SPHERICAL VARIETIES, FUNCTORIALITY, AND QUANTIZATION

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ABSTRACT

We discuss generalizations of the Langlands program, from *reductive groups* to the local and automorphic spectra of *spherical varieties*, and to more general representations arising as "quantizations" of suitable Hamiltonian spaces. To a spherical G-variety X, one associates a *dual group* ${}^L G_X$ and an *L-value* (encoded in a representation of ${}^L G_X$), which conjecturally describe the local and automorphic spectra of the variety. This sets up a problem of functoriality, for any morphism ${}^L G_X \to {}^L G_Y$ of dual groups. We review, and generalize, Langlands' "beyond endoscopy" approach to this problem. Then, we describe the cotangent bundles of quotient stacks of the relative trace formula, and show that transfer operators of functoriality between relative trace formulas in rank 1 can be interpreted as a change of "geometric quantization" for these cotangent stacks.

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1. INTEGRAL REPRESENTATIONS OF L-FUNCTIONS

1.1. Classical periods

1.1.1. In his legendary 1859 paper [66], Riemann proved the functional equation of the zeta function by representing it as the Mellin transform of a theta series

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty y^{\frac{s}{2}} \sum_{n=1}^{\infty} e^{-n^2\pi y}\, d^\times y.$$

The proof used the functional equation of the latter with respect to the substitution $y \leftrightarrow y^{-1}$, previously established by Jacobi and based on the Poisson summation formula.

About 90 years later, Iwasawa, in a short announcement [37], and Tate, in his thesis [83], reformulated this integral in the language of the adeles. The new formulation could be directly applied to the generalizations of the zeta function to arbitrary Dirichlet characters (by Dirichlet), or number fields (by Dedekind) and Grössencharacters (by Hecke), and clarified the meaning of the Euler factors of the zeta function, as Mellin transforms of Schwartz functions on the p-adic completions of $\mathbb{Q}$. Namely, we have an identity

$$\zeta_p(s) = \int_{\mathbb{Q}_p^\times} \Phi_p(x)|x|^s\, d^\times x,$$

where, for finite primes p, $\zeta_p(s) = (1-p^{-s})^{-1}$ and $\Phi_p = 1_{\mathbb{Z}_p}$, the characteristic function of the p-adic integers, is what we will call the *basic Schwartz function* on $\mathbb{Q}_p$; the same interpretation extends to the "Archimedean factor" $\zeta_\infty(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ of the functional equation, with $\mathbb{Q}_\infty = \mathbb{R}$, and Φ_∞ the Gaussian $e^{-\pi x^2}$.

1.1.2. Meanwhile, in 1936–1937, Hecke [34] introduced what is today called the L-function of a modular form, generalized to nonholomorphic automorphic forms by Maass in 1944 [56]. Recast in the adelic language by Jacquet and Langlands in their seminal 1970 work [41], these L-functions, with appropriate Archimedean factors, can be represented as Mellin transforms

$$\int_{k^\times\backslash\mathbb{A}^\times} f\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^s\, d^\times a,$$

where k denotes a number field, $\mathbb{A}$ its ring of adeles, and f is a cuspidal automorphic form on $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A})$.

Shortly after Hecke, Rankin [65] and Selberg [80] discovered an integral representation for the L-function that carries their names, which today is seen as a special case of a Langlands L-function, attached to the tensor product representation

$$\check{G} = \mathrm{GL}_2 \times \mathrm{GL}_2 \xrightarrow{\otimes} \mathrm{GL}_4$$

of the Langlands dual of the group $G = \mathrm{GL}_2 \times \mathrm{GL}_2$. This integral is, on the surface, very different from the Mellin transforms of Riemann and Hecke, as it involves a pair of cusp forms and an Eisenstein series:

$$\int_{\mathrm{GL}_2(\mathbb{A})} f_1(g) f_2(g) E(g,s)\, dg.$$

1.2. The theta series of spherical varieties

1.2.1. The aforementioned works, and their adelic reformulations, led to an explosion of research around L-functions from the 1970s onward, with numerous new integral representations discovered by Godement, Jacquet, Rallis, Piatetski-Shapiro, Gelbart, Shalika, Waldspurger, Ginzburg, Bump, Friedberg, Garrett, and others [19], combining elements from all of the methods above, such as the theta series (from Riemann), the "period integrals" over subgroups (from Hecke), and the Eisenstein series (from Rankin and Selberg).

A uniform approach to many of these methods was proposed in [68]; it relies on the following ingredients:

- A (suitable) *affine spherical variety* X for a group G over a number field k; that is, X is a normal, affine G-variety, with a dense orbit for the Borel subgroup of G. This space is $X = \mathbb{A}^1$ for $G = \mathbb{G}_m$ in Riemann–Iwasawa–Tate theory, $X = \mathrm{GL}_2$ for $G = \mathrm{GL}_1 \times \mathrm{GL}_2$ in Hecke–Jacquet–Langlands theory, and $X = V \times^{\mathrm{GL}_2^{\mathrm{diag}}} (\mathrm{GL}_2 \times \mathrm{GL}_2)$, where V is the standard representation of GL_2, in Rankin–Selberg theory.

- A suitable space of "Schwartz functions" $\mathcal{F}(X(k_v))$ for every completion k_v of k; at almost every place, it contains a distinguished vector $\Phi_{0,v}$, giving rise to a restricted tensor product $\mathcal{F}(X(\mathbb{A})) = \bigotimes_v' \mathcal{F}(X(k_v))$. When X is smooth and k_v is non-Archimedean with ring of integers $\mathfrak{o}_v$, we have $\Phi_{0,v} = 1_{X(\mathfrak{o}_v)}$.

- The X-theta series
$$\Theta : \mathcal{F}\big(X(\mathbb{A})\big) \to C^\infty\big([G]\big),$$
where $[G] = G(k)\backslash G(\mathbb{A})$, given by $\Theta_\Phi(g) := \Theta(\Phi)(g) := \sum_{\gamma \in X(k)} \Phi(\gamma g)$. This generalizes the Jacobi theta series used by Riemann, and many other series of classical analytic number theory, such as Poincaré series (if we allow X to stand for the Whittaker model, which is not just a space but also a nontrivial "line bundle" over it, see § 2.2.5), and Eisenstein series (after we pair a suitable theta series with an automorphic form for some Levi subgroup).

The theta series (for varying inputs Φ) are then integrated against automorphic forms f, and, under some assumptions on the space X, the "period pairing"
$$\langle f, \Theta_\Phi \rangle := \int_{[G]} f(g)\Theta_\Phi(g)\, dg \tag{1.1}$$
is expected to be related to a special value of an L-function of f. This relation will be discussed in Sections 2.3.2–3.1.1.

1.2.2. While it is not the main focus of the present article, it should be mentioned that the main point of the proposal of [68] was to include *singular* affine spherical varieties, in which case the "basic function" of $X(k_v)$ is the "IC function," obtained through the sheaf–function dictionary from the intersection complex of a suitable geometric model of $X(\mathfrak{o}_v)$, see § 3.1.3. This was inspired by the work of Braverman–Kazhdan on the basic affine space [14,16], which goes back to the geometric Langlands program [13] and ideas of Drinfeld.

The conjecture was refined by Ngô [62] for a class of affine embeddings of reductive groups; the IC function, for non-Archimedean local fields in equal characteristic, was defined in [12], where Ngô's conjecture was proven. In recent joint work with Jonathan Wang [78], we have obtained similar results for the IC function of a broad class of spherical varieties, including a straightforward generalization of the Hecke and Rankin–Selberg integral representations to the Langlands L-function associated to the n-fold tensor product representation of the dual group

$$\check{G} = \underbrace{\mathrm{GL}_2 \times \cdots \times \mathrm{GL}_2}_{n \text{ times}} \xrightarrow{\otimes} \mathrm{GL}_{2^n}\,.$$

1.2.3. The elephant in the room, of course, is the global functional equation (and meromorphic continuation), which is not available for these L-functions, yet. In favorable cases, it should arise from a Poisson summation formula for a "Fourier transform"

$$\mathcal{F}\big(X(\mathbb{A})\big) \to \mathcal{F}\big(X^*(\mathbb{A})\big),$$

where X^* is the same variety X, with the G-action twisted by a Chevalley involution. Such a Fourier transform and a Poisson summation formula are often available for smooth affine spherical varieties, which are vector bundles over homogeneous spaces, but are quite mysterious in the singular case. For the moment, they are known for spaces of the form $X =$ the affine closure of $[P, P]\backslash G$, where $P \subset G$ is a parabolic, by the work of Braverman–Kazhdan [14,16] (and its refinement [33]). An extension to $X =$ the affine closure of $U_P\backslash G$, where U_P is the unipotent radical of P, would give rise to the functional equation of normalized Eisenstein series, greatly simplifying and generalizing the theory of L-functions obtained through the Langlands–Shahidi method [50,81]. In recent work, Getz and his collaborators [31,32] have proven a Poisson summation formula for a singular space Y which is not directly related to Eisenstein series—the only example of this sort to date, to my knowledge.

In general, this "Fourier" transform may only be available at the level of trace formulas—see [73,88], as well as the discussion of § 6.4 below.

1.2.4. The "period pairing" (1.1) between theta series coming from spherical varieties and automorphic forms is not general enough to include all known integral representations of L-functions. At the very least, we need to replace the Schwartz space of a spherical G-variety by more general *quantizations of Hamiltonian G-spaces*. In the smooth case, those are affine symplectic G-spaces M, equipped with a moment map $M \to \mathfrak{g}^*$, which generalize the cotangent bundle T^*X of a smooth spherical G-variety. The analog of the "spherical" condition for a Hamiltonian G-space M is that it be *coisotropic*, that is, the Poisson algebra $k(M)^G$ of G-invariant rational functions on M be Poisson-commutative.

An example of such a space, that is not the cotangent bundle of a spherical variety, is a symplectic vector space M under the action of a Howe dual pair G; that is, G is, up to central isogeny, equal to a product $G_1 \times G_2$ of two subgroups of $\mathrm{Sp}(M)$, where G_1 is the commutator of G_2, and vice versa. As "quantization" of M we understand the Weil representation of the metaplectic group $\mathrm{Mp}(M)$ associated with an additive character ψ, restricted to (the metaplectic cover of) G. Theta series and the pairing (1.1) still make sense in this

setting. More general examples mixing the Weil representation with periods are contained in the influential conjectures of Gan, Gross, and Prasad [27].

In an ongoing work with Ben-Zvi and Venkatesh, we describe a class of coisotropic Hamiltonian spaces M whose "quantizations" in the form of theta series are expected to be related to special values of L-functions, and we demonstrate, by means of known examples, that the L-value associated to such a space gives rise to a *dual Hamiltonian space* $\check{M}$ *for the Langlands dual group*. In the context of the geometric Langlands program, this leads to a hierarchy of conjectures, with connections to mathematical physics. We will encounter one of these conjectures in our discussion of unramified L-factors in § 3.2 below.

1.3. Outline of this paper

In Section 2 we introduce the relative Langlands program, up to the conjectural Euler factorization of the period pairings (1.1).

In Section 3 we discuss the relationship between the local unramified Euler factors and special values of L-functions.

In Section 4 we discuss the "beyond endoscopy" approach to functoriality, generalized to the setting of the relative Langlands program.

Section 5 provides a new interpretation for the transfer operators of functoriality studied in [74], based on the concept of quantization. (Proofs for the results of this section will appear in an expanded version of this article on the arXiv.)

Finally, in Section 6 we discuss interesting research directions for the near future.

1.4. Notation and language

- In general, when a variety is defined over a local field F, and there is no danger of confusion, we will use the same letter to denote its F-points, e.g., "a Schwartz function on X" really means "on $X(F)$."
- For a quasiaffine G-variety over a field F, we will denote by X/G the stack quotient, and by $X /\!\!/ G$ the invariant-theoretic quotient $\operatorname{Spec} F[X]^G$.
- A "complex line bundle" on the points of a smooth variety X over a local field F will be
  - when $F = \mathbb{R}$ or $\mathbb{C}$, a complex line bundle on $X(F)$, viewed as a smooth (Nash) manifold;
  - when F is non-Archimedean, a locally constant sheaf of complex vector spaces (l-sheaf) on $X(F)$, for the p-adic (Hausdorff) topology, with 1-dimensional stalks.

  When no confusion arises, we will just say "line bundle" for a complex line bundle; when we want to distinguish it from a line bundle on X in the sense of algebraic geometry, we will say "algebraic line bundle" for the latter.

- An algebraic line bundle L over a smooth F-variety X, where F is a local field, together with a complex number s, give rise to a complex line bundle $|L|^s$ on $X(F)$, by reduction of the corresponding $\mathbb{G}_m$-torsor via the sequence of maps $\mathbb{G}_m(F) \xrightarrow{|\bullet|} \mathbb{R}_+^\times \xrightarrow{x\mapsto x^s} \mathbb{C}^\times$. (When F is non-Archimedean, the absolute value map is discretely-valued, giving rise to the structure of a locally constant sheaf.)

- When $L = \det T^*X$, the line bundle of volume forms on X, the associated complex vector bundle $|L|^s$ is known as the bundle of s-densities. We will, in general, understand the field F as endowed with a Haar measure; this identifies densities, i.e., sections of $|L|$, as measures on $X(F)$. When no confusion arises, we will denote $|dx|$, the density attached to a volume form dx, simply by dx.

- The space of Schwartz functions on the F-points of a variety X, where X is a local field, will be denoted by $\mathcal{F}(X(F))$, the space of Schwartz measures by $\mathcal{S}(X(F))$, and the space of Schwartz half-densities by $\mathcal{D}(X(F))$. These are smooth, compactly supported sections of the corresponding bundles of s-densities, in the non-Archimedean case. In the Archimedean case, they are smooth sections of rapid decay, see [1]. We will also say "test functions/measures," etc., for "Schwartz."

- For an admissible, smooth, complex representation π of a reductive group over a local field, we will denote by $\tilde{\pi}$ its contragredient. When π is unitary, $\tilde{\pi}$ is identified with the complex conjugate $\bar{\pi}$.

- We will generally prefer to replace a hermitian pairing H between functions by the associated bilinear pairing $B(\Phi_1, \Phi_2) = H(\Phi_1, \overline{\Phi_2})$. When H is an inner product, we will sometimes call B, by abuse of language, an "inner product."

- $\mathcal{W}_F$ will denote the Weil group of a local or global field, and $\mathcal{L}_F$ will be the "Langlands group," whose representations should parametrize local and automorphic L-packets. It is the Weil group for Archimedean local fields and global function fields, the Weil–Deligne group for non-Archimedean local fields, and a conjectural extension of the Weil group for number fields.

