FROBENIUS HOMOMORPHISMS IN HIGHER ALGEBRA

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ABSTRACT

We will discuss versions of the Frobenius homomorphism for a ring spectrum R: the *Tate-valued Frobenius* $R \to R^{tC_p}$ and the *Frobenius on topological Hochschild homology* $\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$. Similar to ordinary algebra, these morphisms play an important role in higher algebra and are related to various concepts in stable homotopy theory and algebraic K-theory. We discuss the notion of *perfectness*, which is to say that these morphisms are equivalences, and relate this notion to the Segal conjecture, the red-shift conjecture, and the classification of spaces by stable data.

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In this survey I would like to give an overview of some of the ideas in higher algebra that have been central to my research over the last few years and which will also be crucial for the next years. Those ideas arose in joint work and discussions with a lot of people and many of the aspects are due to them.

The classical *Frobenius homomorphism* for an ordinary commutative ring R and a prime number p is the ring map

$$\varphi_p : R \to R/pR, \quad r \mapsto [r^p].$$

This homomorphism plays a major role in the analysis and structure theory of algebras and schemes in characteristic p. A very important class of $\mathbb{F}_p$-algebras are the *perfect* ones, i.e., those for which the Frobenius homomorphism is an isomorphism. For example, those algebras admit unique torsion-free lifts to $\mathbb{Z}_p$, called the Witt vectors. There are also mixed characteristic versions of perfect $\mathbb{F}_p$-algebras, called perfectoid rings, which have similar deformations. In this survey I will discuss several incarnations of Frobenius homomorphisms in higher algebra and shed light on the role of perfectness in this setting.

I will use the term "higher algebra" to mean "algebra" over the sphere spectrum $\mathbb{S}$, or said differently, the theory of spectra and ring spectra. The reader unfamiliar with the notion of spectra should just think of the sphere spectrum $\mathbb{S}$ as a stable homotopy-theoretic version of the ring of integers $\mathbb{Z}$. Spectra are modules over $\mathbb{S}$, thus a stable homotopy-theoretic version of abelian groups. Similarly, $\mathbb{E}_\infty$-ring spectra are commutative algebras over $\mathbb{S}$, a homotopy coherent version of commutative rings.

It was Waldhausen's vision to apply arithmetic ideas to ring spectra and develop a theory similar to ordinary algebra. He called this branch "brave new algebra" to express this idea. Nowadays there has been a lot of work by many people towards realizing this vision. Many ideas and concept have been transferred from ordinary to higher algebra, including scheme theory, obstruction theory, K-theory, and Hochschild homology. We view the results and ideas that we present in this survey as a further step in this program.

Specifically, we will define and discuss the following two incarnations of Frobenius homomorphisms for a ring spectrum R:

(1) The *Tate-valued Frobenius*

$$\varphi_p : R \to R^{tC_p}.$$

Here R^{tC_p} is the Tate construction which we discuss in Section 1.

(2) The Frobenius on topological Hochschild homology

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p},$$

where $\mathrm{THH}(R)$ is a spectrum associated with R, called *topological Hochschild homology* which generalizes ordinary Hochschild homology.

We will describe how these morphisms are related to many important aspects of stable homotopy theory such as power operations, the characterization of spaces by stable data, and the computation of algebraic K-theory. A major role for this will be played by the notion of perfectness, which is to say that these maps are equivalences (at least in a range of degrees). We will see how perfectness or ring spectra is related to the *Segal conjecture*, various far reaching generalizations of it and the *red-shift conjecture* in algebraic K-theory.

Higher algebra and algebraic K-theory. Let us first say a view general words about the motivation for higher algebra. Since spectra are the central objects in stable homotopy theory, it is clear that gaining an understanding of these objects is the major goal. For example, the homotopy groups of $\mathbb{S}$ are the stable homotopy groups of spheres, which besides their central role in homotopy theory are also closely related to understanding cobordism classes of manifolds by the Pontryagin–Thom construction. Waldhausen's original motivation was to study spaces of diffeomorphism and h-cobordisms, which he managed to relate to the algebraic K-theory groups of certain ring spectra.

More generally, algebraic K-theory is a topic which is a major motivation to study ring spectra. Recall that for an ordinary ring R, Quillen assigned groups $K_*(R)$ for every $* \in \mathbb{N}$, called the algebraic K-theory groups. Those generalized previously defined groups for $* = 0, 1, 2$ and by now play an important role in many areas of mathematics, from geometric topology to arithmetic. In his ICM address 1974, Quillen conjectured a deep relation of the higher K-theory groups to étale cohomology [42], the so-called Lichtenbaum–Quillen conjecture which has later been proven by Voevodsky [48].

By definition, the K-theory groups are the homotopy groups of a spectrum $K(R)$, and it turns out that studying the spectrum $K(R)$ itself is very often more fruitful than just its homotopy groups. For example, Waldhausen observed that Quillen's conjecture could be restated in terms of chromatic homotopy theory, namely that the map $K(R) \to L_1^f K(R)$ induces a p-adic equivalence in high enough degrees (for a prime p and suitable rings R). Here L_1^f is a certain *chromatic* localization, which is a spectral analogue of localizing a ring at a prime number. There are also higher versions of this localization denoted L_n^f for $n \geq 0$ and Rognes' red-shift problem in algebraic K-theory is about whether for certain ring spectra R the corresponding map $K(R) \to L_n^f K(R)$ is a p-adic equivalence in high degrees, see [44] for a more precise formulation. This question can be seen as a natural higher variant of the Lichtenbaum–Quillen conjecture.

In order to study the K-theory spectrum $K(R)$, Bökstedt invented the theory of *topological Hochschild homology* [9], which is the natural generalization of Hochschild homology to the sphere spectrum. Indeed, he defined for every ring R a spectrum $\mathrm{THH}(R)$. Later Bökstedt–Hsiang–Madsen [10] used the definition of THH to define another spectrum $\mathrm{TC}(R)$, called *topological cyclic homology* which comes with a map $K(R) \to \mathrm{TC}(R)$, called the *cyclotomic trace*. This map is a natural generalization of the Chern-character and is often close to an isomorphism by work of many people (see [13] for a very definitive statement). Therefore, if one has a good understanding of topological cyclic homology, one can use this to compute and understand algebraic K-theory. We will explain how the Frobenius on THH is the key structure needed to gain such an understanding.

Overview of this survey. Generally, our aim is not to give a technically precise account or an exhaustive list of applications and results, but rather explain some fundamental ideas and give the reader an insight into the phenomena that can show up. In particular, we do not talk about the concrete computations that have been done using the technology presented here (see, e.g., [21, 43, 45]) since this would go beyond the scope of this article.

We will motivate and complement all the homotopy theoretic construction in higher algebra by the corresponding statements and constructions in ordinary algebra. Therefore most of the sections contain a short paragraph about the "classical" analogue, presented in a way that makes it easier to understand what happens in the higher case.

In Section 1 we introduce for a spectrum X and an action of a finite group G on X the Tate construction X^{tG}. This is a new spectrum whose homotopy groups are closely related to ordinary Tate cohomology. We explain an important theorem of Lin and Gunawardena, which states that for X the sphere spectrum this Tate spectrum is a completion of the sphere spectrum. In Section 2 we construct the Tate diagonal, which is a certain natural map of spectra $X \to (X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X)^{tC_p}$ akin to the diagonal map of a set. This map is the key needed to define all the Frobenius homomorphisms in higher algebra that we discuss in subsequent sections. By a deep result of Rognes–Lunøe-Nielsen (with a small refinement by the author with P. Scholze) this Tate diagonal is often an equivalence. Section 3 contains the construction of the Tate-valued Frobenius $R \to R^{tC_p}$ for an $\mathbb{E}_\infty$-ring spectrum. We also discuss when it gives rise to an equivalence and the notion of perfectness for $\mathbb{E}_\infty$-rings. In Section 4 we introduce the dual version of the Frobenius for $\mathbb{E}_\infty$-coalgebra spectra C. Magically, this gives an endomorphism $\varphi_p : C \to C$ if C is connective and p-complete. We discuss why this endomorphism is the identity for the suspension spectrum $C := (\Sigma^\infty_+ X)^\wedge_p$ of a space X and conjecture that this precisely characterizes suspension spectra among all spectra. In Section 5 we quickly review topological Hochschild homology and then introduce the Frobenius $\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$. We explain how this is gives rise to a cyclotomic structure and how one can define $\mathrm{TC}(R)$ from that. Finally, we elaborate on the role of perfectness of THH, meaning that the Frobenius is an equivalence. Section 6 contains a discussion of cyclotomic spectra with *Frobenius lifts* and the relation between perfectness and boundedness of cyclotomic spectra. We also explain how this relates to the Quillen–Lichtenbaum conjecture following work of Mathew and Hahn–Wilson. In the final Section 7 we explain the Segal conjecture and state a conjecture which would drastically generalize it. This conjecture is the key to prove the conjectures made in Section 4.

All of the section are based on the results and notation introduced in the first two sections. But the reader only interested in the THH-aspects can skip Sections 3 and 4 and jump right into Section 5 and 6. Also Section 7 is independent of Sections 3–6.

1. THE TATE CONSTRUCTION

In this first section we will introduce and study the Tate construction X^{tG} for an action of a finite group G on a spectrum X. The Tate construction is a stable homotopy theoretic version of Tate cohomology, hence the name. We will mostly be interested in the case of the cyclic group C_p with p elements for a prime number p.

The Tate construction in algebra. Let A be an abelian group with an action by a finite group G. In this situation we have a norm map from the coinvariants to the invariants:

$$A_G \to A^G, \quad [a] \mapsto \sum_{g \in G} ga.$$

The zeroth Tate cohomology is the cokernel of this map,

$$\hat{H}^0(G; A) := A^G / A_G.$$

By definition, $\hat{H}^0(G; A)$ comes with a canonical quotient map $\mathrm{can} : A^G \to \hat{H}^0(G; A)$.

