PBW DEGENERATIONS, QUIVER GRASSMANNIANS, AND TORIC VARIETIES

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Dedicated to the memory of Ernest Borisovich Vinberg

ABSTRACT

We present a review on the recently discovered link between the Lie theory, the theory of quiver Grassmannians, and various degenerations of flag varieties. Our starting point is the induced Poincaré–Birkhoff–Witt filtration on the highest weight representations and the corresponding PBW degenerate flag varieties.

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1. INTRODUCTION

The celebrated Poincaré–Birkhoff–Witt theorem claims that there exists a filtration on the universal enveloping of a Lie algebra such that the associated graded algebra is isomorphic to the symmetric algebra. The PBW filtration on the universal enveloping algebra of a nilpotent subalgebra of a simple Lie algebra induces a filtration on the representation space of a highest weight module. The natural problem is to study this filtration and the corresponding graded space. Quite unexpectedly, the problem turned out to be related to numerous representation-theoretic, algebro-geometric, and combinatorial questions. Our goal is to give an overview of the whole story and to describe various links between different parts of the picture. The main objects of study are monomial bases, convex polytopes, flag and Schubert varieties, their degenerations, quiver Grassmannians, and toric varieties.

The paper is organized as follows. In Section 2 we collect representation-theoretic results of algebraic nature. Section 3 is devoted to the geometric representation theory. In Section 4 we discuss combinatorics emerging from the cellular decomposition of the PBW degenerate flag varieties. In Section 5 we describe the link between the Lie theory and the theory of quiver Grassmannians. Finally, Section 6 treats toric degenerations.

Throughout the paper we work over the field of complex numbers.

2. REPRESENTATION THEORY: ALGEBRA

Let $\mathfrak{g}$ be a simple Lie algebra with the set R^+ of positive roots. Let α_i, ω_i, $i = 1, \dots, n-1$ be the simple roots and the fundamental weights. Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the Cartan decomposition. For $\alpha \in R^+$, we denote by $e_\alpha \in \mathfrak{n}^+$ and $f_\alpha \in \mathfrak{n}^-$ the corresponding Chevalley generators. We denote by P^+ the set of dominant integral weights.

Consider the PBW filtration on the universal enveloping algebra $\mathrm{U}(\mathfrak{n}^-)$:

$$\mathrm{U}(\mathfrak{n}^-)_s = \mathrm{span}\{x_1 \cdots x_l : x_i \in \mathfrak{n}^-, l \le s\},$$

for example, $\mathrm{U}(\mathfrak{n}^-)_0 = \mathbb{C} \cdot 1$.

For a dominant integral weight $\lambda = m_1\omega_1 + \cdots + m_{n-1}\omega_{n-1}$, let V_λ be the corresponding irreducible highest weight $\mathfrak{g}$-module with a highest weight vector v_λ. Since $V_\lambda = \mathrm{U}(\mathfrak{n}^-)v_\lambda$, we have an increasing filtration $(V_\lambda)_s$ on V_λ,

$$(V_\lambda)_s = \mathrm{U}(\mathfrak{n}^-)_s v_\lambda.$$

We call this filtration the PBW filtration and study the associated graded space $V_\lambda^a = \mathrm{gr}\, V_\lambda$.

Let us consider an example for fundamental weights in type A. Let V_{ω_1} be the vector representation of $\mathfrak{sl}_n$ with a basis $w_1, \dots, w_n$ and consider $V_{\omega_k} \simeq \Lambda^k V_{\omega_1}$ for $k = 1, \dots, n-1$. Then $(V_{\omega_k})_s$ is spanned by the wedge products $w_{i_1} \wedge \cdots \wedge w_{i_k}$ such that the number of indices a with $i_a > k$ is at most s.

The following holds true [60]:

(1) The action of $\mathrm{U}(\mathfrak{n}^-)$ on V_λ induces the structure of an $S(\mathfrak{n}^-)$-module on V_λ^a and $V_\lambda^a = S(\mathfrak{n}^-)v_\lambda$.

(2) The action of $\mathrm{U}(\mathfrak{n}^+)$ on V_λ induces the structure of a $U(\mathfrak{n}^+)$-module on V_λ^a.

Our aims are to describe V_λ^a as an $S(\mathfrak{n}^-)$-module and to find a basis of V_λ^a. We present the answer in type A. For similar results in other types, see [12, 61, 62, 77, 78, 99].

The positive roots in type A_{n-1} are of the form $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$ with $1 \le i \le j \le n-1$. Recall that a Dyck path is a sequence $\mathbf{p} = (\beta(0), \beta(1), \ldots, \beta(k))$ of positive roots of $\mathfrak{sl}_n$ satisfying the following conditions: if $k = 0$, then $\mathbf{p}$ is of the form $\mathbf{p} = (\alpha_i)$ for some simple root α_i, and if $k \ge 1$, then the first and last elements are simple roots, and if $\beta(s) = \alpha_{p,q}$, then $\beta(s+1) = \alpha_{p,q+1}$ or $\beta(s+1) = \alpha_{p+1,q}$.

Here is an example of a path for $\mathfrak{sl}_6$:

$$(\alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4, \alpha_4 + \alpha_5, \alpha_5).$$

For a multiexponent $\mathbf{s} = \{s_\beta\}_{\beta > 0}$, $s_\beta \in \mathbb{Z}_{\ge 0}$, let $f^{\mathbf{s}} = \prod_{\beta \in R^+} f_\beta^{s_\beta} \in S(\mathfrak{n}^-)$. For an integral dominant $\mathfrak{sl}_n$-weight $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$, let $S(\lambda)$ be the set of all multiexponents $\mathbf{s} = (s_\beta)_{\beta \in R^+} \in \mathbb{Z}_{\ge 0}^{R^+}$ such that for all Dyck paths $\mathbf{p} = (\beta(0), \ldots, \beta(k))$,

$$s_{\beta(0)} + s_{\beta(1)} + \cdots + s_{\beta(k)} \le m_i + m_{i+1} + \cdots + m_j, \tag{2.1}$$

where $\beta(0) = \alpha_i$ and $\beta(k) = \alpha_j$.

The polytopes in $\mathbb{R}_{\ge}^{R^+}$ defined by inequalities (2.1) are referred to as the FFLV polytopes. For their combinatorial properties and connection to the Gelfand–Tsetlin polytopes [75], see [6, 44, 47, 67]. The following theorem holds true [60].

Theorem 2.1. *The vectors $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S(\lambda)$, form a basis of V_λ^a. In addition, $S(\lambda) + S(\mu) = S(\lambda + \mu)$.*

We note that Theorem 2.1 implies that the elements $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S(\lambda)$ form a basis of the classical representation V_λ provided an order of factors is fixed in each monomial $f^{\mathbf{s}}$ (see [121]).

