by a machine. We anticipate that Adams differentials for the k-motivic sphere can be computed through the Postikov–Whitehead tower associated to the Chow t-structure (see [2] for more details).

It would be interesting to compare our approach with methods developed in [38].

8. FURTHER QUESTIONS AND CONJECTURES

We include a few questions and conjectures for future study.

The orders of individual p-primary stable homotopy groups do not follow a clear pattern. However, an empirically observed pattern emerges if we consider the cumulative size of the groups.

Conjecture 8.1 (Stable stems growth conjecture). *Let $f(n)$ be the product of the orders of the p-primary stable homotopy groups in dimensions 1 through n. Then $\log_p f(n) = O(n^2)$.*

The ring spectrum of topological modular forms *tmf* is very useful for computing Adams differentials for the sphere spectrum, since *tmf* detects many classes above a line of slope $1/6$ on the Adams chart. Starting in the high 60s, the Mahowald operator $Ma = \langle g_2, h_0^3, a\rangle$ organizes many more classes just below this line, where a is detected by *tmf*.

Question 8.2 (Mahowald operator detection question). *Does there exist a ring spectrum whose Adams spectral sequence is completely computable such that its E_2-page detects $M^n a$ for all $n > 0$ and all classes a that are detected by tmf?*

Baues, Jibladze, and Nassau [5, 32] described how consideration of the secondary Steenrod algebra leads to the computation of Adams differentials. Recently, Chua [10] has used these ideas to obtain machine-generated values of the Adams d_2-differentials. This allows us to take the Adams E_3-page as given by machine.

Question 8.3 (Automated Adams differential computation question). *Are there effective algorithms that can compute all Adams d_3 or even d_4-differentials in a given range?*

Question 8.4 (Automated Adams–Novikov differential computation question). *Are there effective algorithms that can compute all Adams–Novikov d_3-differentials in a given range?*

Within any Adams filtration on the E_2-page, there is an operation Sq^0 that doubles the internal degree t. The following Conjecture 8.5 is due to Minami.

Conjecture 8.5 (New doomsday conjecture). *For any Sq^0-family*

$$\{x, Sq^0 x, \ldots, (Sq^0)^n x, \ldots\}$$

in the Adams spectral sequence, only finitely many classes survive to the E_∞-page.

Conjecture 8.6 (Stable length conjecture). *Nonzero Adams differentials supported by any Sq^0-family a_n are of the form $d_r(a_n) = c \cdot b_n$ when n is large enough, where b_n is an Sq^0-family and c is a fixed element in* Ext.

In Adams filtrations 1 and 2, the New Doomsday Conjecture is essentially equivalent to the Hopf invariant one problem and the Kervaire invariant one problem, respectively.

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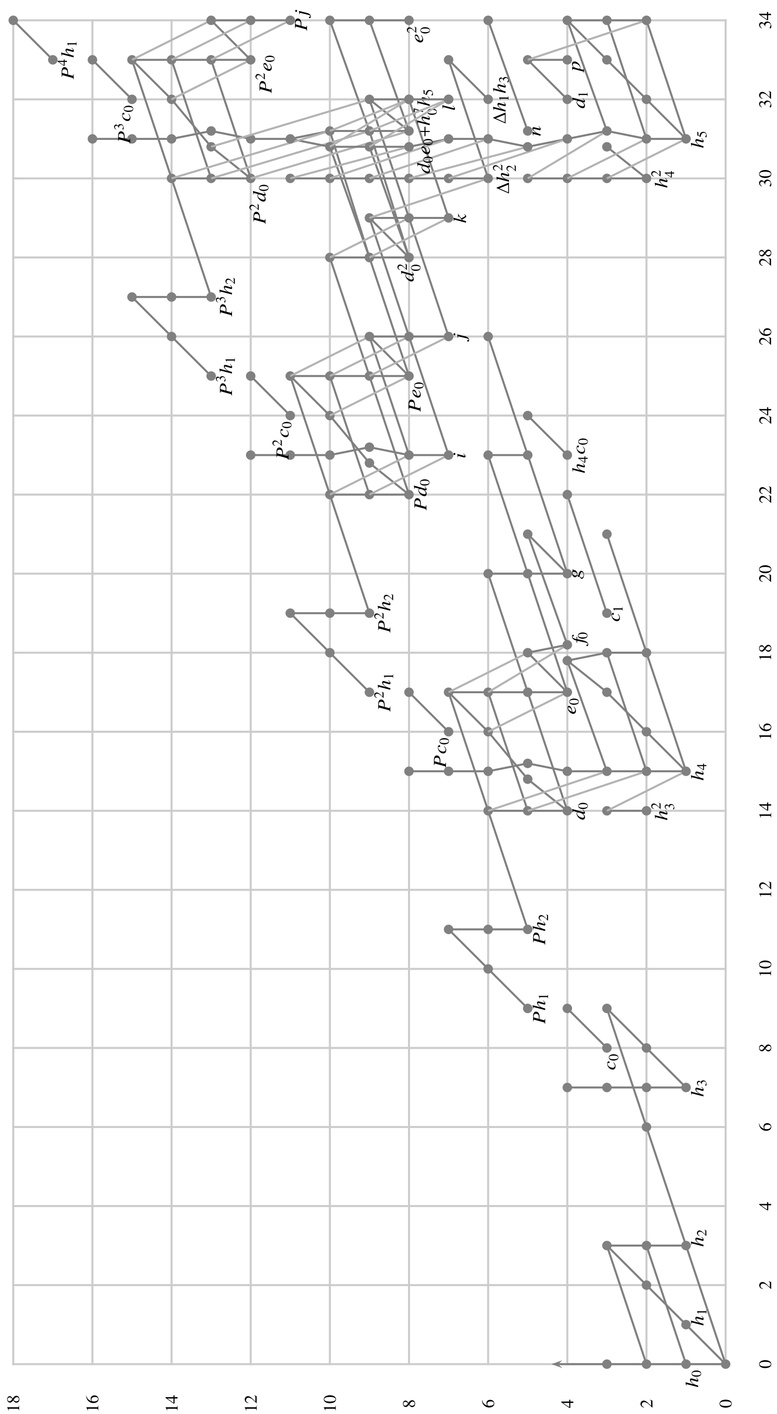