Example 1.1. Assume that G acts trivially on A. Then we get $H^0(G; A) = A/|G|$. To say this a bit more systematically, we note that for the trivial action of G on A we have a natural map

$$\mathrm{triv} : A \xrightarrow{=} A^G \xrightarrow{\mathrm{can}} H^0(G; A), \quad a \mapsto [a],$$

and this map induces the isomorphism $A/|G| \xrightarrow{\cong} H^0(G; A)$.

The "external product"

$$H^0(G; A) \otimes H^0(G; B) \to H^0(G; A \otimes B), \quad [a] \otimes [b] \mapsto [a \otimes b],$$

makes the zeroth Tate cohomology into a lax symmetric monoidal functor. It follows that if A has a ring structure such that G acts through ring homomorphisms, then also $H^0(G; A)$ is a ring.

Example 1.2. For every abelian group A with G-action, we have that multiplication by $|G|$ is zero on $H^0(G; A)$. To see this, note that A is a module over $\mathbb{Z}$ with trivial G-action and so it follows that $H^0(G; A)$ is a $\mathbb{Z}/|G|$-module.

The Tate construction in higher algebra. Let G be a finite group acting on a spectrum X, that is, a functor $BG \to \mathrm{Sp}$. Then we can form the homotopy orbits and the homotopy fixed points X_{hG} and X^{hG} defined as the colimit and limit of this functor. There is a natural norm map

$$\mathrm{Nm}_G : X_{hG} \to X^{hG}$$

such that the composition $X \to X_{hG} \to X^{hG} \to X$ is given by the sum over all the maps $\cdot g : X \to X$ for $g \in G$. The precise definition requires some coherence technology, see, e.g., [32, SECTION 6.16] or [41, CHAPTER 1]. The *Tate construction* is then defined as the cofiber

$$X^{tG} := \mathrm{cofib}(\mathrm{Nm}_G) = X^{hG}/X_{hG}.$$

Example 1.3. Let HA be the Eilenberg–MacLane spectrum of an ordinary abelian group with a G-action. Then we have that $\pi_0(HA^{tG}) = \hat{H}^0(A; G)$. That is the sense in which the

Tate construction for spectra generalizes the construction from the previous section. We can also describe the other homotopy groups as follows:

$$\pi_*(A) = \begin{cases} \ker(A_G \to A^G), & * = 1, \\ H_{*-1}(G; A), & * > 1, \\ H^{-*}(G; A), & * < 0. \end{cases}$$

This follows immediately from the long exact sequence obtained from the defining cofiber sequence. These are the ordinary *Tate cohomology groups*.

By construction, we have a canonical map

$$\mathrm{can} : X^{hG} \to X^{tG}$$

and if X carries the trivial G-action then we can compose this with the map $X \to X^{hG} = X^{BG}$ induced by the projection $BG \to \mathrm{pt}$ to obtain a natural map

$$\mathrm{triv} : X \to X^{tG}.$$

Theorem 1.4 (Lin [28], Gunawardena [17]). *For the sphere spectrum $\mathbb{S}$ equipped with the trivial action of C_p, the map* $\mathrm{triv} : \mathbb{S} \to \mathbb{S}^{tC_p}$ *is a p-completion, i.e., exhibits $\mathbb{S}^{tC_p}$ as the p-complete sphere $\mathbb{S}_p^\wedge$.*

In my opinion, this result is one of the deepest and most striking results in stable homotopy theory. For example, the fact that $\mathbb{S}^{tC_p}$ is connective is already quite surprising if one thinks about Example 1.3. In fact, there is a convergent spectral sequence

$$\hat{H}^i\big(C_p, \pi_j(\mathbb{S})\big) \Rightarrow \pi_{j-i}(\mathbb{S}^{tC_p}).$$

Every single page of this spectral sequence will be periodic, still the result is connective. Theorem 1.4 also shows that the homotopy groups of the Tate construction X^{tC_p} are in contrast to the algebraic case generally not p-torsion groups, as $\pi_0(\mathbb{S}_p^\wedge) = \mathbb{Z}_p$.

From Theorem 1.4 it follows that for any p-adically finite[1] spectrum X with trivial C_p-action the map triv map also induces an equivalence $X_p^\wedge \to X^{tC_p}$ which should be considered as the higher algebra analogue of Example 1.1. For a general spectrum X, this, however, completely fails as the example of $X = H\mathbb{Z}$ already shows. However, the completeness part is still true very generally and should be considered as the "correct" analogue of Example 1.2.

Proposition 1.5 ([41, LEMMA I.2.9.]). *Let X be a bounded below spectrum with C_p-action. Then X^{tC_p} is p-complete.*

The bounded below assumption is crucial here, without that one can easily find counterexamples, e.g., the complex K-theory spectrum KU with trivial C_p-action for which KU^{tC_p} is rational and nontrivial.

1 This means that the p-completion of X is finite over the p-complete sphere. For example, the p-complete sphere itself is an example which is, of course, not finite over $\mathbb{S}$.

Proposition 1.6 ([41, THEOREM I.3.1]). *The functor* $(-)^{tG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$ *admits a canonical lax symmetric monoidal structure.*

In particular, when applied to ring spectra, the Tate construction produces ring spectra. For example, for an ordinary ring R with G-action, we find that $(HR)^{tG}$ is a ring spectrum, which on homotopy groups induces a graded multiplication. This is the multiplication in ordinary Tate cohomology, whose existence I find quite striking.

2. THE TATE DIAGONAL

The goal of this section is to explain that for every spectrum X and every prime p there is a natural map $\Delta_p : X \to (X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X)^{tC_p}$ which we call the *Tate diagonal*. Here C_p-acts by cyclic permutation on the p-fold tensor product $X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X$. This map Δ_p will be the source of all the Frobenius homomorphisms in higher algebra. As usual we want to explain the algebraic analogue first.

The Tate diagonal in algebra. Let A be an abelian group and p a prime number. We consider the map

$$A \to (A \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A)^{C_p}, \quad a \mapsto a \otimes \cdots \otimes a, \tag{2.1}$$

where the target has p tensor factors and C_p acts by cyclic permutations. This map is obviously not additive, as it sends $a_0 + a_1$ to

$$(a_0 + a_1) \otimes \cdots \otimes (a_0 + a_1) = \sum_{(i_0,\dots,i_p)} a_{i_0} \otimes \cdots \otimes a_{i_p},$$

where the sum ranges over all sequences in $\{0,1\}^p$. We see that the deviation from additivity is exactly an element in the image of the norm, so that the composite

$$\Delta_p : A \to (A \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A)^{C_p} \xrightarrow{\text{can}} \hat{H}^0\big(C_p; (A \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A)\big) \tag{2.2}$$

is a group homomorphism. This map is the algebraic version of the Tate diagonal.

Proposition 2.1. *For any abelian group and every prime* p*, the map* Δ_p *induces an isomorphism* $A/p \to \hat{H}^0(C_p; A^{\otimes p})$.

Sketch. The target is a p-torsion group by Example 1.2. Therefore we get an induced map $A/p \to \hat{H}^0(C_p; A^{\otimes p})$. One checks by hand that this map is an isomorphism for $A = \mathbb{Z}/n$ and $A = \mathbb{Z}$.

The key fact to observe is that the construction $\hat{H}^0(C_p; A^{\otimes p})$ commutes with direct sums in A. This can be seen by expanding the expressing for a direct sum and using that $\hat{H}^0(C_p; -)$ vanishes on induced C_p-modules. Therefore we immediately deduce the result for all finitely generated abelian groups A. Finally, the functor also commutes with filtered colimits in A so that it follows for all abelian groups. ■

The Tate diagonal in higher algebra. The higher version of the diagonal (2.2) is incarnated as follows. Considers the functor

$$\mathrm{Sp} \to \mathrm{Sp}, \quad X \mapsto (X^{\otimes p})^{tC_p}. \tag{2.3}$$

This functor admits a lax symmetric monoidal structure induced by the lax symmetric monoidal structure of the Tate construction.

Proposition 2.2. *For any finite spectrum X, there is a map of spectra*

$$\Delta_p : X \to (X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X)^{tC_p}.$$

This map natural and symmetric monoidal in X and unique with respect to these properties.

We will refer to Δ_p as the Tate diagonal. The key observation to prove Proposition 2.2 is that the functor (2.3) is exact. This exactness can be reduced to showing that it preserves binary direct sums $X \oplus Y$ which then amounts to a categorification of the argument given in the last section. Finally, one uses that the identity functor $\mathrm{Sp} \to \mathrm{Sp}$ is the initial lax symmetric monoidal endofunctor **[39]** to deduce Proposition 2.2.

We have the following analogue of Proposition 2.1:

Theorem 2.3 (Lunøe-Nielsen–Rognes **[30]**, Nikolaus–Scholze **[41]**). *For any bounded below spectrum X and any prime p, the Tate diagonal*

$$X \to (X \otimes \cdots \otimes X)^{tC_p}$$

is a p-completion.

Note that for $X = \mathbb{S}$ this theorem reduces to the Theorem of Lin and Gunawardena (Theorem 1.4). Rognes and Lunøe-Nielsen have proven Theorem 2.3 under the additional assumption that X is of finite type over $\mathbb{F}_p$ by relating it to the algebraic "Singer construction" of Li–Singer and Adams–Gunawardena–Miller. In joint work with P. Scholze, we prove the extension to all bounded below spectra. In fact, we deduce that this follows using formal properties of the Tate construction from the case $X = H\mathbb{F}_p$.