Let us describe the Lie algebra $\mathfrak{g}^a$ acting on V_λ^a. As a vector space, $\mathfrak{g}^a$ is isomorphic to $\mathfrak{g}$. The Borel $\mathfrak{b} \subset \mathfrak{g}^a$ is a subalgebra, the nilpotent subalgebra $\mathfrak{n}^- \subset \mathfrak{g}^a$ is an abelian ideal, and $\mathfrak{b}$ acts on the space $\mathfrak{n}^-$ as on the quotient $\mathfrak{g}/\mathfrak{b}$. Then for any $\lambda \in P^+$ the structure of the $\mathfrak{g}$-module on V_λ induces the structures of $\mathfrak{g}^a$ module on V_λ^a.

Note that $V_\lambda^a = S(\mathfrak{n}^-)v_\lambda$ is a cyclic $S(\mathfrak{n}^-)$-module, so we can write $V_\lambda^a \simeq S(\mathfrak{n}^-)/I(\lambda)$, for some ideal $I(\lambda) \subset S(\mathfrak{n}^-)$.

The following theorem holds in types A and C [60]:

Theorem 2.2. $I(\lambda) = S(\mathfrak{n}^-)(\mathrm{U}(\mathfrak{n}^+) \circ \operatorname{span}\{f_\alpha^{(\lambda,\alpha^\vee)+1}, \alpha \in R^+\})$.

This theorem should be understood as a commutative analogue of the well-known description of V_λ as the quotient

$$V_\lambda \simeq \mathrm{U}(\mathfrak{n}^-)/\langle f_\alpha^{(\lambda,\alpha^\vee)+1}, \alpha \in R^+ \rangle$$

(see, for example, [74, 86]).

The proof of the theorems above is based on the following claim available in types A and C [60–62].

Theorem 2.3. *Let $\lambda, \mu \in P^+$. Then*

$$V^a_{\lambda+\mu} \simeq \mathrm{U}(\mathfrak{g}^a)(v_\lambda \otimes v_\mu) \hookrightarrow V^a_\lambda \otimes V^a_\mu$$

as $\mathfrak{g}^a$-modules.

The algebraic and representation theoretic properties of the PBW filtration and the $\mathfrak{g}^a$ action in more general settings are considered in [10–12,32,48,51,52,56,65,70,77,78,99,106,107].

3. REPRESENTATION THEORY: GEOMETRY

Let G be a simple simply-connected Lie group with the Lie algebra $\mathfrak{g}$. Let $B \subset G$ be a Borel subgroup with the Lie algebra $\mathfrak{b}$. Each space V_λ, $\lambda \in P^+$ is equipped with the natural structure of a G-module. Therefore G acts on the projectivization $\mathbb{P}(V_\lambda)$. The (generalized) flag variety $\mathcal{F}_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ is defined as the G-orbit of the line $\mathbb{C}v_\lambda$ (see [73,94]). Each variety $\mathcal{F}_\lambda$ is isomorphic to the quotient of G by the parabolic subgroup leaving the point $\mathbb{C}v_\lambda \in \mathbb{P}(V_\lambda)$ invariant. In particular, for $G = \mathrm{SL}_n$ and a fundamental weight $\lambda = \omega_k$ the flag variety $\mathcal{F}_\lambda$ is isomorphic to the Grassmannian $\mathrm{Gr}_k(n)$. For a regular weight λ, the flag variety $\mathcal{F}_\lambda$ sits inside $\prod_{k=1}^{n-1} \mathrm{Gr}_k(n)$ and consists of chains of embedded subspaces. In what follows, we mostly consider the case $G = \mathrm{SL}_n$ and regular λ, the general type A case can be treated similarly (see [53,54,56,57]). We denote the complete type A_{n-1} flag variety by $\mathcal{F}_n$ (it is known to be independent of a regular weight λ). The variety $\mathcal{F}_n$ admits Plücker embedding into the product of projective spaces $\prod_{k=1}^{n-1} \mathbb{P}(\Lambda^k(\mathbb{C}^n))$. The homogeneous coordinate ring (also known as the Plücker algebra) is a quotient of the polynomial algebra in Plücker variables X_I, $I \subset [n]$ by the quadratic Plücker ideal.

Recall the Lie algebra $\mathfrak{g}^a$ acting on V^a_λ. We now describe the corresponding Lie group G^a. Let $M = \dim \mathfrak{n}$ and let $\mathbb{G}_a$ be the additive group of the field $\mathbb{C}$. The Lie group G^a is a semidirect product $\mathbb{G}_a^M \rtimes B$ of the normal subgroup $\mathbb{G}_a^M$ and the Borel subgroup B. The action by conjugation of B on $\mathbb{G}_a^M$ is induced from the B-action on $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$.

We now define the degenerate flag varieties $\mathcal{F}^a_\lambda$ [54]. Let $[v_\lambda] \in \mathbb{P}(V^a_\lambda)$ be the line $\mathbb{C}v_\lambda$.

Definition 3.1. The variety $\mathcal{F}^a_\lambda \hookrightarrow \mathbb{P}(V^a_\lambda)$ is the closure of the G^a-orbit of $[v_\lambda]$,

$$\mathcal{F}^a_\lambda = \overline{G^a[v_\lambda]} = \overline{\mathbb{G}_a^M[v_\lambda]} \hookrightarrow \mathbb{P}(V^a_\lambda).$$

We note that the orbit $G[v_\lambda] \hookrightarrow \mathbb{P}(V_\lambda)$ coincides with its closure, but the orbit $G^a[v_\lambda]$ does not; in fact, $\mathcal{F}^a_\lambda$ is the so-called $\mathbb{G}_a^M$-variety, see [7,8,82]. Theorem 2.3 implies that in types A and C the varieties $\mathcal{F}^a_\lambda$ depend only on the regularity class of λ, i.e., $\mathcal{F}^a_\lambda$ is isomorphic to $\mathcal{F}^a_\mu$ if and only if the sets of fundamental weights showing up in λ and μ coincide (see [102] for the study of a similar question for Schubert varieties).

In types A and C, we have rather explicit description of the degenerate flag varieties [53,58]. In particular, for $\mathfrak{g} = \mathfrak{sl}_n$ one has $\mathcal{F}^a_{\omega_k} \simeq \mathrm{Gr}_k(n)$. To describe the PBW degenerate flag varieties in type A, we introduce the following notation: let W be an n-dimensional vector space with a basis $w_1, \dots, w_n$. Let us denote by $\mathrm{pr}_k : W \to W$ the projection along w_k. We denote the regular PBW degenerate flag variety by $\mathcal{F}^a_n$. The following theorem holds [53,54] (we use the shorthand notation $[n] = \{1, \dots, n\}$).

