FLOER HOMOLOGY OF 3-MANIFOLDS WITH TORUS BOUNDARY

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ABSTRACT

Manifolds with torus boundary have played a special role in the study of Floer homology for 3-manifolds since the early days of the subject. In joint work with Jonathan Hanselman and Liam Watson, we defined a geometrical Heegaard Floer invariant for 3-manifolds with torus boundary. The invariant is a reformulation of the bordered Floer homology of Lipshitz, Ozsváth, and Thurston, and takes the form of a collection of immersed closed curves (possibly decorated with local systems) in a covering space of the punctured torus. We briefly discuss the construction of the invariant and some applications to the L-space conjecture of Boyer–Gordon–Watson and Juhász. We then describe a generalization to manifolds with sutured boundary, and some applications to the study of satellite knots.

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1. INTRODUCTION

In the seminal papers [16,17], Andreas Floer created two branches of the theory which now bear his name. The first branch is concerned with symplectic geometry, and provides invariants of symplectomorphisms and Lagrangian submanifolds. The second branch, with which we will be concerned, provides invariants of 3-manifolds. In addition to Floer's work, which uses SU_2 instantons, there are many different approaches to defining Floer homology for 3-manifolds, including the monopole Floer homology of Kronheimer and Mrowka [35] or Hutchings' theory of embedded contact homology [29]. We will mainly use the Heegaard Floer homology of Ozsváth and Szabó [44] which is easy to compute and relatively simple to work with from a technical standpoint. Regardless of their definition, all Floer theories assign an abelian group to a closed connected oriented 3-manifold Y. The key question we will be concerned with is:

"What is the Floer homology of a 3-manifold M with $\partial M \simeq T^2$?"

Here our main criterion for defining the Floer homology is that if $Y = M_1 \cup_{T^2} M_2$, we should be able to recover the Floer homology of Y from the Floer homologies of M_1 and M_2.

1.1. The view from 1992

It is illuminating to consider the situation for Floer's original instanton homology, as it was understood 30 years ago. Let $I^*(Y)$ be the homology of a chain complex $\mathrm{CI}^*(Y)$, which is generated by irreducible flat SU_2 connections on Y if things are nice enough. By considering the holonomy representation, we see that the set of flat connections is in bijection with the SU_2 character variety of Y:

$$X_{\mathrm{SU}_2}(Y) = \{\rho : \pi_1(Y) \to \mathrm{SU}_2\}/\,\mathrm{SU}_2$$

where SU_2 acts on the set of representations by conjugation, $(A \cdot \rho)(x) = A\rho(x)A^{-1}$. A representation ρ is *reducible* if its image is contained in an abelian subgroup of SU_2; otherwise, it is *irreducible*. Any reducible representation can be factored as $\pi_1(Y) \to H_1(Y) \to S^1 \subset \mathrm{SU}_2$, so if $H_1(Y) = 0$, the unique irreducible representation is the trivial one.

If $\partial M \simeq T^2$, we can likewise consider the character variety $X_{\mathrm{SU}_2}(M)$. The inclusion $i_* : \pi_1(\partial M) \to \pi_1(M)$ induces a map $i^* : X_{\mathrm{SU}_2}(M) \to X_{\mathrm{SU}_2}(\partial M)$. Since $\pi_1(T^2) \simeq \mathbb{Z}^2$ is abelian, every $\rho : \pi_1(T^2) \to \mathrm{SU}_2$ is reducible. Any two 1-parameter subgroups in SU_2 are conjugate, and the stabilizer of a fixed 1-parameter subgroup is the Weyl group $W = \mathbb{Z}/2$. It follows that

$$X_{\mathrm{SU}_2}(T^2) = \{\rho : \mathbb{Z}^2 \to S^1\}/W = T^2/(\mathbb{Z}/2)$$

is the *pillowcase orbifold* $T^2/(x \sim -x)$. Figure 1 shows the image $i^*(X_{\mathrm{SU}_2}(M))$ for two simple 3-manifolds, namely $M = S^1 \times D^2$ and $M = M_{T_{2,3}}$ – the exterior of the right-hand trefoil knot in S^3.

If $Y = M_1 \cup_{T^2} M_2$, the inclusions $i_{j*} : \pi_1(T^2) \to \pi_2(M_j)$ induce maps $i_j^* : X_{\mathrm{SU}_2}(M_j) \to X_{\mathrm{SU}_2}(T^2)$. The simplest example of such a gluing is a Dehn filling, where $M_2 = S^1 \times D^2$. In this case, it is easy to see:

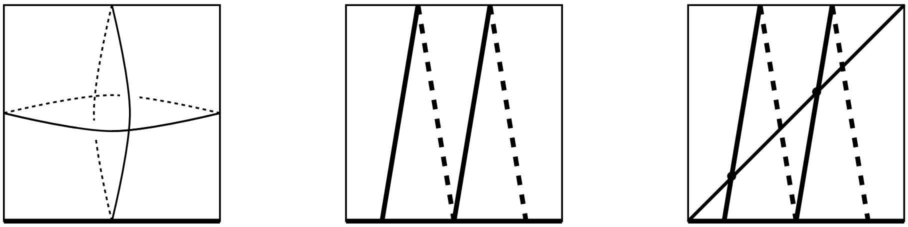


FIGURE 1
The pillowcase orbifold $X_{\mathrm{SU}_2}(T^2)$ is shown on the left. The solid line at the bottom is $i^*(X_{\mathrm{SU}_2}(S^1 \times D^2))$. The middle figure shows the image of $X_{\mathrm{SU}_2}(M_{T_{2,3}})$, which consists of a reducible part (the line at the bottom of the figure) and an irreducible part (the line segment of slope 6). The figure on the right shows the intersection between the two character varieties corresponding to $+1$ surgery on the trefoil.

Lemma 1.1. *If $Y = M_1 \cup_{T^2} (S^1 \times D^2)$, then $X(Y)$ is naturally identified with the fiber product $X_{\mathrm{SU}_2}(M_1) \times_{X_{\mathrm{SU}_2}(T^2)} X_{\mathrm{SU}_2}(S^1 \times D^2)$.*

The Poincaré sphere is the result of $+1$ surgery on $T_{2,3}$. The corresponding fiber product is illustrated on the right-hand side of Figure 1. There are 3 intersection points (circled) between $X_{\mathrm{SU}_2}(M_{T_{2,3}})$ and $X_{\mathrm{SU}_2}(S^1 \times D^2)$, which tells us that $X_{\mathrm{SU}_2}(P)$ consists of two irreducible characters and a single reducible character.

It is tempting to consider the image $i^*(X_{\mathrm{SU}_2}(M)) \subset X_{\mathrm{SU}_2}(\partial M)$ as a proxy for the Floer homology of M. However, a closer consideration of this picture reveals many difficulties:

- How should the reducible flat connections be treated? If Y is a homology sphere, the only reducible connection is the trivial one, which we can afford to ignore. As soon as $H_1(Y) \neq 0$, this is no longer feasible.
- If instead of taking $M_2 = S^1 \times D^2$ we use another 3-manifold, the fiber product in Lemma 1.1 becomes more complicated. Each intersection between irreducible points in $i_1^*(X_{\mathrm{SU}_2}(M_1))$ and $i_2^*(X_{\mathrm{SU}_2}(M_2))$ gives an entire circle of irreducible flat connections in $X_{\mathrm{SU}_2}(Y)$.
- Perhaps most importantly, $X_{\mathrm{SU}_2}(Y)$ is only the set of generators for $\mathrm{CI}^*(Y)$. To compute the homology, we must understand the differential, which involves counting solutions to the SU_2 ASD equation on $Y \times \mathbb{R}$. *A priori,* there is no reason to believe that the character variety should tell us anything about this.