Remark 2.4. For $X = H\mathbb{F}_p$ the Theorem asserts that $(H\mathbb{F}_p \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} H\mathbb{F}_p)^{tC_p}$ is concentrated in degree 0. One can try proving this directly using the Tate spectral sequence, which for example for $p = 2$ takes the form

$$\hat{H}^i\big(C_2, \mathcal{A}_2^\vee\big) \Rightarrow \pi_{j-i}\big((H\mathbb{F}_2 \otimes_{\mathbb{S}} H\mathbb{F}_2)^{tC_2}\big),$$

where $\mathcal{A}_2^\vee$ is the dual Steenrod algebra which carries a C_2-action by the conjugation χ. The dual Steenrod algebra, as well as the conjugation, have been calculated by Milnor **[38]**, but even the invariants of this action are unknown in general (even though it is easy to calculate them in small degrees with a computer). This means that one cannot completely calculate the first page of the spectral sequence. But even in the range where this is possible, it is in my opinion impossible to determine the differentials. Using Theorem 2.3 and the knowledge that almost everything has to cancel out, one can determine the differentials for the first 20 stems or so. This leads to a wild pattern in which I did not see any regularity.

3. THE TATE-VALUED FROBENIUS

Let R be an $\mathbb{E}_\infty$-ring spectrum and p a fixed prime number. The goal of this section is to discuss a variant of the Frobenius homomorphism for ring spectra, called the *Tate-valued Frobenius*, which will be an $\mathbb{E}_\infty$-map $R \to R^{tC_p}$. Here C_p-acts trivially on R.

Recall that for an ordinary commutative ring R the Frobenius homomorphism is the ring morphism

$$R \to R/p, \quad r \mapsto r^p.$$

Using the algebraic Tate-diagonal (2.1), we can write this map as the composite

$$R \xrightarrow{\Delta_p} \hat{H}^0(C_p; R \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R) \xrightarrow{m} \hat{H}^0(C_p; R) \xrightarrow{\cong} R/p,$$

where $m : R \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R \to R$ is the p-fold multiplication map of the ring R, which is C_p-equivariant for the cyclic action on the source and the trivial action on the target.

Definition 3.1 ([41, DEFINITION IV.1.1]). Let R be an $\mathbb{E}_\infty$-ring spectrum. The Tate-valued Frobenius is the composite

$$\varphi_p : R \xrightarrow{\Delta_p} (R \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} R)^{tC_p} \to R^{tC_p},$$

where Δ_p is the Tate diagonal (cf. Proposition 2.2). The Tate-valued Frobenius is by construction a natural map of $\mathbb{E}_\infty$-ring spectra.

Example 3.2. Let HR be the Eilenberg–MacLane spectrum of an ordinary ring R. Then on π_0 the Tate-valued Frobenius induces the ordinary Frobenius $R \to R/p$.

Example 3.3. One can generalize the last example: if R is a connective ring spectrum, then we have a canonical map $d : \pi_0(R^{tC_p}) \to \pi_0(R)/p$ and the composite

$$\pi_0(R) \xrightarrow{\pi_0(\varphi_p)} \pi_0(R^{tC_p}) \xrightarrow{d} \pi_0(R)/p$$

is the Frobenius of $\pi_0(R)$. Note that in contrast to the discrete case, the map d is in general far from an isomorphism; for example, in the case $R = \mathbb{S}$ of the sphere spectrum, it can be identified with the projection $\mathbb{Z}_p \to \mathbb{F}_p$ (using Theorem 1.4).

The Tate-valued Frobenius is closely related to power operations. For example, if $R = C^*(X, \mathbb{F}_2)$ is the $\mathbb{E}_\infty$-ring of $\mathbb{F}_2$-valued cochains on a space X then we have that $\pi_*(C^*(X, \mathbb{F}_2)^{tC_2}) = H^{-*}(X; \mathbb{F}_2)((t))$ with t in homological degree -1. On homotopy groups the Tate-valued Frobenius induces the map

$$H^*(X; \mathbb{F}_2) \to H^*(X; \mathbb{F}_2)((t)), \quad x \mapsto \sum_{i=0}^{\infty} \mathrm{Sq}^i(x) t^{-i},$$

where $\mathrm{Sq}^i : H^*(X; \mathbb{F}_2) \to H^{*+i}(X; \mathbb{F}_2)$ is the ith Steenrod operation, see [41, PROPOSITION IV.1.16] or [49, SECTION 3.5]. Note that this sum is finite by the instability relation $\mathrm{Sq}^i(x) = 0$ for $i > -|x|$ in homological grading. More generally, for an arbitrary $\mathbb{E}_\infty$-algebra R over $\mathbb{F}_2$, we have that the Frobenius induces the map

$$\pi_*(R) \to \pi_*(R)((t)), \quad x \mapsto \sum_{i=-\infty}^{\infty} Q^i(x) t^i,$$

where $Q^i : \pi_*(R) \to \pi_{*+i}(R)$ is the ith Dyer–Lashof operation. This sum is in general not finite, but a Laurent series since $Q^{-i}(x) = 0$ for $i > -|x|$. In fact, one could take this as the definition of the Dyer–Lashof operations and derive the usual properties from it, see Wilson's paper [49] for a very nice discussion employing this perspective.

One can also get the odd primary Steenrod and Dyer–Lashof operations through this perspective. More generally one can relate power operations for algebras over any base $\mathbb{E}_\infty$-algebra to the Tate-valued Frobenius. See [16] for a very nice discussion of general power operations in the language of ring spectra and the relation to Tate spectra.

The Frobenius as an endomorphism. The main difference of the Tate-valued Frobenius to the ordinary Frobenius homomorphism is that the target is in general not p-torsion but rather p-complete as explained around Proposition 1.5. Recall that by Theorem 1.4 we know that if the underlying spectrum of R is a finite spectrum, then the map

$$\mathrm{triv} : R \to R^{tC_p}$$

is a p-completion. Thus using the inverse to this map we can consider the Tate-valued Frobenius as a map

$$\varphi_p : R \to R_p^\wedge$$

for every p-adically finite ring spectrum R. The finiteness assumption is crucial here and in general the Tate-valued Frobenius does not induce an endomorphism. However, sometimes it does, for example, if R is the dual of a connective coalgebra, as we will see in the next section.

Example 3.4. For the sphere spectrum $\mathbb{S}$, we have that φ_p is the completion map

$$\mathbb{S} \to \mathbb{S}_p^\wedge.$$

This is forced, since it is a map of ring spectra. More generally, let k be a finite field of characteristic p. Then there is an $\mathbb{E}_\infty$-ring $\mathbb{S}_{W(k)}$ called the *ring of spherical Witt vectors* uniquely characterized by the property that it is p-complete, flat over $\mathbb{S}_p^\wedge$ and that $\pi_0(\mathbb{S}_{W(k)})$ is the ordinary ring of Witt vectors $W(k)$. This ring spectrum is finite over the p-complete sphere (see, e.g., [33, EXAMPLE 5.2.7] for a discussion of spherical Witt vectors). Thus the Frobenius is an endomorphism

$$\varphi_p : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}$$

which can be identified with the map induced by the Witt vector Frobenius $F : W(k) \to W(k)$. This follows by an obstruction theoretic argument from Example 3.3 since maps between $\mathbb{S}_{W(k)}$ are uniquely determined by their effect on the modulo p-reduction of π_0.

Definition 3.5. We say that a p-complete and p-adically finite $\mathbb{E}_\infty$-ring spectrum R is *perfect* if the map $\varphi_p : R \to R$ is an equivalence.

Example 3.6. Let X be a finite CW complex. Then we consider the mapping spectrum $R := \mathrm{map}(X, \mathbb{S}_p^\wedge)$ which is an $\mathbb{E}_\infty$-algebra. Since the Frobenius is natural and is the identity on the sphere, it follows that it is also the identity on R. In particular R is perfect.

More generally, we can consider a hypercomplete sheaf of spaces on the étale site of $\mathrm{Spec}(\mathbb{F}_p)$, or said differently, a homotopy type X with a (continuous) action of the profinite group $\mathbb{Z}^\wedge$.[2] We will denote this ∞-category by $\mathcal{S}^{\mathbb{Z}^\wedge}$. Then we can form a twisted version $\mathrm{map}_\sigma(X, \mathbb{S}_p^\wedge)$ of $\mathrm{map}(X, \mathbb{S}_p^\wedge)$ as

$$\mathrm{map}_\sigma\big(X, \mathbb{S}_p^\wedge\big) := \mathrm{map}_{\mathbb{Z}^\wedge}(X, \mathbb{S}_{W(\overline{\mathbb{F}}_p)}).$$

In this finite case this p-complete $\mathbb{E}_\infty$-algebra $\mathrm{map}_\sigma(X, \mathbb{S}_p^\wedge)$ is also perfect. In fact, one can deduce from Mandell's theorem [34] that the assignment

$$\mathcal{S}^{\mathbb{Z}^\wedge} \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}), \quad X \mapsto \mathrm{map}_\sigma\big(X, \mathbb{S}_p^\wedge\big)$$

defines an equivalence between the full subcategory of equivariantly p-complete, finite, and simply connected spaces on the left and of perfect $\mathbb{E}_\infty$-algebras R which are p-complete, p-completely finite, and whose reduced $\mathbb{F}_p$-cohomology is simply connected. See [50, SECTION 7] and [31, SECTION 3.5] for similar statements. This should be seen as a classification of a large class of perfect $\mathbb{E}_\infty$-algebras.

4. THE COALGEBRA FROBENIUS

In the last section we have construction the Tate-valued Frobenius $R \to R^{tC_p}$ which could be interpreted as an endomorphism for R finite. In this section we construct a dual morphism for connective coalgebra spectra C which will always be an endomorphism. The construction crucially relies on Theorem 2.3.

