SURFACE AUTOMORPHISMS AND FINITE COVERS

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ABSTRACT

This article surveys recent progress on virtual properties of surface automorphisms.

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1. INTRODUCTION

Self-homeomorphisms of a topological space can be studied through their mapping tori. This very basic observation connects surface automorphisms with 3-manifold theory. In this survey, we focus on recent applications of virtual properties of 3-manifold groups to surface automorphisms and their lifts to finite covers. We collect results and techniques in that direction. We mention some currently open questions, most of which are reformulated from more general 3-manifold versions. We review necessary background to make our exposition accessible to non-expert readers.

Throughout this survey, a *surface* S refers to a connected compact orientable 2-manifold, possibly with boundary, and a *surface automorphism* (S, f) refers to an orientation-preserving self-homeomorphism $f: S \to S$. A *covering* between surface automorphisms $(S', f') \to (S, f)$ refers to an (unramified) covering projection $\kappa: S' \to S$ which is equivariant with respect to the pair of automorphisms (that is, $f \circ \kappa = \kappa \circ f'$). By saying that a covering $(S', f') \to (S, f)$ is finite, regular, characteristic, or so on, we mean that the referred property holds for $S' \to S$.

2. SURFACE AUTOMORPHISMS AFTER NIELSEN AND THURSTON

We recall some aspects about surface automorphisms that have been well developed since the mid-1970s. Our summary puts an emphasis on characterizing dynamical properties of a surface automorphism in terms of the fundamental group of its mapping torus. To this end, we denote the *mapping torus* of any surface automorphism (S, f) as

$$M_f = \frac{S \times \mathbb{R}}{(x, r+1) \sim (f(x), r)},$$

which is topologically a connected compact orientable 3-manifold, with boundary a possibly empty disjoint union of tori. Note that we follow the dynamical convention. It makes sure that translation along the $\mathbb{R}$-factor $S \times \mathbb{R} \to S \times \mathbb{R}: (x, r) \mapsto (x, r+t)$ descends to the (forward) *suspension flow* $\theta_t: M_f \to M_f$, which is a continuous family of self-homeomorphisms parametrized by $t \in \mathbb{R}$, such that θ_0 is the identity. We denote by $\phi_f \in H^1(M_f; \mathbb{Z})$ the distinguished cohomology class homotopically represented by the natural projection $M_f \to \mathbb{R}/\mathbb{Z}$.

2.1. Classification of mapping classes

For surfaces of positive or zero Euler characteristic, the isotopy classes of their automorphisms are easy to describe. When S is a sphere or a disk, any automorphism f of S is isotopic to the identity. When S is an annulus, parametrized as $\mathbb{R}/\mathbb{Z} \times [-1, 1]$, any automorphism f of S is isotopic to either the identity or the involution $(x + \mathbb{Z}, y) \mapsto (-x + \mathbb{Z}, -y)$. When S is a torus, parametrized as $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$, any automorphism f of S is isotopic to a unique linear automorphism represented by a matrix in $\mathrm{SL}(2, \mathbb{Z})$.

In general, the Nielsen–Thurston classification asserts that the isotopy class of any surface automorphism falls into one of three types: periodic, reducible, or pseudo-Anosov. The above description with torus automorphisms provides a prototype of the classification, and the three types correspond to the representing matrix in $\mathrm{SL}(2, \mathbb{Z})$ being elliptic/central, parabolic/central, or hyperbolic, respectively (as a fractional linear transformation on the upper-half complex plane). In general, a surface automorphism is said to be *periodic* if it has finite order under iteration, or *reducible* if it preserves a union of mutually disjoint, essential simple closed curves on the surface. A *pseudo-Anosov* automorphism refers to a surface automorphism (S, f) such that f preserves a pair of (transversely) measured foliations on the interior of S, and rescales the measures by some factors $\lambda > 1$ and λ^{-1}, respectively. Unlike an Anosov torus automorphism, the foliations in the pseudo-Anosov case are allowed to have prong singularities (having prong number ≥ 3 at points in the interior, or ≥ 1 at the ends as punctures). We also require the pair of foliations to be transverse to each other except at a common finite set of singular points. See [8, EXPOSÉ 1].

When S has negative Euler characteristic, an automorphism f of S is periodic up to isotopy if and only if the mapping torus M_f supports the $\mathbf{H}^2 \times \mathbb{R}$ geometry (and hence also the $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ geometry with $\partial S \neq \emptyset$); f is reducible up to isotopy if and only if M_f has nontrivial geometric decomposition; f is pseudo-Anosov up to isotopy if and only if M_f supports the 3-dimensional hyperbolic geometry $\mathbf{H}^3$. Moreover, in the reducible case, a collection of curves on S for reducing f can be obtained by intersecting any essential torus or Klein bottle in M_f (minimally up to isotopy) with the distinguished fiber $S \times \{0\}$. See [3, CHAPTER 1].