Despite these problems, there were reasons for optimism as well. Indeed, $\mathrm{CI}^*(Y)$ was $\mathbb{Z}/8$ graded, and the reduction of this grading to $\mathbb{Z}/2$ agreed with the sign of intersection in the fiber product. Hence the two irreducible generators for $\mathrm{CI}^*(P)$ have the same $\mathbb{Z}/2$ grading, and there were no differentials in the chain complex. Fintushel and Stern [15] showed that the same was true for any Seifert fibered space.

The second, very potent reason was Floer's exact triangle, which related the Floer homologies of different Dehn fillings of M. Suppose $K \subset Y_\infty$ is a null-homologous knot, and let Y_0 and Y_1 be the manifolds obtained by 0 and 1 surgery on K. Then we have:

Theorem 1.2 ([8, 18]). *There is a long exact sequence*

$$\cdots \to I^*(Y_\infty) \to I^*(Y_0) \to I^{*-1}(Y_1) \to I^{*-1}(Y_\infty) \to \cdots .$$

The original motivation for this theorem was the relation between character varieties shown in Figure 2, and the fact that it holds suggests that our naive idea for thinking about the Floer homology of M in terms of the picture provided by the character variety might have something to it after all.

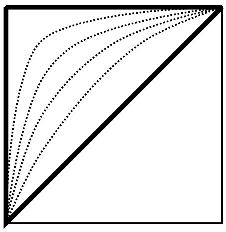


FIGURE 2
The pillowcase with curves corresponding to $+1$, 0, and ∞ surgery slopes. The $+1$ curve can be continuously deformed to the union of the other two, producing a chain complex which computes $\mathrm{HI}^*(Y_1)$ but whose generators are the union of the generators of $\mathrm{CI}^*(Y_0)$ and $\mathrm{CI}^*(Y_\infty)$.

1.2. The modern perspective

In 30 years, we have made a lot of progress. Many of the technical difficulties associated with instanton theory have been simplified or elided, first by the appearance of Seiberg–Witten theory [53] and then by the development of Heegaard Floer homology [44]. The problems associated with reducibles (such as the first two points above) have been addressed in several ways: by working with appropriate equivariant versions of the theory, as in [35]; by restricting to sectors in which reducibles do not appear [36]; or by dividing out by the based gauge group (or something similar) rather than the full gauge group [34].

We will focus on the Heegaard Floer invariant $\widehat{\mathrm{HF}}$, which, roughly speaking, corresponds to the monopole Floer invariant obtained by using the based gauge group instead of the full gauge group. If (Y, z) is a closed, connected, oriented, pointed 3-manifold, $\widehat{\mathrm{HF}}(Y, z)$ is a finite-dimensional vector space over the field $\mathbb{F} = \mathbb{Z}/2$. Some parts of the theory are also known to work with $\mathbb{Z}$ coefficients, but we will stick to $\mathbb{Z}/2$ coefficients throughout. If $z, z' \in Y$, there is a diffeomorphism $\psi : Y \to Y$ with $\psi(z) = z'$, so $\widehat{\mathrm{HF}}(Y, z) \simeq \widehat{\mathrm{HF}}(Y, z')$, but this isomorphism is not canonical, as was first observed by Juhász [32]. (The point z corresponds to the point used to define the based gauge group in Seiberg–Witten theory.)

Suppose that $Y = M_1 \cup_\Sigma M_2$, where Σ is a connected surface containing z. In this situation, the yoga of extended TQFTs suggests that to Σ we should associate an additive category $\mathcal{A}(\Sigma)$, that M_1 and M_2 should determine objects $\mathcal{A}(M_i)$ of $\mathcal{A}(\Sigma)$, and that we should have

$$\widehat{\mathrm{HF}}(Y, z) \cong \mathrm{Hom}\big(\mathcal{A}(M_1), \mathcal{A}(M_2)\big).$$

This picture was realized by Lipshitz, Ozsváth, and Thurston [39] in their seminal work on *bordered Floer homology*. They described the category $\mathcal{A}(\Sigma)$ in terms of algebraic objects which they called Type D and Type A structures. Their work was given a beautiful geometrical interpretation by Auroux [2,3] who showed that $\mathcal{A}(\Sigma)$ can be interpreted as the *partially wrapped Fukaya category* of the symmetric product $\mathrm{Sym}^g(\Sigma - z)$.

Auroux's result is very important from a philosophical standpoint, but its practical applications are limited by the difficulties of working with the Fukaya category of $\mathrm{Sym}^g(\Sigma - z)$. Naively, the objects of the Fukaya category are Lagrangian submanifolds, but in reality one must also consider arbitrary mapping cones built up out of Lagrangians. The algebra required to do this is essentially the same as that of Type A and Type D structures invented by Lipshitz, Ozsváth, and Thurston.

The one exception to this rule is the case $g = 1$, where we can give a simple geometric description of the (compactly supported) Fukaya category of $T^2 - z$. We say that a curve in $T^2 - z$ is *nice* if it is an immersed closed curve which is unobstructed, in the sense that it bounds no monogon in $T^2 - z$. Then we can formulate a version of the (compactly supported) Fukaya category, which we denote by $\mathcal{F}(T^2 - z)$. The objects of $\mathcal{F}(T^2 - z)$ are finite unions of nice curves equipped with local systems, and the group $\mathrm{Hom}(\gamma_1, \gamma_2)$ can be computed combinatorially. In particular, if γ_1 and γ_2 are primitive nonisotopic curves in $T^2 - z$, $\mathrm{Hom}(\gamma_1, \gamma_2)$ is determined by the minimal geometric intersection number $i(\gamma_1, \gamma_2)$.

Let $\overline{\mathcal{F}}(T^2 - z)$ be the set of isomorphism classes of objects in $\mathcal{F}(T^2 - z)$. Together with Hanselman and Watson, we proved the following theorem, which realizes the geometrical hope expressed in the previous section in the context of $\widehat{\mathrm{HF}}$:

Theorem 1.3 ([21,22]). *If (M, z) is a closed, connected, oriented, and pointed 3-manifold with $z \in \partial M \simeq T^2$, there is a well-defined invariant $\widehat{\mathrm{HF}}(M, z) \in \overline{\mathcal{F}}(\partial M - z)$, which satisfies*

$$\widehat{\mathrm{HF}}(M_1 \cup_{T^2} M_2, z) \simeq \mathrm{Hom}\big(\widehat{\mathrm{HF}}(M_1, z), \widehat{\mathrm{HF}}(M_2, z)\big).$$

In this context, the fact that $\widehat{\mathrm{HF}}$ satisfies an exact triangle analogous to that of Theorem 1.2 (proved by Ozsváth and Szabó in [43]) is a consequence of the fact that the lines of slope 0, 1, and ∞ in T^2 form an exact triangle in the Fukaya category. This argument is due to Lipshitz, Ozsváth, and Thurston [39].

The invariant $\widehat{\mathrm{HF}}(M)$ can be effectively computed in many examples; for example, by the work of F. Ye [54], it is known for all but 9 of the 286 orientable 1-cusped hyperbolic manifolds in the SnapPy census of hyperbolic 3-manifolds built from 5 or fewer tetrahedra [11,27].

Some examples of $\widehat{\mathrm{HF}}$ for simple 3-manifolds are shown in Figure 3. Each curve in the figure lives in an infinite cylinder obtained by identifying the dashed lines on the left- and right-hand sides. To pass to the invariant in $T^2 - z$, we divide out by the obvious $\mathbb{Z}$ action. Part (d) shows the pairing between $\widehat{\mathrm{HF}}(M_T)$ and $\widehat{\mathrm{HF}}(S^1 \times D^2)$ corresponding to $+1$ surgery on the trefoil. There is a single intersection point, which matches the fact that $\widehat{\mathrm{HF}}(P) \cong \mathbb{Z}/2$.