The Frobenius homomorphism for ordinary coalgebras. Recall that a cocommutative coalgebra over a field k is a k-module C together with k-linear maps

$$\begin{aligned} \Delta : C &\to C \otimes_k C && \text{(comultiplication)},\\ \varepsilon : C &\to k && \text{(counit)} \end{aligned}$$

satisfying the duals of the axioms of a commutative k-algebra. In fact, one can simply define a coalgebra as an algebra in the opposite category of the category of k-vector spaces.

The dual $C^\vee = \mathrm{Hom}_k(C, k)$ of a coalgebra is always an algebra but the dual $A^\vee$ of an algebra A is in general not a coalgebra unless A is finite dimensional. More precisely, dualization induces a functor

$$(-)^\vee : \mathrm{coAlg}_k^{\mathrm{op}} \to \mathrm{Alg}_k$$

which restricts to an equivalence between finite dimensional coalgebras and algebras.

The fundamental theorem of coalgebras asserts that every coalgebra is the colimit of its finite dimensional subcoalgebras, see [46]. One can use this to show that the category coAlg_k is the ind-category of the category of finite dimensional coalgebras. In particular, it

2 Here continuity is not meant on the naive sense, but in a sense similar to continuous actions of profinite groups on discrete sets. One can make this precise using condensed mathematics.

is opposite equivalent to the category of profinite algebras. From this we can directly deduce the existence of a Frobenius homomorphism for coalgebras over $\mathbb{F}_p$:

Proposition 4.1. *For every cocommutative coalgebra C over $\mathbb{F}_p$ there is a natural map*

$$\varphi_p : C \to C$$

uniquely characterized by the property that its dual $\varphi_p^\vee : C^\vee \to C^\vee$ is the Frobenius homomorphism of the commutative $\mathbb{F}_p$-algebra $C^\vee$.

One can also give a more direct construction of the Frobenius for coalgebras involving the algebraic Tate diagonal. Namely one shows that for every coalgebra over $\mathbb{F}_p$ the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta^p} & (C \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} C)^{C_p} \\ {\scriptstyle\varphi_p}\downarrow & & \downarrow{\scriptstyle\text{can}} \\ C & \xrightarrow{\Delta_p} & \hat{H}^0(C_p; C \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} C) \end{array}$$

commutes, where the upper horizontal map is the p-fold comultiplication of C and the lower horizontal map is the algebraic Tate diagonal. The algebraic Tate diagonal is an isomorphism (Proposition 2.1) so that this diagram defines φ_p, see [40] for a discussion along those lines.

The Frobenius for coalgebra spectra. We now want to discuss the higher algebra analogue of the Frobenius for coalgebras. Thus let C be an $\mathbb{E}_\infty$-coalgebra spectrum, that is, an $\mathbb{E}_\infty$-algebra object in the opposite category of spectra. We can form the dual spectrum $C^\vee = \mathrm{map}(C, \mathbb{S})$ and this will be an $\mathbb{E}_\infty$-ring spectrum. As in the algebraic case, we find that this construction induces a functor of ∞-categories

$$(-)^\vee : \mathrm{coAlg}_{\mathbb{E}_\infty}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$$

which restricts to an equivalence on those full subcategories spanned by objects that are finite over $\mathbb{S}$.

Warning 4.2. The analogue of the fundamental theorem for coalgebras fails over $\mathbb{S}$: not every coalgebra over $\mathbb{S}$ is a filtered colimit of finite coalgebras. Also finite coalgebras are not necessarily compact as objects of $\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ and $\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ is not compactly generated. But it is still a presentable ∞-category and comonadic over the ∞-category of spectra as one can see using the monadicity theorem.

Definition 4.3. For every connective coalgebra $C \in \mathrm{coAlg}_{\mathbb{E}_\infty}(\mathrm{Sp})$, we define a Frobenius morphism $C \to C_p^\wedge$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta^p} & (C \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} C)^{hC_p} \\ {\scriptstyle\varphi_p}\downarrow & & \downarrow{\scriptstyle\text{can}} \\ C_p^\wedge & \xrightarrow{\Delta_p} & (C \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} C)^{tC_p} \end{array} \tag{4.1}$$

commutes. Here we use that the lower horizontal map is an equivalence by Theorem 2.3.

A p-complete, connective $\mathbb{E}_\infty$-coalgebra C over $\mathbb{S}_p^\wedge$ is called perfect if the Frobenius $\varphi_p : C \to C$ is an equivalence.

By definition, φ_p is a map of spectra. However, we can consider it as a map $C_p^\wedge \to C_p^\wedge$ and using the monoidal properties of the Tate diagonal one gets the following result:

Proposition 4.4. *The map $\varphi_p : C_p^\wedge \to C_p^\wedge$ canonically refines to a natural map of co-$\mathbb{E}_\infty$-algebras in p-complete spectra. For finite C this morphisms dualizes to the Frobenius of ring spectra as discussed in Section 3.*

Sketch of proof. A coalgebra in $\mathrm{Sp}_p^\wedge$ is essentially the same as a symmetric monoidal functor

$$\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Sp}_p^\wedge,$$

where Fin is the category of finite sets with disjoint union as symmetric monoidal structure. The opposite category $\mathrm{Fin}^{\mathrm{op}}$ can then naturally be considered as a symmetric monoidal category and thus as a symmetric monoidal ∞-category. It is the free symmetric monoidal ∞-category with a cocommutative coalgebra object (given by the singleton).

Thus for a given p-complete coalgebra $C \in \mathrm{Sp}_p^\wedge$, we get an essentially unique symmetric monoidal functor $\underline{C} : \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Sp}_p^\wedge$ which sends the singleton to C. We can now take the diagram (4.1) and replace all the instances of C by the functor $\underline{C}$ to obtain a diagram

$$\begin{array}{ccc} \underline{C} & \xrightarrow{\Delta^p} & (\underline{C} \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} \underline{C})^{hC_p} \\ {\scriptstyle \varphi_p}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ \underline{C} & \xrightarrow{\Delta_p} & (\underline{C} \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} \underline{C})^{tC_p} \end{array}$$

of functors $\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Sp}_p^\wedge$. The functors are not strong symmetric monoidal, but still lax symmetric monoidal, and all the maps admit the structure of symmetric monoidal transformations. Therefore it follows that the left vertical morphism is also symmetric monoidal which shows the first claim. We skip the argument for the second. ■

We conclude this section by a conjectural further refinement of the Frobenius for coalgebras.

Conjecture 4.5. *The Frobenius refines to an action of the monoidal category* $\mathrm{B}\mathbb{N}$ *on the ∞-category* $\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathrm{Sp}_p^\wedge)$ *of p-complete $\mathbb{E}_\infty$-coalgebras.*

Here $\mathrm{B}\mathbb{N}$ is the category with a single object and the natural numbers as endomorphisms. This category is itself symmetric monoidal since $\mathbb{N}$ is abelian.

Remark 4.6. An action of $\mathrm{B}\mathbb{N}$ is the same as an $\mathbb{E}_2$-map[3]

$$\rho : \mathbb{N} \to Z := Z\big(\mathrm{coAlg}_{\mathbb{E}_\infty}(\mathrm{Sp}_p^\wedge)\big),$$

3 By $\mathbb{E}_2$-monoid we mean $\mathbb{E}_2$-algebra in the ∞-category $\mathcal{S}$ of spaces.

where the target is the center of the ∞-category of coalgebras, by which we mean the $\mathbb{E}_2$-monoid of endomorphisms of the identity functor $\mathrm{id} : \mathrm{coAlg}_{\mathbb{E}_\infty}(\mathrm{Sp}_p^\wedge) \to \mathrm{coAlg}_{\mathbb{E}_\infty}(\mathrm{Sp}_p^\wedge)$. Of course, the map ρ should send the generator of $\mathbb{N}$ to the Frobenius. This determines it as an $\mathbb{E}_1$-map, since $\mathbb{N}$ is the free $\mathbb{E}_1$-space on a single generator. So the conjecture is about a refinement of the $\mathbb{E}_1$-map given by the Frobenius to an $\mathbb{E}_2$-map. Informally, this means that the Frobenius has to coherently commute with itself.

One can try to understand such a question by obstruction theory: the $\mathbb{E}_2$-monoid $\mathbb{N}$ admits a cell structure

$$\mathbb{N} = \underrightarrow{\mathrm{colim}}(M_1 \to M_2 \to M_3 \to M_4 \to \cdots)$$

in which M_1 is the free $\mathbb{E}_2$-monoid on a single generator in degree 0 and each M_{n+1} is obtained from M_n by attaching an $\mathbb{E}_2$-cell of dimension $2n$. This can be seen applying arguments similar to those used in [2, APPENDIX B] and we learned this statement from Achim Krause. After group completion the induced $\mathbb{E}_2$-cell structure on $\mathbb{Z} = \Omega^2\mathbb{C}\mathrm{P}^\infty$ corresponds to the standard cell structure of $\mathbb{C}\mathrm{P}^\infty$.

The Frobenius defines a map $M_1 \to Z$, and we get iteratively for $n \geq 1$ a sequence of obstructions $o_n \in \pi_{2n-1}(Z)$. One could hope that all these homotopy groups vanish, but we have no insight into whether this might be true. However, if we restrict to perfect coalgebras this could hold, at least it does under additional finiteness conditions, by the results outlined at the end of Section 3.