2.2. Periodic orbit classes and indices

Given a surface automorphism, one could freely ask if there are any fixed points. Nielsen's fixed point theory is more than to answer yes or no. The general theory applies to continuous self-maps of compact connected simplicial complexes. Instead of considering individual fixed points, which may disappear or duplicate under homotopy, one will consider abstract fixed point classes, and distinguish finitely many essential ones from the others. Then the essential fixed point classes will depend only on the homotopy class of the self-map, and each of them will guarantee at least one distinct fixed point. Below we follow the mapping torus approach as suggested by B. Jiang in the survey [13]; see also [11] for more detail.

Let (S, f) be a surface automorphism. Denote by $\mathrm{Fix}(f) \subset S$ the set of fixed points. For any $x \in \mathrm{Fix}(f)$, we obtain a 1-periodic trajectory (of the suspension flow) $\gamma_x\colon \mathbb{R}/\mathbb{Z} \to M_f$ (coming from the line $x \times \mathbb{R}$ in $S \times \mathbb{R}$). We say that $x, y \in \mathrm{Fix}(f)$ are *of the same* fixed point class if γ_x and γ_y are freely homotopic in M_f. More abstractly, a *fixed point class* of f can be defined as a free homotopy loop γ in M_f, such that $\phi_f(\gamma) = \langle \phi_f, [\gamma] \rangle$ equals 1. Every fixed point class $\mathbf{p}$ has a well-defined *fixed point index* $\mathrm{ind}(f; \mathbf{p}) \in \mathbb{Z}$, which can be described as follows.

Note that $\mathrm{Fix}(f) \subset S$ is a union of mutually disjoint isolated connected closed subsets, with only finitely many components (since S is compact). The subset $\mathrm{Fix}(f; \mathbf{p}) \subset \mathrm{Fix}(f)$ of fixed point class $\mathbf{p}$ is a subunion of those components. If $\mathrm{Fix}(f; \mathbf{p})$ is empty, $\mathrm{ind}(f; \mathbf{p})$ equals zero. Otherwise, take (a smooth structure of S and) a smooth homotopy perturbation $\tilde{f}$ of f, supported in an open neighborhood U of $\mathrm{Fix}(f; \mathbf{p})$ away from the rest of $\mathrm{Fix}(f)$; make sure that $\tilde{f}$ has only nondegenerate fixed points in U (that is, for any $x \in \mathrm{Fix}(\tilde{f}) \cap U$, the tangent map $\mathrm{d}\tilde{f}|_x\colon T_xS \to T_xS$ has no eigenvalue 1). Then $\mathrm{ind}(f; \mathbf{p})$

can be calculated as the sum of the sign $+1$ or -1 of $\det(\mathrm{id} - \mathrm{d}\tilde{f}|_x)$, where x ranges over $\mathrm{Fix}(\tilde{f}) \cap U$.

A fixed point class is said to be *essential* if its index is nonzero. In particular, every essential fixed point class is represented by a fixed point of f. If S is closed and if f is pseudo-Anosov, every fixed point represents a distinct essential fixed point class. In this case, there are simple rules for telling the fixed point index. When a fixed point x is a k-prong singularity (of either of the invariant foliations), its index equals $1-k$ if f preserves every prong at x, otherwise its index equals 1; when x is not singular, its index equals -1 or 1 according as f preserves or reverses an orientation of the leaf through x. For general surface automorphisms, it is also possible to characterize all the essential fixed point classes and their index, in terms of normal forms in the Nielsen–Thurston classification [14].

For any positive integer $m \in \mathbb{N}$, an *m-periodic orbit class* of f can be defined as a free homotopy loop γ in M_f with $\phi_f(\gamma) = m$. The *m-index* of the m-periodic class is defined as the sum of the fixed point indices of γ' with respect to f^m, where γ' ranges over all the free homotopically distinct lifts of γ to the m-cyclic cover M_{f^m}. (The indices of different lifts are actually equal, so the summation only counts one value with suitable multiplicity.) Finally, *essential* m-periodic orbit classes are those of nonzero m-index.