Theorem 3.2. *One has*

$$\mathcal{F}^a_n \simeq \bigl\{(V_1, \dots, V_{n-1}) : V_k \in \mathrm{Gr}_k(W), k \in [n]; \mathrm{pr}_{k+1} V_k \subset V_{k+1}, k \in [n-1]\bigr\}.$$

Using this description, one proves the following theorem [29–31] (see also [96]).

Theorem 3.3. *The variety $\mathcal{F}^a_n$ is isomorphic to a Schubert variety for the group* SL_{2n-1}.

The symplectic PBW degenerations are described in [58] (see also [16]).

For a partition $\lambda = (\lambda_1 \ge \dots \ge \lambda_{n-1} \ge 0)$, we denote by Y_λ the corresponding Young diagram. Recall that the classical SL_n flag variety admits an embedding to the product of Grassmannians. The corresponding homogeneous coordinate ring (the Plücker algebra) is generated by the Plücker variables X_I, $I \subset [n]$ and is known to be isomorphic to the direct sum $\bigoplus_{\lambda \in P^+} V^*_\lambda$ (see [73]). There is a one-to-one bijection between the Plücker variables and columns filled with numbers from $[n]$ (the numbers increase from top to bottom). Then the semistandard Young tableaux provide a basis of the homogeneous coordinate ring of SL_n/B (one takes the product of Plücker variables, corresponding to the columns of a tableau). Similar result holds true in the PBW degenerate situation.

We denote by μ_j the length of the jth column of a diagram.

Definition 3.4. A semistandard PBW-tableau of shape λ is a filling $T_{i,j}$ of the Young diagram Y_λ with numbers $1, \dots, n$. The number $T_{i,j} \in \{1, \dots, n\}$ is attached to the box (i, j). The filling $T_{i,j}$ has to satisfy the following properties:

(1) if $T_{i,j} \le \mu_j$, then $T_{i,j} = i$;

(2) if $i_1 < i_2$ and $T_{i_1,j} \ne i_1$, then $T_{i_1,j} > T_{i_2,j}$;

(3) for any $j > 1$ and any i, there exists $i_1 \ge i$ such that $T_{i_1,j-1} \ge T_{i,j}$.

One can show that the number of shape λ semistandard PBW-tableaux is equal to $\dim V_\lambda$. Moreover, the following theorem holds [54] (see also [63,79]).

Theorem 3.5. *The homogeneous coordinate ring of $\mathcal{F}^a_n$ (also known as the PBW degenerate Plücker algebra) is isomorphic to the direct sum of dual PBW degenerate modules $(V^a_\lambda)^*$, $\lambda \in P^+$. The ideal of relations is quadratic and is generated by degenerate Plücker relations. The PBW semistandard tableaux parametrize a basis in the coordinate ring.*

Certain infinite-dimensional analogues of the results described above are obtained in [59,108]. However, this direction has not been seriously pursued so far.

4. TOPOLOGY AND COMBINATORICS

In this section we describe a cellular decomposition of the type A complete PBW degenerate flag varieties $\mathcal{F}_n^a$ (see [16, 53, 58] for a more general picture).

Let us fix an n-dimensional vector space W with a basis $w_1, \dots, w_n$. Let $\mathbf{I} = (I_1, \dots, I_{n-1})$ be a collection of subsets of the set $[n]$ such that $|I_k| = k$. We denote by $p_\mathbf{I} \in \prod_{k=1}^{n-1} \mathrm{Gr}_k(W)$ a point in the product of Grassmann varieties such that the kth component is equal to the linear span of w_i with $i \in I_k$. Theorem 3.2 implies that $p_\mathbf{I} \in \mathcal{F}_n^a$ if and only if

$$I_k \subset I_{k+1} \cup \{k+1\} \quad \text{for all } k = 1, \dots, n-2. \tag{4.1}$$

The following theorem is proved in [53].

Theorem 4.1. *The G^a orbits of the points $p_\mathbf{I}$ provide a cellular decomposition of $\mathcal{F}_n^a$.*

A natural problem is to compute the Euler characteristic and Poincaré polynomial of $\mathcal{F}_n^a$. The answer is given in terms of the normalized median Genocchi numbers and the Dellac configurations.

The normalized median Genocchi numbers h_n, $n = 0, 1, 2, \dots$ form a sequence which starts with $1, 1, 2, 7, 38, 295$ [1]. The earliest definition was given by Dellac in [36] (see also [13, 15, 17, 40–42, 55, 80, 93, 120, 122]). Consider a rectangle with n columns and $2n$ rows. It contains $n \times 2n$ boxes labeled by pairs (l, j), where $l = 1, \dots, n$ is the number of a column and $j = 1, \dots, 2n$ is the number of a row. A Dellac configuration D is a subset of boxes, subject to the following conditions:

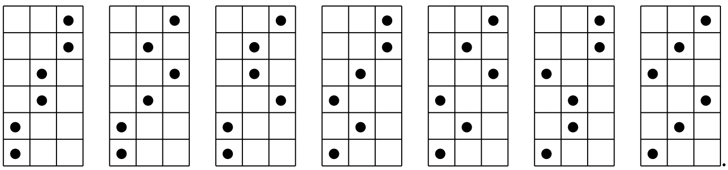
(1) each column contains exactly two boxes from D,

(2) each row contains exactly one box from D,

(3) if the (l, j)th box is in D, then $l \le j \le n + l$.

Let DC_n be the set of such configurations. Then the number of elements in DC_n is equal to h_n.

We list all Dellac's configurations for $n = 3$.



The importance of the median Genocchi numbers comes from the following theorem [53].

Theorem 4.2. *The number of collections $\mathbf{I}$ subject to conditions (4.1) is equal to the normalized median Genocchi number h_n. The Euler characteristic of $\mathcal{F}_n^a$ is equal to h_n.*

An explicit formula for the numbers h_n is available (see [25]), namely

$$h_n = \sum_{f_0,\dots,f_n \ge 0} \prod_{k=1}^{n} \binom{1+f_{k-1}}{f_k} \prod_{k=0}^{n-1} \binom{1+f_{k+1}}{f_k} \tag{4.2}$$

with $f_0 = f_n = 0$.