In recent work [50], Allen Yuan has proven a version of this conjecture, namely he shows that it is true when we restrict to the full subcategory spanned by those coalgebras whose underlying spectrum is p-adically finite. Equivalently, this can be then be translated into the dual setting of $\mathbb{E}_\infty$-algebras by Proposition 4.4. Yuan really proves the corresponding statement about the Frobenius of finite $\mathbb{E}_\infty$-algebras. His proof crucially relies on the Segal conjecture, that we will discuss in Section 7 below, specifically Theorem 7.8. In order to apply similar techniques to prove Conjecture 4.5 in its full generality, one would have to prove the version of the Segal conjecture for the norm that we propose as Conjecture 7.10.

Boolean coalgebras. In this section we want to talk about a specific class of coalgebras over the sphere, which we call Boolean coalgebras. The definition is conditional to Conjecture 4.5. We first give the corresponding definition for ordinary coalgebras.

Definition 4.7. Let C be an (ordinary) coalgebra over $\mathbb{F}_p$. Then it is called Boolean (or p-Boolean if we want to emphasize the prime p) if the Frobenius endomorphism $\varphi_p : C \to C$ is the identity.

The name Boolean is motivated by the fact that the dual notion (i.e., algebras over $\mathbb{F}_p$ with $\varphi_p = \mathrm{id}$) is for $p = 2$ the same as a Boolean algebra in the sense of logic.

Proposition 4.8 (coStone duality). *The category of p-Boolean coalgebras is equivalent to the category of sets. The equivalence is given by sending a coalgebra C to the set of coalgebra morphisms from $\mathbb{F}_p$ to C, i.e., the set of grouplike elements in C. Conversely, a set S is sent to the coalgebra $\bigoplus_S \mathbb{F}_p$ with the pointwise comultiplication.*

Now we would like to generalize this to higher algebra, i.e., to give a definition of Boolean coalgebra spectra. The definition of those is conditional to Conjecture 4.5, which was about the existence of an $B\mathbb{N}$-action on the ∞-category of p-complete coalgebras, which is pointwise given by the Frobenius endomorphism $\varphi_p : C \to C$. Under suitable finiteness conditions, this in fact follows from the results of Yuan.

Definition 4.9. A p-Boolean coalgebra spectrum is a fixed point for the action of $B\mathbb{N}$ on the ∞-category of $\mathbb{E}_\infty$-coalgebras. A (global) Boolean coalgebra spectrum is an $\mathbb{E}_\infty$-coalgebra C over $\mathbb{S}$ together with a refinement to a p-Boolean coalgebra spectrum for every prime p.

Concretely, a p-Boolean coalgebra is a p-complete Boolean $\mathbb{E}_\infty$-coalgebra C together with a *coherent* equivalence between the Frobenius endomorphism $\varphi_p : C \to C$ and the identity $\mathrm{id} : C \to C$.

Example 4.10. The sphere $\mathbb{S}$ refines uniquely to a Boolean coalgebra. Therefore for every space X the suspension spectrum $\Sigma^\infty_+ X$ also does, as it can be considered as the colimit of the constant X-indexed diagram in Boolean coalgebras with value the sphere. This way we get a unique refinement of the functor Σ^∞_+ through Boolean coalgebras.

Now we conjecture that these Boolean coalgebras can be used to describe spaces algebraically.

Conjecture 4.11. *The functor Σ^∞_+ induces an equivalence between simply connected objects in the ∞-category of spaces and in the ∞-category of Boolean coalgebras (here simply connected relative to the counit map). The inverse is given by sending a Boolean coalgebra C to the mapping space from $\mathbb{S}$ to C.*

This conjecture came up in discussions with Heuts and Klein based on a theorem of Heuts giving an inductive description of spaces using Tate coalgebras [23]. The version for finite simply connected spaces has been implemented by Yuan based on a similar theorem of Mandell [34, 35] (using $\mathbb{E}_\infty$-algebras over $\mathbb{Z}$).

Conjecture 4.11 relies on the generalized Segal conjecture that we will explain later (Conjecture 7.10) for two reasons: first, to define the notion of Boolean coalgebras, but also to analyze the adjunction counit of the functors between spaces and Boolean coalgebras. We finally note that one can also make a more general conjecture about a description of p-adic perfect coalgebras in terms of p-complete spaces with a (suitably continuous) action of the profinite integers $\mathbb{Z}^\wedge$. In the finite case, this can again be deduced from Mandell's results [34, 35] similar to the result at the end of Section 3. See also the discussion in [50, SECTION 7].

5. THE FROBENIUS ON THH

In this section we will discuss a further instance of the Frobenius operator in higher algebra, namely the Frobenius operator on topological Hochschild homology. We first recall that topological Hochschild homology is a variant of Hochschild homology, namely the variant where one works relative to the sphere spectrum $\mathbb{S}$.

The motivation to consider THH and refined invariants such as TC is that they are good approximations to algebraic K-theory. More precisely, for any ring R there is a natural map $K(R) \to \mathrm{TC}(R)$, called the cyclotomic trace, which is often close to an isomorphism. For example, it is a seminal result of Mathew–Clausen–Morrow [13] based on previous work by McCarthy and many others that for a p-adic ring R the spectrum $\mathrm{TC}(R)$ is after p-adic completion equivalent to étale K-theory. The proof of this result relies crucially on the Frobenius-perspective on cyclotomic spectra that we will explain in this section.

THH and Bökstedt's theorem. Let us review the definition and basic properties of THH here. For more details, see, e.g., [27] or the YouTube lectures based on this document.

Definition 5.1. Let R be a ring spectrum, not necessarily commutative. Then $\mathrm{THH}(R)$ is defined as the relative tensor product

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{\mathrm{op}}} R.$$

Here R^{op} is R equipped with the opposite multiplication so that left $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$-modules are R-bimodules. Then we can view R as a left module over $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$ via its canonical bimodule structure, but also as a right module over $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$ using the flip-involution on $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$.

We could have worked over other bases than the sphere $\mathbb{S}$ by simply taking the tensor product $R \otimes R^{\mathrm{op}}$ over other bases. In particular, if R is an algebra spectrum over the Eilenberg–MacLane spectrum $H\mathbb{Z}$, e.g., if R is itself the Eilenberg–MacLane spectrum of an ordinary ring, then we can form the relative tensor product

$$R \otimes_{R \otimes_{H\mathbb{Z}} R^{\mathrm{op}}} R,$$

and it turns out that this is equivalent to ordinary Hochschild homology $\mathrm{HH}(R)$.[4]

There is a canonical map $R \to \mathrm{THH}(R)$ induced by the inclusion of R into any of the two tensor factors. If R is an $\mathbb{E}_\infty$-ring spectrum then $\mathrm{THH}(R)$ also inherits a natural $\mathbb{E}_\infty$-structure, in particular the homotopy groups $\mathrm{THH}_*(R)$ are a graded commutative ring. The map $R \to \mathrm{THH}(R)$ refines to a map of $\mathbb{E}_\infty$-maps in this case. The most important result about THH is Bökstedts calculation of THH for the finite field $\mathbb{F}_p$, which we implicitly consider as an Eilenberg–MacLane spectrum.

Theorem 5.2 (Bökstedt [9]). *We have that*

$$\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[x], \quad |x| = 2.$$

This result is central to almost everything that has been done with THH. It is the basis of essentially all the K-theory computations that have been carried out using trace methods, such as the seminal computations of Hesselholt–Madsen [20]. It is used to deduce the very important computations of Bhatt–Morrow–Scholze for perfectoid rings [6]. These

4 Really one gets a derived variant sometimes called Shukla-homology, but we will not distinguish that here.

are the basis of the relation of THH to p-adic Hodge theory and prismatic cohomology (pioneered by Lars Hesselholt). Using Bökstedt's theorem combined with the relation of THH to K-theory, one can even deduce classical Bott periodicity for complex K-theory (see [22, SECTION 1.3.2] for details). There are also computations related to the red-shift conjecture pioneered by Ausoni–Rognes [3] and with a recent breakthrough by Hahn–Wilson [18] which we will describe in Section 6. These result also dependent on the computation of $\mathrm{THH}(\mathbb{F}_p)$.

The way Theorem 5.2 was originally proven is by computing the $\mathbb{F}_p$-homology of the spectrum $\mathrm{THH}(\mathbb{F}_p)$ using a specific spectral sequence:

$$\mathrm{HH}\big(\pi_*(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p)/\mathbb{F}_p\big) \Rightarrow \pi_*\big(\mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{S}} \mathbb{F}_p\big).$$

Here $\pi_*(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p) = \mathcal{A}_p^\vee$ is the dual Steenrod algebra, which has been calculated by Milnor [38]. It has a number of polynomial and exterior generators. For the computation of Bökstedt, one needs a bit more information about the dual Steenrod algebra, namely its Dyer–Lashof operations which have been calculated by Steinberger [11, CHAPTER 3, THEOREMS 2.2 AND 2.3]. The interesting thing about these calculations for the dual Steenrod algebra is that if one takes these operations into account, then the homotopy ring of the dual Steenrod algebra is completely generated by a single element in $\pi_1(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p)$. One can in fact combine all these computations into a more conceptual statement as follows (see [26, SECTION 1.1] for a review):

Theorem 5.3 (Milnor, Steinberger, Araki-Kudo, Dyer-Lashof, etc.). *The ring spectrum $\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p$ is, as an $\mathbb{E}_2$-algebra over $\mathbb{F}_p$, free on a generator in degree* 1.

This result really is a combination of all the computations above. For example, Milnor's computation of the dual Steenrod algebra can easily be deduced from it. Once one has phrased the statement about the dual Steenrod algebra in this way it is very easy to deduce Bökstedt's theorem from it. But one can also do the opposite: Theorem 5.3 follows formally from Theorem 5.2 using a version of bar–cobar duality, see, e.g., [26] for details.