- We adopt the "Weil group" convention for L-groups of reductive groups, ${}^L G = \check{G} \rtimes \mathcal{W}_F$; the dual group $\check{G}$ is identified with the set of its complex points.

2. THE RELATIVE LANGLANDS CONJECTURES

2.1. The local and global spectrum of a spherical variety

2.1.1. To understand the relationship between period pairings (1.1) and L-functions, one needs to start by understanding the phenomenon of *distinction*, highlighted by the groundbreaking work of Jacquet and his collaborators [38, 42]. A naive formulation of this phenomenon goes as follows:

The local and global spectrum of a spherical G-variety X only contain representations with Langlands parameters in a certain subgroup ${}^L G_X \subset {}^L G$ of the L-group of G.

To make sense of this statement, we need to explain "the local and global spectrum of a G-variety." Then, we need to talk about the L-group ${}^L G_X$. Finally, the statement needs to be corrected, for some "nontempered" varieties X, replacing Langlands parameters by appropriate Arthur parameters.

Let R denote either a local field, or the adelic points of a global field. In order to define the local and global spectrum of a G-variety X (defined over the corresponding field), we will introduce the Plancherel formula and the relative trace formula. These decompose certain distributions—or rather, generalized functions—on the R-points of $X \times X$, invariant under the diagonal action of G. For the purposes of the Langlands program, it turns out to be more natural to think of them as generalized functions on the R-points of the quotient stack $\mathcal{X} = (X \times X)/G^{\mathrm{diag}}$, which naturally includes "pure inner forms" of the pair (G, X).

2.1.2. Let F be a local field. The space $L^2(X)$ is the Hilbert space completion of the space $\mathcal{D}(X)$ of Schwartz half-densities on $X(F)$, with respect to the L^2-inner product, and furnishes a unitary representation of G. By the Plancherel decomposition, there are a measure μ_X on the unitary dual $\hat{G}$ of G and a measurable family of linear forms

$$J_\pi : \mathcal{D}(X \times X) \to \mathbb{C}$$

such that:

- for μ_X-almost every π, J_π factors as $\mathcal{D}(X \times X) \to \pi \hat{\otimes} \bar{\pi} \to \mathbb{C}$, and
- for all $\Phi \in \mathcal{D}(X \times X)$, we have

$$\int_{X^{\mathrm{diag}}} \Phi = \int_{\hat{G}} J_\pi(\Phi) \mu_X(\pi). \tag{2.1}$$

A linear form satisfying the first property above will be called a *relative character*. The product $J_\bullet \mu_X$, which can be thought of as a measure valued in the space of functionals on $\mathcal{D}(X \times X)$, is uniquely defined. Moreover, the relative characters J_π are invariant under the diagonal action of $G = G(F)$; thus, they factor through the coinvariant space $\mathcal{D}(X \times X)_G =$ the quotient of $\mathcal{D}(X \times X)$ by the (closed, in the Archimedean case) subspace generated by elements of the form $f - g \cdot f$, where $g \cdot f$ denotes the action of $g \in G$ on f by diagonal translation.

Let us assume that X carries a positive G-invariant measure dx, and use it to identify functions, half-densities, and measures on X through the G-equivariant maps $\Phi \mapsto \Phi(dx)^{\frac{1}{2}} \mapsto \Phi dx$ (and similarly on $X \times X$). Then, the coinvariant space $\mathcal{D}(X \times X)_G \simeq \mathcal{S}(X \times X)_G$ is more naturally understood as a subspace of the *Schwartz space of the quotient stack* $\mathcal{X} := (X \times X)/G$ [71]. This Schwartz space is really a complex of vector spaces, but here we will focus only on its zeroth cohomology, which has the explicit

description

$$\mathcal{S}(\mathfrak{X}) = \bigoplus_{\alpha} \mathcal{S}(X^\alpha \times X^\alpha)_{G^\alpha}. \tag{2.2}$$

Here, α runs over isomorphism classes of G-torsors (parametrized by the Galois cohomology set $H^1(\Gamma_F, G)$, where Γ_F is the Galois group of a separable closure of F); if R^α is a representative of a class α, we let $G^\alpha = \mathrm{Aut}_G(R^\alpha)$, and $X^\alpha = X \times^G R^\alpha$, a G^α-space. In other words, G^α is what is called a "pure inner form" of G, and X^α can similarly be called a "pure inner form" of X, if its set of F-points is nonempty.

The Plancherel formula (2.1) extends to $\mathcal{S}(\mathfrak{X})$, with a measure $\mu_{\mathfrak{X}}$, on the right-hand side, on the union of the unitary duals of the pure inner forms G^α. The support $\Pi_{\mathfrak{X}}$ of this measure (avoiding redundancy—i.e., the support of the canonical linear form-valued measure $J_\bullet \mu_{\mathfrak{X}}$) can be called the *local (L^2-)spectrum of the quotient stack* $\mathfrak{X}$.

2.1.3. The global (automorphic) spectrum of X (or rather, again, of the stack $\mathfrak{X} = (X \times X)/G^{\mathrm{diag}}$) can be defined through the relative trace formula of Jacquet. This is a generalization of the Arthur–Selberg trace formula, and an automorphic analog of the local Plancherel formula. Its definition uses the theta series encountered in § 1.2.1, therefore we assume here that X, defined over a global field k, is quasiaffine, so that $X(k)$ is discrete in the adelic points $X(\mathbb{A})$. As before, we write $[G] = G(k)\backslash G(\mathbb{A})$ for the automorphic quotient space.

Roughly speaking, the relative trace formula is the Plancherel formula for $L^2([G])$, applied to the inner product of two theta series for X, i.e., decomposing the functional

$$\begin{aligned}\mathrm{RTF}_X : \mathcal{F}\big(X(\mathbb{A})\big) \otimes \mathcal{F}\big(X(\mathbb{A})\big) \ni \Phi_1 \otimes \Phi_2 &\mapsto \Theta_{\Phi_1} \otimes \Theta_{\Phi_2} \\ &\mapsto \int_{[G]} \Theta_{\Phi_1}(g)\Theta_{\Phi_2}(g)\, dg \in \mathbb{C}. \end{aligned} \tag{2.3}$$

This naive point of view requires some caution:

- The inner product on the right-hand side of (2.3) does not, in general, converge, and needs to be regularized. Depending on X, there may be a *canonical* way to regularize it, described in [71, §6]. In many cases of interest, though, notably in the case of the Arthur–Selberg trace formula (where $X = H$, a reductive group, and $G = H \times H$), a canonical regularization is not available, and it takes the mastery of Arthur's work [6] to engineer an invariant expression. Such work has not yet been done in the general setting of the relative trace formula.

- We can again choose a G-invariant measure on $X(\mathbb{A})$ (e.g., Tamagawa measure) to identify functions with measures, and understand the $G(\mathbb{A})^{\mathrm{diag}}$-invariant functional (2.3) as a functional on $\mathcal{S}(X \times X(\mathbb{A}))_{G(\mathbb{A})}$. As in the local case, this space is a subspace of the global Schwartz space of the stack $\mathfrak{X} = (X \times X)/G^{\mathrm{diag}}$,

$$\mathcal{S}\big(\mathfrak{X}(\mathbb{A})\big) = \bigotimes_{v}{}' \mathcal{S}\big(\mathfrak{X}(k_v)\big),$$

and the relative trace formula should be defined as a functional on the bigger space,

$$\mathrm{RTF}_{\mathfrak{X}} = \sum_{\alpha} \mathrm{RTF}_{X^\alpha},$$

where now α runs over isomorphism classes of G-torsors over the global field k.

Ignoring the regularization issue, if we could apply the Plancherel formula for $\bigoplus_\alpha L^2([G^\alpha])$ to the pairing (2.3), we would obtain the *spectral side* of the relative trace formula,

$$\mathrm{RTF}_{\mathfrak{X}} = \int J_\pi^{\mathrm{aut}} \mu_{\mathfrak{X}}^{\mathrm{aut}}(\pi), \tag{2.4}$$

where the product $J_\bullet^{\mathrm{aut}} \mu_{\mathfrak{X}}^{\mathrm{aut}}$ is a measure on the union $\bigsqcup_\alpha \widehat{G^\alpha}^{\mathrm{aut}}$ of (L^2-)automorphic spectra of the pure inner forms of G, valued in linear forms on $\mathcal{S}(\mathfrak{X}(\mathbb{A}))$.

The global (automorphic) spectrum of $\mathfrak{X}$ is defined as the support of $J_\bullet^{\mathrm{aut}} \mu_{\mathfrak{X}}^{\mathrm{aut}}$. Clearly, this definition is incomplete, as it relies on overcoming the aforementioned issues of regularization, and developing a spectral decomposition for the relative trace formula.

2.2. The Langlands dual group

2.2.1. The local and global spectrum of a spherical variety X are conjecturally governed by the *L-group* ${}^L G_X$ of X. We owe this dual group to the insights developed by Nadler in his thesis [60], and in his joint work with Gaitsgory [25]. They realized that the "little Weyl group" of a spherical G-variety (defined by Brion in [18], and generalizing the little Weyl group of a symmetric space) corresponds to a subgroup $\check{G}_X \subset \check{G}$ of the Langlands dual group of G, and gives rise to a form of the geometric Satake isomorphism for the spherical variety. In [77], it was proposed that this dual group comes equipped with a distinguished morphism

$$\check{G}_X \times \mathrm{SL}_2 \to \check{G} \tag{2.5}$$

that governs the harmonic analysis of X, in a way that will be described below. Since Gaitsgory and Nadler did not fully identify their dual group $\check{G}_X$ (constructed in a Tannakian way), an independent description of a morphism (2.5) was achieved by Knop and Schalke [47]; we can take this as the definition of the dual group, for what follows. Finally, for the purposes of the Langlands program, we need an L-group, in the form of an extension

$$1 \to \check{G}_X \to {}^L G_X \to \mathcal{W}_F \to 1.$$

The correct definition of this L-group, when G is not split, is not completely understood yet, although it is probably within reach. For what follows, we will assume such an L-group, and an extension of the homomorphism (2.5) to the L-groups, in the sense that the conjectures to be stated should hold for an appropriate definition of ${}^L G_X$.

2.2.2. We briefly describe one way to characterize the root datum of the dual group $\check{G}_X$: As in the case of reductive groups, the first step is to describe a canonical maximal torus, which in turn is dual to an "abstract Cartan" group. Let A be the abstract Cartan group of G; it is canonically equal to the reductive quotient of any Borel subgroup of G. Fix such a Borel subgroup B, with unipotent radical N, and let X° be the open B-orbit. On the quotient $X^\circ /\!\!/ N$, B acts through a quotient A_X of A; this is *the Cartan group of* X, and it can be seen to be independent of B, in the sense that any two choices induce canonical tori up to a canonical isomorphism. (These definitions assume that B is defined over the base field, but by Galois descent the Cartan groups A and A_X are defined over the field, even if B is not.)

The quotient $A \to A_X$ gives rise to a morphism of dual tori $\check{A}_X \to \check{A}$, which could have nontrivial (finite) kernel. The image of this morphism is the canonical maximal torus of the Gaitsgory–Nadler dual group $\check{G}_X$. (We caution the reader that in [77] the group $\check{G}_X$ was not necessarily defined as a subgroup of $\check{G}$, and had $\check{A}_X$ as its maximal torus.)

It is slightly harder to define the little Weyl group W_X. Once this is done, the coroots of $\check{G}_X$, which will be called the *normalized spherical roots* of X, are uniquely determined up to multiple, and that multiple is fixed by the following axiom:

> *A normalized spherical root is either a root of G, or the sum of two strongly orthogonal roots, i.e., two roots whose linear span contains no other roots but their multiples.*

2.2.3. For the Weyl group, there are many equivalent definitions. Most relevant to our purposes, when X is defined over a field F in characteristic zero, is the following one, due to Knop [45]: We may assume that X is smooth and that F is algebraically closed (since $\check{G}_X$ only depends on the open G-orbit over the algebraic closure). Consider the cotangent space $M = T^*X$, equipped with the moment map $\mu : M \to \mathfrak{g}^*$. If $\mathfrak{a}^*$ denotes the dual Lie algebra of the Cartan of G, Chevalley's isomorphism identifies the invariant-theoretic quotient $\mathfrak{g}^* /\!\!/ G$ with $\mathfrak{a}^* /\!\!/ W$. The *polarized cotangent bundle*

$$\hat{M} := M \times_{\mathfrak{a}^* /\!\!/ W} \mathfrak{a}^*$$

is not, in general, irreducible. Knop describes a distinguished irreducible component $\hat{M}^\circ$ living over the dual Lie algebra $\mathfrak{a}_X^* \subset \mathfrak{a}^*$ of A_X, and shows that the map $\hat{M}^\circ \to M$ is generically a Galois cover with covering group a subquotient W_X of the Weyl group; this is the little Weyl group of X [45, §6].

For later use, we mention a related result of Knop, still in the homogeneous case. Let $\mathfrak{g}_X^* =$ the normalization of the image of the moment map *in* M (i.e., the spectrum of the integral closure of the image of $F[\mathfrak{g}^*]$ in $F[M]$). The composition $\hat{M}^\circ \to \mathfrak{a}_X^* \to \mathfrak{c}_X^* := \mathfrak{a}_X^* /\!\!/ W_X$ factors through a map $\mu_G : M \to \mathfrak{g}_X^* \to \mathfrak{c}_X^*$, called the *invariant moment map*, and identifies $\mathfrak{c}_X^*$ with the invariant-theoretic quotient $M /\!\!/ G$ [45, KOROLLAR 7.2].

2.2.4. Finally, the restriction of the map (2.5) to SL_2, which we will call the "Arthur-SL_2" of X, is determined by the conjugacy class of parabolics of the form

$$P(X) = \{g \in G \mid X^\circ g = X^\circ\},$$

where X° is the open orbit for a Borel subgroup B. In the quasiaffine case, $P(X)$ is the largest parabolic such that all highest weight vectors in $k[X]$ are $P(X)$-eigenvectors. To this class is canonically associated a standard Levi subgroup $\check{L}(X)$ of $\check{G}$, and the Arthur-SL_2 of X is a principal $\mathrm{SL}_2 \to \check{L}(X)$.

2.2.5. In order to keep the discussion that follows as simple as possible, let us single out a convenient class of spherical varieties. We will say that a spherical G-variety X is *excellent* if it is affine, homogeneous, and the kernel of the map $A \to A_X$ is connected (equivalently, the map $\check{A}_X \to \check{A}$ of dual tori is injective).