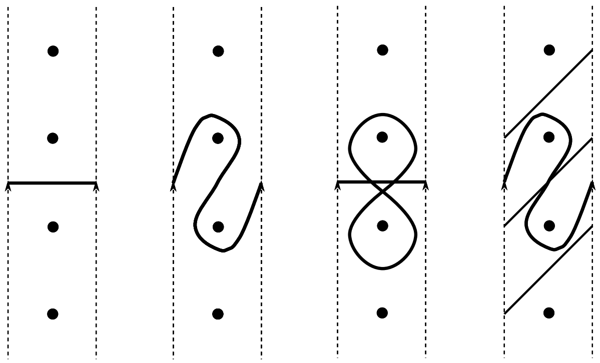


FIGURE 3

Heegaard Floer invariant $\widehat{\mathrm{HF}}$ of some simple manifolds, including (a) $S^1 \times D^2$, (b) the exterior of the right-hand trefoil, and (c) the exterior of the figure-eight knot. The dots indicate punctures.

We close this section with a question about instanton Floer homology. Kronheimer and Mrowka have defined [36] an invariant $I^\sharp(Y)$ which is an instanton analog of $\widehat{\mathrm{HF}}(Y)$ and is conjectured to be isomorphic to it. It is thus very natural to ask:

Question 1.4. Is there an instanton analog of the invariant $\widehat{\mathrm{HF}}(M)$, and, if so, can it be related directly to $X_{\mathrm{SU}_2}(M)$?

It is probably too much to ask for a direct relation with the character variety in every case, but one might still hope for it in some simple examples. (See [22] for something along these lines using the Seiberg–Witten moduli space.)

In the sections that follow, we will briefly explain how the bordered Floer homology of a manifold with torus boundary can be reinterpreted to define the invariant $\widehat{\mathrm{HF}}(M)$, describe some applications of the theorem, and discuss generalizations and further directions.

2. CONSTRUCTION AND PROPERTIES OF THE INVARIANT

2.1. The Fukaya category

We begin with a brief and imprecise account of the Fukaya category. For more careful discussions, we refer the reader to [4, 51, 52]. Suppose that (M, ω) is an exact symplectic manifold. Taken naively, objects of the Fukaya category $\mathcal{F}(M)$ are Lagrangian submanifolds $L_i \subset M$, and $\mathrm{Hom}(L_1, L_2) = \mathrm{HF}(L_1, L_2)$ is Lagrangian Floer homology—the other sort of homology invented by Floer. The Floer chain complex is generated by intersections between L_1 and L_2, and the differential is given by counting J-holomorphic disks with respect to a compatible almost-complex structure. This is an oversimplification for many reasons: first, $\mathcal{F}(M)$ is an A_∞-category, with higher morphisms given by counts of holomorphic polygons with higher numbers of sides; and second, $\mathcal{F}(M)$ is triangulated, so a typical object is actually a *twisted complex*—an iterated mapping cone built out of geometric Lagrangians.

All this extra structure may seem daunting to the newcomer, but it has its advantages. Although there are usually infinitely many different Lagrangians in M, in many cases it is possible to show that every object of $\mathcal{F}(M)$ is isomorphic to a twisted complex built up out of a finite number of Lagrangians $L_1, \dots, L_n$. In this case the L_i are said to generate the Fukaya category. This is easiest to arrange in the case where both M and the L_i are noncompact. In fact, M should be a Liouville manifold, so that near infinity it looks like the symplectization of a contact manifold N. In this situation, we need to be more careful about what is meant by $\mathrm{Hom}(L_i, L_j)$. The correct answer turns out to be the *wrapped Floer homology*, in which we replace L_i by its image under a flow determined by the Reeb flow on N. More generally, we can consider the *partially wrapped Floer homology* [4] in which the flow is stopped on some $X \subset N$.

If $L_1, \dots, L_n$ generate, we define $\mathcal{L} = \bigoplus_i L_i$, and consider the A_∞ algebra $\mathcal{A} = \mathrm{End}(\mathcal{L})$. If L is an object of $\mathcal{F}(M)$, we can consider $\mathcal{M}_L = \mathrm{Hom}(\mathcal{L}, L)$, which is an A_∞ module over $\mathcal{A}$. By the Yoneda embedding lemma, L and $\mathcal{M}_L$ carry the same information [2].

2.2. Bordered Floer homology

Next, we discuss the work of Lipshitz, Ozsváth, and Thurston [39].

Definition 2.1. Let Σ be a closed, connected, and oriented surface. A *parametrization* $\mathcal{P}$ of Σ is a minimal handle decomposition of Σ, together with a choice of basepoint z on the boundary of the 2-handle. A *bordered 3-manifold* $(M, \mathcal{P})$ is a compact, connected, and oriented 3-manifold M, together with a parametrization $\mathcal{P}$ of ∂M.

Up to isotopy, $\mathcal{P}$ is specified by the position of the 2-handle and the cocores of its 1-handles. These form a system of disjoint arcs $\alpha_1, \dots, \alpha_{2g} \subset \Sigma$ with ends on the boundary of the 2-handle. To a parametrized surface $(\Sigma, \mathcal{P})$, Lipshitz, Ozsváth, and Thurston associate an explicit A_∞ algebra $\mathcal{A}(\mathcal{P})$. (In fact, $\mathcal{A}(\mathcal{P})$ is a dga: $\mu_i = 0$ for all $i > 2$.) They also define the notions of Type D and Type A structures over $\mathcal{A}(\mathcal{P})$. In the language of Section 2.1, a (bounded) Type D structure is essentially a twisted complex over the category determined by $\mathcal{A}(P)$. The Type A structure corresponding to a Type D structure $\mathcal{D}$ is essentially $\mathrm{Hom}(\mathcal{A}, \mathcal{D})$. Thus the relation between Type D and Type A structures is the same as the relation between L and $\mathcal{M}_L$ in the Fukaya category. The main theorem of bordered Floer homology is:

Theorem 2.2 ([38, 39]). *A bordered 3-manifold $(M, \mathcal{P})$ determines a Type D structure $\widehat{\mathrm{CFD}}(M, \mathcal{P})$ over $\mathcal{A}(\mathcal{P})$ which is well defined up to quasiisomorphism. If $Y = M_1 \cup_\Sigma M_2$ and $\mathcal{P}$ is a parametrization of Σ, then $\widehat{\mathrm{HF}}(Y) \cong \mathrm{Hom}(\widehat{\mathrm{CFD}}(M_1, \mathcal{P}), \widehat{\mathrm{CFD}}(M_2, \mathcal{P}))$.*

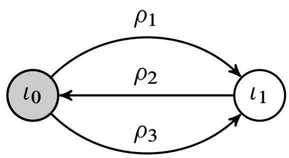
Suppose Σ is a parametrized surface, and let $\Sigma_0 \subset \Sigma$ be the complement of the 2-handle. If $I = \{i_1, \ldots, i_g\}$ is a g-element subset of $\{1, \ldots, 2g\}$, we define L_I to be the image of $\alpha_{i_i} \times \cdots \times \alpha_{i_g}$ in $\mathrm{Sym}^g \Sigma_0$. The L_I are noncompact Lagrangian submanifolds of $M = \mathrm{Sym}^g \Sigma_0$. In addition, the point $z \in \partial \Sigma_0$ determines a stop X_z for M. Auroux proved:

Theorem 2.3 ([2, 3]). *The L_I generate $\mathcal{F}_{X_z}(\mathrm{Sym}^g(\Sigma_0))$, and $\mathcal{A}(\mathcal{P}) \cong \mathrm{End}\, \mathcal{L}$, where $\mathcal{L} = \oplus L_I$.*

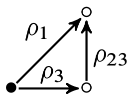
Hence $\widehat{\mathrm{CFD}}(M, \mathcal{P})$ determines an object of $\mathcal{F}_{X_z}(\mathrm{Sym}^g(\Sigma_0))$. A priori, this object is neither compact nor geometric—it is a twisted complex built up out of noncompact Lagrangians.