FIGURE 1

The Adams E_2-page in dimensions 0–34

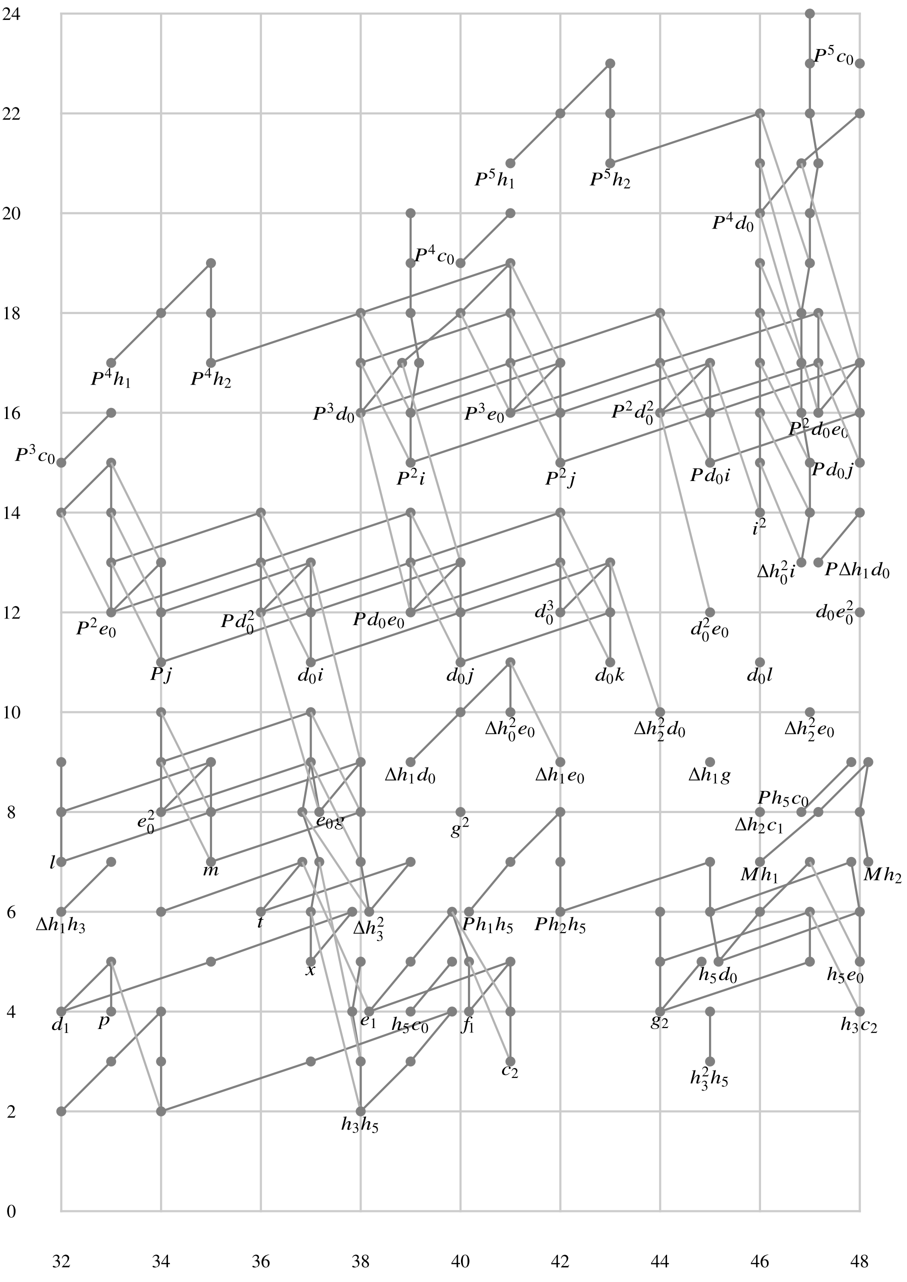

FIGURE 2

The Adams E_2-page in dimensions 32–48

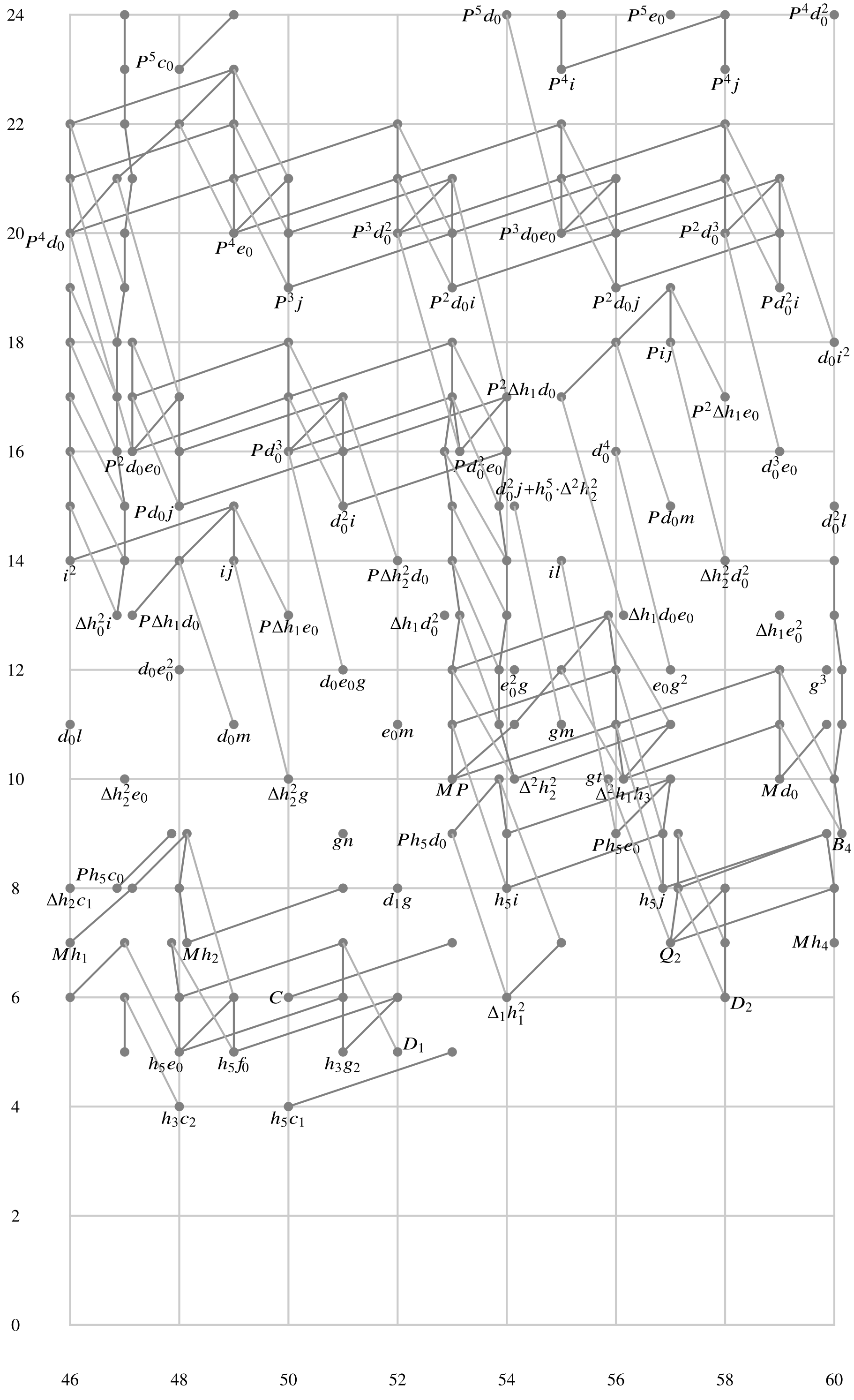

FIGURE 3