The Frobenius on THH. Now we describe extra structure on THH, namely a Frobenius homomorphism. Let us note that while we are mostly interested in commutative rings the Frobenius homomorphism even exist in the noncommutative setting. To understand this, let us first understand the corresponding construction in ordinary algebra.

Assume R is an ordinary, not necessarily commutative ring. Then we have

$$R \otimes_{R \otimes_{\mathbb{Z}} R^{\mathrm{op}}} R = R/[R, R],$$

where $[R, R] \subseteq R$ is the subgroup additively generated by commutators $rs - sr$. This quotient $R/[R, R]$ is the algebraic analogue of THH. In fact, it is isomorphic to π_0 of $\mathrm{THH}(R)$.

Observation 5.4. *The map*

$$R/[R, R] \to \big(R/[R, R]\big)/p = R/\big([R, R] + pR\big), \quad [r] \mapsto \big[r^p\big],$$

is a well-defined group homomorphism.

This is the algebraic version of the Frobenius that we will construct on THH now. Before we can discuss the Frobenius on THH, we have to introduce another bit of structure: THH carries a natural action of the circle group $\mathbb{T} = \mathrm{U}(1)$. This goes back to Connes and is a homotopy-theoretic incarnation of the Connes' operator. We will take this as a black-box here, see [41] for a careful construction. For an $\mathbb{E}_\infty$-ring R, this $\mathbb{T}$-action on $\mathrm{THH}(R)$ is an action through $\mathbb{E}_\infty$-homomorphisms and it is a result of McClure, Schwänzel, and Vogt that $\mathrm{THH}(R)$ is initial among $\mathbb{E}_\infty$-rings under R with an action of $\mathbb{T}$ [37].

For any action of $\mathbb{T}$, we get an induced C_p-action by restriction to the subgroup $C_p \subseteq \mathbb{T}$ of pth roots. Then we have the following result:

Proposition 5.5. *For any ring spectrum R, there is a map*

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

called the cyclotomic Frobenius, which is a natural and symmetric monoidal transformation. Moreover, it is $\mathbb{T}$-equivariant where the target carries the residual action by $\mathbb{T}/C_p \cong \mathbb{T}$.

Proof sketch. Let us sketch the construction of φ_p in the case of an $\mathbb{E}_\infty$-ring R. There is a unique extension of the $\mathbb{E}_\infty$-map $R \to \mathrm{THH}(R)$ to a C_p-equivariant $\mathbb{E}_\infty$-map

$$R^{\otimes p} = R \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} R \to \mathrm{THH}(R)$$

using that the source is the free $\mathbb{E}_\infty$-ring with an action of C_p. In particular, we get an induced map $(R \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} R)^{tC_p} \to \mathrm{THH}(R)^{tC_p}$. Then the Frobenius φ_p is the unique $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-map rendering the diagram

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{THH}(R) \\ \downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \varphi_p} \\ (R \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} R)^{tC_p} & \longrightarrow & \mathrm{THH}(R)^{tC_p} \end{array}$$

commutative. Such a map exists by the result of McClure, Schwänzel, and Vogt that $\mathrm{THH}(R)$ is the initial $\mathbb{E}_\infty$-ring under R with a $\mathbb{T}$-action. ∎

The map $\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ is a refinement of the map of Observation 5.4 in the sense that for an ordinary ring R on π_0 it recovers the map of Observation 5.4.

Definition 5.6. A cyclotomic spectrum is a spectrum X with $\mathbb{T}$-action and for every prime p a $\mathbb{T}$-equivariant map $X \to X^{tC_p}$.

Using this definition we can rephrase Proposition 5.5 as the existence of a natural cyclotomic structure on $\mathrm{THH}(R)$. The main content of the paper [41] is a discussion of the theory of cyclotomic spectra and of topological cyclic homology from this perspective. In particular topological cyclic homology $\mathrm{TC}(R)$ of a connective ring spectrum R can be computed as the mapping spectrum in the stable ∞-category of cyclotomic spectra from $\mathrm{THH}(\mathbb{S})$ to $\mathrm{THH}(R)$. Prior to that, $\mathrm{TC}(R)$ was defined using point set models of $\mathrm{THH}(R)$ and genuine equivariant homotopy theory.

Remark 5.7. For an $\mathbb{E}_\infty$-ring R the Tate-valued Frobenius $\varphi_p : R \to R^{tC_p}$ constructed in Section 3 and the map $\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ of Proposition 5.5 are related in the following sense: there is an $\mathbb{E}_\infty$-map $b : \mathrm{THH}(R) \to R$. This can be constructed as the unique $\mathbb{T}$-equivariant map (with trivial action on the target) extending the identity $R \to R$. Then the following diagram commutes:

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{THH}(R) \\ \downarrow{\scriptstyle \varphi_p} & & \downarrow{\scriptstyle \varphi_p} \\ R^{tC_p} & \xleftarrow{b^{tC_p}} & \mathrm{THH}(R)^{tC_p} \end{array}$$

so that the Frobenius on THH refines the Tate-valued Frobenius.

Perfectness of THH. Recall that we have seen that for a p-adically finite spectrum X the map $\mathrm{triv} : X \to X^{tC_p}$ is a p-completion (Theorem 1.4). If X is bounded below then the map $\Delta_p : X \to (X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X)^{tC_p}$ is a p-completion. In this section we will see that in many situations also the cyclotomic Frobenius

$$\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

is an equivalence, at least in large degrees. We capture this in the following definition:

Definition 5.8. We say that a cyclotomic spectrum X is *eventually perfect* (or it satisfies the *Segal conjecture*) if the map

$$X_p^\wedge \to X^{tC_p}$$

is an equivalence in sufficiently large degrees $* \gg 0$.

It turns out that there are many cases of eventually perfect cyclotomic spectra. We try to give a short (and incomplete list) to illustrate this.

Example 5.9 (Bökstedt–Madsen). For $R = \mathbb{F}_p$, the map

$$\varphi_p : \mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(\mathbb{F}_p)^{tC_p}$$

is an equivalence in degrees ≥ -1. The map

$$\varphi_p : \mathrm{THH}(\mathbb{Z}_p) \to \mathrm{THH}(\mathbb{Z}_p)^{tC_p}$$

is an equivalence in degrees ≥ 0. This can be considerably generalized to see that $\mathrm{THH}(R)$ is eventually perfect for DVRs of mixed characteristic with perfect residue field [20], for smooth algebras in positive characteristic [19, PROP. 6.6], and for torsion-free excellent noetherian rings R with R/p finitely generated over its pth powers [36]. Finally, Bhatt–Morrow–Scholze show that also for R a perfectoid ring the spectrum $\mathrm{THH}(R)$ is eventually perfect.

Example 5.10 (Rognes–Lunøe-Nielsen). For $R = \mathrm{MU}$, the complex cobordism spectrum, or $R = \mathrm{BP}$, the Brown–Peterson spectrum (which is a retract of p-localized MU), the map

$$\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

is a p-completion [29].

Example 5.11 (Ausoni–Rognes [3] for $n = 1$, Hahn–Wilson [18] for general n). Let F be any finite type $(n+1)$-complex. Then the cyclotomic spectrum $F \otimes_{\mathbb{S}} \mathrm{THH}(\mathrm{BP}\langle n\rangle)$ is eventually perfect, in other words, the map

$$F_*\big(\mathrm{THH}\big(\mathrm{BP}\langle n\rangle\big)\big) \to F_*\big(\mathrm{THH}\big(\mathrm{BP}\langle n\rangle\big)^{tC_p}\big)$$

is an equivalence for $* \gg 0$.

Remark 5.12. Spectrum $\mathrm{THH}(R)$ is the geometric realization (i.e., colimit) of a simplicial spectrum $\mathrm{THH}_\bullet(R) = R^{\otimes(\bullet+1)}$, the *cyclic Bar complex*. By a p-fold edgewise subdivision we see that this is also the colimit over $\mathrm{sd}_p\mathrm{THH}_\bullet(R) = R^{\otimes p(\bullet+1)}$. One can show that applying the Tate diagonal levelwise extends to a map of simplicial objects $\mathrm{THH}_\bullet(R) \to (\mathrm{sd}_p\mathrm{THH}_\bullet(R))^{tC_p}$. In this picture, the THH-Frobenius φ_p is induced by the map

$$\mathrm{THH}(R) = \big|\mathrm{THH}_\bullet(R)\big| \xrightarrow{\Delta_p} \big|\big(\mathrm{sd}_p\mathrm{THH}_\bullet(R)\big)^{tC_p}\big| \xrightarrow{i} \big|\mathrm{sd}_p\mathrm{THH}_\bullet(R)\big|^{tC_p} = \mathrm{THH}(R)^{tC_p},$$

where i is the canonical interchange map. For a connective ring spectrum R, the first map Δ_p is a p-adic equivalence by Theorem 2.3. Thus the question whether THH is (eventually) perfect is equivalent to the question whether i is an equivalence (in a range).

6. FROBENIUS LIFTS AND TR

In this section we will talk about Frobenius lifts for cyclotomic spectra. We fix a prime p and focus on this prime. One can also set up a global theory for all primes, but we will not get into this here.

Frobenius lifts in algebra. Recall that for an ordinary commutative ring R a Frobenius lift (for the fixed prime p) is a ring homomorphism $F_p : R \to R$ such that the composite

$$R \xrightarrow{F_p} R \xrightarrow{\mathrm{can}} R/p$$

is the Frobenius homomorphism. Here can is the canonical projection. If R is p-torsion free this precisely captures the notion of a δ-ring (also known as p-typical λ-ring or λ_p-ring). If R has p-torsion this not quite true on the nose. In this case one should define a Frobenius lift as an endomorphism $F_p : R \to R$ *together* with a homotopy between

$$R \xrightarrow{F_p} R \xrightarrow{\mathrm{can}} R \otimes^L_{\mathbb{Z}} \mathbb{F}_p \quad \text{and} \quad R \xrightarrow{\varphi_p} R \otimes^L_{\mathbb{Z}} \mathbb{F}_p$$

considered as maps of animated commutative rings (also known as simplicial commutative rings). A Frobenius lift on R in this sense is then always equivalent to the structure of a δ-ring on R, see [7, REMARK 2.5].