We also need to enlarge the class of spherical varieties, in order to include objects such as the Whittaker model. The Whittaker model is the space $N\backslash G$, where G is quasisplit and N is a maximal unipotent subgroup, endowed with a nondegenerate character $\psi : N(F) \to \mathbb{C}^\times$. This character defines, by induction, a complex line bundle L_ψ over $N\backslash G$, and the Whittaker model consists of sections of this line bundle. In the sequel, when we say that Y is "the Whittaker model," we will mean the space $N\backslash G$ together with this line bundle, and we will be using the Schwartz space notation $\mathcal{F}(Y)$, $\mathcal{S}(Y)$, etc., to denote Schwartz sections (resp. measures) valued in this line bundle. For a more general discussion of "Whittaker induction," see [77, §2.6].

2.3. Conjectures

2.3.1. Let X be defined over a local field F, and let $\Pi_{\mathfrak{X}}$ be the set of L^2-distinguished representations of X and its pure inner forms, as in § 2.1.2. We assume, for simplicity, that X carries an invariant measure, to identify measures with half-densities. The "relative local Langlands conjecture" of my work with Venkatesh [77, §16] states:

Conjecture. *Let $\mu_{{}^L G_X}$ be the natural measure on the set of tempered local Langlands parameters into ${}^L G_X$. There is a decomposition of the inner product on $\mathcal{D}(X \times X)$,*

$$\int_{X^{\mathrm{diag}}} \Phi = \int J_\phi(\Phi)\mu_{{}^L G_X}(\phi), \tag{2.6}$$

where the "stable relative characters" J_ϕ are linear combinations of relative characters for representations belonging to Arthur packets with parameter

$$\mathcal{L}_F \times \mathrm{SL}_2 \xrightarrow{\phi\times \mathrm{Id}} {}^L G_X \times \mathrm{SL}_2 \xrightarrow{(2.5)} {}^L G. \tag{2.7}$$

For the "natural measure" on such parameters, see [77, §16]. Comparing with the Plancherel formula (2.1), the conjecture implies that the local L^2-spectrum $\Pi_{\mathfrak{X}}$ of $\mathfrak{X}$ belongs to the union of Arthur packets with parameters of the form (2.7). Developing a Plancherel formula for X in terms of discrete-mod-center spectra of its "boundary degenerations," as in [22, 77], reduces the conjecture to discrete spectra.

When G is quasisplit, acts faithfully on X, and the map $\check{A}_X \to \check{A}$ is injective (§ 2.2.2), one would expect the functionals J_ϕ of (2.6), after summing over all pure inner forms of X, to be nonzero. In a broad range of individual cases, including the Gan–Gross–Prasad conjectures [27] and other cases considered by D. Prasad [64] and C. Wan [87], we have much more precise conjectures about how many and which elements in the given Arthur packets are distinguished. A number of cases have been proven by Waldspurger, Mœglin, Beuzart-Plessis, Gan, Ichino, and others [9, 28, 58, 86], and by Mœglin–Renard [57] for symmetric spaces over $\mathbb{R}$.

Besides the question of L^2-distinction, one can ask the question of smooth distinction: Which irreducible representations embed as $\pi \hookrightarrow C^\infty(X)$? The general answer to this question is less understood.

2.3.2. Now, let X be defined over a global field k, and let $\Pi^{\mathrm{aut}}_{\mathfrak{X}}$ be the automorphic spectrum of $\mathfrak{X}$, as in § 2.1.3. We recall that its definition is, in general, conditional on developing the spectral decomposition of the relative trace formula. Nonetheless, one can often restrict to parts of the spectrum where the full relative trace formula is not needed; for example, if π is a discrete automorphic representation where the period pairing (1.1) is absolutely convergent, the corresponding functional-valued measure $J^{\mathrm{aut}}_\bullet \mu^{\mathrm{aut}}_{\mathfrak{X}}$ of (2.4), applied to a test function $\Phi_1 \otimes \Phi_2$ on $(X \times X)(\mathbb{A})$, should have the meaning of

$$J^{\mathrm{aut}}_\pi(\Phi_1 \otimes \Phi_2)\mu^{\mathrm{aut}}_{\mathfrak{X}}(\pi) = \sum_f \left(\int_{[G]} \Theta_{\Phi_1} f\right)\left(\int_{[G]} \Theta_{\Phi_2} \bar{f}\right),$$

where f runs over an orthonormal basis of π.

A landmark in our understanding of these global relative characters was the paper [36] of Ichino and Ikeda, generalizing the formula of Waldspurger [85] to a precise conjectural Euler factorization of J^{aut}_π, in the case of orthogonal Gross–Prasad periods. The conjectures of Ichino and Ikeda gave rise to the realization that there was a general pattern in the Euler factorization of automorphic periods, and were quickly adapted to other cases. Unlike the orthogonal case, which remains open, the conjecture for unitary Gross–Prasad periods has been proven in [8, 10, 90], its analog for Whittaker periods of metaplectic and unitary groups was proven in [53, 54, 59], and there are significant partial results in many other cases.

A generalization of the Ichino–Ikeda conjecture to a wide range of spherical periods (satisfying certain conditions) was proposed in [77]. As in the local case, it lacks the precision of conjectures known in special cases, hence leaving an open problem that should be addressed in the near future. On the other hand, the conjecture of [77] makes clear the connection between the (global) relative trace formula and the (local) Plancherel formula. I will formulate a variant of this conjecture here, using the hypothetical notion of global Arthur parameters, and being a bit vague on choices of measures (see [77, §17] for some hints). Its formulation also relies on Conjecture 3.1.1 below, expressing the local Plancherel density of the basic function $\Phi_{0,v} \in \mathcal{F}(X(k_v))$ at almost every place v in terms of a local L-value $L_X(\phi_v) := L(\phi_v, \rho_X, 0)$, where ϕ_v is a local unramified Langlands parameter into ${}^L G_X$ and $\rho_X : {}^L G_X \to \mathrm{GL}(V_X)$ is a certain representation of the L-group of X.

Conjecture. *There is a decomposition*

$$\mathrm{RTF}_{\mathfrak{X}} = \int J_{\phi}^{\mathrm{aut}} \mu_X^{\mathrm{aut}}(\phi),$$

where μ_X^{aut} *is a measure on the set of global Arthur parameters which factor as*

$$\mathcal{L}_k \times \mathrm{SL}_2 \xrightarrow{\phi} {}^L G_X \times \mathrm{SL}_2 \xrightarrow{(2.5)} {}^L G$$

(with ϕ *lying over the identity map for the projections to* SL_2*), and* J_{ϕ}^{aut} *is a sum of relative characters* $\mathcal{S}(\mathfrak{X}(\mathbb{A})) \to \pi \hat{\otimes} \bar{\pi} \to \mathbb{C}$ *for automorphic representations* π *belonging to the corresponding Arthur packet.*

Moreover, when $\mathfrak{X}$ *is stable, the restriction of* $J_{\phi}^{\mathrm{aut}} \mu_X^{\mathrm{aut}}(\phi)$ *to the* most tempered *Arthur type* ($\phi(\mathrm{SL}_2) = \mathrm{SL}_2$)*,* away from the poles of $L_X(\phi|_{\mathcal{L}_k})$ *is equal to*

$$\frac{1}{|S_\phi|} \prod_v{}' J_{\phi_v} \cdot \mu_{{}^L G_X}(\phi), \tag{2.8}$$

where $\mu_{{}^L G_X}$ *is the natural measure on the set of such parameters,* S_ϕ *is the stabilizer of* ϕ *in* $\check{G}_X$*, and the factors* J_{ϕ_v} *of the Euler product are the local Plancherel relative characters of Conjecture* 2.3.1.

"Stable," here, means that the stabilizers of generic points have trivial Galois cohomology; one can drop this assumption, replacing $\mathrm{RTF}_{\mathfrak{X}}$ by its (properly defined) stable analog. The Euler product of the conjecture needs to be understood, outside of a finite set S of places, as the partial L-value $L_X^S(\phi)/L^S(\phi, \check{\mathfrak{g}}_X, 1)$, according to Conjecture 3.1.1 below. The conjecture can be generalized to other quantizations of suitable Hamiltonian spaces, such as the theta correspondence, where it was shown in [72] to follow from a version of the Rallis inner product formula proven in [29,89]. The conjecture is compatible with earlier results and methods for computing period integrals, such as the "unfolding" method [77, §18], or the work of Jacquet and Feigon–Lapid–Offen on unitary periods [7,23,39].

3. THE L-VALUE OF A SPHERICAL VARIETY

3.1. Plancherel density of the basic function

3.1.1. It is a very interesting problem to relate the Euler factors of (2.8)—that is, the local Plancherel densities—to special values of local L-functions at every place, including ramified and Archimedean ones. However, we will confine ourselves here to the calculation of the local Plancherel density of the *basic function* $\Phi_0 \in \mathcal{F}(X(F))$, for a local non-Archimedean field F. We assume that G, X are defined over the integers $\mathfrak{o}$ of F, with G reductive, and recall that the basic function is equal to $1_{X(\mathfrak{o})}$, when X is smooth and affine; in general, it is the "IC function," see § 3.1.3 below. We assume that the map $\check{A}_X \to \check{A}$ is injective (§ 2.2.2).

For simplicity of presentation, we will assume that G is split, so that the maximal compact subgroup $\check{A}_X^1 \subset \check{A}_X$ is identified with the group of unramified unitary characters of A_X. The unramified representations appearing in Conjecture 2.3.1 are those obtained by

unitary induction of those characters from the parabolic $P(X)$ (§ 2.2.4) through the quotient $P(X) \to A_X$, and the "natural measure" of the conjecture, restricted to unramified parameters (i.e., to $\check{A}^1_X / W_X$), reads

$$\mu_{\check{G}_X}(\phi) = \frac{L(\phi, \check{\mathfrak{g}}_X/\check{\mathfrak{a}}_X, 1)}{L(\phi, \check{\mathfrak{g}}_X/\check{\mathfrak{a}}_X, 0)} d_{\mathrm{Haar}}\phi.$$

Conjecture. *Let X be an affine spherical variety satisfying the conditions above, with a good model over $\mathfrak{o}$, and Φ_0 its basic function. There is a representation $\rho_X : {}^L G_X \to \mathrm{GL}(V_X)$ such that, setting $L_X(\phi) = L(\phi, \rho_X, 0)$, the Plancherel decomposition of Φ_0 reads*

$$\|\Phi_0\|^2 = \int_{\check{A}^1_X / W_X} \frac{L_X(\phi)}{L(\phi, \check{\mathfrak{g}}_X, 1)} \mu_{\check{G}_X}(\phi). \tag{3.1}$$

We refrain from giving details on the precise normalization of Haar measures, or the precise meaning to "good model;" at a minimum, the conjecture should be valid at almost every place for any global model. This, in particular, will identify almost every Euler factor of Conjecture 2.3.2 as a quotient of special values of local L-functions. Note that the "true" point of evaluation of L_X is not 0, but is encoded in ρ_X, which is a representation of the full L-group. Here, this representation would factor through the unramified quotient, and the "true" point of evaluation depends on the action of Frobenius.

The relation between local L-values and Plancherel densities is a fascinating one. On the surface, it is just the outcome of a local integral. For example, when $X(\mathfrak{o}) = H(\mathfrak{o})\backslash G(\mathfrak{o})$, the value $J_\phi(1_{X(\mathfrak{o})})$ is given by the following Ichino–Ikeda local period, in the so-called *strongly tempered* cases where it is convergent:

$$J_\phi(1_{X(\mathfrak{o})}) = \int_H m_\phi(h) dh,$$

where m_ϕ is the zonal spherical function (= unramified matrix coefficient with value 1 at the identity) for the unramified representation with Satake parameter ϕ. This calculation, however, has a conceptual meaning, in terms of both harmonic analysis and geometry. We will only attempt to give a flavor of the richness of the topic here.

3.1.2. The study of the Plancherel density of the basic function is a topic with a long history. The mainstream method for calculating it is the Casselman–Shalika method [20,21], and it is essentially equivalent to the problem of calculating eigenvectors for the unramified (spherical) Hecke algebra $\mathcal{H}(G(F), G(\mathfrak{o}))$ on the space $C^\infty(X(F))$.

The calculation was related to the structure theory of spherical varieties in [70], for G split. Here, we will formulate the result in the special case when X is an excellent spherical variety (§ 2.2.5) with $\check{G}_X = \check{G}$. Fix a Borel subgroup $B \subset G$, with unipotent radical N. The important geometric invariants determining the L-value are the *colors* of the spherical variety X: those are the B-stable prime divisors on X, over the algebraic closure. For simplicity, we will assume all those divisors to be defined over F. Each such divisor D induces a valuation on the function field $F(X)$, which we restrict to the multiplicative group of nonzero B-eigenfunctions. This gives rise to a homomorphism factoring as

$$F(X)^{(B)} \to X^\bullet(A_X) \to \mathbb{Z},$$

i.e., an element $\check{v}_D \in X_\bullet(A_X)$, the character group of the dual torus $\check{A}_X$ (which here is equal to $\check{A}$). By [78, COROLLARY 7.3.4], the weights $\check{v}_D$ are all minuscule. Let V_X be the smallest self-dual (algebraic) representation of $\check{G}$ which contains all those weights (with multiplicity, if some of the $\check{v}_D$'s coincide). For an alternative interpretation of this representation, in terms of the structure of the Hamiltonian space T^*X, see Theorem 6.2.2 below. The following was proven in [70] under some assumptions, and in [78] in general:

Theorem. *The Plancherel density of the basic function $1_{X(\mathfrak{o})}$ is given by*

$$J_\phi(1_{X(\mathfrak{o})} \otimes 1_{X(\mathfrak{o})})\mu_{\check{G}}(\phi) = \frac{L(\phi, V_X, \frac{1}{2})}{L(\phi, \check{\mathfrak{g}}, 1)}\mu_{\check{G}}(\phi). \tag{3.2}$$

Note that, for simplicity, we have assumed that $\check{G}_X = \check{G}$. The point $\frac{1}{2}$ of evaluation changes in the general case.

3.1.3. The case of *singular* affine varieties X was undertaken in [78], for the cases with $\check{G}_X = \check{G}$. As mentioned, here one needs to work in a geometric setting, assuming that F is a local field in equal characteristic, $F \simeq \mathbb{F}_q((t))$, with G, X defined over $\mathbb{F}_q$. (There are ad hoc ways to transfer the results to mixed characteristic, but it would be nice to see a direct geometric approach.) The basic function Φ_0 is then defined as the Frobenius trace on the stalks of the intersection complex of finite-dimensional formal models of L^+X, the formal arc space of X [12].