2.3. Homological directions

Let (S, f) be a surface automorphism. As every periodic orbit class $\mathbf{p}$ is free-homotopically represented by a periodic trajectory in M_f, its homology class is a well-defined element $[\mathbf{p}] \in H_1(M_f;\mathbb{Z})$. Passing to real coefficients, there is a unique minimal convex cone in $H_1(M_f;\mathbb{R})$ (formed by linear rays emanating from the origin) that contains all the homology classes of the essential periodic trajectories. Fried shows that this cone is polyhedral. In other words, it is the convex hull of finitely many extreme rays. If f is pseudo-Anosov, this cone has codimension zero in $H_1(M_f;\mathbb{R})$. The directions of rays in the cone are called the (essential) *homological directions* of the suspension flow. Fried's cone of homological directions is exactly dual to the Thurston's fibered cone that contains $\phi_f \in H^1(M_f;\mathbb{R})$, as we elaborate below, based on Fried's exposition [8, EXPOSÉ 14].

We recall that the Thurston norm is defined for any compact connected orientable 3-manifold N as a seminorm on the real linear space $H^1(N;\mathbb{R})$. It is nondegenerate if N supports the 3-dimensional hyperbolic geometry (of finite volume). It is characterized by the property that for any integral cohomology class $\phi \in H^1(N;\mathbb{Z})$, the Thurston norm of ϕ is the minimum of the complexity among all properly embedded oriented surfaces $(S, \partial S) \subset (N, \partial N)$ homologous to the Poincaré–Lefschetz dual of ϕ in $H_2(N, \partial N;\mathbb{Z})$. Here, the complexity of S refers to $\sum_{i=1}^{k} \max(0, -\chi(S_i))$, where $S_1, \ldots, S_k$ enumerate the connected components of S.

The unit ball of the Thurston norm of N is a (possibly noncompact) convex polyhedron of codimension zero in $H^1(N;\mathbb{R})$, central symmetric about the origin. Its dual is a (possibly positive codimensional) compact convex polyhedron in $H_1(N;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(H^1(N;\mathbb{R}),\mathbb{R})$. Moreover, if $\phi \in H^1(N;\mathbb{Z})$ is fibered (that is, homotopically represented by a bundle projection onto the circle with surface fibers), Thurston shows that ϕ

is contained in the cone over an open codimension-one face the Thurston norm unit ball, in which the integral classes are all fibered. Such cones are called the *fibered cones* of N, over the *fibered faces* of the Thurston norm unit ball in $H^1(N;\mathbb{R})$. The fibered faces are dual to a collection of vertices in the dual of the Thurston norm unit ball in $H_1(N;\mathbb{R})$, which we may reasonably call the *flow vertices*.

For the mapping torus M_f of a surface automorphism (S, f), we find a distinguished fibered cone in $H^1(M_f;\mathbb{R})$ that contains the distinguished cohomology class ϕ_f. When S has negative Euler characteristic, the corresponding flow vertex can be figured out as $-e_f/2$ in $H^2(M_f, \partial M_f;\mathbb{R}) \cong H_1(M_f;\mathbb{R})$, where $e_f \in H^2(M_f, \partial M_f;\mathbb{Z})$ denotes the relative Euler class of the (oriented, ∂-transverse) vertical tangent bundle of M_f with respect to its fibering over $\mathbb{R}/\mathbb{Z}$, oriented compatibly to make $\phi_f(e_f) = \chi(S)$. We can naturally identify the tangent space of $H_1(M_f;\mathbb{R})$ at the opposite vertex $e_f/2$ as $H_1(M_f;\mathbb{R})$. Then Fried's cone of homological directions consists exactly of those tangent vectors at $e_f/2$ pointing into (the corner of) the polytope dual to the Thurston norm unit ball.

2.4. Various zeta functions

For any surface automorphism (S, f), the *Nielsen zeta function* is a useful tool for analyzing the iteration dynamics. It can be defined by the following expression:

$$\zeta_{N,f}(t) = \exp\left(\sum_{m=1}^{\infty} \frac{N(f^m)}{m} \cdot t^m\right)$$

where $N(f^m)$ denotes the number of essential fixed points of f^m, called the *Nielsen number* of f^m. When f is pseudo-Anosov with stretch factor $\lambda > 1$, the Nielsen numbers $N(f^m)$ grow exponentially as

$$\varlimsup_{m\to\infty} N(f^m)^{1/m} = \lambda.$$

More generally, the above limit superior is equal to the maximum stretch factor among the pseudo-Anosov components in the Nielsen–Thurston decomposition, or 1 if all the components are periodic. In other characterizations, the logarithm of that value is known to be the mapping-class topological entropy of f. In particular, $\zeta_{N,f}(t)$ converges absolutely as a complex analytic function in t in a neighborhood of 0. It is known that $\zeta_{N,f}(t)$ is a radical of a rational function in t near 0.