An example of a ring with a Frobenius lift is given by the ring of p-typical Witt vectors $W(R)$ for any ring R. The Frobenius lift $F : W(R) \to W(R)$ is the Witt vector Frobenius. By definition, the Witt vectors come with a reduction map $W(R) \to R$, and we have:

Proposition 6.1. *For any commutative ring R, the ring of p-typical Witt vectors $W(R)$ is the universal ring with Frobenius lift over R, i.e., the cofree δ-ring on R.*

In fact, the Witt vectors have additional structure, namely a map $V : W(R) \to W(R)$ called *Verschiebung*, which is additive but not multiplicative. It instead satisfies the relations

$$FV = p \cdot \mathrm{id} \quad \text{and} \quad V\big(F(a)b\big) = aV(b) \quad \text{for all } a, b \in W(R).$$

Moreover, we have that $W(R)/V = R$ and that $W(R)$ is derived complete with respect to the filtration induced by V, i.e., the derived inverse limit $\mathrm{R}\varprojlim_V W(R)$ vanishes.

One can express this a bit more systematically as follows: an abelian group A with operators $F, V : A \to A$ such that $FV = p \cdot \mathrm{id}$ is called a (p-typical) *Cartier module*. On the category of Cartier modules, there is a tensor product $\boxtimes$ and the Witt vectors form an algebra with respect to that, see [1, SECTION 4.2] for these facts.

Cyclotomic spectra with Frobenius lift. Now we would like to find analogous statements to the statements of the last paragraph for cyclotomic spectra. We fix a prime p throughout this section, and everything will depend on p.

Definition 6.2. Let X be a cyclotomic spectrum. A Frobenius lift is a $\mathbb{T}$-equivariant map $F : X \to X^{hC_p}$ together with an equivalence $\mathrm{can} \circ F \simeq \varphi_p$ of $\mathbb{T}$-equivariant maps $X \to X^{tC_p}$.

Now we have an analogue to the algebraic situation, i.e., the analogue of the Witt vectors:

Proposition 6.3 ([27]). *For every cyclotomic spectrum X, there is a universal cyclotomic spectrum* $\mathrm{TR}(X) \to X$ *with Frobenius lift. If $X = \mathrm{THH}(R)$ for a connective $\mathbb{E}_\infty$-ring spectrum R, then $\mathrm{TR}(X)$ is also an $\mathbb{E}_\infty$-ring spectrum and the ring $\pi_0\mathrm{TR}(X)$ is canonically isomorphic to the ring $W(\pi_0 R)$ of p-typical Witt vectors.*

This spectrum $\mathrm{TR}(X)$ can be described explicitly by an iterated pullback and has appeared in the theory of cyclotomic spectra much earlier (in fact, it was used to define TC in the first place by Bökstedt–Hsiang–Madsen [10]). By a result of Blumberg–Mandell, the functor $X \mapsto \mathrm{TR}(X)$ is even corepresentable on the stable ∞-category of cyclotomic spectra [8]. We shall not need these facts here, but there is the following abstract characterization:

Proposition 6.4 (Nikolaus–Antieau [1]). *For every cyclotomic spectrum X, there is a $\mathbb{T}$-equivariant map*

$$V : \mathrm{TR}(X)_{hC_p} \to \mathrm{TR}(X)$$

called Verschiebung *such that the cofiber of V is equivalent to X and the composite $F \circ V$:* $\mathrm{TR}(X)_{hC_2} \to \mathrm{TR}(X)^{hC_2}$ *is canonically identified with the C_2-norm. If X is additionally bounded below, then we have a pullback square of spectra with $\mathbb{T}$-action of the form*

$$\begin{array}{ccc} \mathrm{TR}(X) & \longrightarrow & X \\ \downarrow F & & \downarrow \varphi_p \\ \mathrm{TR}(X)^{hC_p} & \xrightarrow{\mathrm{can}} & \mathrm{TR}(X)^{tC_p} \simeq X^{tC_p}. \end{array} \tag{6.1}$$

From the pullback square (6.1), we see that the cyclotomic spectrum X is eventually perfect precisely if the map $F : \mathrm{TR}(X) \to \mathrm{TR}(X)^{hC_p}$ is an equivalence in high degrees. This will be important in the next section.

Definition 6.5. A *topological Cartier module* is a spectrum M with $\mathbb{T}$-action, $\mathbb{T}$-equivariant maps $V : M_{hC_p} \to M$, $F : M \to M^{hC_p}$ and a $\mathbb{T}$-equivariant equivalence of the composite $F \circ V$ to the C_p-norm.

With this language, Proposition 6.4 can be summarized by saying that for every cyclotomic spectrum X the spectrum $\mathrm{TR}(X)$ is a topological Cartier module. Moreover, one can show that the functor TR induces an equivalence between the ∞-category of bounded below cyclotomic spectra and the ∞-category of bounded below topological Cartier modules which are complete with respect to the Verschiebung, see [1].

Example 6.6. For the spectrum $X = \mathrm{THH}(\mathbb{F}_p)$, one can compute the spectrum $\mathrm{TR}(X)$ and finds that it is given by the Eilenberg–MacLane spectrum $H\mathbb{Z}_p$. The $\mathbb{T}$-action is (necessarily) trivial and the Frobenius is given by the map $F : H\mathbb{Z}_p \to H\mathbb{Z}_p^{hC_p}$ which is the identity on π_0. The Verschiebung is the map $V : (H\mathbb{Z}_p)_{hC_p} \to H\mathbb{Z}_p$ which is multiplication by p on π_0. The pullback then takes the form

$$\begin{array}{ccc} H\mathbb{Z}_p & \longrightarrow & \mathrm{THH}(\mathbb{F}_p) \\ \Big\downarrow F & & \Big\downarrow \varphi_p \\ H\mathbb{Z}_p^{hC_p} & \xrightarrow{\text{can}} & \mathbb{Z}_p^{tC_p} \end{array} \tag{6.2}$$

and implies that $\mathrm{THH}(\mathbb{F}_p)$ is equivalent to the connective cover of $H\mathbb{Z}_p^{tC_p}$. In fact, one can reverse the logic here: since $\pi_0(\mathrm{TR}(\mathbb{F}_p)) = W(\mathbb{F}_p) = \mathbb{Z}_p$ by Proposition 6.3, we see that Bökstedt's theorem (Theorem 5.2) is equivalent to the assertion that $\mathrm{TR}(\mathbb{F}_p)$ has no higher homotopy groups, i.e., is 0-truncated. As explained around Theorem 5.3, this is then also equivalent to the combination of Milnor's and Steinberger's computations and implies topological Bott periodicity.

One can attempt to prove the truncatedness of $\mathrm{TR}(\mathbb{F}_p)$ directly to give new proofs of all of these theorems. Such a direct proof is given in [5] using the theory of polynomial functors and the fact that TR can be evaluated on polynomial functors.

Boundedness of TR. Now we shall relate the perfectness of THH to the red-shift conjecture and computations of algebraic K-theory. This is based on recent work of Mathew and Hahn–Wilson. We first review a bit more of the theory of cyclotomic spectra. The stable ∞-category of cyclotomic spectra carries a t-structure, defined as follows:

Definition 6.7. A cyclotomic spectrum X is called *(cyclotomically) n-connective* if the underlying spectrum is n-connective and *cyclotomically n-truncated* if every map $Y \to X$ where Y is $(n+1)$-connective is nullhomotopic (as a map of cyclotomic spectra). We say that X is *cyclotomically bounded* if it is cyclotomically bounded above and below, i.e., n-connective for some $n \gg 0$ and k-truncated for some $k \gg 0$.

It is not hard to see that this defines a t-structure and as usual the heart consists of those cyclotomic spectra which are cyclotomically 0-connective and 0-truncated.

Remark 6.8. Note that we are a bit loose with the adjective "cyclotomic" for connective things, since this simply means that the underlying spectrum is connective and there is no danger of confusion. For truncated objects, we will be careful though, since this cyclotomic notion of truncatedness is completely different to the truncatedness of the underlying spectrum X. For example, $\mathrm{THH}(\mathbb{F}_p)$ is cyclotomically 0-truncated as one can see from the following theorem together with Example 6.6, but the underlying spectrum is not truncated at all (recall Theorem 5.2).

Theorem 6.9 (Antieau–Nikolaus [1]). *A p-complete cyclotomic spectrum is cyclotomically bounded precisely if the spectrum* $\mathrm{TR}(X)$ *is bounded as a spectrum. In this case X is eventually perfect (see Definition 5.8) and the p-completion of* $\mathrm{TC}(X)$ *is bounded as a spectrum. The heart of the t-structure is equivalent to the abelian category of derived V-complete Cartier modules.*

One reason why we care about the boundedness of TR and TC is its relation to the red shift conjecture in algebraic K-theory, which we want to outline now (following the recent paper [18] of Hahn–Wilson). The first ingredient is the following result, which is a consequence of work of Mahowald–Rezk:

Proposition 6.10 (Hahn–Wilson [18]). *Assume that for a connective ring spectrum R and a type $(n+2)$-complex F the spectrum $F \otimes_{\mathbb{S}} \mathrm{THH}(R)$ is cyclotomically bounded and $\pi_i(R_p^\wedge)$ is finitely generated for each i. Then the map*

$$\mathrm{TC}(R) \to L_{n+1}^f \mathrm{TC}(R)$$

is an equivalence in degrees $* \gg 0$.