Let us discuss the special case when X is the affine closure $\operatorname{Spec} \mathbb{F}_q[X^\bullet]$ of its open G-orbit $X^\bullet$. Colors, here, do not need to be minuscule, but one can still define V_X as before. We have the following generalization of Theorem 3.1.2:

Theorem. *There is a representation V'_X of $\check{A}$, with the same weights as V_X and W-invariant multiplicities, such that the Plancherel density of the basic function Φ_0 is*

$$J_\phi(\Phi_0 \otimes \Phi_0)\mu_{\check{G}}(\phi) = \frac{L(\phi, V'_X, \frac{1}{2})}{L(\phi, \check{\mathfrak{g}}, 1)}\mu_{\check{G}}(\phi). \tag{3.3}$$

Of course, we expect that $V'_X = V_X$. This is automatic in the minuscule case. For example [78, EXAMPLE 1.1.3], there is a family of varieties X_n, $n \in \mathbb{N}$, which gives rise to the generalization of the Hecke and Rankin–Selberg integrals, mentioned in § 1.2.2.

3.2. Derived Satake equivalence for spherical varieties

Ongoing joint work with Ben-Zvi and Venkatesh has revealed deeper relations between periods and L-functions; currently, those can be formulated over function fields and their completions. In the local setting, $F \simeq \mathbb{F}_q((t))$, Conjecture 3.1.1 should be obtained by applying the sheaf–function dictionary to a categorical statement, along the following lines:

We retain the assumptions of the previous subsection, with X and G defined over $\mathbb{F}_q$, and also assume X to be smooth. We denote formal loop and arc spaces by L, resp. L^+, so that $LX(\mathbb{F}_q) = X(F)$, $L^+G(\mathbb{F}_q) = G(\mathfrak{o})$. For appropriate measures, the left-hand side of (3.1), and, more generally, the pairing of two $G(\mathfrak{o})$-invariant functions f, g obtained via the sheaf–function dictionary from objects $\mathscr{F}, \mathscr{G}$ in the bounded derived category

$\mathrm{Shv}(LX/L^+G)$ of constructible ℓ-adic étale sheaves on LX/L^+G can be computed as the (alternating) trace of geometric Frobenius on a derived homomorphism complex:

$$\int_{X(F)} f(x)g(x)dx = \mathrm{tr}\big(\mathrm{Frob}_q, \mathrm{Hom}(\mathscr{F}, D\mathscr{G})^*\big),$$

where D = Verdier dual. The pairing is really a finite sum, and makes sense over $\overline{\mathbb{Q}_\ell}$.

The right-hand side of (3.1), through a simple application of the Weyl integration formula, can be interpreted as the Frobenius trace on

$$\mathbb{C}[V_X]^{\check{G}_X} = \mathbb{C}[\check{M}]^{\check{G}},$$

where V_X is the space of the representation ρ_X, and we have set

$$\check{M} = V_X \times^{\check{G}_X} \check{G}.$$

The empirical observation is that the space $\check{M}$ *has a natural symplectic structure*, and, moreover, that *the assignment* $M = T^*X \to \check{M}$ *is involutive*, although, to make sense of this, one needs to allow for more general coisotropic Hamiltonian spaces, as mentioned in § 1.2.4. For the categorical analog of Conjecture 3.1.1, we need to *shear* the ring $\mathbb{C}[\check{M}]$ into a dg-algebra $\mathbb{C}[\check{M}]^{/\!/}$, with zero differentials, in degrees related to the action of Frobenius in ρ_X.

Conjecture. *Fix an isomorphism* $\mathbb{C} = \overline{\mathbb{Q}_\ell}$. *There is an equivalence of triangulated* $\mathbb{C}$*-linear categories*

$$\mathrm{Shv}(LX/L^+G) \xrightarrow{\sim} D^{/\!/}_{\mathrm{per}}(\check{M}/\check{G}),$$

where $D^{/\!/}_{\mathrm{per}}(\check{M}/\check{G})$ *denotes the full triangulated subcategory, generated by perfect complexes, of the category of* $\check{G}$*-equivariant differential graded* $\mathbb{C}[\check{M}]^{/\!/}$*-modules localized by quasi-isomorphisms.*

This generalizes the derived Satake equivalence of Bezrukavnikov–Finkelberg [11]; it should be compatible with it, under the action of $\mathrm{Shv}(L^+G\backslash LG/L^+G)$ on the left, and the moment map $\check{M} \to \check{\mathfrak{g}}^*$ on the right. There is a similar, categorical version of the global Conjecture 2.3.2, for which I defer to the upcoming article.

4. BEYOND ENDOSCOPY

4.1. Relative functoriality

4.1.1. Let X, Y be two spherical varieties (for possibly different groups G, G'), and let r be a morphism of their L-groups, $r : {}^LG_X \to {}^LG_Y$. According to the relative local Langlands conjecture of § 2.3.1, it should give rise to a map

$$\{X\text{-distinguished } L\text{-packets}\} \to \{Y\text{-distinguished } L\text{-packets}\},$$

at least for L-packets distinguished in the L^2-sense.

A basic tenet of Langlands' "beyond endoscopy" proposal [51], generalized to the relative setting, states that the resulting map of stable relative characters $J^X_{\phi_1} \mapsto J^Y_{\phi_2}$ should be realized as the adjoint of a "transfer operator" between spaces of stable test measures,

$$\mathcal{T} : \mathcal{S}(\mathcal{Y})^{\mathrm{st}} \to \mathcal{S}(\mathcal{X})^{\mathrm{st}}, \tag{4.1}$$

where $\mathfrak{Y}$ denotes the stack $(Y \times Y)/G'$, and $\mathfrak{X} = (X \times X)/G$. In most cases, one can take "stable" to mean the image of the canonical pushforward map

$$\mathcal{S}(\mathfrak{X}) \to \text{Measures}\big((X \times X) /\!\!/ G\big).$$

When the map $\mathcal{S}(\mathfrak{X}) \to \mathcal{S}(\mathfrak{X})^{\text{st}}$ is an isomorphism (e.g., for the Kuznetsov formula), we will be dropping the exponent "st."

In the group case, this operator has been studied by Langlands [52] and Johnstone [43] when X is a torus and Y is GL_n. Understanding these transfer operators could be considered as the basic problem of functoriality, at least in the local setting.

4.1.2. In the global setting, one would have to find a way to employ these transfer operators in a comparison of relative trace formulas. Langlands' proposal, generalized to our setting, is to extract the part of the automorphic spectrum of $\mathfrak{Y}$ that is in the image of the functorial lift from $\mathfrak{X}$ from the (stable) relative trace formula for $\mathfrak{Y}$ by means of poles of L-functions.

The question of whether it is possible to identify the spectrum of $\mathfrak{X}$ by orders of poles of L-functions has been studied and is known to have a negative answer, in general [5]. Other difficulties with this proposal include the isolation of the tempered part of the spectrum; a lot of hard work has gone into this problem, already for the case of GL_2 [2–4, 24].

4.2. An example: symmetric square lift

4.2.1. Rather than speculating on how to overcome these difficulties, it may be more instructive to look at a variant of the idea, which was applied successfully in the thesis of Venkatesh [84], and to understand what the structure of local transfer operators can tell us about the global problem. Here, $X = T$ is a 1-dimensional torus over a global field k (the kernel of the norm map for a quadratic etale algebra E/k whose quadratic idele class character we will denote by η), and Y is the Whittaker model of the group $G = \mathbb{G}_m \times \mathrm{SL}_2$, so that $\check{G}_Y = \check{G}$. There is a morphism of L-groups $r : {}^L T \to {}^L G$, whose image stabilizes a vector under the product of the standard representation of $\mathbb{G}_m$ with the adjoint representation of PGL_2 (that is, the symmetric-square representation of GL_2, as it factors through $\mathrm{GL}_2 \to \check{G} = \mathbb{G}_m \times \mathrm{PGL}_2 \to \mathrm{GL}_3$). Let $\mathbb{Z}/2$ act on T by inversion. The local transfer operator for this morphism was computed in [76]:

Theorem. *Let G, T as above be defined over a local field F. There is a transfer operator*

$$\mathcal{T} : \mathcal{S}(N, \psi \backslash G / N, \psi) \to \mathcal{S}(T)^{\mathbb{Z}/2},$$

such that the pullback of every unitary character of T is the Kuznetsov relative character of its functorial lift. In natural coordinates (r, t) for $N \backslash G /\!\!/ N \simeq \mathbb{G}_m \times \mathbb{A}^1$, it is given by

$$(da)^{-1} \mathcal{T} f(a) = (dt)^{-1} \lambda(\eta, \psi) \int_r \int_x f\left(r, \frac{t}{x}\right) \eta(xrt) \psi(x)\, dx, \tag{4.2}$$

where $a \in T$ and $t = t(a)$ is its image through an isomorphism $T /\!\!/ (\mathbb{Z}/2) \simeq \mathbb{A}^1 \simeq N \backslash \mathrm{SL}_2 /\!\!/ N$, and $\lambda(\eta, \psi)$ is a constant.

What does this theorem tell us about how to extract from the relative trace formula of Y (that is, the Kuznetsov formula of G) the part of the spectrum that is due to the torus T? Venkatesh [84] performs this extraction in two steps, a Poisson summation formula followed by taking the pole of a zeta integral. As explained in [76, §10], the adelic reformulation of the first step is the Poisson summation formula for the Fourier transform corresponding to the inner integral of (4.2), while the second step is a global version of the Mellin transform represented by the outer integral.

4.2.2. Thus, we see that understanding the local transfer operators can guide our steps for the global "beyond endoscopy" comparisons of trace formulas. Another example of such a comparison is that between the Kuznetsov formula of $G = \mathrm{GL}_2$ and the Selberg trace formula for the same group. The local transfer operator for this comparison was computed in [75, §4], and is given by a simple Fourier tranform (see Theorem 5.3.5). Restricted to holomorphic cusp forms, this global comparison via a Poisson summation formula was performed in the thesis of Zeev Rudnick [67], about 10 years before Langlands' "beyond endoscopy" proposal. A generalization of this comparison to the full Kuznetsov formula, and for GL_n with n arbitrary, is the object of ongoing joint work with Chen Wan.

4.2.3. It has hopefully become clear that understanding the transfer operators is of paramount importance for the problem of functoriality. In [74], I showed that these operators have a very uniform form, for spherical varieties of rank 1. In the remainder of this paper, I would like to propose a reinterpretation of this work, which provides an understanding of those transfer operators as "change of Schrödinger model/geometric quantization" associated to a symplectic group scheme.

5. TRANSFER OPERATORS AND QUANTIZATION

The goal of this section is to recast the transfer operators of functoriality, studied in [74], in the language of quantization. The idea that quantization should have something to do with functoriality is not new; V. Lafforgue suggested it several years ago (private communication), in order to interpret the Rankin–Selberg method, and the functoriality kernels of L. Lafforgue [49]. Here, however, we apply this idea in a different setting: the setting of "beyond endoscopy," and of the quotient stacks showing up in the relative trace formula—the hope being that these operators of functoriality will always exist in this setting, even if they do not exist for the spaces "upstairs."

Geometric quantization was introduced by Kostant and Souriau [48, 82], following the work of Kirillov on the orbit method [44]. Since the notion of quantization for measures on stacks that we need has not been developed yet, we will take a phenomenological approach, with ad hoc definitions that provide the desired reformulation of the results of [74].

5.1. Cotangent space of the RTF stack

5.1.1. The groundbreaking work of Friedrich Knop has shown that, although spherical varieties can be very different from each other, their *cotangent bundles* are quite similar.

This will be the basis of our considerations, when we try to relate cotangent bundles of different quotient stacks of the form $(X \times X)/G$.

For the rest of this paper we will assume, for simplicity, that all groups are split, defined over a field F in characteristic zero. The results of Knop, then, recalled in § 2.2.3, hold verbatim over F. We assume that X is smooth and quasiaffine, and set again $\mu : M = T^*X \to \mathfrak{g}^*$ for the cotangent bundle and its moment map.

The ring of regular functions $F[M]$ has a Poisson structure. Knop has studied the subalgebra $F[M]^G$ of G-invariants; when X is spherical, this subalgebra is Poisson-commutative, and can be naturally identified with the algebra of regular functions on the affine space $\mathfrak{c}_X^*$, defined in § 2.2.3. Hence, regular functions on $\mathfrak{c}_X^*$ pull back to a Poisson-commuting algebra of G-invariant functions ("Hamiltonians") on M. One can ask whether the corresponding Hamiltonian vector fields can be integrated to the action of an abelian group scheme J_X (over $\mathfrak{c}_X^*$) of G-automorphisms of M, and Knop has answered this in the affirmative [46].

5.1.2. More precisely, J_X is "the group scheme of regular centralizers in the split reductive group G_X dual to $\check{G}_X$." When $\check{G}_X$ is adjoint, so that G_X is simply connected, the group scheme J_X has an explicit description as

$$(\mathrm{Res}_{\mathfrak{a}_X^*/\mathfrak{c}_X^*} T^*A_X)^{W_X} \tag{5.1}$$

(see [61, §2.4]), where $\mathrm{Res}_{\mathfrak{a}_X^*/\mathfrak{c}_X^*}$ denotes Weil restriction of scalars from $\mathfrak{a}_X^*$ to $\mathfrak{c}_X^*$.

In general, the group scheme J_X acting on M is an open subgroup scheme of (5.1), which depends not only on the pair (A_X, W_X), but also on the root datum of X. Knop defines a slightly different root datum than ours in [46, §6], giving the maximal possible subgroup scheme acting on M. For our purposes, we will be content with taking $J_X =$ the open subgroup scheme of (5.1) that corresponds to the set of normalized spherical roots, see § 2.2.2. This can be described as the regular centralizer group scheme of G_X, and is the complement of a divisor in (5.1), see [46, THEOREM 7.7], [61, §2.4].

5.1.3. Let us discuss the rank-1 cases. Consider first the case $\check{G}_X = \mathrm{PGL}_2$, so that $A_X = \mathbb{G}_m$, $W_X = \mathbb{Z}/2$, and the normalized spherical root is twice the generator of the character lattice. (The isomorphism $A_X \simeq \mathbb{G}_m$ is canonical, if we require positive roots to correspond to positive powers.) Then, $J_X = J_{\mathrm{SL}_2}$ is given by the restriction of scalars (5.1), which can explicitly be described as follows: Identify $\mathfrak{a}_X^* = \mathfrak{g}_m^*$ with the affine line, with coordinate $\sigma =$ the differential of the identity cocharacter, and set $\xi = \sigma^2$, a coordinate on $\mathfrak{c}_X^*$. We can write

$$J_X = \mathrm{Spec}\, F[t_0, t_1, \xi]/(t_0^2 - \xi t_1^2 - 1),$$

so that the canonical base change map

$$J_X \bullet \mathfrak{a}_X^* \to T^*A_X = T^*\mathbb{G}_m = \mathbb{G}_m \times \mathfrak{g}_m^*,$$

where by $\bullet$ we denote fiber product over $\mathfrak{c}_X^*$, is given by $(t_0, t_1, \sigma) \mapsto (t_0 + \sigma t_1, \sigma)$.