The *Lefschetz zeta function* $\zeta_{L,f}(t)$ of (S, f) is defined using the Lefschetz numbers $L(f^m)$ instead of the Nielsen numbers $N(f^m)$. This makes $\zeta_{L,f}(t)$ easier to calculate than $\zeta_{N,f}(t)$. Indeed, recall that $L(f^m)$ is equal to the alternating sum of the traces of f_*^m on $H_*(S;\mathbb{Q})$. It follows that $\zeta_{L,f}(t)$ is equal to $t^{-\chi(S)}$ divided by the alternating product of the characteristic polynomials of f_* on $H_*(S;\mathbb{Q})$. The resulting form can be recognized as (a representative of) the Reidemeister torsion of M_f with respect to ϕ_f. This is an instance of a general connection between twisted Lefschetz zeta functions and twisted Reidemeister torsions.

Let R be a commutative domain that contains $\mathbb{Z}$. Suppose that $\rho\colon \pi_1(M_f) \to \mathrm{GL}(n, R)$ is a linear representation over R^n. The *twisted Lefschetz zeta function* of (S, f)

with respect to ρ is defined as

$$\zeta^{\rho}_{L,f}(t) = \exp\left(\sum_{m=1}^{\infty} \frac{\sum_{\mathbf{p}} \chi_{\rho}(\mathbf{p}) \cdot \mathrm{ind}_m(f;\mathbf{p})}{m} \cdot t^m\right),$$

where $\mathbf{p}$ ranges over all the m-periodic orbit classes; being a free homotopy loop, $\mathbf{p}$ also represents a conjugacy class in $\pi_1(M_f)$, so the character χ_ρ of ρ can be evaluated at $\mathbf{p}$, as the trace of ρ evaluated at any group element in that conjugacy class; the notation $\mathrm{ind}_m(f;\mathbf{p})$ stands for the m-index of $\mathbf{p}$ with respect to f (so the summation over $\mathbf{p}$ is essentially finite); finally, the whole expression is understood as a formal power series over the field of fractions $F(R)$ in an indeterminate t, with $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$. In particular, $\zeta^{\rho}_{L,f}(t)$ is invariant under homotopy of f and conjugation of ρ. On the other hand, we simply recall that the *twisted Reidemeister torsion* $\tau^{\rho,\phi_f}_{M_f}(t)$ of M_f with respect to ϕ_f and ρ is well defined as an element in $F(R[t,t^{-1}]) = F(R)(t)$ up to units of $R[t,t^{-1}]$ (that is, up to factors in the multiplicative subgroup $(R[t,t^{-1}])^{\times} = R^{\times} \times t^{\mathbb{Z}}$); see [9]. Under the above assumptions, $\zeta^{\rho}_{L,f}(t)$ agrees with the power series expansion in t of a unique rational function over $F(R)$, and the identity

$$\tau^{\rho,\phi_f}_{M_f}(t) \doteq \zeta^{\rho}_{L,f}(t)$$

holds up to units of $R[t,t^{-1}]$; see [12] (and also [17, LEMMA 8.2]).

3. VIRTUAL HOMOLOGICAL EIGENVALUES

Let (S,f) be a surface automorphism. Since $H_1(S;\mathbb{Z})$ is a finitely generated free abelian group, the induced linear automorphism $f_*\colon H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z})$ has a characteristic polynomial, which we denote as

$$\Delta_f(t) = \det_{\mathbb{Z}[t]}(t \cdot \mathrm{id} - f_*).$$

This is a monic polynomial over $\mathbb{Z}$ with the property $\Delta_f(1) = \pm 1$. If S has genus g and h boundary components, $\Delta_f(t)$ factorizes as the product of a reciprocal polynomial of degree $2g$ and cyclotomic factors of total degree $\max(0, h-1)$, because f preserves ∂S and descends to an automorphism of the closed surface obtained by filling ∂S with disks. Moreover, $\Delta_f(t)$ can be recognized as the (first) Alexander polynomial of M_f with respect to ϕ_f, the latter being well-defined in $\mathbb{Z}[t,t^{-1}]$ up to units.

A *homological eigenvalue* of a surface automorphism (S,f) refers to a complex root of the polynomial $\Delta_f(t)$, and a *virtual homological eigenvalue* of (S,f) refers to a complex root of the polynomial $\Delta_{f'}(t)$ where $(S',f') \to (S,f)$ is some finite covering. We are interested in a general question as to which complex values may occur as virtual homological eigenvalues of a given surface automorphism. Moreover, how do they reflect the dynamical complexity of its isotopy class?