2.3. The torus

Up to isotopy, the torus T^2 has a unique parametrization, $\mathcal{P}$, as shown in Figure 4. The corresponding algebra $\mathcal{A}(T^2) = \mathcal{A}(\mathcal{P})$ is a quotient of the quiver algebra generated by the quiver below by the quadratic relations $\rho_2\rho_1 = \rho_3\rho_2 = 0$. Geometrically speaking, the arrows in the quiver correspond to the labeled arcs in on the boundary of the punctured torus, as shown in Figure 4. Composition is given by concatenation (when possible) and is 0 otherwise. We write $\rho_1\rho_2 = \rho_{12}$, etc.



A Type D structure over $\mathcal{A}(T^2)$ can be represented by a decorated graph, whose vertices are labeled by idempotents of $\mathcal{A}$ (we use $\bullet$ for L_0, and $\circ$ for L_1) and whose edges are labeled by morphisms. The labels on the edges determine the differential D in the twisted chain complex, which must satisfy $D^2 = 0$. Here is an example of a twisted complex built out of three objects—one copy of L_0 and two of L_1:



The key step in the proof of Theorem 1.3 is an algebraic structure theorem, which shows that every Type D structure over $\mathcal{A}(T^2)$ is homotopy equivalent to that with a particularly nice form.

Definition 2.4. A Type D structure over is a *loop* if its underlying graph (forgetting labels and orientations) is a cycle. More generally, a Type D structure with graph G is a *loop with*

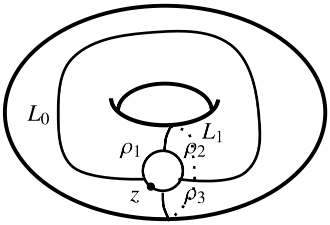


FIGURE 4
Parametrization of T^2, showing the 2-handle (large circle), α arcs (labeled L_0 and L_1), and basepoint.

a local system of dimension k if there is a loop L and a map $\pi : G \to L$ which preserves the labels edges and vertices and is a k-to-1 covering map away from one edge of L.

The torus algebra is a quotient of a slightly larger algebra $\widetilde{\mathcal{A}}$, which is obtained by adding in a generator ρ_0 corresponding to the arc that runs over the basepoint z, and setting any word that contains two copies of ρ_0 to 0, as well as the usual quadratic relations. An important result due to Lipshitz, Ozsváth, and Thurston is that if $\partial M = T^2$, then $\widehat{\mathrm{CFD}}(M)$ is extendable, that is, we can add in additional arrows labeled by elements of $\widetilde{\mathcal{A}}$ so that

$$D^2 = \sum_{j=0}^{3} \rho_j \rho_{j+1} \rho_{j+2} \rho_{j+3},$$

where the subscripts are to be interpreted modulo 4. The main technical result of [21] is:

Theorem 2.5. *An extendable Type D structure over* $\mathcal{A}$ *is homotopy equivalent to a disjoint union of loops with local systems.*

The first theorem of this type was proved by Haiden, Katzarkov, and Kontsevich [19], who showed that any twisted complex over $\mathcal{A}(T^2)$ (or more generally, the algebra associated to the Fukaya category of a higher genus surface) is a direct sum of loops with local systems and chains. The key role that such loops play in the study of bordered Floer homology was first observed by Hanselman and Watson in [23]. In [21] we give an effective algorithm for reducing an arbitrary extendable Type D structure to a disjoint union of loops. Alternately, one can appeal to [19], and then use the fact that the Type D structure is extendable to rule out the presence of any chains.

The final step in the proof of Theorem 1.3 is to associate a geometric loop $\gamma_{\mathcal{D}}$ (a closed curve in $T^2 \setminus z$) to a loop-type Type D structure $\mathcal{D}$, and show that

$$\mathrm{Hom}(\mathcal{D}_1, \mathcal{D}_2) \cong \mathrm{HF}(\gamma_{\mathcal{D}_1}, \gamma_{\mathcal{D}_2}).$$

Here the left-hand side is Hom in the category of Type D structures, and the right-hand side is an appropriately formulated version of Floer homology in $T^2 - z$; $\Gamma_{\mathcal{D}}$ is constructed by taking a straight line segment for each object in the loop, and joining the ends of consecutive objects according to the label on the arrow that joins them. There are two ways to do this.

In [19], the authors use the noncompact Lagrangians L_0, L_1, and join them by arcs along the boundary. In [21] we take a dual approach, using the compact arcs coming from the cores of the 1-handles and joining them by curves along the boundary of the 0-handle.

2.4. Spin^c structures and the Alexander polynomial

In this section, we review three basic properties of $\widehat{\mathrm{HF}}$ for closed 3-manifolds, and explain their generalization to manifolds with torus boundary. First, it is well known that $\widehat{\mathrm{HF}}(Y)$ can be decomposed according to the set of Spin^c structures on Y:

$$\widehat{\mathrm{HF}}(Y) = \bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} \widehat{\mathrm{HF}}(Y,\mathfrak{s}).$$

The same statement is true when M has torus boundary:

$$\widehat{\mathrm{HF}}(M) = \bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(M)} \widehat{\mathrm{HF}}(M,\mathfrak{s}),$$

where now the direct sum is taken in the Fukaya category, where it is given by disjoint union of curves. More interestingly, each summand $\widehat{\mathrm{HF}}(M,\mathfrak{s})$ can be lifted to a covering space of $\partial M - z$. To be precise, let $\overline{T}_M$ be the covering space of $T_M = \partial M$ whose fundamental group is the kernel of the composite map $\pi_1(\partial M) \to H_1(\partial M) \to H_1(M)$. Let $T_M^\circ = \partial M - z$, and define $\overline{T}_M^\circ$ to be its preimage in $\overline{T}_M$. It is shown in [21] that $\widehat{\mathrm{HF}}(M,\mathfrak{s})$ lifts to $\overline{T}_M^\circ$. For example, if $H_1(M) \cong \mathbb{Z}$, $\overline{T}_M^\circ$ is a punctured cylinder like those shown in Figure 3. There is a unique $\mathfrak{s} \in \mathrm{Spin}^c(M,\partial M)$, and the curves shown in Figure 3 are $\widehat{\mathrm{HF}}(M,\mathfrak{s})$ for their respective M's.

Suppose $Y = M_1 \cup_{T^2} M_2$. By considering the pairing of the lifted curves for $\widehat{\mathrm{HF}}(M_1)$ and $\widehat{\mathrm{HF}}(M_2)$, together with the action of the deck group, one can recover the Spin^c decomposition on $\widehat{\mathrm{HF}}(Y)$. Figure 5(b) illustrates this computation for 0 surgery on the torus knot $T(2,5)$.

Second, $\widehat{\mathrm{HF}}(Y)$ carries a natural $\mathbb{Z}/2$ grading. For manifolds with torus boundary, we have the following analog:

Proposition 2.6. *If M is a 3-manifold with torus boundary, then there is natural orientation on $\widehat{\mathrm{HF}}(M)$. If $Y = M_1 \cup_{T^2} M_2$ and x is a generator of $\widehat{\mathrm{HF}}(Y)$ corresponding to an intersection point of $\widehat{\mathrm{HF}}(M_1)$ and $\widehat{\mathrm{HF}}(M_2)$, then the $\mathbb{Z}/2$ grading of $x \in \widehat{\mathrm{HF}}(M_1) \cap \widehat{\mathrm{HF}}(M_2)$ is given by the sign of intersection of $\widehat{\mathrm{HF}}(M_1)$ and $\widehat{\mathrm{HF}}(M_2)$ at x.*

Finally, it is well known (going back to Casson [1]) that the Euler characteristic of Floer homology is related to the Alexander polynomial. We describe this relation in our context, restricting to the case where $H_1(M) = \mathbb{Z}$ for simplicity. Let $\pi : \overline{T}_M \to \partial M$ be the projection. The set $\pi^{-1}(z)$ can be naturally identified with $\mathbb{Z}$ by the action of the deck group. The space $\overline{T}_M$ has two ends: a positive end to which the z_n converge as $n \to \infty$ and a negative end to which the z_n converge as $n \to -\infty$. Let η_n be a path from z_n to the negative end, and define a_n to be the signed intersection number of η_n with $\widehat{\mathrm{HF}}(M)$. (Since $\widehat{\mathrm{HF}}(M)$ is compact, $z_n = 0$ for $n \ll 0$.) Then we have:

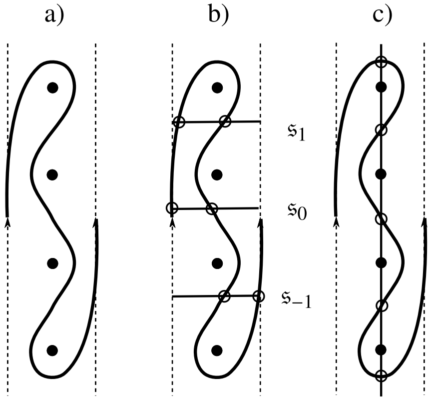


FIGURE 5
Some computations with the (2, 5) torus knot: (a) $\widehat{\mathrm{HF}}(M_{T(2,5)})$, (b) $\widehat{\mathrm{HF}}$ of 0-surgery on $T(2,5)$ has dimension 2 in each of the Spin^c structures $\mathfrak{s}_{-1}, \mathfrak{s}_0$, and $\mathfrak{s}_1$, and (c) $\widehat{\mathrm{HFK}}(T(2,5), i)$ has dimension 1 for $i = -2, -1, 0, 1, 2$.

Proposition 2.7 ([21]). *If* $\Delta(M) \in \mathbb{Z}[t^{\pm 1}]$ *is the Alexander polynomial, then*

$$\frac{\Delta_M}{1-t} \sim \sum_{n\in\mathbb{Z}} a_n t^n.$$

Here both sides are to be interpreted as Laurent series, and $\sim$ indicates equality up to multiplication by some power of t. The quantity on the left-hand side is the *Milnor torsion* of M. For example, by referring to Figure 3, one can easily compute that

$$\frac{\Delta(T)}{1-t} \sim t^{-1} + t + t^3 + t^4 + t^5 + \cdots,$$

corresponding to the well-known fact that $\Delta(T) = t^{-1} - 1 + t$.

2.5. Knot Floer homology

The definition of Heegaard Floer homology for closed 3-manifolds can be generalized to give an invariant of a pair $K \subset Y$, where K is a knot in Y. This invariant is called knot Floer homology (written $\widehat{\mathrm{HFK}}(K)$), and was discovered by Ozsváth and Szabó [42] and independently by the author [48]. Two basic properties of knot Floer homology are:

- If $b_1(Y) = 0$, $\widehat{\mathrm{HFK}}(K)$ splits as a direct sum $\widehat{\mathrm{HFK}}(K) = \bigoplus_{i\in\mathbb{Z}} \widehat{\mathrm{HFK}}(K,i)$. The grading i is called the *Alexander grading* and satisfies

$$\sum_i \chi\big(\widehat{\mathrm{HFK}}(K,i)\big) \cdot t^i \sim \Delta_K(t).$$

- If $b_1(Y) = 0$, there are two spectral sequences with E_1 term $\widehat{\mathrm{HFK}}(K)$ which converge to $\widehat{\mathrm{HF}}(Y)$. In one sequence, the differentials decrease the Alexander grading, while in the other they increase it.

If $\partial M = T^2$ and α is a simple closed curve on ∂M, we can form the Dehn filling $M(\alpha) = M \cup_\phi S^1 \times D^2$, where $\phi_*([\partial D^2]) = [\alpha]$. We consider the *core knot*

$$K_\alpha = S^1 \times 0 \subset S^1 \times D^2 \subset M(\alpha)$$

whose complement is again M. Let L'_α be the (noncompact) Lagrangian in $\partial M \setminus z$ which consists of a line of slope α passing through z. Then we have, the following result, which is essentially due to Lipshitz, Ozsváth, and Thurston:

Proposition 2.8. $\widehat{\mathrm{HFK}}(K_\alpha) \simeq \mathrm{HF}(\widehat{\mathrm{HF}}(M), L'_\alpha)$. *Conversely, if $K \subset S^3$ and we understand both sets of differentials on $\widehat{\mathrm{HFK}}(K)$, we can reconstruct $\widehat{\mathrm{HF}}(M_K)$.*

Since $\widehat{\mathrm{HF}}(M)$ is compact, the pairing $\mathrm{HF}(\widehat{\mathrm{HF}}(M), L'_\alpha)$ can still be computed by taking the minimal number of intersections between the two curves. By way of comparison, if L_α is the compact line of slope α in ∂M, then $\mathrm{HF}(\widehat{\mathrm{HF}}(M), L'_\alpha) \cong \widehat{\mathrm{HF}}(M(\alpha))$.

As in the previous section, we can understand the Alexander grading by passing to the lift $\widehat{\mathrm{HF}}(M, \mathfrak{s}) \subset \overline{T}^\circ_M$. For simplicity, we again restrict to the case where $H_1(M) \cong \mathbb{Z}$. The Lagrangian L'_α is homeomorphic to an open interval, so the set of lifts to $\overline{T}_M$ can be labeled as $L_{\alpha,i}$ for $i \in \mathbb{Z}$. With an appropriate choice of labeling,

$$\widehat{\mathrm{HFK}}(K_\alpha, i) \cong \mathrm{HF}\big(\widehat{\mathrm{HF}}(M), L_{\alpha,i}\big).$$

From this perspective, the spectral sequences from $\widehat{\mathrm{HFK}}(K_\alpha)$ to $\widehat{\mathrm{HF}}(M(\alpha))$ arise from the fact that if we push $\bigcup_i L'_{\alpha,i}$ off of the preimage of z, we get $\pi^{-1}(L_\alpha)$. The fact that there are two such sequences corresponds that we can push either to the left or to the right. From a more algebraic point of view, as an object of the Fukaya category, $\pi^{-1}(L_\alpha)$ is isomorphic to the filtered complex

$$\cdots \xrightarrow{\rho_{12}} L'_{\alpha,2} \xrightarrow{\rho_{12}} L'_{\alpha,1} \xrightarrow{\rho_{12}} L'_{\alpha,0} \xrightarrow{\rho_{12}} L'_{\alpha,-1} \xrightarrow{\rho_{12}} L'_{\alpha,-2} \xrightarrow{\rho_{12}} \cdots$$

whose associated grading is $\bigoplus_{i\in\mathbb{Z}} L'_{\alpha,i}$.

3. FLOER SIMPLE MANIFOLDS AND THE L-SPACE GLUING THEOREM

3.1. Floer simple manifolds

We say that Y is an *L-space* if Y is a rational homology sphere and $\dim \widehat{\mathrm{HF}}(Y, \mathfrak{s}) = 1$ for each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Since $\chi(\widehat{\mathrm{HF}}(Y, \mathfrak{s}) = 1$, this is as small as it can be, and L-spaces are the closed manifolds with the simplest possible Floer homology. In this section, we discuss the analogous notion for manifolds with torus boundary.

If M is such a manifold, let $\mathrm{Sl}(M)$ be the set of possible Dehn filling slopes on M. Then $\mathrm{Sl}(M)$ is naturally identified with the projective space on $H_1(\partial M; \mathbb{Z})$. By choosing a basis of $H_1(M; \mathbb{Z})$, we can identify $\mathrm{Sl}(M)$ with the rational projective space $\mathbb{QP}^1$. Let

$$\mathcal{L}(M) = \big\{\alpha \in \mathrm{Sl}(M) \mid M(\alpha) \text{ is an L-space}\big\}$$

be the set of L-space Dehn filling slopes of M. With S. Rasmussen, we proved

Theorem 3.1 ([49]). *If $\partial M \cong T^2$ and $b_1(M) = 1$, $\mathcal{L}(M)$ is one of the following:*

- *the empty set,*
- *a single point*
- *a closed interval with rational endpoints in $\mathbb{QP}^1$, or*
- *$\mathbb{QP}^1 \setminus [\ell]$ where ℓ is the rational longitude.*

Definition 3.2. Manifold M is *Floer simple* if $\mathcal{L}(M)$ contains more than one element. In this case we call $\mathcal{L}(M)$ the *L-space interval* and write $\mathcal{L}^\circ(M)$ for its interior.