The symplectic form is given by

$$\omega = \frac{dt_1 \wedge d\xi}{2t_0} = \frac{dt_0 \wedge d\xi}{2\xi t_1} = dt_0 \wedge d(t_1^{-1}).$$

It is immediate to check that this is regular and nondegenerate everywhere on J_X.

On the other hand, when the normalized root datum of X is that of PGL_2 (i.e., the normalized spherical root is a generator of the character lattice), the fiber of (5.1) over the nilpotent point $0 \in \mathfrak{c}_X^*$ is isomorphic to $\mathbb{G}_a \times \{\pm 1\}$, but the fiber of J_X is just $\mathbb{G}_a$.

5.1.4. Returning to the general case, the Lie algebra of J_X is canonically isomorphic to the cotangent space of $\mathfrak{c}_X^*$. Thus, the Hamiltonian vector fields associated to $F[\mathfrak{c}_X^*]$ give rise to a homomorphism from $\mathrm{Lie}(J_X)$ to G-invariant vector fields along the fibers of $M \to \mathfrak{g}_X^*$ (notation as in § 2.2.2). Knop has shown [46] that these vector fields integrate to an action of J_X on $M = T^*X$ over $\mathfrak{g}_X^*$, commuting with the action of G.

Moreover, over a dense open subset $\mathring{\mathfrak{c}}_X^* \subset \mathfrak{c}_X^*$, the map $M \to \mathfrak{g}_X^*$ is a J_X-torsor, and the action of J_X arises from the stabilizers of points of $\mathfrak{g}_X^*$, or even of $\mathfrak{g}^*$, in G, i.e., the stabilizer G_z of a generic point $z \in \mathfrak{g}^*$ in the image of the moment map acts transitively on the fiber through a map $G_z \to J_X$.

5.1.5. In order to study the relative trace formula for a stack of the form $\mathfrak{X} = (X \times X)/G$, I propose to use its cotangent stack

$$T^*\mathfrak{X} = (T^*X \times_{\mathfrak{g}^*} T^*X)/G.$$

(Strictly speaking, the fiber product over $\mathfrak{g}^*$ should taken with respect to the moment map and its negative, $(\mu, -\mu)$, so that the quotient above corresponds to symplectic reduction with respect to the diagonal action of G. We apply multiplication by -1 on the fibers of one factor, in order to have fiber product with respect to (μ, μ), which is notationally simpler.)

The fiber product, here, should be taken in the derived sense, turning this quotient into a symplectic derived stack. Although the derived features are likely to be important in the future, for the purposes of the current paper we will ignore them. Then, the fiber product over $\mathfrak{g}^*$ coincides with the fiber product over $\mathfrak{g}_X^*$ over a dense open $\mathring{\mathfrak{c}}_X^* \subset \mathfrak{c}_X^*$, which we will take small enough so that it also has the properties of § 5.1.4.

Knop's theory, now, gives a very satisfactory description of a dense substack. Namely, consider the diagonal embedding $T^*X \hookrightarrow T^*X \times_{\mathfrak{g}^*} T^*X$, which by the action of J_X on the first variable gives rise to

$$J_X \bullet T^*X \to T^*X \times_{\mathfrak{g}^*} T^*X, \tag{5.2}$$

where again by $\bullet$ denotes $\times_{\mathfrak{c}_X^*}$. Given that $T^*X \to \mathfrak{g}^*$ is generically a J_X-torsor over its image, this map is birational into an irreducible component of the right hand side.

Taking quotients by the G-action in (5.2), and using the invariant moment map $T^*X \to \mathfrak{c}_X^*$, we obtain a correspondence

$$J_X \leftarrow (J_X \bullet T^*X)/G \to T^*\mathfrak{X} = (T^*X \times_{\mathfrak{g}^*} T^*X)/G. \tag{5.3}$$

Over the dense open subset $\mathring{\mathfrak{c}}^*_X$, the right arrow is an isomorphism, and the left arrow is an isomorphism if we ignore stabilizers. We would like to think of (5.3) as saying that the symplectic stack $T^*\mathfrak{X}$ is "birational" to the symplectic group scheme J_X. Of course, this is a very naive notion of birationality since, even over a good open dense subset, we are ignoring stacky and derived structures in $T^*\mathfrak{X}$. Nonetheless, even this weak correspondence is quite remarkable, since J_X depends only on the dual group of X. *It is not too far-fetched to imagine that this correspondence plays a key role in functoriality.*

5.2. Rank 1 spherical varieties

5.2.1. We will now specialize to spherical varieties $X = H\backslash G$ with G and H split reductive groups, whose dual group is $\check{G}_X = \mathrm{PGL}_2$ or SL_2. The group scheme J_X was described in § 5.1.3.

In this setting, [74] gave explicit formulas for transfer operators (4.1) between $\mathcal{Y}$ = the Kuznetsov stack for the group with dual group $\check{G}_X$ (see § 5.2.2 below), and $\mathfrak{X} = (X \times X)/G^{\mathrm{diag}}$. These operators transfer spaces of test measures to each other, but in a number of cases, studied in [30,75,76], properties such as the transfer of characters or the appropriate fundamental lemma are also known; thus, there is enough evidence to believe that these are the "correct" operators of functoriality for these comparisons.

The simplest form of transfer operators appears when the normalized root datum of X is simply connected, i.e., $\check{G}_X = \mathrm{PGL}_2$. For some varieties, this is possible to achieve by passing to a finite cover; the only cases where this can be done lead to the following spaces:

$$X = \mathrm{SO}_{2n-1} \setminus \mathrm{SO}_{2n},\ \mathrm{Spin}_7 \setminus \mathrm{Spin}_8,\ \mathrm{G}_2 \setminus \mathrm{Spin}_7\,. \tag{5.4}$$

We will call them "cases of type G," because the base case is the group variety $\mathrm{SO}_3 \setminus \mathrm{SO}_4 \simeq \mathrm{SL}_2$. (Here, the $\mathrm{SO}_7 \setminus \mathrm{SO}_8 \simeq \mathrm{Spin}_7 \setminus \mathrm{Spin}_8 \simeq \mathrm{G}_2 \setminus \mathrm{Spin}_7$ as varieties, but with the action twisted by the triality automorphism of Spin_8, for the second, and restricted to the subgroup Spin_7, for the third.)

The remaining cases of spherical varieties satisfying our assumptions are

$$\begin{aligned} X = {} & \mathrm{GL}_n \setminus \mathrm{PGL}_{n+1},\ \mathrm{SO}_{2n} \setminus \mathrm{SO}_{2n+1},\ \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \setminus \mathrm{Sp}_{2n}\ (\text{with } n \geq 2), \\ & \mathrm{Spin}_9 \setminus F_4,\ \mathrm{SL}_3 \setminus G_2. \end{aligned} \tag{5.5}$$

We will call them "cases of type T," because in the base case $\mathbb{G}_m \setminus \mathrm{PGL}_2$ the stabilizer is a torus. (Again, $\mathrm{SL}_3 \setminus G_2 \simeq \mathrm{SO}_6 \setminus \mathrm{SO}_7$, but with the action restricted to $G_2 \subset \mathrm{SO}_7$.)

In those cases, the normalized root datum of X is that of PGL_2, and, as we will see, it will be necessary to "lift" our description of transfer operators to the root datum of GL_2.

5.2.2. Let X be as above. Let G' be the split reductive group with the same dual group as X, that is, $G' = \mathrm{SL}_2$ for the varieties of (5.4) and $G' = \mathrm{PGL}_2$ for the varieties of (5.5). Let $N \subset G'$ be the upper triangular unipotent subgroup, identified with the additive group $\mathbb{G}_a$, fix a nontrivial character ψ of F, and let Y be the Whittaker model of G' with respect to (N, ψ), § 2.2.5. Let Y^- be the Whittaker model with respect to the inverse character, ψ^{-1}.

We will symbolically write $\mathcal{Y} = (Y \times Y^-)/G'$ for the "Kuznetsov stack," but we will really treat it not as an abstract quotient stack, but as one equipped with the line bundle defined by the Whittaker character. More precisely, this symbol will only find a rigorous meaning in its Schwartz space $\mathcal{S}(\mathcal{Y})$, which we define to be the G'^{diag}-coinvariant space of the space $\mathcal{S}(Y \times Y^-)$ of Whittaker Schwartz measures.

Having identified N with $\mathbb{G}_a$, we let $M' = (f + \mathfrak{n}^\perp) \times^N G'$ be the *Whittaker cotangent bundle*, where f is a nilpotent element of $(\mathfrak{g}')^*$ that is equal to the identity functional on $\mathfrak{n} = \mathbb{G}_a$. The corresponding bundle for Y^- is $M'^- = (-f + \mathfrak{n}^\perp) \times^N G'$. Both come equipped with natural moment maps to $\mathfrak{g}'^*$, which we will indiscriminately denote by μ. We now define the *Kuznetsov cotangent stack* as

$$T^*\mathcal{Y} = \left(M' \times_{\mu,\mathfrak{g}'^*,(-\mu)} M'^-\right)/G' \simeq (M' \times_{\mu,\mathfrak{g}'^*,\mu} M')/G'.$$

Note that the invariant-theoretic quotient $\mathfrak{g}' /\!\!/ G'$ is canonically identified with $\mathfrak{c}_X^*$, and the group scheme of regular centralizers in G' is canonically identified with J_X. It is well known that M' is a J_X-torsor over the regular subset of $\mathfrak{g}'^*$; in fact, a Kostant section provides a section for this torsor. Therefore, the same considerations that led us to (5.3) hold, but here we have an exact isomorphism

$$J_X \simeq T^*\mathcal{Y}. \tag{5.6}$$

Our hope, now, is to demonstrate the following idea:

> The cotangent stacks $T^*\mathcal{X}$ and $T^*\mathcal{Y}$ being roughly isomorphic to J_X (by (5.3), (5.6)), there is a transfer operator of functoriality
>
> $$\mathcal{T} : \mathcal{S}(\mathcal{Y}) \to \mathcal{S}(\mathcal{X})^{\mathrm{st}},$$
>
> corresponding to a "change of geometric quantization" for J_X.

5.2.3. Quantization, of course, is as much of a science as an art, and the reader should not expect a rigorous formulation of this hope in this article. In particular, the type of geometric quantization that we need (suitable for encoding measures on stacks) has not, to my knowledge, been developed, yet. Therefore, the real content of the results that follow is already contained in [74]; but we will dress them up in an ad hoc language of quantization, in order to exhibit some deeper structure that seems to be lying behind them.

We will also assume, from now on, that our base field is $F = \mathbb{R}$, in order to use the language of line bundles with connection, and will write the chosen additive character as $\psi(x) = e^{i\hbar x}$, where $\hbar$ is a nonzero real constant. The final results, contained in Theorems 5.3.5 and 5.4.3, are valid and were proven in [74] over an arbitrary local field in characteristic 0, just by an obvious translation of the formulas. We will only care to describe transfer operators up to an absolute scalar; therefore, we will feel free to choose measures that only modify the result by a scalar, without commenting on those choices.

5.3. Geometric quantization for type G

5.3.1. The process of geometric quantization on a (real) symplectic manifold (M, ω) consists in fixing a (complex) Hermitian vector bundle L, equipped with a con-

nection ∇ whose curvature is $i\hbar\omega$, as well as a Lagrangian foliation $\mathscr{F}$, such that the space of leaves $M/\mathscr{F}$ is a Hausdorff manifold. Then, one attaches to these data the vector space $\mathcal{D}_{\mathscr{F}}(M, L)$ of smooth half-densities on $M/\mathscr{F}$ valued in the space of sections of L over M that are constant along the foliation $\mathscr{F}$ (with respect to the connection). This space has a canonical inner product (namely, the L^2-inner product over $M/\mathscr{F}$), giving rise to a Hilbert space, by completion.

To reformulate the results of [74] in this language, we will now recast the space $\mathcal{S}(\mathcal{X})^{\mathrm{st}}$ of stable test measures for the relative trace formula as a space of half-densities on the quotient of the group scheme J_X by a Lagrangian foliation, valued in a line bundle L_X.

5.3.2. Fix a rank-1 space X "of type G," i.e., in the list (5.4).

Consider the composition of maps

$$J_X \bullet T^*X \to T^*X \times_{\mathfrak{g}^*} T^*X \to X \times X \tag{5.7}$$

induced from (5.2). There is a natural scaling $\mathbb{G}_m$-action on the left-hand side, under which this composition is invariant, and if we consider the "projectivization" of the space on the left (= remove the zero section in T^*X and divide by $\mathbb{G}_m$), it was shown in [74, §3] that the resulting map

$$\mathbb{P}\big(J_X \bullet T^*X\big) \to X \times X \tag{5.8}$$

is generically an isomorphism. More precisely, in the type-G cases it is an isomorphism over $\mathbb{P}(J_X^\circ \bullet T^*X)$, where $J_X^\circ \subset J_X$ is the complement of the divisor given by the homogeneous equation $t_1 = 0$. (This is a combination of Propositions 3.3.2, 3.5.1 in [74], and the fact that those spaces have an involutive G-automorphism.)

The invariant-theoretic quotient $X \times X \to (X \times X) /\!\!/ G$ is an affine line, and its composition with (5.7) is the map $J_X \bullet T^*X \to J_X \to \mathbb{A}^1$ that remembers only t_0 from the triple (t_0, t_1, ξ) [74, PROPOSITION 3.4.2]. We notice that the level sets of t_0 on J_X° form a Lagrangian foliation; we will call this foliation "vertical," and denote it by $\mathscr{F}_{\mathrm{ver}}$.