The Adams E_2-page in dimensions 46–60

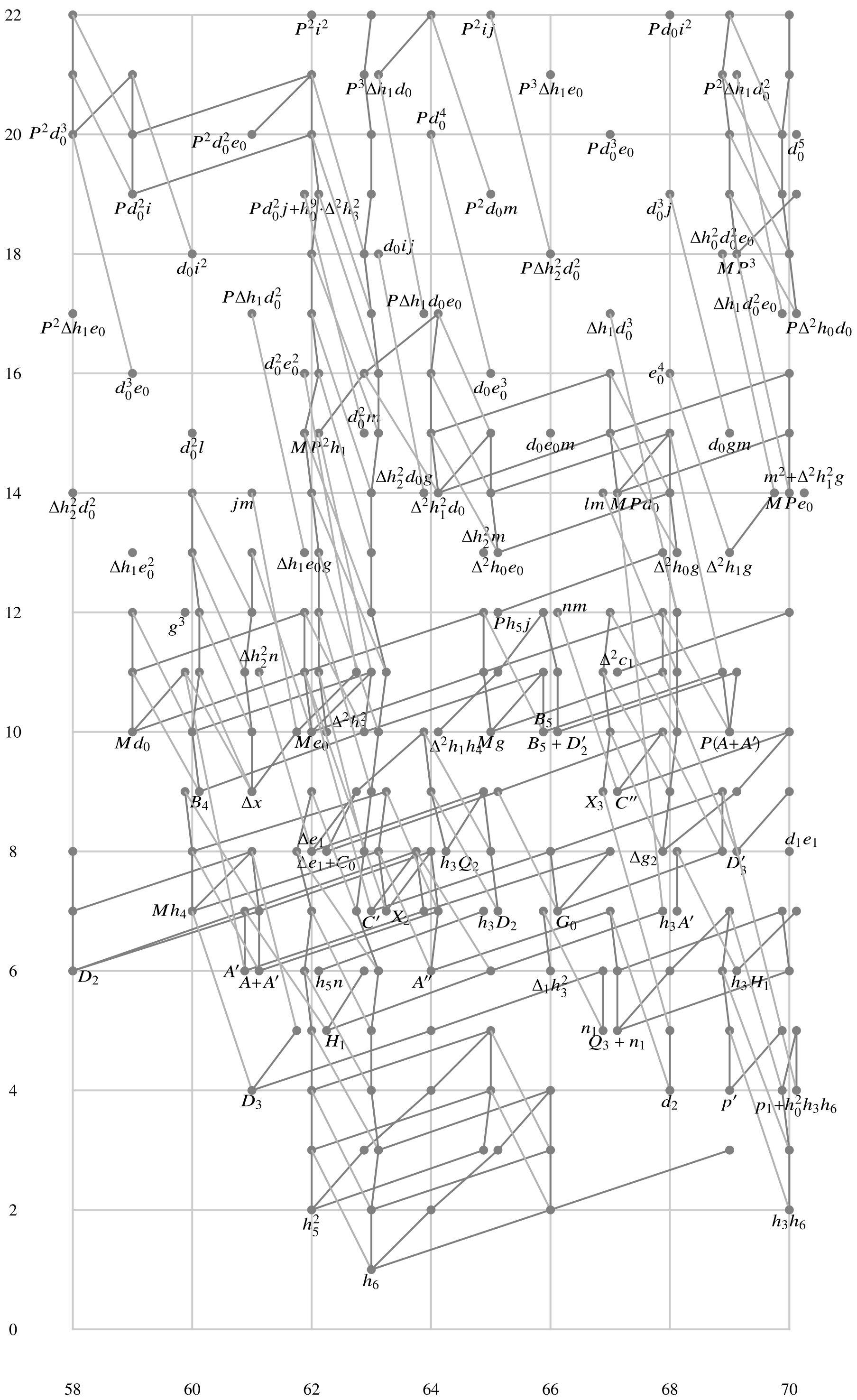

FIGURE 4

The Adams E_2-page in dimensions 58–70

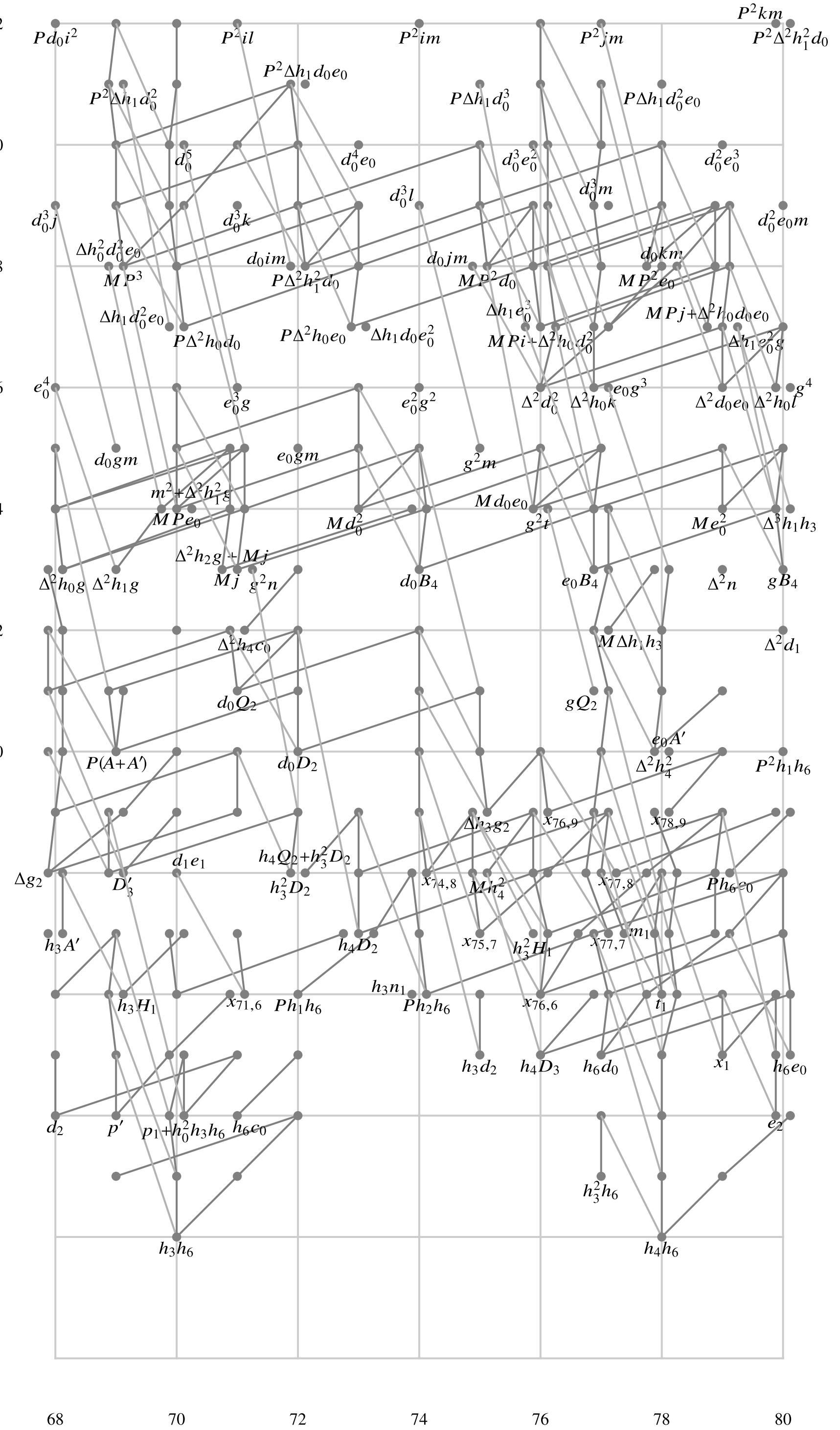

FIGURE 5