If M is Floer simple, $\widehat{\mathrm{HF}}(M)$ is easy to describe. If $\alpha \in Sl(M)$, let $n_\alpha = \alpha \cdot \ell$ and consider the map $p_\alpha : T_M \to \mathbb{R}/(n_\alpha)$ given by $p_\alpha(x) = \alpha \cdot x$. For $\mathfrak{s} \in \mathrm{Spin}^c(M)$, let $\gamma_{M,\mathfrak{s}}$ be curve obtained by pulling $\widehat{\mathrm{HF}}(M, \mathfrak{s})$ tight. Then we have:

Proposition 3.3 ([21]). *Manifold M is Floer simple with $\alpha \in \mathcal{L}^\circ(M)$ if and only if p_α maps $\gamma_{M,\mathfrak{s}}$ bijectively to $\mathbb{R}/(n_\alpha)$ bijectively for all $\mathfrak{s} \in \mathrm{Spin}^c(M)$.*

If M is Seifert fibered with $b_1 = 1$, then M is Floer simple, and the class of the fiber slope is in $\mathcal{L}^\circ(Y)$. But many hyperbolic 3-manifolds are Floer simple as well. In [12], Dunfield studied Burton's census [9] of all 59,068 1-cusped hyperbolic 3-manifolds which have $b_1 = 1$ and admit an ideal triangulation with 9 or fewer tetrahedra. He found that 50,598 of them were Floer simple and 8,352 were not, leaving only 118 where he was unable to decide. It is natural to ask if the condition of being Floer simple has any geometrical meaning. By applying the fibration detection theorem of Ni [41], it is easy to see

Proposition 3.4. *If M is a Floer simple manifold with $H_1(M) \cong \mathbb{Z}$, then M fibers over the circle.*

Conversely, we could ask if there is a geometric characterization of a the monodromy of a fibered Floer simple manifold. To make the question precise, write $\mathrm{Mod}_{g,1}$ for the mapping class group of a genus g surface with one puncture. For $\phi \in \mathrm{Mod}_{g,1}$, let M_ϕ be the mapping torus of ϕ, and define $\mathcal{FS}_g = \{\phi \in \mathrm{Mod}_{g,1} \mid M_\phi \text{ is Floer simple}\}$.

Question 3.5. Describe $\mathcal{FS}_g$ as a subset of $\mathrm{Mod}_{g,1}$.

When $g = 1$, $\mathrm{Mod}_{1,1} \cong \mathrm{SL}_2(\mathbb{Z})$. Using Baldwin's work on the Floer homology of genus-one fibered manifolds [5], it is not difficult to see

$$\mathcal{FS}_1 = \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid \mathrm{tr}\, A \leq 1\}.$$

In other words, $A \in \mathrm{SL}_2(\mathbb{Z})$ is a Floer simple monodromy if A is elliptic or negatively hyperbolic or parabolic, but not if A is positively parabolic or hyperbolic. For $g > 1$, virtually nothing is known. It would be interesting to know if $g = 1$ is typical (in the sense that $\mathcal{FS}_g$ forms a large subset of $\mathrm{Mod}_{g,1}$) or atypical (in the sense that $\mathcal{FS}_g$ is relatively sparse.)

In another direction, if M is Floer simple, $\mathcal{L}(M)$ forms a distinguished interval in the circle of slopes. If we know a single point of $\mathcal{L}^\circ(M)$, the entire interval can be determined from the Turaev torsion of M [49], but from a purely homological perspective, it is difficult to say anything about $\mathcal{L}(M)$. We have a distinguished slope given by the homological longitude ℓ which is not contained in $\mathcal{L}(M)$, but otherwise $\mathrm{Sl}(M)$ looks quite homogenous. If M is Seifert fibered, then the fiber slope is a distinguished element of $\mathcal{L}(M)$. When M is a 1-cusped hyperbolic manifold, there is also a lot more geometry available: the hyperbolic metric on M naturally induces a flat metric (the cusp metric) on ∂M or, equivalently, on $H_1(\partial M)$, and we can talk about shortest geodesics, length of curves, the degeneracy slope, etc.

Question 3.6. If M is hyperbolic, can $\mathcal{L}(M)$ be related to the geometry of the induced metric on the cusp?

In this direction, there is an interesting unpublished observation of T. Brown, who pointed out that for many (but not all) manifolds in Burton's census, $\ell^{\perp}$ (the slope orthogonal to the homological longitude with respect to the cusp metric) is contained in $\mathcal{L}(M)$.

3.2. L-space gluings

Our work on the immersed curve picture for Floer homology arose out of earlier attempts [23,49] to understand the L-space conjecture of Boyer–Gordon–Watson and Juhász. This conjecture posits a surprising relation between Heegaard Floer homology and the fundamental group. To be precise, we say that a nontrivial group G is left orderable if there is a total order $<$ on G satisfying $gx < gy$ whenever $x < y$. (By convention, the trivial group is not left orderable.)

Conjecture 3.7 (The L-space conjecture [7,31]). *If Y is prime, the following statements are equivalent:*

- *Y is an not an L-space,*
- *$\pi_1(Y)$ is left-orderable,*
- *Y admits a coorientable taut foliation.*

The notion of the L-space interval, along with similar sets for fibered and non-left orderable fillings, was first introduced by Boyer and Clay [6], who used it to study the L-space conjecture for graph manifolds. Building on their work, we proved

Theorem 3.8 ([20,50]). *The L-space conjecture holds for graph manifolds.*

There are two independent proofs of this theorem—one of them by S. Rasmussen [50], and the other by Hansel, Watson, and Rasmussen.[2] Based on their work, we conjectured the following *L-space Gluing Theorem*, which was proved in [21].

Theorem 3.9. *$Y = M_1 \cup_{T^2} M_2$ is an L-space if and only if $\mathcal{L}^{\circ}(M_1) \cup \mathcal{L}^{\circ}(M_2) = \mathrm{Sl}(T^2)$. (In particular, if Y is an L-space, both M_1 and M_2 must be Floer simple.)*

When M_1 and M_2 are Floer simple, partial results in this direction were obtained in [23, 49], but the proof of the full result relies in an essential way on Theorem 1.3. As a corollary, we were able to reprove the following result of Eftekhary:

Corollary 3.10 ([14]). *An L-space homology sphere cannot contain an incompressible torus.*

4. LINKS, SATELLITES, AND SUTURES

The results of Theorem 1.3 can be extended without much difficulty to describe a broad class of manifolds with sutured boundary. We describe an application of this method to the knot Floer homology of satellite knots.

4.1. Sutured manifolds

Sutured Floer homology is a very important variant of Heegaard Floer homology introduced by Juhasz [30]. A *balanced sutured manifold* is a compact, oriented 3-manifold with boundary, together with a multicurve γ (the suture) which divides ∂M into two parts R_+ and R_- such that (a) $\chi(R_+) = \chi(R_-)$ and (b) each component of ∂M contains at least one component of γ. If (M, γ) is a balanced sutured manifold, its sutured Floer homology $\mathrm{SFH}(M, \gamma)$ is a vector space over $\mathbb{Z}/2$.

Both $\widehat{\mathrm{HF}}$ and $\widehat{\mathrm{HFK}}$ appear as special cases of SFH. If (Y, z) is a connected, pointed 3-manifold, we define $Y(1) \subset Y$ to be the complement of a small ball in centered at z, and let $\gamma \subset \partial Y$ be a simple closed curve on ∂Y. Then $\mathrm{SFH}(Y(1), \gamma) \cong \widehat{\mathrm{HF}}(Y)$. More generally, if $Y(n)$ is the analogous manifold where we remove n balls, we have

$$\mathrm{SFH}\big(Y(n), \gamma\big) \cong \widehat{\mathrm{HF}}(Y) \otimes H^*(S^1)^{\otimes n}.$$

Similarly, if $K \subset Y$ has meridian μ, we let $M_K \subset Y$ be the exterior of K and define γ_μ to be two parallel copies of μ in ∂M_K. Then $\mathrm{SFH}(M_K, \gamma_\mu) \cong \widehat{\mathrm{HFK}}(K)$.