We start with the following well-known, simple observation.

Theorem 3.1. *If a surface automorphism has no pseudo-Anosov type components in its Nielsen–Thurston decomposition, then its virtual homological eigenvalues are all roots of unity.*

Proof. The condition implies that some finite iterate of the given surface automorphism is isotopic to a finite product of left- or right-hand Dehn twists along mutually disjoint simple closed curves. Then the characteristic polynomial of that iterate is a power of $t-1$. The conclusion follows because the condition also holds for any finite covering of the given surface automorphism. ■

There are many pseudo-Anosov automorphisms on any surface of negative Euler characteristic, such that the induced homological action is trivial. However, when (S, f) is a pseudo-Anosov automorphism with transversely orientable invariant foliations, the stretch factor $\lambda > 1$ must occur as a homological eigenvalue. Moreover, if every singularity of the invariant foliations is formed with an even number of prongs, one may achieve the transverse orientability condition by passing to a covering (S', f') of degree at most 2, so λ still occurs as a virtual homological eigenvalue. Note that the same trick does not apply when there are singularities of odd prong numbers, because they locally obstruct the transverse orientability, and they lift locally homeomorphically to any covering.

The above facts lead to the first part of the following theorem. The second part is truly surprising, known as the gap theorem due to C. T. McMullen [22]. It is proved by comparing the Teichmüller metric on the Teichmüller space and the Kobayashi metric on Siegel spaces associated to finite covers.

Theorem 3.2. *Let (S, f) be a pseudo-Anosov automorphism with stretch factor $\lambda > 1$.*

(1) *If the invariant foliations of (S, f) have no singularities of odd prong numbers, then λ is a virtual homological eigenvalue of (S, f).*

(2) *Otherwise, there exists some constant $1 < r < \lambda$, depending only on (S, f), such that every virtual homological eigenvalue μ of (S, f) satisfies $|\mu| < r$.*

McMullen conjectured the converse of Theorem 3.1. The converse has been proved by the author [18], as the following theorem. The proof relies on the virtual specialization of hyperbolic 3-manifold groups.

Theorem 3.3. *If a surface automorphism has a pseudo-Anosov type component in its Nielsen–Thurston decomposition, then it has a virtual homological eigenvalue outside the complex unit circle.*

Remark 3.4. An analogous conjecture for outer automorphisms of finitely generated free groups is proved by A. Hadari [10]. The desired finite-index normal subgroup therein is constructed using nilpotent quotients. Hadari's result also implies Theorem 3.3 for surfaces with nonempty boundary.

An effective version of Theorem 3.3 is yet unknown. We pose the following question, as analogous to the Kojima–McShane inequality regarding pseudo-Anosov stretch factors [15].

Question 3.5. Suppose that (S, f) is an automorphism of a surface of negative Euler characteristic. Does the following inequality hold as μ ranges over all virtual homological eigenvalues of (S, f):

$$\sup_{\mu} \log|\mu| \geq \frac{1}{3\pi \cdot |\chi(S)|} \cdot \mathrm{Vol}(M_f)?$$

Here, $\mathrm{Vol}(M_f)$ denotes the Gromov norm of M_f times the volume of a regular ideal hyperbolic tetrahedron.

We mention another upper-bound estimate regarding the distribution of virtual homological eigenvalues. It is a quick consequence of a theorem due to T. T. Q. Lê [16]. Recall that the (multiplicative) Mahler measure of a nonzero complex polynomial $P(t) = c \cdot \prod_{j=1}^{d}(t - \xi_j)$ refers to the positive value $\mathbb{M}(P) = |c| \cdot \prod_{j=1}^{d} \max(1, |\xi_j|)$. In particular, $\mathbb{M}(P) \geq 1$ holds for any nonzero $P(t) \in \mathbb{Z}[t]$.

Theorem 3.6. *Suppose that* $\cdots \to (S'_n, f'_n) \to \cdots \to (S'_1, f'_1)$ *is a cofinal tower of surface automorphisms which are regular finite coverings of* (S, f). *Then*

$$\overline{\lim_{n\to\infty}} \frac{\log \mathbb{M}(\Delta_n)}{[S'_n : S]} \leq \frac{1}{6\pi} \cdot \mathrm{Vol}(M_f),$$

where Δ_n *denotes the characteristic polynomial of* f'_{n*} *on* $H_1(S'_n; \mathbb{Z})$.