5.3.3. Let $d = \dim X$. The short version of the story that follows is that we replace the element $f \in \mathcal{S}(\mathcal{X})^{\mathrm{st}}$ (a measure in the variable t_0) by

$$f(t_0)(dt_0)^{-\frac{1}{2}}|t_1|^{-\frac{d}{2}+1}, \tag{5.9}$$

obtaining a half-density on $J_X^\circ/\mathscr{F}_{\mathrm{ver}}$ valued in the line bundle L_X whose sections are functions on $J_X^\circ/\mathscr{F}_{\mathrm{ver}}$ multiplied by the factor $|t_1|^{-\frac{d}{2}+1}$. More precisely, L_X will be identified with the trivial line bundle on J_X°, but endowed with a connection

$$\nabla^X = \nabla^0 + d\log|t_1|^{\frac{d}{2}-1} - i\hbar t_1^{-1}dt_0 = \nabla^0 + \left(\frac{d}{2} - 1\right)t_1^{-1}dt_1 - i\hbar t_1^{-1}dt_0, \tag{5.10}$$

with curvature $i\hbar\omega$, where ∇^0 is the standard flat connection, so that its parallel sections along the vertical foliation are as described.

Presented this way, this connection is completely unmotivated. In § 5.3.6 below, we will discuss a more natural description of the pair (L_X, ∇^X). Continuing, for now, in this ad

hoc fashion, (5.9) defines a map

$$\mathcal{S}(\mathcal{X})^{\mathrm{st}} \to D_{\mathrm{hor}}(J_X^\circ, L_X), \tag{5.11}$$

where D_{hor} (with regular font) denotes *continuous* (not necessarily Schwartz, or even smooth) "horizontal" half-densities valued in L_X (i.e., half-densities on $J_X^\circ/\mathscr{F}_{\mathrm{ver}}$ valued in the descent of L_X by parallel transport). The image of this map will be denoted by $\mathcal{D}(\mathcal{X})$.

5.3.4. We consider another Lagrangian foliation $\mathscr{F}_{\mathrm{hor}}$ on J_X°, which we will call "horizontal": its leaves are the level sets of t_1. For the line bundle with connection (L_X, ∇^X), as above, flat sections along horizontal leaves are simply functions of $t_1 \neq 0$, multiplied by the factor $\psi(\frac{t_0}{t_1})$; note that this description is independent of the dimension d used to define ∇^X.

We now propose to think of the space of test measures $\mathcal{S}(\mathcal{Y})$ for the Kuznetsov formula as a subspace $\mathcal{D}(\mathcal{Y}) \subset D_{\mathrm{ver}}(J_X^\circ, L_X)$, where D_{ver} denotes continuous "vertical" half-densities (i.e., half-densities on $J_X^\circ/\mathscr{F}_{\mathrm{hor}}$) valued in L_X. First of all, consider the map

$$J_X \bullet M' \xrightarrow{\sim} M' \times_{\mathfrak{g}'^*} M',$$

with notation as in § 5.2.2, where the action of J_X is again on the first copy of M'.

Lemma. *The composition of the map above with* $M' \times_{\mathfrak{g}'^*} M' \to Y \times Y \to (Y \times Y) /\!\!/ G$ *is the map that only remembers the coordinate* t_1 *of* J_X.

This is the reason why the foliation $\mathscr{F}_{\mathrm{hor}}$ is relevant to the Kuznetsov formula.

There is a natural pullback from Whittaker functions on $Y \times Y^-$ to scalar-valued functions on $M' \times_{\mathfrak{g}'^*} M'$, as follows: Thinking of elements of $\mathcal{F}(Y)$ (that is, Whittaker functions) as sections of a line bundle L_ψ over $Y = N\backslash G'$ (and similarly for Y^-, just replacing ψ by ψ^{-1}), we note that the line bundle $L_\psi \boxtimes L_{\psi^{-1}}$ is canonically trivial over the diagonal $Y^{\mathrm{diag}} \subset Y \times Y^-$. There is now a unique trivialization of its pullback to $J_X \bullet M'$ which coincides with the canonical one over the diagonal, and is equivariant with respect to the action of $J_X \times G'$. More explicitly, if we use a Kostant section to identify $T^*Y \simeq \mathrm{c}_X^* \times G'$, and the negative of that section for Y^-, we pull back Whittaker functions to scalar-valued functions on T^*Y via the projection to G', and then restrict to $T^*Y \times_{\mu,\mathfrak{g}'^*,(-\mu)} T^*Y^- \simeq M' \times_{\mathfrak{g}'^*} M'$.

The short version of the story, now, is that we fix a G'-invariant measure on $N\backslash G'$, use it to identify Schwartz (Whittaker) measures on $Y \times Y^-$ with Schwartz (Whittaker) functions, pull them back to scalar-valued functions on $J_X^\circ \bullet M' \simeq J_X^\circ \times G'$, and integrate them against a chosen Haar measure on G'. This gives functions on J_X° that, as can be easily confirmed, correspond to sections of L_X, flat along the leaves of $\mathscr{F}_{\mathrm{hor}}$; further multiplying them by the factor $|t_1|^{\frac{1}{2}} dt_1^{\frac{1}{2}}$ gives rise to an element of $D_{\mathrm{ver}}(J_X^\circ, L_X)$. This descends to an injective map

$$\mathcal{S}(\mathcal{Y}) \to D_{\mathrm{ver}}(J_X^\circ, L_X), \tag{5.12}$$

whose image will be denoted by $\mathcal{D}(\mathcal{Y})$. Again, the factor $|t_1|^{\frac{1}{2}} dt_1^{\frac{1}{2}}$ seems unmotivated, and we will attempt to explain it in § 5.3.9 below, after formulating the main theorem.

5.3.5. To recap, we have defined a line bundle L_X on J_X°, endowed with a connection ∇^X with curvature $i\hbar\omega$, "vertical" and "horizontal" foliations $\mathscr{F}_{\mathrm{ver}}$, $\mathscr{F}_{\mathrm{hor}}$ on J_X°, and have identified the spaces $\mathcal{S}(\mathfrak{X})^{\mathrm{st}}$, $\mathcal{S}(\mathfrak{Y})$ of test measures for the corresponding quotients with spaces $\mathcal{D}(\mathfrak{X})$, $\mathfrak{D}(\mathfrak{Y})$ of "horizontal" and "vertical" half-densities for (L_X, J_X°). The main result [74, THEOREM 1.3.1] for the transfer operator in this case can now be formulated as follows:

Theorem. *There is an injective operator*

$$\mathcal{T} : \mathcal{D}(\mathfrak{Y}) \to \mathcal{D}(\mathfrak{X})$$

given by integration along the leaves of the vertical foliation:

$$\begin{aligned} &D_{\mathrm{ver}}(J_X^\circ, L_X) \dashrightarrow D_{\mathrm{hor}}(J_X^\circ, L_X), \\ &\mathcal{T}\varphi(j) = \int_{\mathscr{F}_{\mathrm{ver},j}} T_{j,j'}\big(\varphi(j')\big)\big|\omega(j')\big|^{\frac{1}{2}}, \end{aligned} \tag{5.13}$$

where $\mathscr{F}_{\mathrm{ver},j}$ denotes the leaf of $\mathscr{F}_{\mathrm{hor}}$ through the point j, and $T_{j,j'}$ denotes parallel transport from the fiber of L_X over j' to the fiber over j' along this leaf.

Its inverse $\mathcal{T}^{-1}$, valued in an enlargement $\mathcal{D}^-_{L_X}(\mathfrak{Y}) \supset \mathcal{D}(\mathfrak{Y})$ described in [74, §1.3]*, is given by integration along the leaves of the horizontal foliation:*

$$D_{\mathrm{hor}}(J_X^\circ, L_X) \dashrightarrow D_{\mathrm{ver}}(J_X^\circ, L_X),$$

Note that a horizontal half-density on J_X°, multiplied by the half-density $|\omega|^{\frac{1}{2}}$, gives rise to a vertical half-density valued in the bundle of densities on the leaves of $\mathscr{F}_{\mathrm{hor}}$; thus, it makes sense to integrate it along these leaves, obtaining a half-density on $J_X^\circ/\mathscr{F}_{\mathrm{hor}}$. This, of course, is completely analogous to the canonical intertwiners for the Schrödinger models quantizing a symplectic vector space [55].

5.3.6. The line bundle L_X, with its connection, admits a more natural description as the dual to *a line bundle of half-densities on the fibers of the invariant moment map* $\mu_G : M = T^*X \to \mathfrak{c}_X^*$. The map (5.11), then, admits a more natural description as descending, up to a choice of invariant measure on X, from a map from Schwartz half-densities on $X \times X$,

$$\mathcal{D}(X \times X) \to D_{\mathrm{hor}}(J_X^\circ, L_X). \tag{5.14}$$

Let us see how this works.

It will be convenient to choose a section s of the invariant moment map μ_G. Such a section exists in the cases $X = H\backslash G$ of (5.4) when G and H are split; it suffices to check the case of $\mathrm{SO}_{2n-1} \setminus \mathrm{SO}_{2n}$, and we refrain from attempting to give an abstract argument. See § 5.4.7 for a further discussion of this issue.

Obviously, the section s has image in the smooth locus of the map μ_G, which implies that the fibers of this map are transversal to the section. If $\mathcal{O}_\xi$ denotes the fiber over $\xi \in \mathfrak{c}_X^*$, let D_X be the algebraic line bundle over $\mathfrak{c}_X^*$ whose fiber over ξ is the determinant of the tangent space of $\mathcal{O}_\xi$ at $s(\xi)$. Let $L_X = |D_X|^{\frac{1}{2}}$, a complex line bundle whose fiber over ξ is dual to the space of Haar half-densities on this tangent space. By pullback, we will also consider L_X as a line bundle over J_X.

For $\xi \neq 0$, the fibers of the invariant moment map μ_G are G-orbits, therefore the tangent space of $\mathcal{O}_\xi$ is $\mathfrak{g}/\mathfrak{g}_{s(\xi)}$, and the fiber of L_X over ξ is the complex line $|\det \mathfrak{g} \otimes \det \mathfrak{g}^*_{s(\xi)}|^{\frac{1}{2}}$. Its dual is the line of invariant half-densities on $\mathcal{O}_\xi$.

There is a natural way to trivialize the bundle L_X, up to a scalar. It uses the fact that the stabilizers $G_{s(\xi)}$, for $\xi \neq 0$, are isomorphic over the algebraic closure. Thus, G-conjugacy gives canonical isomorphisms between the complex line bundles $|\det \mathfrak{g}_{s(\xi)}|$, and this allows us to uniformly fix an invariant measure $d\dot{g}$ on all orbits $\mathcal{O}_\xi$, for $\xi \neq 0$, see [74, §4]. Then, by [74, THEOREM 4.0.3]:

Proposition. *For a suitable choice of $d\dot{g}$ as above, and the canonical measure dz on T^*X induced by the symplectic form, we have the integration formula*

$$\int_{T^*X} \Phi(z)dz = \int_{\mathfrak{c}_X^*} |\xi|^{\frac{d}{2}-1} \int_{\mathcal{O}_\xi} \Phi\big(s(\xi)\dot{g}\big)\, d\dot{g}\, d\xi.$$

Hence, the family of Haar half-densities $\xi \mapsto (|\xi|^{\frac{d}{2}-1} d\dot{g})^{\frac{1}{2}}$ on the orbits $\mathcal{O}_\xi$, for $\xi \neq 0$, extends to a nonvanishing half-density on the fiber over 0. We now use this family (depending up to a constant on our choice of $d\dot{g}$) to trivialize L_X^*, hence also L_X, i.e., we have an isomorphism

$$L_X \simeq \underline{\mathbb{C}} \tag{5.15}$$

with the trivial line bundle. Moreover, the proposition above shows that a nonzero element of the fiber of L_X^* over 0 corresponds to a unique half-density on $\mathcal{O}_0$, obtained as the limit of G-invariant half-densities over the fibers $\mathcal{O}_\xi$ with $\xi \neq 0$. Hence, each element in the total space of L_X^* gives rise to a half-density on the corresponding fiber of μ_G.

5.3.7. We can now define the map (5.14). Let $\varphi \in \mathcal{D}(X \times X)$. The product $\varphi \cdot (dt_0)^{-\frac{1}{2}}$ restricts to a half-density on each fiber of the smooth locus of the invariant-theoretic quotient $X \times X \to (X \times X) /\!\!/ G$. The idea is to integrate this half-density, but for that purpose we need to turn it into a measure. We will do so after pulling it back to J_X° via the maps

$$J_X^\circ \bullet (T^*X \smallsetminus X) \to \mathbb{P}(J_X^\circ \bullet T^*X) \to X \times X,$$

where the second arrow is (5.8), an isomorphism onto its image.

For every $j \in J_X^\circ$ with image $\xi(j) \in \mathfrak{c}_X^*$, the map (5.7) restricts to a map $\{j\} \bullet \mathcal{O}_{\xi(j)} \to X \times X$ that is, by (5.8), an isomorphism onto its image (up to removing the zero section of T^*X, if $\xi(j) = 0$). Thus, the pullback of $\varphi \cdot (dt_0)^{-\frac{1}{2}}$ induces a half-density on $\mathcal{O}_{\xi(j)}$. Multiplying by the half-density corresponding to an element of the fiber of L_X^* over $\xi(j)$ gives rise to a measure, which we can integrate. This way, we get a canonical map

$$\mathcal{D}(X \times X)(dt_0)^{-\frac{1}{2}} \times \underline{L_X^*} \to \mathbb{C},$$

where $\underline{L_X^*}$ denotes the total space of the line bundle L_X^* over J_X°. This corresponds to a map

$$\mathcal{D}(X \times X)(dt_0)^{-\frac{1}{2}} \to \Gamma(J_X^\circ, L_X), \tag{5.16}$$

where the right-hand side denotes (continuous) sections of L_X over J_X°.

5.3.8. The image of $\varphi \cdot (dt_0)^{-\frac{1}{2}}$ under this map has an invariance property: namely, its values at different points j with the same value of t_0 "coincide." To make sense of this, we need to endow L_X with a connection, whose parallel sections along vertical Lagrangians descend to duals of half-densities on the corresponding G-orbits on $X \times X$. To explicate this, consider the integration formula of [74, THEOREM 4.0.2]:

Proposition. *For a function Φ and an invariant measure dx on $X \times X$, we have*

$$\int_{X\times X} \Phi(x)\, dx = \int_{(X\times X)/\!\!/ G} |t_0^2 - 1|^{\frac{d}{2}-1} \int_{\mathcal{O}_{t_0}} \Phi(\dot{g})\, d\dot{g}\, dt_0. \tag{5.17}$$

Here, we have denoted by $\mathcal{O}_{t_0}$ the preimage of $t_0 \in (X \times X) /\!\!/ G$ in $X \times X$, using similar notation as for the preimages of points of c_X^* in T^*X. The reason is that, as above, for $j \in J_X^\circ$ with $\xi(j) \neq 0$, we can identify the G-orbit $\mathcal{O}_{\xi(j)}$ with the image of $\{j\} \bullet \mathcal{O}_{\xi(j)}$ in $X \times X$, which is equal to $\mathcal{O}_{t_0(j)}$. The measure $d\dot{g}$ in the Proposition, then, is the same measure on $\mathcal{O}_{\xi(j)}$ as that used to trivialize the bundle L_X in § 5.3.6.