In order to compute the Poincaré polynomial of $\mathcal{F}_n^a$, we define a length $l(D)$ of a Dellac configuration D as the number of pairs (l_1, j_1), (l_2, j_2) such that the boxes (l_1, j_1) and (l_2, j_2) are both in D and $l_1 < l_2$, $j_1 > j_2$. This definition resembles the definition of the length of a permutation. We note that in the classical case the complex dimension of the cell attached to a permutation σ in a flag variety is equal to the number of pairs $j_1 < j_2$ such that $\sigma(j_1) > \sigma(j_2)$, which equals the length of σ. One has [53]:

Theorem 4.3. *The complex dimension of the cell in $\mathcal{F}_n^a$ containing a point $p_{\mathbf{I}}$ is equal to $l(D)$. Thus the Poincaré polynomial $h_n(q) = P_{\mathcal{F}_n^a}(q)$ is given by $h_n(q) = \sum_{D \in DC_n} q^{l(D)}$.*

The first four polynomials $h_n(q)$ are as follows:

$$h_1(q) = 1, \quad h_2(q) = 1 + q,$$
$$h_3(q) = 1 + 2q + 3q^2 + q^3,$$
$$h_4(q) = 1 + 3q + 7q^2 + 10q^3 + 10q^4 + 6q^5 + q^6.$$

The following (fermionic type) formula for the polynomials $h_n(q)$ is obtained in [25] using the geometry of quiver Grassmannians:

$$h_n(q) = \sum_{f_1,\dots,f_{n-1} \ge 0} q^{\sum_{k=1}^{n-1}(k-f_k)(1-f_k+f_{k+1})} \prod_{k=1}^{n} \binom{1+f_{k-1}}{f_k}_q \prod_{k=0}^{n-1} \binom{1+f_{k+1}}{f_k}_q \tag{4.3}$$

(we assume $f_0 = f_n = 0$). The formula is given in terms of the q-binomial coefficients

$$\binom{m}{n}_q = \frac{m_q!}{n_q!(m-n)_q!}, \quad m_q! = \prod_{i=1}^{m} \frac{1-q^i}{1-q}.$$

5. QUIVER GRASSMANNIANS

Theorem 3.2 provides a link between the PBW degenerate flag varieties and quiver Grassmannians. Let Q be a quiver with the set of vertices Q_0 and the set of arrows Q_1. For two vectors $\mathbf{e}, \mathbf{d} \in \mathbb{Z}^{Q_0}$, we denote by $\langle \mathbf{e}, \mathbf{d} \rangle$ the value of the Euler from of the quiver. For a Q module M and a dimension vector $\mathbf{e} \in \mathbb{Z}_{\ge 0}^{Q_0}$, we denote by $\mathrm{Gr}_{\mathbf{e}}(M)$ the quiver Grassmannian consisting of $\mathbf{e}$-dimensional subrepresentations of M. For more details on the quiver representation theory, see [9,34,35,113]. The general theory of quiver Grassmannians can be found in [21] (see also [2,3,20,22,85,97,109,110]).

Now let Q be an equioriented type A_{n-1} quiver. We label the vertices by the numbers from 1 to n. Then the set Q_1 consists of arrows $i \to i+1$, $i \in [n-1]$. The indecomposable representations of Q are labeled by pairs $1 \le i \le j \le n$; the representation $U_{i,j}$ is supported on vertices from i to j and is one-dimensional at each vertex. The projective indecomposable representations are given by $P_k = U_{k,n}$ and the injective indecomposables are $I_k = U_{1,k}$. In particular, the path algebra A of Q is isomorphic to the direct sum $\bigoplus_{k=1}^{n-1} P_k$ and the dual A^* is the direct sum $\bigoplus_{k=1}^{n-1} I_k$ of all indecomposable injectives.

By the very definition, the classical complete flag variety SL_n/B is isomorphic to the quiver Grassmannian $\mathrm{Gr}_{\dim A}(P_1^{\oplus n})$. The following observation was made in [25]:

$$\mathcal{F}_n^a \simeq \mathrm{Gr}_{\dim A}(A \oplus A^*). \tag{5.1}$$

The realization (5.1) provides additional tools for the study of algebro-geometric and combinatorial properties of the degenerate flag varieties (see [25–27,29]). In particular, one recovers and generalizes [27,28] the Bott–Samelson type construction for the resolution of singularities of $\mathcal{F}_n^a$ [57] (see also [89,112] for further generalizations). The resolution is constructed as a quiver Grassmannian for a larger quiver attached to Q.

Since the degenerate flag varieties have many nice properties, it is natural to study the quiver Grassmannians $\mathrm{Gr}_{\dim P}(P \oplus I)$ for arbitrary projective representation P and an injective representation I and a Dynkin quiver Q (the so-called well-behaved quiver Grassmannians). We summarize the main properties of these quiver Grassmannians in the following theorem (see [25,26]).

Theorem 5.1. *Let P and I be a projective and an injective representations of a Dynkin quiver Q. Then the quiver Grassmannian $X = \mathrm{Gr}_{\dim P}(P \oplus I)$ has the following properties:*

(1) $\dim X = \langle \dim P, \dim I\rangle$,

(2) *X is irreducible and normal,*

(3) *X is locally a complete intersection,*

(4) *there exists an algebraic group $G \subset \mathrm{Aut}(P \oplus I)$ acting on X with finitely many orbits.*

For a dimension vector $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$, let $R_{\mathbf{d}}$ be the variety of Q-representations of dimension $\mathbf{d}$. The group $\mathrm{G}L_{\mathbf{d}} = \prod_{i\in Q_0} \mathrm{G}L_{d_i}$ acts on $R_{\mathbf{d}}$ by base change and the orbits are parameterized by the isoclasses of $\mathbf{d}$-dimensional representations of Q. The closure of orbits induces the degeneration order on the set of isoclasses. Fixing a dimension vector $\mathbf{e}$, we obtain a family $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d})$ of $\mathbf{e}$-dimensional quiver Grassmannians over the representation space $R_{\mathbf{d}}$ (the so-called universal quiver Grassmannian). Let us denote the projection map $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) \to R_{\mathbf{d}}$ by $p_{\mathbf{e},\mathbf{d}}$.