The Adams E_2-page in dimensions 68–80

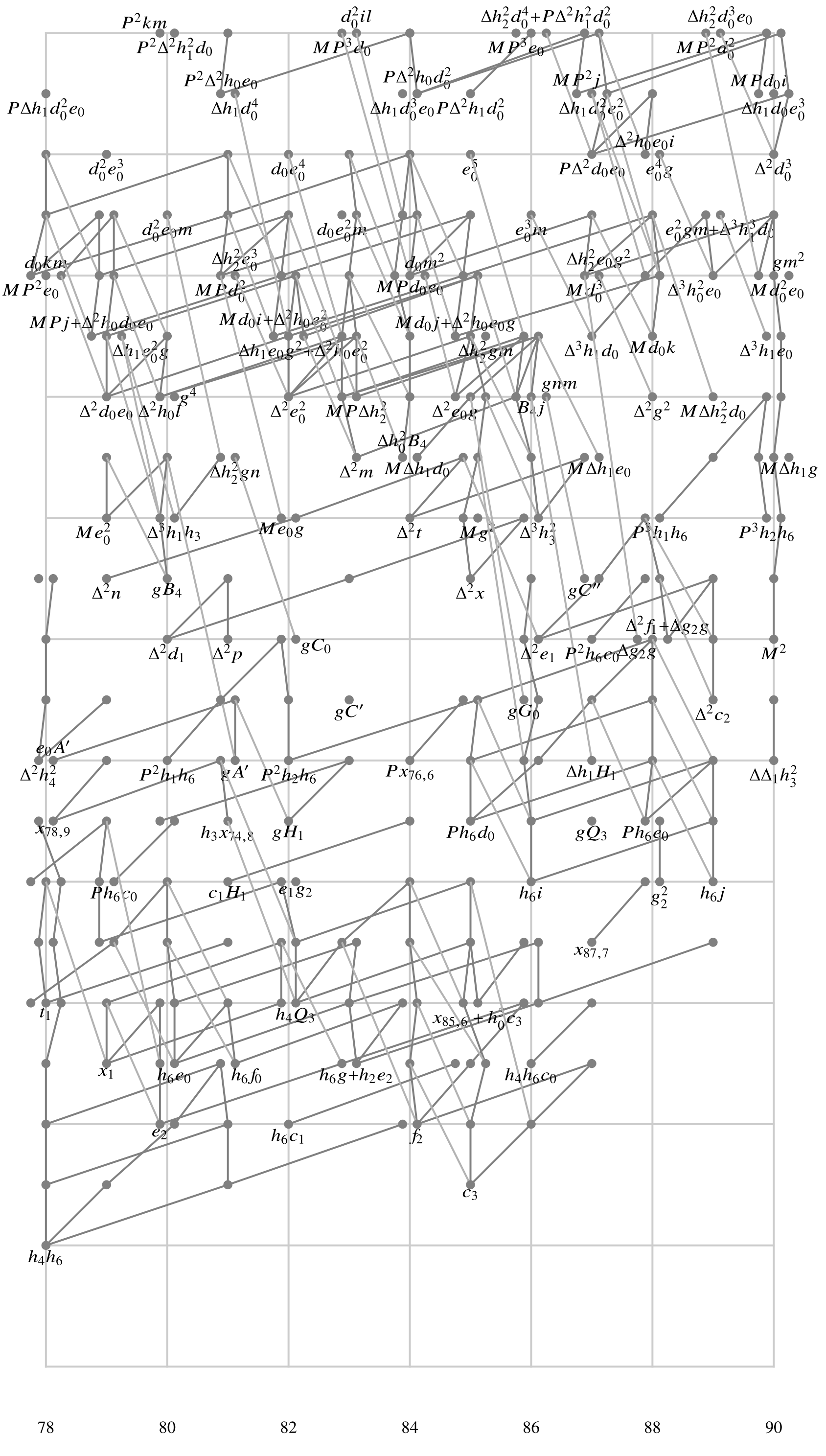

FIGURE 6

The Adams E_2-page in dimensions 78–90

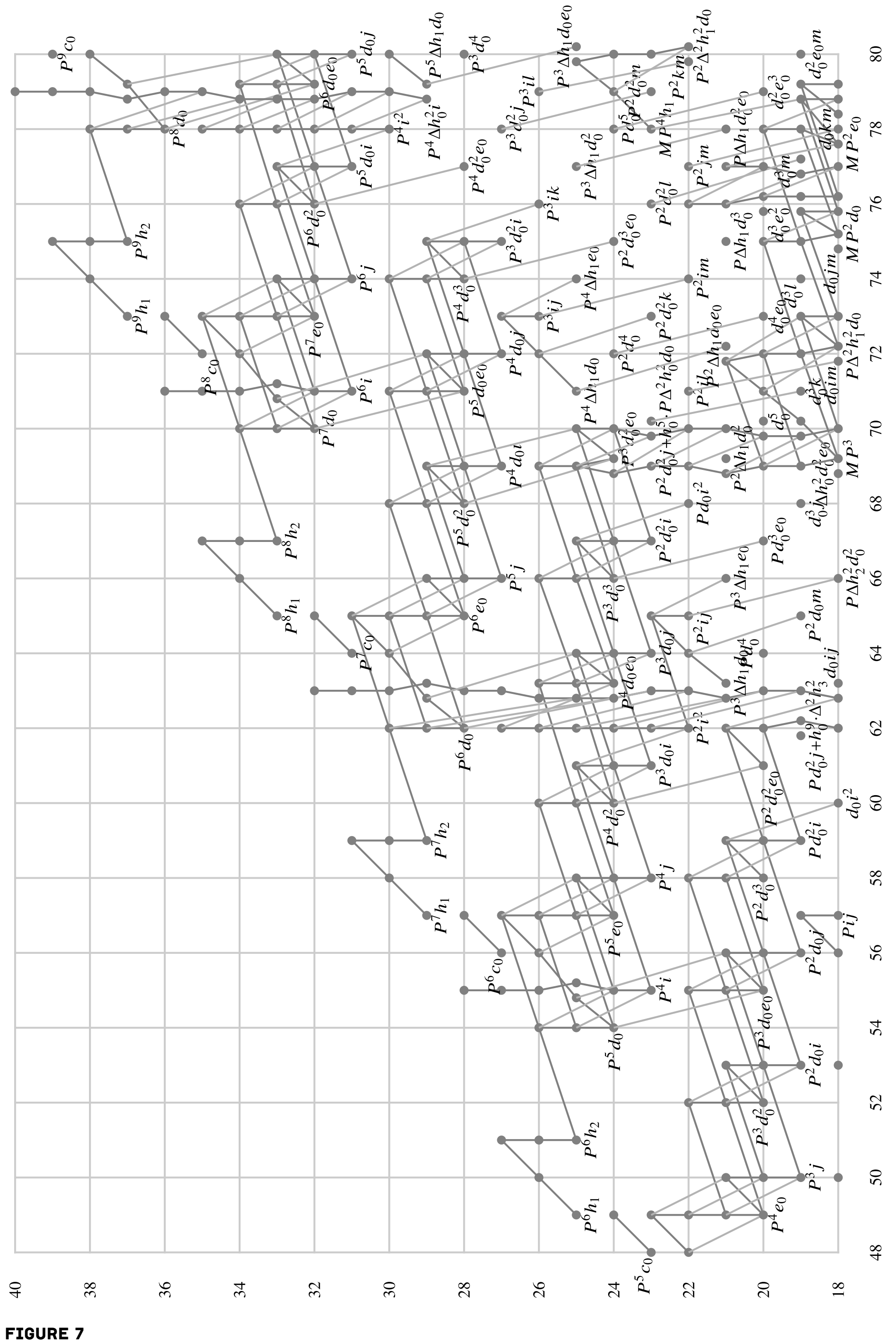

FIGURE 7