The proposition above tells us that, if for every j with $\xi(j) \neq 0$ we multiply $\varphi(dt_0)^{-\frac{1}{2}}$ by the half-density $(|t_0^2(j) - 1|^{\frac{d}{2}-1} d\dot{g})^{\frac{1}{2}}$, the integral will depend only on the function $t_0(j)$ of j. On the other hand, the trivialization (5.15) of L_X uses the half-density $(|\xi|^{\frac{d}{2}-1} d\dot{g})^{\frac{1}{2}}$. The quotient of the two is $|t_1|^{\frac{d}{2}-1}$. We conclude that the map (5.16), composed with the trivialization (5.15), gives rise to *functions f on J_X° such that $|t_1|^{\frac{d}{2}-1} f$ is constant along fibers of t_0*. This explains the definition of ∇^X in (5.10), and completes the construction of the map (5.14).

5.3.9. In a similar way, we define a line bundle L_Y on J_X, pulled back from c_X^*, as the dual of the line bundle of G'-invariant half-densities on the fibers of $M' = T^*Y \to \mathrm{c}_X^*$. Here, the fibers are G'-torsors, hence fixing a Haar half-density on G' gives rise to a trivialization

$$L_Y \xrightarrow{\sim} \underline{\mathbb{C}}. \tag{5.18}$$

Through the trivializations (5.15) and (5.18), the line bundles L_X and L_Y are identified.

Recall from § 5.3.4 that the description of "vertical" half-densities for (L_X, ∇^X) is the same in every case (does not depend on the dimension d of X). We can now define a map

$$\mathcal{D}(Y \times Y^-) \to D_{\mathrm{ver}}(J_Y^\circ, L_Y), \tag{5.19}$$

in a completely analogous way to (5.14), using also the scalar-valued pullback from Whittaker functions on $Y \times Y^-$ to scalar-valued functions on $M' \times_{\mathfrak{g}'^*} M'$, described in § 5.3.4. To describe it explicitly, consider the integration formula for functions on Y^2, analogous to (5.17),

$$\int_{Y\times Y} \big|\Phi(y)\big|\, dy = \int_{(Y\times Y)/\!\!/ G'} |t_1| \int_{G'} \big|\Phi(\tilde{t}_1 g)\big|\, dg\, dt_1 \tag{5.20}$$

(where $\tilde{t_1}$ denotes any lift of t_1 to $Y \times Y$). Symbolically, we can write the $G' \times G'$-invariant measure dy on $Y \times Y$ as $dg \times |t_1| dt_1$. Similarly, we can write the spaces of Schwartz measures and half-densities as

$$\mathcal{S}(Y \times Y) = \mathcal{F}(Y \times Y) \cdot \big(dg \times |t_1| dt_1\big),$$
$$\mathcal{D}(Y \times Y) = \mathcal{F}(Y \times Y) \cdot \big(dg \times |t_1| dt_1\big)^{\frac{1}{2}}.$$

Choosing the half-density $(dg)^{\frac{1}{2}}$ to define the trivialization (5.18), the image of a Schwartz half-density $\Phi \cdot (dg \times |t_1| dt_1)^{\frac{1}{2}}$ under (5.19) is precisely the image of the Schwartz measure $\Phi \cdot (dg \times |t_1| dt_1)$ described in the definition of (5.12), giving a natural meaning to that definition.

5.4. Geometric quantization for type *T*

5.4.1. In the cases (5.5) of "type T," the analog of Theorem 5.3.5 does not directly hold. In turns out, however, that there is a similar interpretation of transfer operators, if we pass from J_X, the group scheme of regular centralizers in PGL_2, to $\tilde{J_X}$ = the group scheme of regular centralizers in GL_2. It lives over $\tilde{\mathfrak{c}_X}^* := \mathfrak{gl}_2^* /\!\!/ \mathrm{GL}_2$.

If we write $\mathrm{GL}_2 = (\mathrm{SL}_2 \times \mathbb{G}_m)/\mu_2$, use coordinates (t_0, t_1, ξ), as before, for J_{SL_2}, and coordinates (z, τ) for $T^*\mathbb{G}_m = \mathbb{G}_m \times \mathfrak{g}_m^*$, we obtain

$$\tilde{J_X} = \mathrm{Spec}\, F\big[t_0, t_1, \xi, z^{\pm 1}, \tau\big]^{\mu_2}/(t_0^2 - \xi t_1^2 - 1),$$

where $-1 \in \mu_2$ acts by $(t_0, t_1, \xi, z, \tau) \mapsto (-t_0, -t_1, \xi, -z, \tau)$. The map to $J_X = J_{\mathrm{PGL}_2}$ is then obtained by symplectic reduction modulo $\mathbb{G}_m$, and the symplectic form on $\tilde{J_X}$ reads

$$\omega = dt_0 \wedge d(t_1^{-1}) + d^\times z \wedge d\tau,$$

where the notation is $d^\times z := d \log z = \frac{dz}{z}$.

5.4.2. To motivate the passage to $\tilde{J_X}$, we should first look at the simple case $X = \mathbb{G}_m \backslash \mathrm{PGL}_2$. This space is a quotient of $\tilde{X} = \mathbb{G}_m \backslash \mathrm{GL}_2$, where $\mathbb{G}_m$ is embedded as the general linear group of a 1-dimensional subspace, and one can think of $\mathcal{S}((X \times X)/G)$ as the $\mathbb{G}_m$-coinvariants of the space $\mathcal{S}((\tilde{X} \times \tilde{X})/\tilde{G})$, where $\mathbb{G}_m$ stands for the center of $\tilde{G} = \mathrm{GL}_2$,

$$\mathcal{S}\big((X \times X)/G\big) = \mathcal{S}\big((\tilde{X} \times \tilde{X})/\tilde{G}\big)_{\mathbb{G}_m}.$$

In this setting, one can easily study a transfer operator

$$\mathcal{T} : \mathcal{S}(\tilde{\mathfrak{Y}}) \to \mathcal{S}(\tilde{\mathfrak{X}}),$$

where $\tilde{\mathfrak{X}} = (\tilde{X} \times \tilde{X})/\tilde{G}$, and $\tilde{\mathfrak{Y}}$ is the "Kuznetsov quotient stack" of GL_2, via the "unfolding" method. The "unfolding" method [77, §9.5] gives rise to an explicit morphism of Schwartz half-densities

$$\mathcal{U} : \mathcal{D}(\tilde{Y}) \to \mathcal{D}(\tilde{X}) \tag{5.21}$$

(where $\tilde{Y}$ denotes the Whittaker model of GL_2), which extends to an L^2-isometry.

We can repeat the earlier constructions to identify the spaces of test measures above with spaces of "vertical," resp. "horizontal," half-densities on $\tilde{J_X}^\circ$ = the complement of $t_1 = 0$, valued in a line bundle L_X (with connection). Here, the corresponding foliations are determined by the following:

Lemma. *The invariant-theoretic quotient* $(\tilde{X} \times \tilde{X}) /\!\!/ \tilde{G} = \mathbb{G}_m \backslash \mathrm{GL}_2 /\!\!/ \mathbb{G}_m$ *is a two-dimensional affine space. One can choose the coordinates* (x, y) *so that the resulting map*

$$\tilde{J_X} \to \tilde{J_X} \bullet T^*\tilde{X} \to T^*\tilde{X} \times_{\tilde{\mathfrak{g}}^*} T^*\tilde{X} \to (\tilde{X} \times \tilde{X}) /\!\!/ \tilde{G}$$

(the first arrow again by choosing a section of $T^*\tilde{X} \to \tilde{\mathfrak{c}_X}^*$*) is given by*

$$(t_0, t_1, \xi, z, \tau) \mapsto \big(x = z^{-1}(t_0 - \tau t_1), y = z(t_0 + \tau t_1)\big). \tag{5.22}$$

The invariant-theoretic quotient for the Kuznetsov formula of GL_2*, composed with the analogous map from* $\tilde{J_X}$*,*

$$\tilde{J_X} \to \tilde{J_X} \bullet T^*\tilde{Y} \to N \backslash \tilde{G} /\!\!/ N$$

is the map that remembers all even-order monomials in the coordinates t_1 *and* $z^{\pm 1}$*.*

Note that the map $\tilde{J_X} \to (\tilde{X} \times \tilde{X}) /\!\!/ \tilde{G}$ is smooth, when restricted to $\tilde{J_X}^\circ$ = the complement of the divisor $t_1 = 0$. The "vertical" foliation $\mathscr{F}_{\mathrm{ver}}$ is defined as the set of fibers of this map. The "horizontal" foliation $\mathscr{F}_{\mathrm{hor}}$ on $\tilde{J_X}^\circ$ is defined by the level sets of $(t_1 z, z^2)$.

Remark. The passage to $(X \times X) /\!\!/ G = \mathbb{A}^1$ is given by the coordinate

$$(x, y) \mapsto c := xy = t_0^2 - \tau^2 t_1^2 = (\xi - \tau^2) t_1^2 + 1. \tag{5.23}$$

5.4.3. Still in the case of $\tilde{X} = \mathbb{G}_m \backslash \mathrm{GL}_2$, defining line bundles L_X, L_Y over $\tilde{J_X}^\circ$ exactly as before, we can repeat the constructions of the maps (5.14) and (5.19) for $\tilde{G}$, to identify test measures as spaces

$$\mathcal{D}(\tilde{\mathfrak{X}}) \hookrightarrow D_{\mathrm{hor}}(\tilde{J_X}^\circ, L_X),$$
$$\mathcal{D}(\tilde{\mathfrak{Y}}) \hookrightarrow D_{\mathrm{ver}}(\tilde{J_X}^\circ, L_Y)$$

of "horizontal," resp. "vertical," half-densities valued in those line bundles.

With the appropriate identification $L_X \simeq L_Y$ over $\tilde{J_X}^\circ$ (which we will present for the general case in § 5.4.4), we can now descend the unfolding map (5.21), applied to $\mathcal{D}(\tilde{Y}) \otimes \mathcal{D}(\tilde{Y}^-)$, to coinvariants for the diagonal action of $\tilde{G}$, obtaining a transfer operator, which can be explicitly described, along the lines of **[69, THEOREM 5.4]**:

Theorem. *The transfer operator*

$$\tilde{\mathcal{T}} : \mathcal{D}(\tilde{\mathfrak{Y}}) \xrightarrow{\sim} \mathcal{D}(\tilde{\mathfrak{X}})$$

is the operator of integration along the leaves of the vertical foliation:

$$D_{\mathrm{ver}}(\tilde{J_X}^\circ, L_X) \dashrightarrow D_{\mathrm{hor}}(\tilde{J_X}^\circ, L_X).$$

The transfer operator

$$\mathcal{T} : \mathcal{D}(\mathfrak{Y}) \to \mathcal{D}(\mathfrak{X})$$

is the descent of $\tilde{\mathcal{T}}$ *to* $\mathbb{G}_m$*-coinvariants.*

5.4.4. Let, now, X be a general space from the list (5.5). The idea is to generalize the statement of Theorem 5.4.3 for the transfer operator $\mathcal{T}$, *even though the space X does not, in general, admit a cover such as $\tilde{X}$.* (Such a cover exists, more generally, for $X = \mathrm{GL}_n \setminus \mathrm{PGL}_{n+1}$, and can be used to motivate some of the definitions that follow.)

An important feature of the general case is that we will extend the map $J_X \to (X \times X) /\!\!/ G \simeq \mathbb{A}^1$ to a map $\tilde{J}_X \to \mathbb{A}^1$, given by the same coordinate c, (5.23), as in the case of $\mathbb{G}_m \setminus \mathrm{GL}_2$, and will define the vertical and horizontal foliations on $\tilde{J}_X{}^\circ$ as in § 5.4.2, e.g., the vertical foliation consists of level sets of the pair of functions (x, y) of (5.22). We define the complex line bundle L_X over $\mathfrak{c}_X^*$ as in § 5.3.6, and *extend it to a line bundle L_X on $\tilde{\mathfrak{c}_X}{}^*$*, by pullback along the map

$$\tilde{\mathfrak{c}_X}{}^* \ni (\xi, \tau) \mapsto \xi - \tau^2 \in \mathfrak{c}_X^*. \tag{5.24}$$

By pullback from $\tilde{\mathfrak{c}_X}{}^*$, this also becomes a line bundle over $\tilde{J}_X$. We endow it with the same trivialization (5.15) as before.

Roughly speaking, now, if we fix a Haar measure on $F^\times$, the map (5.24) allows us to pull back elements of $D_{\mathrm{hor}}(J_X^\circ, L_X)$ to $\mathbb{G}_m$-invariant elements of $D_{\mathrm{hor}}(\tilde{J}_X{}^\circ, L_X)$, thus obtaining maps

$$\mathcal{D}(X \times X) \to D_{\mathrm{hor}}(J_X^\circ, L_X) \to D_{\mathrm{hor}}(\tilde{J}_X{}^\circ, L_X)^{\mathbb{G}_m}. \tag{5.25}$$

(Fixing a Haar measure on $F^\times$ allows the switch from the coinvariants of Theorem 5.4.3 to invariants.) However, there is one more twist, which is not seen in the cases of $X = \mathrm{GL}_n \setminus \mathrm{PGL}_{n+1}$, but is needed in the general case. Namely, instead of $\mathbb{G}_m$-invariants, one needs *twisted* invariants with respect to a character of $\mathbb{G}_m$ (that is, of $F^\times$).