We are interested in the case when Q is the equioriented type A_{n-1} quiver, $\mathbf{d} = (n, \dots, n)$ and $\mathbf{e} = (1, 2, \dots, n-1)$. Then both the classical and the PBW degenerate flag varieties are isomorphic to the fibers of $p_{\mathbf{e},\mathbf{d}}$. It is thus natural to ask about the properties of the whole family. The $\mathrm{G}L_{\mathbf{d}}$ orbits on $R_{\mathbf{d}}$ are parametrized by the tuples $\mathbf{r}$ of ranks $r_{i,j}$ of the compositions of the maps between the ith and jth vertices. We define three rank tuples $\mathbf{r}^0$, $\mathbf{r}^1$, and $\mathbf{r}^2$ by

$$r_{i,j}^0 = n+1, \quad r_{i,j}^1 = n+1-(j-i), \quad r_{i,j}^2 = n-(j-i).$$

Then the corresponding representations of Q are given by $M^0 = P_1^{\oplus n}$, $M^1 = A \oplus A^*$, and

$$M^2 = \bigoplus_{k=1}^{n-1} P_k \oplus \bigoplus_{k=1}^{n-2} I_k \oplus S,$$

where S is the direct sum of all simple modules of Q. One has $\mathrm{SL}_n / B \simeq \mathrm{Gr}_{\mathbf{e}}(M^0)$, $\mathcal{F}_n^a \simeq \mathrm{Gr}_{\mathbf{e}}(M^1)$. In [23] we prove the following theorem:

Theorem 5.2. (a) *The quiver Grassmannian $p_{\mathbf{e},\mathbf{d}}^{-1}(M^2)$ is of expected dimension $n(n-1)/2$. It is reducible and the number of irreducible components is equal to the nth Catalan number.*

(b) *The flat irreducible locus of* $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d})$ *consists of the fibers $p_{\mathbf{e},\mathbf{d}}^{-1}(M)$ such that M degenerates to M^1.*

(c) *The flat locus of* $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d})$ *consists of the fibers $p_{\mathbf{e},\mathbf{d}}^{-1}(M)$ such that M degenerates to M^2.*

The case of partial flag varieties is considered in [24].

6. TORIC DEGENERATIONS

As explained in Section 5, the degeneration of the classical flag variety into the PBW degenerate flag variety can be considered within a family of quiver Grassmannians over the representation space of the quiver. In particular, the study of other degenerations (intermediate and deeper ones) leads to the new and interesting results and examples. Yet another direction is to make a connection between the PBW degeneration and toric degenerations [33] of flag varieties (the latter attracted a lot of attention in the last two decades, see [4, 5, 14, 18, 19, 45, 46, 64, 81, 95]). One of the most famous examples is a flat degeneration of $\mathcal{F}_n$ into the toric variety with the Newton polytope being the Gelfand–Tsetlin polytope [75, 76, 92, 119]. We are able to prove the following theorem.

Theorem 6.1. *The complete flag variety $\mathcal{F}_n = \mathrm{SL}_n / B$ admits a flat degeneration to the toric variety corresponding to a FFLV polytope with regular highest weight. This degeneration factors through the PBW degeneration.*

The GT and FFLV polytopes are identified with the order and chain polytopes of a certain poset (see [6, 47, 100, 103, 118]). Several proofs of Theorem 6.1 are available. Essentially, there are three different approaches:

(1) via the representation space V_λ,

(2) via the Gröbner theory for the Plücker ideal,

(3) via the SAGBI theory for the Plücker algebra.

The first approach is utilized in [43, 48, 63]. The approach is similar to the PBW degeneration construction: instead of attaching degree one to each Chevalley generator, one uses a weight system, attaching weight $a_{i,j}$ to the generators $f_{\alpha_{i,j}}$ for all positive roots. For certain weight systems, one gets a filtration on the universal enveloping algebra, which leads to a filtered (and then graded) representation space and degenerate flag variety. The following theorem holds (see [48,63]).

Theorem 6.2. *Consider the PBW filtration with the weight system* $a_{i,j} = (j - i + 1)(n - j)$. *Then*

(1) *in the associated graded space the nonzero monomials in* f_α *form a basis,*

(2) *the associated graded space is acted upon by the symmetric algebra* $S(\mathfrak{n}^-)$ *and the degenerate flag variety is a* $\mathbb{G}_a^{n(n-1)/2}$ *variety,*

(3) *the corresponding degenerate flag variety is toric with the Newton polytope being the FFLV polytope.*

Instead of working with the representation space, one may start with the algebraic variety $\mathcal{F}_n$ from the very beginning. As an intermediate step one considers the theory of Newton–Okounkov (NO) bodies [88,105]. The connection between the NO bodies and toric degenerations is used in many papers, see, e.g., [5,49,81,87]. The following holds true.

Theorem 6.3. *The toric variety attached to the FFLV polytope can be constructed as a Newton–Okounkov body for certain valuations. The valuations are obtained via Lie theory* [63] *or geometrically* [71,72,90,91].

Recall the Plücker coordinates X_I, the quadratic Plücker ideal defining the flag variety $\mathcal{F}_n$ inside the product of the projectivized fundamental representations and the Plücker algebra (the quotient by the Plücker ideal). There are two general constructions leading to the degenerations of algebraic varieties: Gröbner theory for the defining ideals [83,104] (see also [43,116,117] for the tropical version) and the SAGBI (subalgebra analogues of the Gröbner bases for ideals) theory [111] (see also [37,83,84]). The former construction works with the defining ideals, attaching certain degrees to the variables, and the latter deals with the quotient algebras, using certain monomial orders. In our setting the following claims hold (see [43,101]).

Theorem 6.4. *There exists a maximal cone in the Gröbner fan of the Plücker ideal such that a general point corresponds to the monomial ideal defined by the PBW semistandard tableaux. There exists a monomial order on the set of Plücker variables such that the monomials in Plücker variables corresponding to the PBW semistandard tableaux form a SAGBI basis of the Plücker algebra.*

Let us close with the remark that it would be very interesting to construct and study toric degenerations for affine flag varieties [94] and semiinfinite flag varieties [50,68]. The first steps in this direction were made in [114,115]. From the representation theory point of view, this would lead to new constructions of bases and character formulas for the integrable representations of affine algebras and global Weyl and Demazure modules for current algebras [38, 39, 66, 98].

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REFERENCES

[1] The On-Line Encyclopedia of Integer Sequences, A000366, https://oeis.org/.

[2] S. Abeasis and A. Del, Fra, Degenerations for the representations of quiver of type A_m. *J. Algebra* **93** (1985), no. 2, 376–412.

[3] S. Abeasis and A. Del Fra, Degenerations for the representations of an equioriented quiver of type A_m. *Boll. Unione Mat. Ital. Suppl.* **2** (1984), 81–172.

[4] V. Alexeev and M. Brion, Toric degenerations of spherical varieties. *Selecta Math. (N.S.)* **10** (2004), no. 4, 453–478.

[5] D. Anderson, Okounkov bodies and toric degenerations. *Math. Ann.* **356** (2013), 1183–1202.

[6] F. Ardila, T. Bliem, and D. Salazar, Gelfand–Tsetlin polytopes and Feigin–Fourier–Littelmann–Vinberg polytopes as marked poset polytopes. *J. Combin. Theory Ser. A* **118** (2011), no. 8, 2454–2462.

[7] I. Arzhantsev, Flag varieties as equivariant compactifications of $\mathbb{G}_a^n$. *Proc. Amer. Math. Soc.* **139** (2011), no. 3, 783–786.

[8] I. Arzhantsev and E. Sharoiko, Hassett–Tschinkel correspondence: modality and projective hypersurfaces. *J. Algebra* **348** (2011), no. 1, 217–232.