In order to use this result, one needs an efficient way of verifying the boundedness of a cyclotomic spectrum which is provided by the following criterion.

Proposition 6.11 (Mathew [36], Hahn–Wilson [18]). *Assume a cyclotomic spectrum X is bounded below, p-power torsion and eventually perfect. Then the following are equivalent:*

(1) *X is cyclotomically bounded, i.e.,* TR *is bounded.*

(2) *The $\mathbb{T}$-spectrum X^{tC_p} is $\mathbb{T}$-nilpotent, i.e., lies in the thick subcategory generated by free $\mathbb{T}$-spectra.*

(3) *For $* \gg 0$ the maps $\pi_*(\mathrm{can}) : \pi_*(X^{hC_{p^k}}) \to \pi_*(X^{tC_{p^k}})$ with $1 \leq k \leq \infty$ are zero.*

Especially condition (3) can be verified in practice. Using this, Hahn–Wilson prove that $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ for every $\mathbb{E}_3$-form of $\mathrm{BP}\langle n\rangle$ satisfies the assumptions of Proposition 6.10. From this, together they deduce the following groundbreaking result.

Theorem 6.12 (Hahn–Wilson **[18]**). *For every $\mathbb{E}_3$-form of* $\mathrm{BP}\langle n\rangle$*, the morphisms*

$$\mathrm{TC}\big(\mathrm{BP}\langle n\rangle\big) \to L_{n+1}^{f}\big(\mathrm{TC}\big(\mathrm{BP}\langle n\rangle\big)\big),$$
$$K\big(\mathrm{BP}\langle n\rangle\big) \to L_{n+1}^{f}\big(K\big(\mathrm{BP}\langle n\rangle\big)\big)$$

are after p-completion equivalences in degrees $* \gg 0$.

Remark 6.13. The thick subcategory of $\mathbb{T}$-nilpotent spectra is in fact a tensor ideal. Therefore Mathew concludes from Proposition 6.11 and the results of Hahn–Wilson that, for a connective $\mathrm{BP}\langle n\rangle$-algebra spectrum R and a finite type $(n+2)$-complex F, the cyclotomic spectrum $F \otimes \mathrm{THH}(R)$ is cyclotomically bounded precisely if it is eventually perfect **[18, PROPOSITION 3.3.7]**.

7. THE SEGAL CONJECTURE

In this last section we want to discuss the Segal conjecture also known as Segal's Burnside ring conjecture. Despite its name, this is a theorem that was proven by Carlsson **[12]**. We will conjecture a generalization of this theorem.

In order to do this, we first have to discuss a variant of the Tate construction called the *proper Tate construction*. We will first sketch the algebraic counterpart as usual.

The proper Tate construction in algebra. Let A be an abelian group with G-action and $H \subseteq G$ a subgroup of G. Then we have a relative norm map

$$A^H \to A^G, \quad a \mapsto \sum_{[g]\in G/H} ga.$$

We can form the quotient of A^G by all these relative norms

$$\frac{A^G}{\bigoplus_{H \subsetneq G} A^H}, \tag{7.1}$$

where H ranges trough all proper subgroups of H. This quotient equals the zeroth Tate construction if $G = C_p$ for a prime p but differs in general.

Example 7.1. If G has an index n-subgroup then the quotient (7.1) is n-torsion, as one easily verifies. Applying this observation to Sylow subgroups, we see that the quotient (7.1) vanishes if G is not a p-group. If G is a p-group which acts trivially on A then the quotient (7.1) is isomorphic to A/p.

The proper Tate construction in higher algebra. Now we want to make the analogous construction for spectra. Let X be a spectrum with G-action. Then for any subgroup $H \subseteq G$ there is a similar relative norm map

$$\mathrm{Nm}_H^G : X^{hH} \to X^{hG}$$

and these maps Nm_H^G are compatible as H ranges through all subgroups of G. To make this compatibility precise, we consider the orbit category Orb_G consisting of all G-orbits, i.e., transitive G-sets. All of these orbits are of the form G/H for a subgroup $H \subseteq G$. The precise compatibility statement that we need is the following statement, which follows, for example, from the work of Barwick [4]:

Proposition 7.2. *For any spectrum X with G-action, there is a canonical functor*

$$\mathrm{Orb}_G \to \mathrm{Sp}$$

which sends G/H to X^{hH} and a map in the orbit category to a relative norm.

Definition 7.3. For a spectrum X with an action by a finite group G, we define the proper Tate construction $X^{\varphi G}$ as the cofiber

$$\mathrm{colim}_{H \subsetneq G} X^{hH} \to X^{hG},$$

where the colimit is indexed over the full subcategory of Orb_G consisting of the orbits G/H for proper subgroups H of G (which we abusively denote by $H \subsetneq G$).

Example 7.4. For $G = C_p$ with p prime, the proper Tate construction agrees with the Tate construction, i.e., $X^{\varphi C_p} = X^{tC_p}$.

Example 7.5. For $G = C_{p^n}$ with $n > 0$, a cofinality argument shows that the colimit in Definition 7.3 is equivalent to the colimit over the full subcategory of Orb_G spanned by the orbit with p-elements. Thus we find that

$$X^{\varphi(C_{p^n})} \simeq \mathrm{cofib}\big((X^{hC_{p^{n-1}}})_{hC_p} \to X^{hC_{p^n}}\big) \simeq (X^{hC_{p^{n-1}}})^{tC_p},$$

where the action of C_p on $X^{hC_{p^{n-1}}}$ is the "residual" action under the identification $C_p = C_{p^n}/C_{p^{n-1}}$.

Warning 7.6. For any abelian group A with G-action, there is a canonical map

$$A^G/\oplus_{H \subsetneq G} A^H \to \pi_0(HA^{\varphi G}),$$

where HA is the Eilenberg–MacLane spectrum associated with A. In contrast to the case for $G = C_p$, this map is in general not an isomorphism.

Similar to the Tate construction, the proper Tate construction refines to a lax symmetric monoidal functor

$$(-)^{\varphi G} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$$

and, if X is equipped with the trivial G-action, there is a natural map

$$\mathrm{triv} : X \to X^{\varphi G}.$$

Moreover, there are homotopy-theoretic versions of the statements in Example 7.1:

Proposition 7.7 ([41]). *Let X be a spectrum with G action. Then we have $X^{\varphi G} = 0$, unless G is a p-group.*

Now the following deep result is a generalization of the theorem of Lin and Gunawardena (Theorem 1.4) and a higher analogue of the second part of Example 7.1.

Theorem 7.8 (Carlsson [12]). *For any finite p-group G and every finite spectrum X with trivial G-action, the map* $\mathrm{triv} : X \to X^{\varphi G}$ *is a p-completion.*

Remark 7.9. This theorem is equivalent to the Segal conjecture which states that for any group G the canonical map

$$\mathbb{S}^G \to \mathbb{S}^{hG}$$

is a completion. Here $\mathbb{S}^G$ is the spectrum obtained by the group completion of the category of finite G-sets. Its π_0 is given by the Burnside ring and the relevant completion is the completion with respect to the augmentation ideal. In this form the conjecture was inspired by the Atiyah–Segal completion theorem which states that a similar map in complex K-theory is a completion.

Now similar to the Tate diagonal discussed in Section 2, there is a unique natural and symmetric monoidal map $\Delta_G : X \to (X^{\otimes G})^{\varphi G}$ for every finite group G. Note that such a map does not exist if we replace the proper Tate construction $(-)^{\varphi G}$ by the actual Tate construction $(-)^{tG}$.

Conjecture 7.10. *For any finite p-group G and any bounded below spectrum X, the Tate diagonal*

$$X \to (X^{\otimes G})^{\varphi G}$$

is a p-completion.

This conjecture reduces for $X = \mathbb{S}$ to Theorem 7.8. For $G = C_p$, it reduces to Theorem 2.3 above. One can formally deduce the case $G = C_{p^n}$ from this case using the Tate orbit lemma of [41] which implies that for any bounded below spectrum X with C_{p^n}-action we have $X^{\varphi C_{p^n}} = (X^{tC_p})^{\varphi C_{p^{n-1}}}$. The key step to verify Conjecture 7.10 in general is the case of G an elementary abelian p-group. We have been informed that Håkon Bergsaker is very close proving this conjecture using the continuous Adams spectral sequence and an Ext-calculation based on the Singer construction similar to the case of the ordinary Segal conjecture.

Remark 7.11. Conjecture 7.10 is equivalent to the assertion that for any finite group G and any bounded below spectrum X the map

$$N_e^G(X)^G \to (X^{\otimes G})^{hG}$$

from the fixed points of the Hill–Hopkins–Ravenel norm to the homotopy fixed points is a completion (at the augmentation ideal of the Burnside ring). Equivalently, the HHR-norm is Borel complete after this completion. In this language, it becomes clear that this is a direct analogue of the classical Burnside ring conjecture.

A consequence would be that for any finite group G and every connective spectrum X, the spectrum $(X^{\otimes G})^{hG}$ is connective and π_0 is a completion of π_0 of the norm. The latter has an explicit algebraic expression in terms of $\pi_0(X)$. This was described in [24, 47] but remains somewhat inexplicit. In the case that the group G is given by C_{p^n}, this algebraic expression is given by the Witt vectors of $\mathbb{Z}$ with values in the abelian group $\pi_0(X)$ defined and discussed in [14], see also [25] for a similar description. For a general group G, the group π_0 of the norm should similarly be a certain (yet to be defined) version of Dress–Siebeneicher's Witt–Burnside ring [15] for the group G with coefficients in $\pi_0(X)$.

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