The Adams E_2-page in dimensions 48–80 in high filtration

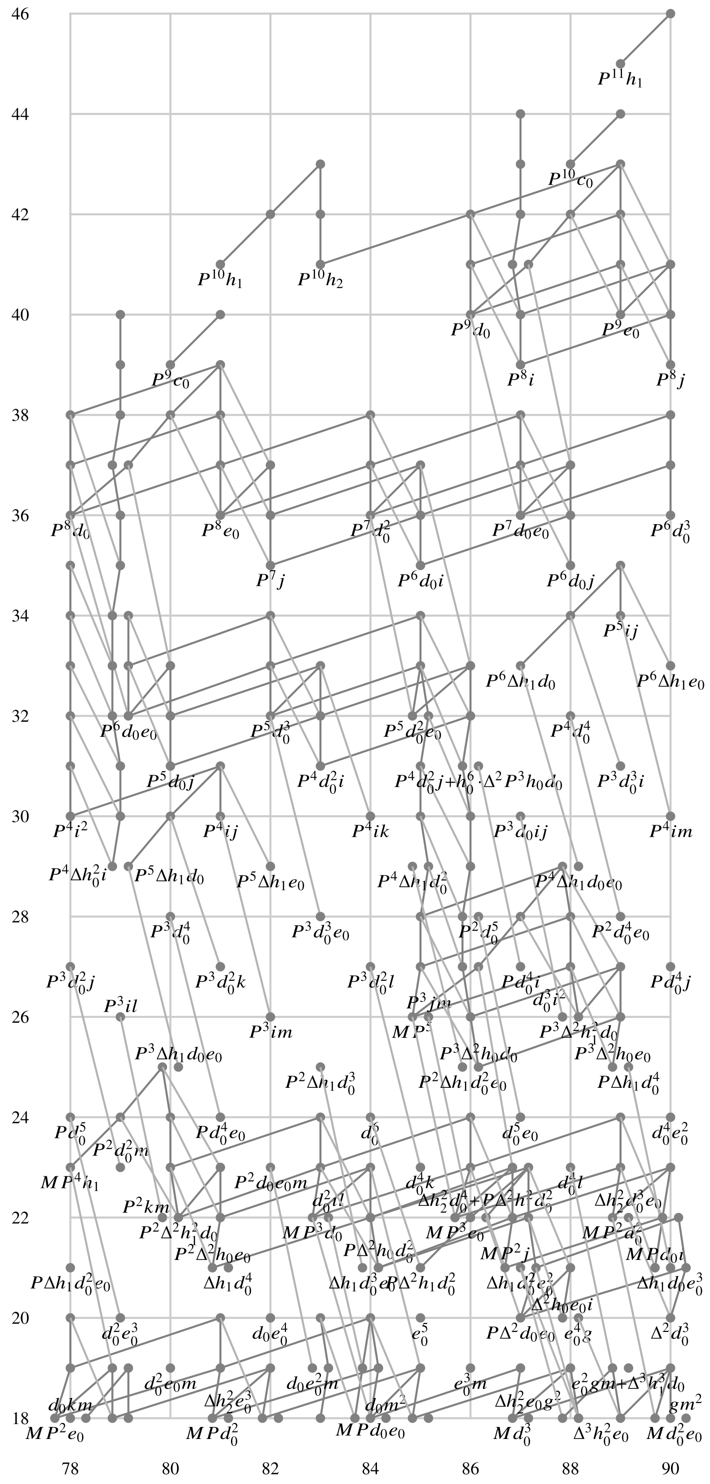

FIGURE 8

The Adams E_2-page in dimensions 78–90 in high filtration

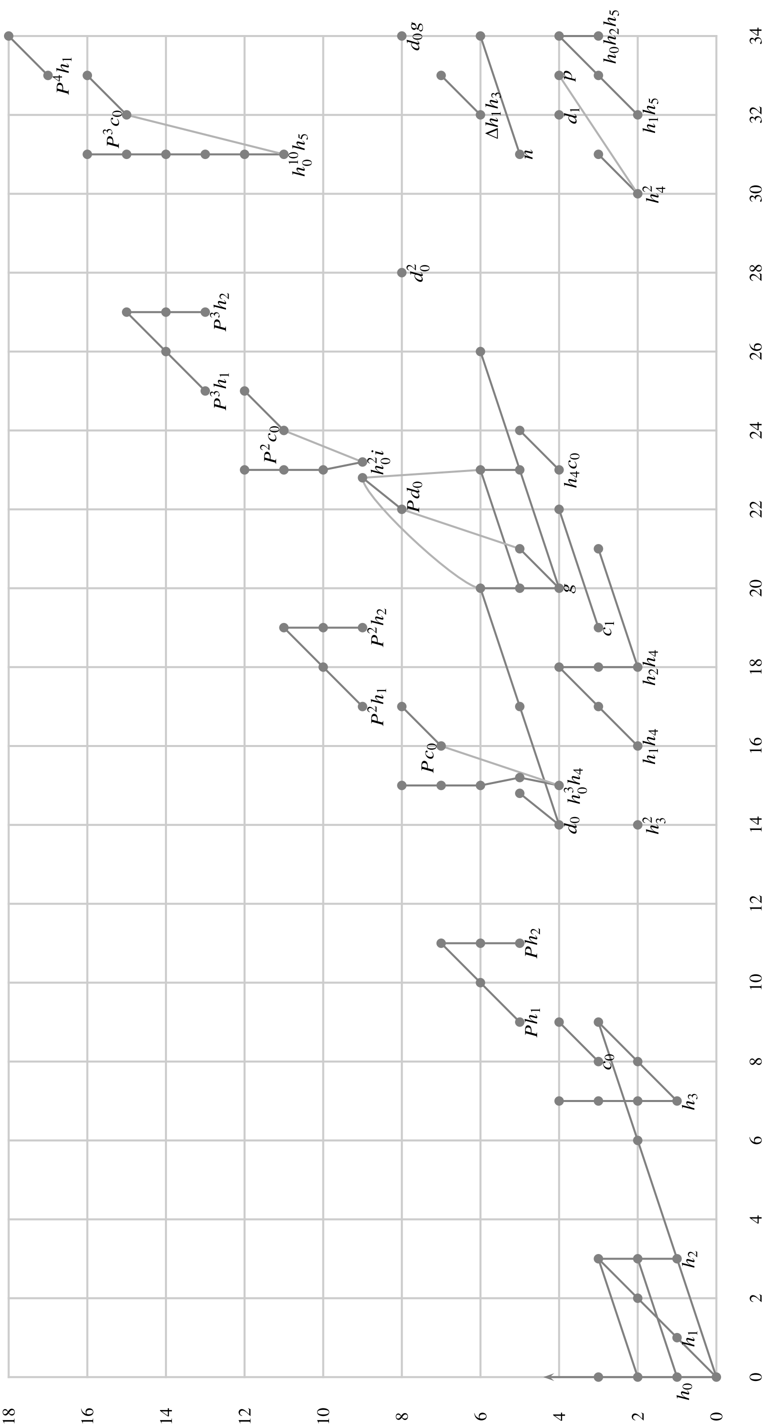

FIGURE 9