[9] I. Assem, D. Simson, and A. Skowronski, *Elements of the representation theory of associative algebras, Vol. 1. Techniques of representation theory*. London Math. Soc. Stud. Texts 65, Cambridge University Press, Cambridge, 2006.

[10] T. Backhaus and C. Desczyk, PBW filtration: Feigin–Fourier–Littelmann modules via Hasse diagrams. *J. Lie Theory* **25** (2015), no. 3, 818–856.

[11] T. Backhaus, X. Fang, and G. Fourier, Degree cones and monomial bases of Lie algebras and quantum groups. *Glasg. Math. J.* **59** (2017), no. 3, 595–621.

[12] T. Backhaus and D. Kus, The PBW filtration and convex polytopes in type B. *J. Pure Appl. Algebra* **223** (2019), no. 1, 245–276.

[13] D. Barsky, Congruences pour les nombres de Genocchi de 2e espèce. *Groupe Étude Anal. Ultramétr., 8e année* (1980/81), no. 34, 13 pp.

[14] V. Batyrev, I. Ciocan-Fontanine, B. Kim, and D. Van Straten, Mirror symmetry and toric degenerations of partial flag manifolds. *Acta Math.* **184** (2000), no. 1, 1–39.

[15] A. Bigeni, Combinatorial study of Dellac configurations and q-extended normalized median Genocchi numbers. *Electron. J. Combin.* **21** (2014), no. 2, 2.32, 27 pp.

[16] A. Bigeni and E. Feigin, Symmetric Dellac configurations and symplectic/orthogonal flag varieties. *Linear Algebra Appl.* **573** (2019), 54–79.

[17] A. Bigeni and E. Feigin, Symmetric Dellac configurations. *J. Integer Seq.* **23** (2020), no. 4, 20.4.6, 32 pp.

[18] L. Bossinger, X. Fang, G. Fourier, M. Hering, and M. Lanini, Toric degenerations of $Gr(2,n)$ and $Gr(3,6)$ via plabic graphs. *Ann. Comb.* **22** (2018), no. 3, 491–512.

[19] P. Caldero, Toric degenerations of Schubert varieties. *Transform. Groups* **7** (2002), no. 1, 51–60.

[20] P. Caldero and M. Reineke, On the quiver Grassmannian in the acyclic case. *J. Pure Appl. Algebra* **212** (2008), no. 11, 2369–2380.

[21] G. Cerulli Irelli, *Three lectures on quiver Grassmannians*. Contemp. Math. 758, American mathematical Society, 2020.

[22] G. Cerulli Irelli, F. Esposito, H. Franzen, and M. Reineke, Cellular decomposition and algebraicity of cohomology for quiver Grassmannians. *Adv. Math.* **379** (2021), 107544, 47 pp.

[23] G. Cerulli Irelli, X. Fang, E. Feigin, G. Fourier, and M. Reineke, Linear degeneration of flag varieties. *Math. Z.* **287** (2017), no. 1–2, 615–654.

[24] G. Cerulli Irelli, X. Fang, E. Feigin, G. Fourier, and M. Reineke, Linear degenerations of flag varieties: partial flags, defining equations, and group actions. *Math. Z.* **296** (2020), no. 1–2, 453–477.

[25] G. Cerulli Irelli, E. Feigin, and M. Reineke, Quiver Grassmannians and degenerate flag varieties. *Algebra Number Theory* **6** (2012), no. 1, 165–194.

[26] G. Cerulli Irelli, E. Feigin, and M. Reineke, Degenerate flag varieties: moment graphs and Schröder numbers. *J. Algebraic Combin.* **38** (2013), no. 1.

[27] G. Cerulli Irelli, E. Feigin, and M. Reineke, Desingularization of quiver Grassmannians for Dynkin quivers. *Adv. Math.* **245** (2013), 182–207.

[28] G. Cerulli Irelli, E. Feigin, and M. Reineke, Homological approach to the Hernandez–Leclerc construction and quiver varieties. *Represent. Theory* **18** (2014), 1–14.

[29] G. Cerulli Irelli, E. Feigin, and M. Reineke, Schubert Quiver Grassmannians. *Algebr. Represent. Theory* **20** (2017), no. 1, 147–161.

[30] G. Cerulli Irelli and M. Lanini, Degenerate flag varieties of type A and C are Schubert varieties. *Int. Math. Res. Not.* **15** (2015), 6353–6374.

[31] G. Cerulli Irelli, M. Lanini, and P. Littelmann, Degenerate flag varieties and Schubert varieties: a characteristic free approach. *Pacific J. Math.* **284** (2016), no. 2, 283–308.

[32] I. Cherednik and E. Feigin, Extremal part of the PBW-filtration and E-polynomials. *Adv. Math.* **282** (2015), 220–264.

[33] D. Cox, J. Little, and H. Schenck, *Toric varieties*. Grad. Stud. Math. 124, American Mathematical Society, Providence, RI, 2011.

[34] W. Crawley-Boevey, Lectures on representations of quivers. Preprint, 1992. https://www.math.uni-bielefeld.de/~wcrawley/quivlecs.pdf.

[35] W. Crawley-Boevey, More lectures on representations of quivers. Preprint, 1992. https://www.math.uni-bielefeld.de/~wcrawley/morequivlecs.pdf.

[36] H. Dellac, Problem 1735. *L'Intermédiaire des Math.* **7** (1900), 9–10.

[37] C. De Concini, D. Eisenbud, and C. Procesi, *Hodge algebras*. Astérisque 91, Société Mathématique de France, Paris, 1982, 87 pp.

[38] I. Dumanski and E. Feigin, Reduced arc schemes for Veronese embeddings and global Demazure modules. 2019, arXiv:1912.07988.

[39] I. Dumanski, E. Feigin, and M. Finkelberg, Beilinson–Drinfeld Schubert varieties and global Demazure modules. *Forum Math. Sigma* **9** (2021), 1–25.

[40] D. Dumont, Interprétations combinatoires des nombres de Genocchi. *Duke Math. J.* **41** (1974), 305–318.

[41] D. Dumont and A. Randrianarivony, Dérangements et nombres de Genocchi. *Discrete Math.* **132** (1994), 37–49.

[42] D. Dumont and G. Viennot, A combinatorial interpretation of the Seidel generation of Genocchi numbers. *Discrete Math.* **6** (1980), 77–87.

[43] X. Fang, E. Feigin, G. Fourier, and I. Makhlin, Weighted PBW degenerations and tropical flag varieties. *Commun. Contemp. Math.* **21** (2019), no. 1, 1850016, 27 pp.

[44] X. Fang and G. Fourier, Marked chain-order polytopes. *European J. Combin.* **58** (2016), 267–282.