Zarev [55] extended the framework of bordered Floer homology to the class of *bordered sutured manifolds*. Such a manifold consists of a compact oriented 3-manifold M together with a decomposition $\partial M = F \cup R$, where F is the bordered (or glueable) part of the boundary, and R is sutured, that is, it is equipped with a multicurve γ which divides R into two parts R_+ and R_-. We do not impose a condition on the Euler characteristic of $R_\pm$, but do require that each boundary component of R intersects both R_+ and R_-.

Following Auroux, Zarev's bordered sutured Floer homology can be interpreted as defining an object in a partially wrapped Fukaya category of $\mathrm{Sym}^k(F)$ for some k, where the set of stops used to define the wrapping is determined by the set of intersections of R_+ with ∂R. The usual bordered Floer homology corresponds to the special case where R is a small disk centered at z which is divided in half by a single arc.

Suppose that F is a once punctured torus. When R_+ intersects ∂F in a single interval and $\chi(R_+) = \chi(R_-)$, the bordered sutured Floer homology can be interpreted as an object of the partially wrapped Fukaya category of (a covering space of) F. Rather than discussing this situation in generality, we will focus on a special case. Suppose that $\partial M \simeq T^2_a \cup T^2_b$, and give M the bordered sutured where $R = R_\mu$ consists of all of T^2_b, equipped with a pair of parallel sutures of slope μ, together with a disk in T^2_a which is divided in half by a single suture. Then we have

Proposition 4.1. *The pair (M, R_μ) determines $\widehat{\mathrm{HF}}(M, R_\mu)$, which is a compactly supported object of $\mathcal{F}(T^2_a - z)$. As in the closed case, $\widehat{\mathrm{HF}}(M, R_\mu)$ can be represented by a union of immersed closed curves equipped with local systems.*

Zarev's gluing theorem [55] then implies that if M' is a manifold with torus boundary and $\phi : T^2_a \to \partial M$ is an orientation reversing diffeomorphism, then

$$\mathrm{HF}\big(\widehat{\mathrm{HF}}(M'), \widehat{\mathrm{HF}}(M, R_\mu)\big) \cong \mathrm{SFH}(M' \cup_\phi M, \gamma_\mu) \otimes H^*(S^1). \tag{4.1}$$

The extra factor of $H^*(S^1)$ appears because the sutured manifold obtained by gluing (M, R_μ) and (M', R_z) together has *three* boundary components—there is an extra bubble in the middle coming from the two sutured disks. To get $M' \cup_\phi M$ without the bubble, we would need to use a bordered sutured manifold $(M(\eta), R_{\eta,\mu})$ which is constructed by choosing a framed path η from a point on γ_μ to z, removing a tubular neighborhood of η, and using the framing to extend the sutures over the boundary of the tubular neighborhood. The difference between (M, R_μ) and $(M(\eta), R_{\eta,\mu})$ is illustrated in Figure 6.

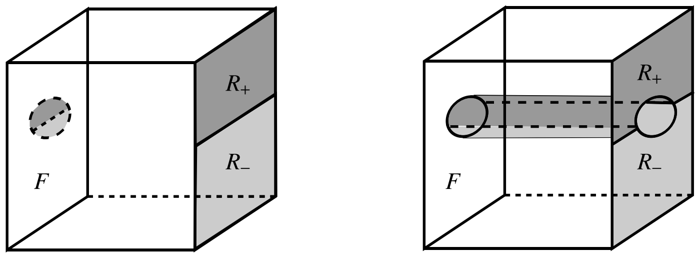


FIGURE 6
The bordered sutured manifolds (M, R_μ) (on the left) and $(M(\eta), R_{\eta,\mu})$ (on the right). The left- and right-hand faces of the cubes are part of the boundary of M. All the other faces are in the interior of M.

4.2. Link complements

We now specialize to the situation where $M = M_L$ is the exterior of a 2-component link $L \subset S^3$, and $\mu = \mu_2$ is the meridian of the second component of L. (We label the meridians and longitude of L_i by μ_i, λ_i.) Let $(M_L, \gamma_{\mu_1,\mu_2})$ be the sutured manifold with meridinal sutures on both boundary components. Then

$$\mathrm{SFH}(M_L, \gamma_{\mu_1,\mu_2}) = \widehat{\mathrm{HFL}}(L)$$

is the link Floer homology defined by Ozsváth and Szabó in [45]. In the same way that we can compute $\widehat{\mathrm{HF}}(M_K)$ from $\widehat{\mathrm{HFK}}(K)$ (and the differentials on it) when K is a knot in S^3, we can compute $\widehat{\mathrm{HF}}(M_L, R_{\mu_2})$ from $\widehat{\mathrm{HFL}}(L)$ (and the differentials on it) when L_2 is the unknot in S^3.

We will describe some examples, but before doing so, we pause to discuss Spin^c structures and lifts. As before, the Floer homology decomposes as

$$\widehat{\mathrm{HF}}(M_L, R_\mu) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M_L, R_\mu)} \widehat{\mathrm{HF}}(M_L, R_\mu, \mathfrak{s}).$$

Here $\mathrm{Spin}^c(M_L, R_\mu)$ is an affine set modeled on $\operatorname{coker} i_{a*}$, where $i_{a*} : \pi_1(F) \to H_1(M_L)$. It is easy to see that $\operatorname{coker} i_{a*} \cong \mathbb{Z}/n$, where n is the linking number of L. Also as before, $\widehat{\mathrm{HF}}(M_L, R_\mu, \mathfrak{s})$ lifts to the covering space $\overline{T}^\circ_M$ of $T^\circ_M := F$ given by $\pi_1(\overline{T}_M) = \ker i_{a*}$. Note that $\overline{T}^\circ_M$ is also determined by the linking number: it is the universal abelian cover of T°_M if $n \neq 0$, but an infinite punctured cylinder if $n = 0$.

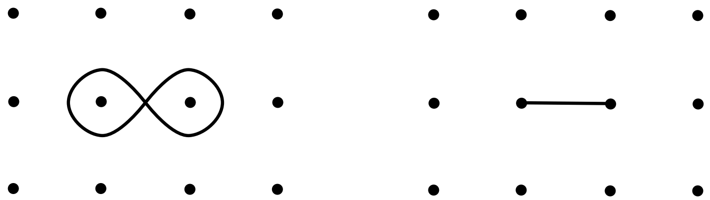


FIGURE 7
Curves when L is the Hopf link: $\widehat{\mathrm{HF}}(M_L, R_\mu)$ (left) and $\widehat{\mathrm{HF}}(M_L(\eta), R_{\eta,\mu})$ (right).

Example 4.2. If L is the Hopf link, then $M_L = T^2 \times I$. The linking number is 1, so there is a unique Spin^c structure and $\overline{T}_M$ is the universal abelian cover of T_M; $\widehat{\mathrm{HF}}(M_L, R_\mu) = \gamma_1$ is shown on the left-hand side of Figure 7. In this case there is a canonical choice of the path η, namely $z \times I$. With this choice, $\widehat{\mathrm{HF}}(M_L(\eta), R_{\eta,\mu}) = \gamma_2$ is shown on the right. The vector defined by the line segment is $\lambda_1 = \mu_2$ (the homology class of the suture.) Note that γ_1 is obtained by "inflating" γ_2 to form a figure-eight. It is not hard to see that $\mathrm{HF}(\gamma, \gamma_1) = HF(\gamma, \gamma_2) \otimes H^*(S^1)$ for any closed curve γ, as predicted by equation (4.1).