Proof. Note that $\mathbb{M}(\Delta_{g^m}) = \mathbb{M}(\Delta_g)^m$ for any $g = f'_n$ and any $m \in \mathbb{N}$. We can find some sequence $m_n \in \mathbb{N}$, such that the mapping tori M''_n of $(f'_n)^{m_n}$ form a cofinal tower of regular finite coverings over M_f. Then apply [16, THEOREM 1.1]. ■

The estimate in Theorem 3.6 would become an equality if the homological torsion growth conjecture could be proved to that generality [4, 20] (see also [16, CONJECTURE 1.3]). That would also imply a positive answer to Question 3.5.

4. DETERMINING PROPERTIES USING FINITE QUOTIENT ACTIONS

Let (S, f) be a surface automorphism. Fix a base point of S for speaking of $\pi_1(S)$. Using any path from the base point to its image under f, we can construct an automorphism of $\pi_1(S)$. Different choices of the path only affect the construction by inner automorphisms of $\pi_1(S)$. Therefore, for any characteristic subgroup K of $\pi_1(S)$, (S, f) induces a well-defined outer automorphism of the quotient group $\pi_1(S)/K$, which we denote as $[f]_K \in \mathrm{Out}(\pi_1(S)/K)$.

Theorem 4.1. *Let* (S, f_A) *and* (S, f_B) *be automorphisms of a closed surface. If* $[f_A]_K$ *is conjugate to* $[f_B]_K$ *in* $\mathrm{Out}(\pi_1(S)/K)$ *for every characteristic finite index subgroup* K *of* $\pi_1(S)$, *then* f_A *and* f_B *are of identical type in the Nielsen–Thurston classification.*

Remark 4.2. (1) Theorem 4.1 is a consequence of a theorem due to H. Wilton and P. A. Zalesskii [28]. They prove that the profinite group completion detects the geometric decomposition of any finitely generated 3-manifold group. In fact, their result implies that f_A and f_B as in Theorem 4.1 have isomorphic Nielsen–Thurston decomposition graph decorated with vertex types (as being periodic or pseudo-Anosov). See [17, SECTION 12] for an exposition.

(2) The condition in Theorem 4.1 defines an equivalence relation on the set of automorphisms, which passes to an equivalence relation on the mapping class group $\mathrm{Mod}(S)$. Equivalent mapping classes in this sense are said to be *procongruently conjugate* (thinking of any K as a "principal congruence subgroup" in $\pi_1(S)$ by analogy). Any procongruent conjugacy class in $\mathrm{Mod}(S)$ is a disjoint union of conjugacy classes in $\mathrm{Mod}(S)$. Being procongruently conjugate is equivalent as having conjugate image under the natural homomorphism $\mathrm{Mod}(S) \to \mathrm{Out}(\widehat{\pi})$, where $\widehat{\pi}$ denotes the profinite completion of $\pi = \pi_1(S)$. The homomorphism naturally factors through the profinite completion of $\mathrm{Out}(\pi)$, which is very different from $\mathrm{Out}(\widehat{\pi})$ in general. See [17, SECTION 3] for detailed discussion.

The following theorems are proved in [17].

Theorem 4.3. *Let (S, f_A) and (S, f_B) be pseudo-Anosov automorphisms of a closed surface of genus ≥ 2. If $[f_A]_K$ is conjugate to $[f_B]_K$ in $\mathrm{Out}(\pi_1(S)/K)$ for every characteristic finite index subgroup K of $\pi_1(S)$, then f_A and f_B have identical stretch factor and their invariant foliations have identical number of singularities of each prong number. In fact, f_A and f_B have identical number of fixed points of each index.*

Theorem 4.4. *Let (S, f_B) be an automorphism of a closed surface. Then there exists a finite collection $\mathcal{B}$ of automorphisms of S with the following property. If (S, f_A) is any automorphism, such that $[f_A]_K$ is conjugate to $[f_B]_K$ in $\mathrm{Out}(\pi_1(S)/K)$ for every characteristic finite index subgroup K of $\pi_1(S)$, then f_A is isotopic to a topological conjugate of some $f_b \in \mathcal{B}$.*

Remark 4.5. Theorem 4.4 is equivalent to saying that every procongruent conjugacy class in $\mathrm{Mod}(S)$ is the disjoint union of finitely many conjugacy classes.

In the pseudo-Anosov case, the finiteness follows immediately from Theorem 4.3 (and Theorem 4.1), together with the well-known finiteness of pseudo-Anosov automorphisms with uniformly bounded stretch factor. See also [19] for a more recent finiteness result regarding profinite completions of finite-volume hyperbolic 3-manifold groups.