[45] X. Fang, G. Fourier, and P. Littelmann, Essential bases and toric degenerations arising from birational sequences. *Adv. Math.* **312** (2017), 107–149.

[46] X. Fang, G. Fourier, and P. Littelmann, On toric degenerations of flag varieties. In *Representation theory – current trends and perspectives*, pp. 187–232, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2017.

[47] X. Fang, G. Fourier, J.-P. Litza, and C. Pegel, A Continuous Family of Marked Poset Polytopes. *SIAM J. Discrete Math.* **34** (2020), no. 1, 611–639.

[48] X. Fang, G. Fourier, and M. Reineke, PBW-Filtration on quantum groups of type A_n. *J. Algebra* **449** (2016), 321–345.

[49] X. Fang and P. Littelmann, From standard monomial theory to semi-toric degenerations via Newton–Okounkov bodies. *Trans. Moscow Math. Soc.* **78** (2017), 275–297.

[50] B. Feigin and E. Frenkel, Affine Kac–Moody algebras and semi-infinite flag manifolds. *Comm. Math. Phys.* **128** (1990), 161–189.

[51] E. Feigin, The PBW filtration, Demazure modules and toroidal current algebras. *SIGMA* **4** (2008), 070, 21 pp.

[52] E. Feigin, The PBW filtration. *Represent. Theory* **13** (2009), 165–181.

[53] E. Feigin, Degenerate flag varieties and the median Genocchi numbers. *Math. Res. Lett.* **18** (2011), no. 6, 1–16.

[54] E. Feigin, $\mathbb{G}_a^M$ degeneration of flag varieties. *Selecta Math.* **18** (2012), no. 3, 513–537.

[55] E. Feigin, The median Genocchi numbers, Q-analogues and continued fractions. *European J. Combin.* **33** (2012), 1913–1918.

[56] E. Feigin, Degenerate SL_n: representations and flag varieties. *Funct. Anal. Appl.* **48** (2014), no. 1, 59–71.

[57] E. Feigin and M. Finkelberg, Degenerate flag varieties of type A: Frobenius splitting and BWB theorem. *Math. Z.* **275** (2013), no. 1–2, 55–77.

[58] E. Feigin, M. Finkelberg, and P. Littelmann, Symplectic degenerate flag varieties. *Canad. J. Math.* **66** (2014), no. 6, 1250–1286.

[59] E. Feigin, M. Finkelberg, and M. Reineke, Degenerate affine Grassmannians and loop quivers. *Kyoto J. Math.* **57** (2017), no. 2, 445–474.

[60] E. Feigin, G. Fourier, and P. Littelmann, PBW filtration and bases for irreducible modules in type A_n. *Transform. Groups* **16** (2011), no. 1, 71–89.

[61] E. Feigin, G. Fourier, and P. Littelmann, PBW filtration and bases for symplectic Lie algebras. *Int. Math. Res. Not. IMRN* **24** (2011), 5760–5784.

[62] E. Feigin, G. Fourier, and P. Littelmann, PBW-filtration over $\mathbb{Z}$ and compatible bases for $V_{\mathbb{Z}}(\lambda)$ in type A_n and C_n. In *Symmetries, integrable systems and representations*, pp. 35–63, Springer Proc. Math. Stat. 40, Springer, Heidelberg, 2013.

[63] E. Feigin, G. Fourier, and P. Littelmann, Favourable modules: filtrations, polytopes, Newton–Okounkov bodies and flat degenerations. *Transform. Groups* **22** (2017), no. 2, 321–352.

[64] E. Feigin, M. Lanini, and A. Pütz, Totally nonnegative Grassmannians, Grassmann necklaces and quiver Grassmannians. 2021, arXiv:2108.10236.

[65] E. Feigin and I. Makedonskyi, Nonsymmetric Macdonald polynomials, Demazure modules and PBW filtration. *J. Combin. Theory Ser. A* (2015), 60–84.

[66] E. Feigin and I. Makedonskyi, Semi-infinite Plücker relations and Weyl modules. *Int. Math. Res. Not.* **14** (2020), 4357–4394.

[67] E. Feigin and I. Makhlin, Vertices of FFLV polytopes. *J. Algebraic Combin.* **45** (2017), no. 4, 1083–1110.

[68] M. Finkelberg and I. Mirković, Semi-infinite flags. I. Case of global curve $\mathbb{P}^1$. Differential topology, infinite-dimensional Lie algebras, and applications. In *Differential topology, infinite-dimensional Lie algebras, and applications*, pp. 81–112, Amer. Math. Soc. Transl. Ser. 2 194, Amer. Math. Soc., Providence, RI, 1999.

[69] S. Fomin and A. Zelevinsky, Cluster algebras I: foundations. *J. Acad. Mark. Sci.* **15** (2002), no. 2, 497–529.

[70] G. Fourier and D. Kus, PBW degenerations of Lie superalgebras and their typical representations. *J. Lie Theory* **31** (2021), no. 2, 313–334.

[71] N. Fujita, Newton–Okounkov polytopes of flag varieties and marked chain-order polytopes. 2021, arXiv:2104.09929.

[72] N. Fujita and A. Higashitani, Newton–Okounkov bodies of flag varieties and combinatorial mutations. *Int. Math. Res. Not. IMRN* **12** (2021), 9567–9607.

[73] W. Fulton, *Young tableaux, with applications to representation theory and geometry*. Cambridge University Press, 1997.

[74] W. Fulton and J. Harris, *Representation theory. A first course*. Undergrad. Texts Math. Read. Math. 129, Springer, New York, 1991.

[75] I. Gelfand and M. Tsetlin, Finite dimensional representations of the group of unimodular matrices. *Dokl. Akad. Nauk USSR* **71** (1950), no. 5, 825–828.

[76] N. Gonciulea and V. Lakshmibai, Degenerations of flag and Schubert varieties to toric varieties. *Transform. Groups* **1** (1996), no. 3, 215–248.

[77] A. Gornitskii, Essential signatures and canonical bases of irreducible representations of the group G_2. *Math. Notes* **97** (2015), no. 1, 30–41.

[78] A. Gornitskii, Essential signatures and monomial bases for B_n and D_n. *J. Lie Theory* **29** (2019), no. 1, 277–302.

[79] C. Hague, Degenerate coordinate rings of flag varieties and Frobenius splitting. *Selecta Math. (N.S.)* **20** (2014), no. 3, 823–838.

[80] G.-N. Han and J. Zeng, On a q-sequence that generalizes the median Genocchi numbers. *Ann. Sci. Math. Québec* **23** (1999), 63–72.

[81] M. Harada and K. Kaveh, Integrable systems, toric degenerations and Okounkov bodies. *Invent. Math.* **202** (2015), no. 3, 927–985.