Example 4.3. If L is the $(2, 4)$ torus link, the linking number is 2, and there are 2 distinct Spin^c structures. In each case $\widehat{\mathrm{HF}}(M_L, R_\mu, \mathfrak{s})$ consists of a single figure-eight obtained by inflating a line segment. In one Spin^c structure the segment represents the vector μ_1, and in the other it represents the vector $\mu_1 + \lambda_1$.

Example 4.4. If L is the positive Whitehead link, the linking number is 0, so we have Spin^c structures $\mathfrak{s}_i$ for $i \in \mathbb{Z}$. In $\mathfrak{s}_{\pm 1}$, we have a single figure-eight representing μ_1, while in $\mathfrak{s}_0$, we have two figure-eights representing $\mu_1 + \lambda_1$ and μ_1, respectively.

More generally, the we can make the same calculations when L is a 2-bridge link. In this case, both components of L are unknots (so we are in the situation where can compute

$\widehat{\mathrm{HF}}(M, R_\mu)$ from $\widehat{\mathrm{HFL}}(L)$), and L is alternating, so $\widehat{\mathrm{HFL}}(L)$ can be computed by a theorem of Ozsváth and Szabó. We deduce:

Theorem 4.5. *If M is the complement of a 2-bridge link L, then $\widehat{\mathrm{HF}}(M_L, R_\mu)$ is a collection of figure-eights determined by the multivariable Alexander polynomial, signature, and linking number of L.*

We expect that in this case there should be a natural choice of curve η for which $\widehat{\mathrm{HF}}(M_L(\eta), R_{\eta,\mu})$ is a collection of line segments which are the "cores" of the figure-eights in $\widehat{\mathrm{HF}}(M, R_\mu)$.

4.3. Satellites

Suppose $L \subset S^3$ is a 2-component link, where L_1 is the unknot. If $C \subset S^3$ is a knot, we choose $\phi : \partial M_{L_1} \to \partial M_C$ with $\phi_*(l_1) = \mu_C$ and $\phi_*(\mu_1) = \lambda_C$ Then $M_C \cup_\phi M_{L_1} \cong S^3$, so the image of L_2 in this union is knot in S^3. It is called the *satellite knot* $C(P)$, where C is the *companion* and $P := L_2$ is the *pattern*.

There is a well-known formula for the Alexander polynomial of a satellite,

$$\Delta_{C(P)}(t) = \Delta_C(t^n)\Delta_P(t),$$

where n is the winding number of P (its homology class in the solid torus) and $\Delta_P(t)$ is the single-variable Alexander polynomial of $P \subset S^3$. It is thus very natural to ask whether there is a formula for the knot Floer homology of $C(P)$.

The knot Floer homology of satellites has been studied extensively, starting with the work of Eftekhary [13] and Hedden [25, 26], and including important contributions by Hom [28] and Levine [37]. More recently, Chen [10] gave a very interesting method for computing $\widehat{\mathrm{HFK}}(K(P))$ when P is a component of a 2-bridge link.

The method described above gives an alternate approach to the same problem. From equation (4.1) above, it is clear that to compute $\widehat{\mathrm{HFK}}(C(P))$, it suffices to understand $\widehat{\mathrm{HF}}(M_L, R_\mu)$. Hence if L is a 2-bridge link, Theorem 4.5 implies that there is a formula for $\widehat{\mathrm{HFK}}$ of the satellite, in the sense that there is a finite set of slopes $\alpha_i \in \mathrm{Sl}(M_K)$ such that

$$\widehat{\mathrm{HFK}}\big(C(P)\big) \cong \bigoplus_i \widehat{\mathrm{HFK}}(K_{\alpha_i})).$$

The slopes α_i are determined by the multivariable Alexander polynomial, signature, and linking number of L.

Chen's method also makes use of the curve invariant $\widehat{\mathrm{HF}}(M_K)$, but in a rather different way. (For example, he is able to compute $\tau(C(P))$, which the method above does not allow us to do.) It would be interesting to understand how the two approaches are related.

Ideally, one would like to compute the full curve invariant $\widehat{\mathrm{HF}}(M_{C(P)}$ rather than just the knot Floer homology. It is unknown how to do this in general, but Hanselman and Watson have given a very beautiful description of how to do this for cables [24].

5. FURTHER DEVELOPMENTS AND QUESTIONS

5.1. Tangles

There are other situations in which Zarev's bordered Floer sutured homology gives an invariant which lives in the Fukaya category of a surface. One of the most interesting invariants corresponds to the case of 2-strand tangles. This situation has been studied by Zibrowius, who proved

Theorem 5.1 ([58]). *If T is a 2-strand tangle in B^3, then there is a well-defined invariant $\widehat{\mathrm{HFT}}(T)$ which takes the form of a collection of immersed closed curves with local systems in the 4-punctured sphere. If $L = T_1 \cup T_2$, where T_1 and T_2 are such tangles, then $\widehat{\mathrm{HFL}}(L)$ can be computed by pairing the curves $\widehat{\mathrm{HFT}}(T_1)$ and $\widehat{\mathrm{HFT}}(T_2)$.*

Other tangle invariants analogous to bordered Floer homology have been developed by Ozsváth and Szabó [46] and Petkova and Vertesi [47].

Unlike the case of a manifold with torus boundary (where relatively few restrictions on the form of the curve invariant are known), Zibrowius was able to prove some very strong constraints on the form of the curves that appear in $\widehat{\mathrm{HFT}}(T)$. This enabled him to answer a long-standing question about the effect of mutation on the total dimension of knot Floer homology.

Theorem 5.2 ([57]). *If K_1 and K_2 are mutant knots, then* $\dim \widehat{\mathrm{HFK}}(K_1) = \dim \widehat{\mathrm{HFK}}(K_2)$.

More recently, Kotelskiy, Watson, and Zibrowius have introduced some similar interpretations of the Khovanov homology of a 4-ended tangle T [33]. At the level of polynomials, the Jones polynomial of a 4-ended tangle is not so different from its $\mathfrak{sl}(n)$/HOMFLY-PT polynomial. (Both live in 2-dimensional vector spaces.) Hence it is natural to ask:

Question 5.3. Can the $\mathfrak{sl}(n)$ homology of a 4-ended tangle be interpreted as a curve invariant?

5.2. Cobordisms and extended TQFTs

Although we have not discussed it here, $\widehat{\mathrm{HF}}$ fits into the structure of a (relative) $3+1$ dimensional TQFT, as established by Zemke [56]. A cobordism $(W, \eta) : (Y_0, z_0) \to (Y_1, z_1)$ induces a map $F_{W,\eta} : \widehat{\mathrm{HF}}(Y_0, z_0) \to \widehat{\mathrm{HF}}(Y_1, z_1)$. It is an important foundational problem to show that the structure of bordered Floer homology can be extended to give a (pointed) extended TQFT, so that we associate a category $\mathcal{A}(\Sigma, z)$ to a pointed surface Σ, an object of that category $\mathcal{A}(M, z)$ to a pointed 3-manifold M with $\partial M \cong \Sigma$, and a morphism $\mathcal{A}(M_0, z) \to \mathcal{A}(M_1, z)$ to a cobordism with corners $W : M_0 \to M_1$. The lower-dimensional parts of this structure have already been established by Lipshitz–Ozsváth–Thurston, and it is not difficult to understand what the cobordism maps should be. The real work is in showing that they are well defined and satisfy an appropriate gluing theorem.

5.3. HF^-

For closed 3-manifolds, $\widehat{\mathrm{HF}}$ is part of a larger package that also includes the equivariant homologies HF^+ and HF^-. One might hope to understand what these invariants mean for a manifold with torus boundary. Lipshitz, Ozsváth, and Thurston are in the process of developing a bordered theory for HF^- (see [40] for a first installment), and it will be interesting to see whether and how this can be interpreted in terms of curves and the Fukaya category. Some ideas for knot Floer homology have already been developed by Hanselman.

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