Example 4.6. Let S be the torus $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$. The mapping class group $\mathrm{Mod}(S)$ can be identified with $\mathrm{SL}(2, \mathbb{Z})$. In 1972, P. F. Stebe [26] discovered a pair of matrices

$$\begin{bmatrix} 188 & 275 \\ 121 & 177 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 188 & 11 \\ 3025 & 177 \end{bmatrix}$$

which are not conjugate in $\mathrm{SL}(2, \mathbb{Z})$, or in $\mathrm{GL}(2, \mathbb{Z})$, but are conjugate in $\mathrm{GL}(2, \mathbb{Z}/N\mathbb{Z})$ for any natural number N.

The above example shows that the finiteness in Theorem 4.4 cannot be improved to uniqueness, in general. Nevertheless, we pose the following question:

Question 4.7. Let S be a surface of negative Euler characteristic. If (S, f_A) and (S, f_B) are pseudo-Anosov automorphisms, such that $[f_A]_K$ and $[f_B]_K$ are conjugate in $\mathrm{Out}(\pi_1(S)/K)$ for every characteristic finite index subgroup K of $\pi_1(S)$, is it true that $[f_A]$ and $[f_B]$ are conjugate in $\mathrm{Out}(\pi_1(S))$?

For a one-holed torus, M. R. Bridson, A. W. Reid, and H. Wilton have answered Question 4.7 affirmatively [6]. More generally, if one could prove that finite-volume hyperbolic 3-manifold groups are profinitely rigid among 3-manifold groups, a positive answer to Question 4.7 should follow from the $\widehat{\mathbb{Z}}^{\times}$-regularity of profinite isomorphisms as in [19]. See [25] for a survey of the profinite rigidity problem, and [5] for some recent evidence in finite-volume hyperbolic 3-manifold groups.

Question 4.8. Input a pair of surface automorphisms (S, f_A) and (S, f_B). Is there an algorithm to certify the statement that for all characteristic finite index subgroups K of $\pi_1(S)$, $[f_A]_K$ and $[f_B]_K$ are conjugate in $\mathrm{Out}(\pi_1(S)/K)$?

Question 4.9. Let (S, f) be a pseudo-Anosov automorphism on a closed surface of genus ≥ 2. Is it possible to characterize the Heegaard Floer homology $\mathrm{HF}^+(M_f)$ [23, 24] in terms of $[f] \in \mathrm{Out}(\widehat{\pi})$, where $\widehat{\pi}$ denotes the profinite completion of $\pi = \pi_1(S)$?

5. MISCELLANEOUS ON FIBERED CONES

Let (S, f) be a surface automorphism. For any regular finite cover M' of the mapping torus M_f, the pullback ϕ' of the distinguished cohomology class ϕ_f lies in the interior of a unique fibered cone in $H^1(M'; \mathbb{R})$, which we simply refer to as the *distinguished fibered cone* of M'. The induced action of deck transformations on $H^1(M'; \mathbb{R})$ fixes ϕ', and hence preserves the distinguished fibered cone.

Theorem 5.1. *If (S, f) is a pseudo-Anosov automorphism on a surface of negative Euler characteristic, then for any natural number n, there exists a finite regular cover M' of M_f, such that the distinguished fibered cone of M' has at least n distinct deck transformation orbits of codimension-one faces.*

Remark 5.2.

(1) Theorem 5.1 is a key ingredient in the proof of Theorem 3.3. For the case with $\partial S = \emptyset$, see [18, PROBLEM 1.5] for an outline in dual terms of cones of homological directions; see also [19, SECTION 6.2] for a more detailed proof. The case with $\partial S \neq \emptyset$ can be derived easily using a well-known hyperbolic Dehn filling trick.

(2) By virtual specialization, every finite-volume hyperbolic 3-manifold is virtually fibered, and has unbounded virtual first Betti numbers [2]. Moreover, the virtual numbers of fibered cones are unbounded [1]. Theorem 5.1 shows that any fibered cone can virtually become as complicated as you want.

Question 5.3. Let (S, f) be a pseudo-Anosov automorphism of a surface of negative Euler characteristic. For any primitive periodic trajectory γ of $\pi_1(M_f)$, does there always exist a regular finite cover M' of M_f such that the homological direction along $[\gamma']$ is extreme on the distinguished cone of homological directions of M'? Here, γ' denotes any preimage component of γ in M'', and $[\gamma']$ denotes its homology class in $H_1(M;\mathbb{R})$.

In other words, is every primitive periodic trajectory covered by some virtual periodic trajectory in an extreme virtual homological direction?