[82] B. Hassett and Yu. Tschinkel, Geometry of equivariant compactifications of $\mathbb{G}_a^n$. *Int. Math. Res. Not.* **20** (1999), 1211–1230.

[83] J. Herzog and T. Hibi, *Monomial ideals*. Grad. Texts in Math. 260, Springer, London, 2011.

[84] T. Hibi, Distributive lattices, affine semigroup rings and algebras with straightening laws. In *Commutative algebra and combinatorics*, pp. 93–109 Adv. Stud. Pure Math. 11, North-Holland, Amsterdam, 1987.

[85] A. Hubery, Irreducible components of quiver Grassmannians. *Trans. Amer. Math. Soc.* **369** (2017), no. 2, 1395–1458.

[86] A. Joseph, On the Demazure character formula. *Ann. Sci. Éc. Norm. Supér.* (1985), 389–419.

[87] K. Kaveh, Crystal bases and Newton–Okounkov bodies. *Duke Math. J.* **164** (2015), 2461–2506.

[88] K. Kaveh and A. G. Khovanskii, Newton–Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. of Math.* **176** (2012), no. 2, 925–978.

[89] B. Keller and S. Scherotzke, Desingularizations of quiver Grassmannians via graded quiver varieties. 2013, arXiv:1305.7502.

[90] V. Kiritchenko, Newton–Okounkov polytopes of flag varieties. *Transform. Groups* **22** (2017), no. 2, 387–402.

[91] V. Kiritchenko, Newton–Okounkov polytopes of flag varieties for classical groups. *Arnold Math. J.* **5** (2019), no. 2–3, 355–371.

[92] M. Kogan and E. Miller, Toric degeneration of Schubert varieties and Gelfand–Tsetlin polytopes. *Adv. in Math.* **193** (2015), 1–17.

[93] G. Kreweras, Sur les permutations comptées par les nombres de Genocchi de 1-ière et 2-ième espèce. *European J. Combin.* **18** (1997), 49–58.

[94] S. Kumar, *Kac–Moody groups, their flag varieties and representation theory*. Progr. Math. 204, Birkhäuser Boston, Inc., Boston, MA, 2002.

[95] V. Lakshmibai, Degenerations of flag varieties to toric varieties. *C. R. Acad. Sci. Paris* **321** (1995), 1229–1234.

[96] M. Lanini and E. Strickland, Cohomology of the flag variety under PBW degenerations. *Transform. Groups* **24** (2019), no. 3, 835–844.

[97] O. Lorscheid and T. Weist, Plücker relations for quiver Grassmannians. *Algebr. Represent. Theory* **22** (2019), no. 1, 211–218.

[98] I. Makedonskyi, Semi-infinite Plücker relations and arcs over toric degeneration. 2020, arXiv:2006.04172.

[99] I. Makhlin, FFLV-type monomial bases for type B. *Algebraic Combin.* **2** (2019), no. 2, 305–322.

[100] I. Makhlin, Gelfand–Tsetlin degenerations of representations and flag varieties. *Transform. Groups* (2020). DOI 10.1007/s00031-020-09622-z.

[101] I. Makhlin, Gröbner fans of Hibi ideals, generalized Hibi ideals and flag varieties. 2020, arXiv:2003.02916.

[102] I. Makhlin, PBW degenerate Schubert varieties: Cartan components and counterexamples. *Algebr. Represent. Theory* **23** (2020), no. 6, 2315–2330.

[103] A. Molev and O. Yakimova, Monomial bases an branching rules. *Transform. Groups* **26** (2021), 995–1024.

[104] T. Mora and L. Robbiano, The Gröbner fan of an ideal. *J. Symbolic Comput.* **6** (1988), 183–208.

[105] A. Okounkov, Multiplicities and Newton polytopes. In *Kirillov's seminar on representation theory*, pp. 231–244, Amer. Math. Soc. Transl. Ser. 2 181, Amer. Math. Soc., Providence, RI, 1998.

[106] D. Panyushev and O. Yakimova, A remarkable contraction of semi-simple Lie algebras. *Ann. Inst. Fourier (Grenoble)* **62** (2012), no. 6, 2053–2068.

[107] D. Panyushev and O. Yakimova, Parabolic contractions of semi-simple Lie algebras and their invariants. *Selecta Math.* **19** (2013), no. 3, 699–717.

[108] A. Pütz, Degenerate affine flag varieties and quiver Grassmannians. *Algebr. Represent. Theory* (2020). DOI 10.1007/s10468-020-10012-y.

[109] M. Reineke, Every projective variety is a quiver Grassmannian. *Algebr. Represent. Theory* **16** (2013), 1313–1314.

[110] C. M. Ringel, Quiver Grassmannians for wild acyclic quivers. *Proc. Amer. Math. Soc.* **146** (2018), no. 5, 1873–1877.

[111] L. Robbiano and M. Sweedler, Subalgebra Bases. In *Proc. Commutative Algebra (Salvador, 1988)*, pp. 61–87, Lecture Notes in Math. 1430, Springer, Berlin, 1990.

[112] S. Scherotzke, Desingularization of Quiver Grassmannians via Nakajima Categories. *Algebr. Represent. Theory* **20** (2017), 231–243.

[113] R. Schiffler, *Quiver representations*. CMS Books Math., Springer, 2014.

[114] F. Sottile, Real rational curves in Grassmannians. *J. Amer. Math. Soc.* **13** (2000), 333–341.

[115] F. Sottile and B. Sturmfels, A sagbi basis for the quantum Grassmannian. *J. Pure Appl. Algebra* **158** (2001), no. 2–3, 347–366.

[116] D. Speyer and B. Sturmfels, The tropical Grassmannian. *Adv. Geom.* **4** (2003), 389–411.

[117] D. Speyer and L. Williams, The tropical totally positive Grassmannian. *J. Algebraic Combin.* **22** (2005), no. 2, 189–210.

[118] R. P. Stanley, Two poset polytopes. *Discrete Comput. Geom.* **1** (1986), no. 1, 9–23.

[119] B. Sturmfels, *Algorithms in invariant theory*. Texts Monogr. Symbol. Comput., Springer, Vienna, 1993.

[120] G. Viennot, Interprétations combinatoires des nombres d'Euler et de Genocchi. In *Seminar on Number Theory*, Univ. Bordeaux I, Talence, 1981/1982, no. 11, 94 pp.

[121] E. Vinberg, *On some canonical bases of representation spaces of simple Lie algebras, conference talk*, Bielefeld, 2005.

[122] J. Zeng and J. Zhou, A q-analog of the Seidel generation of Genocchi numbers. *European J. Combin.* (2006), 364–381.

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