The Adams E_∞-page in dimensions 0–34

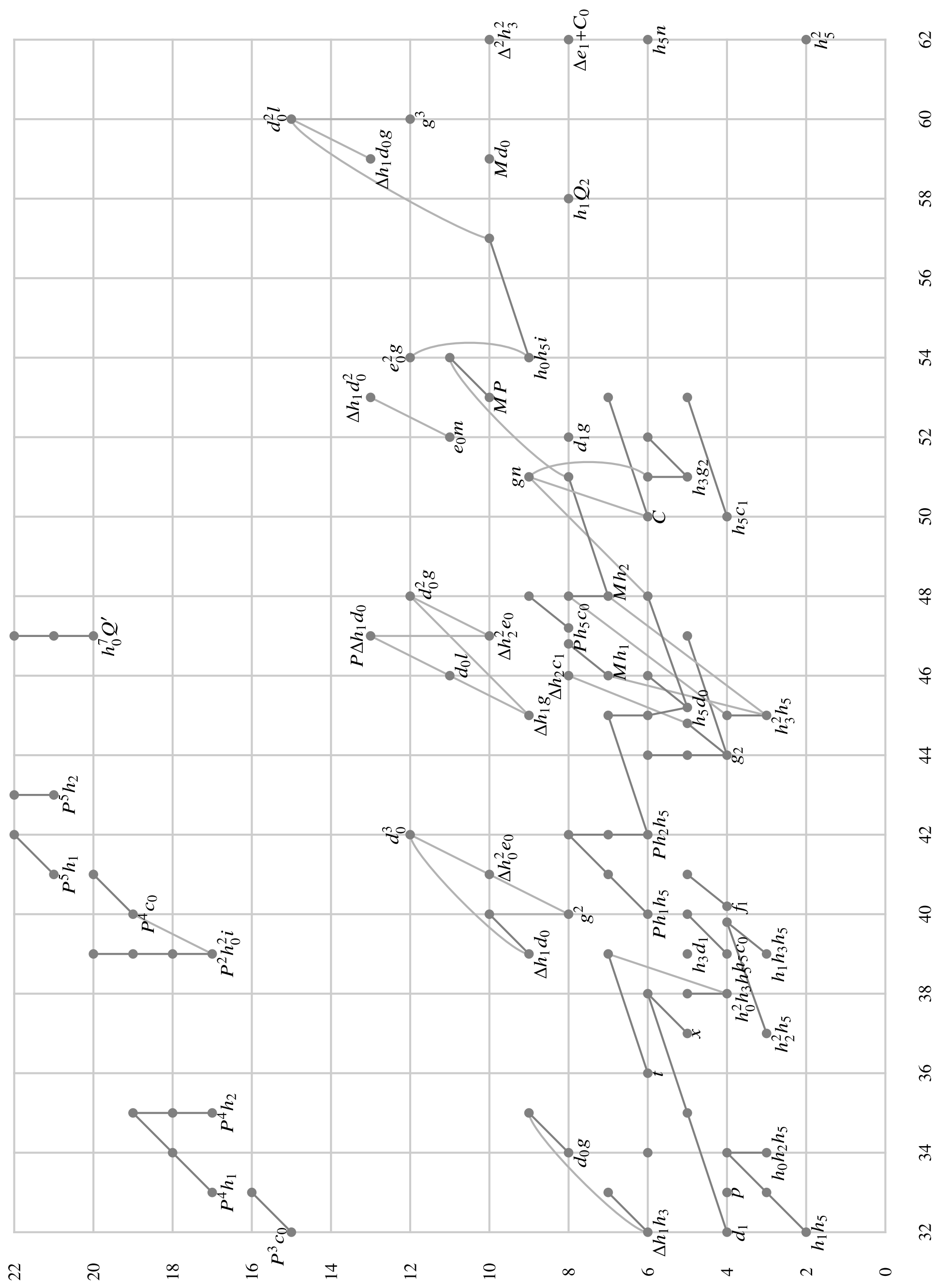

FIGURE 10
The Adams E_∞-page in dimensions 32–62

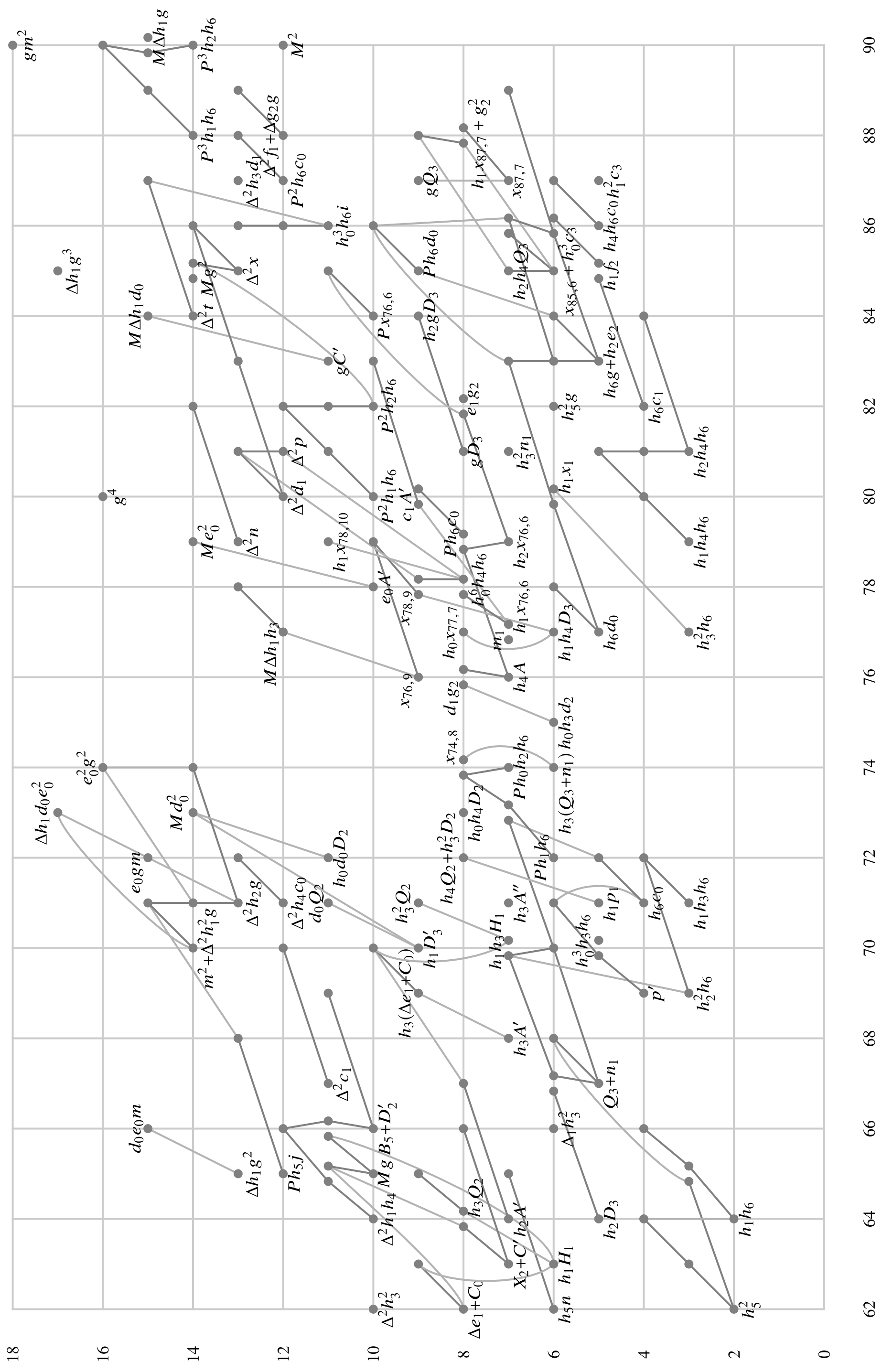

FIGURE 11
The Adams E_∞-page in dimensions 62–90

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DANIEL C. ISAKSEN

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA, isaksen@wayne.edu

GUOZHEN WANG

Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200433, China, wangguozhen@fudan.edu.cn

ZHOULI XU

Department of Mathematics, University of California San Diego, La Jolla, CA 92093, USA, xuzhouli@ucsd.edu