Question 5.4. Let N be an orientable finite-volume hyperbolic 3-manifold. For any embedded closed geodesic γ, does there always exist a finite cover N' of N with a fibered class ϕ', such that some preimage component γ' of γ is freely homotopic to a periodic trajectory of the pseudo-Anosov suspension flow on N' dual to ϕ'?

In other words, is every primitive closed geodesic covered by some (essential) virtual periodic trajectory, with respect to some virtual fibering?

Question 5.5. Let N be an orientable finite-volume hyperbolic 3-manifold with cusps. In similar brief words, is every peripheral slope covered by some virtual slope of degeneracy, with respect to some virtual fibering?

Any cohomology class $\psi \in H^1(M_f;\mathbb{Z})$ in the distinguished fibered cone is homotopically represented by a bundle projection $M_f \to \mathbb{R}/\mathbb{Z}$. The monodromy of that bundle defines a surface automorphism whose mapping torus is homeomorphic to M_f. If f is pseudo-Anosov, then the surface automorphism (S^ψ, f^ψ) associated to ψ is also pseudo-Anosov, and its suspension flow is isotopic to θ_t up to parametrization.

Fried showed that the stretch factor $\lambda\colon \psi \mapsto \lambda(f^\psi)$ extends to a continuous function on the distinguished fibered cone of ϕ_f valued in $(1,+\infty)$, such that $\lambda(r\psi) = \lambda(\psi)^r$ holds for any $r > 0$. Moreover, restricted to the corresponding fibered face of the Thurston norm unit ball, $1/\log\lambda$ is a strictly concave function, and converges zero as ψ tends to the boundary. McMullen introduced a Teichmüller polynomial in the group ring $\Theta \in \mathbb{Z}H$, where H denotes the free abelianization of $\pi_1(M_f)$. One may think of Θ as a multivariable Laurent polynomial by fixing a basis of H, and Θ can be characterized by the property that $\lambda(\psi)$ is the maximum modulus among the zeros of the ψ-specialization of Θ (that is, $\sum_{h\in H} a_h t^{\psi(h)}$ in $\mathbb{Z}[t,t^{-1}]$, denoting $\Theta = \sum_{h\in H} a_h h$ in $\mathbb{Z}H$ and $\psi\colon H \to \mathbb{Z}$ in $H^1(M_f;\mathbb{Z})$). With the Teichmüller polynomial, McMullen reproved the above properties of Fried's stretch factor function λ, and went on to ask if the unique minimum of λ on the distinguished fibered face is achieved at a rational point (that is, a point in $H^1(M_f;\mathbb{Q})$) [21].

H. Sun exhibits examples where the rationality holds, and other more generic examples where the rationality fails [27]. The following theorem summarizes some properties of the stretch factor function as discovered in [27].

Theorem 5.6. *If (S, f) is a pseudo-Anosov automorphism, then the stretch factor minimizing point ψ_0 on the distinguished fibered face of M_f is either rational or transcendental. Moreover, for any finite cover M' of M_f, then the stretch factor minimizing point on the distinguished fibered face of M' is the pullback of ψ_0 divided by $[M' : M_f]$.*

As is implied by Theorem 5.6, the rationality/transcendence of the stretch factor minimizing point is a property that depends only the fibered cone, and is a commensurability class invariant with respect to cone in a certain sense.

We record the following question as suggested in Sun [27].

Question 5.7. Let N be a finite-volume hyperbolic 3-manifold. Does there always exist a finite cover N' of N, such that N' has a fibered face on which the stretch factor function is minimized at a transcendental point?

In [7], D. Calegari, H. Sun, and S. Wang initiate a systematic study of commensurability relations of surface automorphisms. A pair of surface automorphisms (S_A, f_A) and (S_B, f_B) are said to be *commensurable* if (S_A, f_A^k) and (S_B, f_B^l) admit a common finite covering surface automorphism for some natural numbers nonzero integers k and l. This is equivalent to saying that the mapping tori M_A of (S_A, f_A) and M_B of (S_B, f_B) admit a common finite cover such that the distinguished cohomology classes ϕ_A and ϕ_B are pulled back to rationally commensurable cohomology classes.

With natural extension of the terminology to 2-orbifold automorphisms, Calegari, Sun, and Wang prove the following theorem.

Theorem 5.8. *The commensurability class of any pseudo-Anosov surface automorphism has a unique (possibly orbifold) minimal member.*

Remark 5.9. Theorem 5.8 contrasts the well-known fact that the commensurability class of any arithmetic hyperbolic 3-manifold has infinitely many orbifold minimal members.

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