5.4.5. To introduce this final piece of the puzzle, we recall from **[74]** that the space $X \times X$ has two closed G-orbits of codimension larger than 1: the diagonal X^{diag} (whose codimension we keep denoting by d), and a second closed G-orbit, whose codimension we will denote by d'. We define a character of $\mathbb{G}_m$ by $\chi_{d'} : z \mapsto |z|^{-\frac{d'}{2}+1}$. We will then understand the space of test densities for $(X \times X)/G$ as a subspace of the $(\mathbb{G}_m, \chi_{d'})$-equivariant elements of $D_{\mathrm{hor}}(\tilde{J}_X{}^\circ, L_X)$, by composing (5.25) with multiplication by

$$|y|^{-\frac{d'}{2}+1} = |z(t_0 + \tau t_1)|^{-\frac{d'}{2}+1},$$

obtaining a map

$$\mathcal{D}(X \times X) \to D_{\mathrm{hor}}(\tilde{J}_X{}^\circ, L_X)^{(\mathbb{G}_m, \chi_{d'})}. \tag{5.26}$$

The image $\mathcal{D}(\mathfrak{X})$ is identified, as before, with the space $\mathcal{S}(\mathfrak{X})^{\mathrm{st}}$ of stable test measures, if we fix an invariant measure on $X \times X$. To summarize, the map $\mathcal{S}(\mathfrak{X})^{\mathrm{st}} \to \mathcal{D}(\mathfrak{X})$ takes a measure $f(c)$ to

$$f(c)(dc)^{-\frac{1}{2}} |y|^{-\frac{d'}{2}+1} |t_1|^{-\frac{d}{2}+1} (d^\times z)^{\frac{1}{2}}, \tag{5.27}$$

in the trivialization (5.15), where y, c are given by (5.22) and (5.23).

Remark. The most convincing argument for the relevance of the character $\chi_{d'}$ is **[74, PROPOSITION 6.1.5]**, describing orbital integrals in the neighborhood of $c = xy = 0$ in terms of

$\mathbb{G}_m$-orbital integrals on the (x, y)-plane, twisted by this character. However, a more conceptual understanding of it would be highly desirable.

5.4.6. We also replace half-densities for the Kuznetsov formula of $G' = \mathrm{PGL}_2$ by half-densities for GL_2 with central character $\chi_{d'}$, namely, we define an embedding

$$\mathcal{D}(\mathcal{Y}) \hookrightarrow D_{\mathrm{ver}}(\tilde{J}_X^{\circ}, L_X)^{(\mathbb{G}_m, \chi_{d'})}$$

simply by multiplying the embedding $\mathcal{D}(\mathcal{Y}) \hookrightarrow D_{\mathrm{ver}}(J_X^{\circ}, L_Y)$ of § 5.3.9 by the factor $|z|^{-(\frac{d'}{2}-1)}(d^{\times}z)^{\frac{1}{2}}$, and use the same trivialization (5.18) to identify $L_Y \simeq \underline{\mathbb{C}} \simeq L_X$.

The main result **[74, THEOREM 1.3.1]** for the transfer operator in this case can now be formulated as follows:

Theorem. *There is an injective operator*

$$\mathcal{T} : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})$$

given by integration along the leaves of the vertical foliation:

$$D_{\mathrm{ver}}(\tilde{J}_X^{\circ}, L_X) \dashrightarrow D_{\mathrm{hor}}(\tilde{J}_X^{\circ}, L_X).$$

Its inverse $\mathcal{T}^{-1}$, valued in an enlargement $\mathcal{D}^{-}_{L_X}(\mathcal{Y}) \supset \mathcal{D}(\mathcal{Y})$ described in **[74, §1.3]**, *is given by integration along the leaves of the horizontal foliation:*

$$D_{\mathrm{hor}}(\tilde{J}_X^{\circ}, L_X) \dashrightarrow D_{\mathrm{ver}}(\tilde{J}_X^{\circ}, L_X),$$

5.4.7. I finish this section with a brief discussion of a case where a section $\mathrm{c}_X^* \to T^*X$ does not exist. Let $X = T\backslash \mathrm{PGL}_2$, where T is a nonsplit torus, splitting over a quadratic extension E/F. In this case, the transfer operator

$$\mathcal{T} : \mathcal{S}(\mathcal{Y}) \to \mathcal{S}(\mathcal{X})$$

was computed in **[69]**, and can be described as follows:

Instead of defining $\tilde{J}_X$ to be the group scheme of regular centralizers in GL_2, define it to be the $\mathrm{Gal}(E/F)$-twist of that, determined by the automorphism $(t_0, t_1, \xi, z, \tau) \mapsto (t_0, t_1, \xi, z^{-1}, -\tau)$; that is, $\tilde{J}_X$ will be isomorphic to the group scheme of regular centralizers in the quasisplit unitary group U_2. The transfer operator, now, it obtained as in Theorem 5.4.3, by descending the operator of integration along the leaves of the corresponding "vertical" foliation on $\tilde{J}_X$ to U_1-coinvariants.

Both the Schwartz space and the descent to U_1-coinvariants, here, need to be understood in a sophisticated, "stacky" way. Namely, the full space $\mathcal{S}(\mathcal{X})$ includes a "pure inner form" as in (2.2),

$$\mathcal{S}(\mathcal{X}) = \mathcal{S}(X \times X)_G \oplus \mathcal{S}(X^{\alpha} \times X^{\alpha})_{G^{\alpha}},$$

where $X^{\alpha} \simeq T\backslash G^{\alpha}$, with $G^{\alpha} = PD^{\times}$, the projective multiplicative group of the quaternion division algebra. Similarly, the U_1-coinvariants of $D_{\mathrm{ver}}(\tilde{J}_X^{\circ}, L_X)$, $D_{\mathrm{hor}}(\tilde{J}_X^{\circ}, L_X)$ need to be understood in a stacky way. Explicitly, recall that in the split case the space $D_{\mathrm{hor}}(\tilde{J}_X^{\circ}, L_X)$ was the space of half-densities on the (x, y)-plane (in coordinates (5.22)), with $\mathbb{G}_m$ acting as $z \cdot (x, y) = (z^{-1}x, zy)$. In the nonsplit case, the (x, y)-plane becomes

the space $V = \mathrm{Res}_{E/F}\,\mathbb{G}_a$, and instead of U_1-coinvariants of the space $\mathcal{D}(V(F))$, one needs to consider the direct sum

$$\mathcal{D}\big(V(F)\big)_{U_1} \oplus \mathcal{D}\big(V'(F)\big)_{U_1},$$

where V' is the twist of V by the nontrivial U_1-torsor. The same interpretation is needed for "stacky" U_1-coinvariants of the space $D_{\mathrm{ver}}(\tilde{J_X}^{\circ}, L_X)$ (with the coordinates $a = zt_1$, $b = z^{-1}t_1$ for the leaves of the horizontal foliation interchanged by the Galois action), and the operator of "integration along vertical half-densities"—essentially, a Fourier transform from the Galois-twisted (x, y)-plane to the Galois-twisted (a^{-1}, b^{-1})-plane—naturally descends to give the transfer operator $\mathcal{T}$ of [69, THEOREM 5.1].

6. PROBLEMS FOR THE NEAR FUTURE

6.1. The relative Langlands conjectures

The relative Langlands conjectures presented in Sections 2.3.1–2.3.2 do not have the precision of the local conjectures of Gan–Gross–Prasad [27], or the global conjectures of Ichino–Ikeda [36]. Moreover, an extension of those conjectures to Arthur packets not appearing in the L^2-decomposition is required, as in [26].

It is therefore an important problem to refine the existing conjectures. It is also a fascinating one: as always, finding a way to blend several known cases into a uniform theory can lead to new insights about the nature of the problems. The geometric relative Langlands conjectures proposed in joint work with Ben-Zvi and Venkatesh can probably assist in this direction, providing a geometric spectral answer to automorphic problems, which can then be translated to number theory by decategorifying.

6.2. Transfer operators in higher rank

The most important problem "beyond endoscopy," in my view, is to understand transfer operators in higher rank, and for morphisms of L-groups ${}^L G_X \to {}^L G_Y$ that are not isomorphisms. Regarding the latter, despite the traditional emphasis on the Arthur–Selberg trace formula, it might be better, as first observed by Sarnak [79], to try to compare Kuznetsov formulas, which, according to the results of [74], seem to be the "base cases" for every comparison. This can be "explained" by the simple structure (5.6) of the Kuznetsov cotangent stack.

If the ideas discussed in this paper have any merit, understanding transfer operators in terms of "quantization" would involve several steps, including the following:

6.2.1. Develop a theory of "geometric quantization" for derived symplectic stacks, whose output includes the Schwartz spaces of stacks defined in [71]. The cases presented here, and in particular the construction of the maps (5.14) and (5.26), could provide some hints on how to do that, but the various twists involved need to be better understood.

6.2.2. Obtain a better understanding of the structure of coisotropic Hamiltonian spaces and the cotangent stacks appearing in the relative trace formula. The diagram (5.3),

arising from the work of Knop, which was interpreted as a "birational" description of $T^*\mathcal{X}$, does not capture the difference between $T^*\mathcal{X}$, and the Kuznetsov cotangent stack with the same L-group. This difference seems to be significant for the structure of transfer operators and for spaces of test measures. For example, the enlarged spaces $\mathcal{S}^-_{L_X}(\mathcal{Y})$ of test measures for the Kuznetsov formula in Theorems 5.3.5 and 5.4.3 should be seen as quantizations of $T^*\mathcal{X}$, which is strictly larger than the Kuznetsov stack $T^*\mathcal{Y}$. This difference can also explain "Galois twists" of transfer operators, as in the example of § 5.4.7, where $T^*\mathcal{X}$ failed to admit an analog of a Kostant section.

A baby case of the idea that embeddings of Hamiltonian spaces correspond to enlarged Schwartz spaces is Iwasawa–Tate theory, where the embedding $T^*\mathbb{G}_m \hookrightarrow T^*\mathbb{A}^1$ corresponds to the enlargement $\mathcal{S}(F^\times) \hookrightarrow \mathcal{S}(F)$. This can be generalized to toric stacks, as described in [63, §5.2]. For a torus A, a collection of coweights $\mu : \mathbb{G}_m^r \to A$ defines a stack $\overline{A}_\mu := \mathbb{A}^r / \ker(\mu)$ with an action of A. The *dual Hamiltonian space* of such a stack is defined as the symplectic $\check{A}$-vector space $\check{M}$ with weights $\{\pm\mu\}$.

To give an example of the descriptions of Hamiltonian spaces envisioned here, in an ongoing joint work with Ben-Zvi and Venkatesh we take a step beyond Knop's theory, modeling the most regular locus of $M = T^*X$, up to codimension 2, on the analog of a "toric stack" for the group scheme J_X. Our result confirms an observation of V. Lafforgue, shared in private communication several years ago. For example, for spherical varieties X with $\check{G}_X = \check{G}$ and $\check{A}_X = \check{A}$, our description uses the toric stack $\overline{A_X}$ corresponding to the dual Hamiltonian space $\check{M} = V_X$ of the spherical variety, described in terms of its colors in § 3.1.2.

Theorem. *In the setting above, there is an action of W_X on the toric cotangent stack $T^*\overline{A_X}$, and an open dense subset $\mathfrak{c}_X^{*\prime} \subset \mathfrak{c}_X^*$, whose complement has codimension ≥ 2, such that the restriction $M' \subset M$ to the image of $\mathfrak{c}_X^{*\prime}$ under a Kostant section $\kappa : \mathfrak{c}_X^* \to \mathfrak{g}^*$ admits a J_X-equivariant symplectomorphism*

$$M' \simeq (\mathrm{Res}_{\mathfrak{a}_X^*/\mathfrak{c}_X^*} T^*\overline{A_X})^{W_X} \times_{\mathfrak{c}_X^*} \mathfrak{c}_X^{*\prime}.$$

6.2.3. Use cotangent spaces to understand transfer operators. As we saw in the discussion of cases of type T in § 5.4, a "naive" change of geometric quantization on J_X did not give the correct transfer operators; instead, one has to pass to the group scheme $\tilde{J}_X$ associated with the root datum of GL_2, producing a 2-dimensional Fourier transform.

Ongoing joint work with C. Wan, comparing the Kuznetsov to the Arthur–Selberg trace formula for GL_n, suggests that, in higher-rank cases, the transfer operator for a comparison to a Kuznetsov quotient with the same dual group might be given by an r-dimensional integration, where r is, roughly, half the dimension of the nonzero weight spaces of the representation V_X (§ 3.1.1) of the dual group. In view of the discussion of § 6.2.2, this seems to be closely related to the structure of $T^*\mathcal{X}$. For example, in the setting of Theorem 6.2.2 (such as in the case of Gan–Gross–Prasad periods), one could speculate that the transfer operator to the Kuznetsov formula is somehow a "descent" of a Fourier transform in $\frac{\dim \check{M}}{2}$-dimensions. In this "dream," the following three objects would be closely related: the L-value L_X associated to a spherical variety (encoded in a dual Hamiltonian space $\check{M}$), the fine structure of

the Hamiltonian space $M = T^*X$ (whose quantization is the space of test measures on X), and the transfer operator between the relative trace formula of X and the Kuznetsov formula with the same dual group.

6.3. Poisson summation formula

Understanding the local transfer operators should be followed by a global comparison of trace formulas. For example, comparing the relative trace formula for any spherical variety X with the Kuznetsov formula with the same dual group should amount to commutativity of the diagram

$$\begin{array}{ccc} \mathcal{S}^{-}_{L_X}(\mathfrak{Y}(\mathbb{A})) & \xrightarrow{\mathcal{T}} & \mathcal{S}(\mathfrak{X}(\mathbb{A}))^{\mathrm{st}}, \\ & {\scriptstyle \mathrm{RTF}_{\mathfrak{Y}}} \searrow \quad \swarrow {\scriptstyle \mathrm{RTF}_{\mathfrak{X}}} & \\ & \mathbb{C} & \end{array} \tag{6.1}$$

where $\mathcal{S}^{-}_{L_X}(\mathfrak{Y}(\mathbb{A}))$ is a suitable enlarged space of test measures for the Kuznetsov formula, related to the L-value L_X of X.

Having a formula for the transfer operator in terms of Fourier transforms (as in Theorems 5.3.5 and 5.4.3) gives hope of employing the Poisson summation formula to establish commutativity of (6.1). However, this is far from straightforward, as the spaces of stable test measures are nonstandard. In Altuğ's work [2], the approximate functional equation was used for the trace formula of $X = \mathrm{GL}_2$, obtaining an expression similar to the Kuznetsov formula (in particular, containing Kloosterman sums), but not quite equal to it.

A different approach was introduced in [73], for the case $X = T\backslash \mathrm{PGL}_2$. It is based on the idea of *deforming* spaces of test measures and transfer operators in analytic families depending on a parameter s (which moves the point of evaluation of L_X), so that in some domain for s the Poisson summation formula is valid. It is likely that this method can be applied more generally, but it requires a better understanding of the idea of deforming spaces of test measures (orbital integrals).

6.4. Hankel transforms

In the recent literature on automorphic forms, the term "Hankel transforms" has been used to describe two distinct conjectural notions:

- The nonlinear Fourier transforms $\mathcal{S}_\rho(G(F)) \to \mathcal{S}_{\rho^*}(G(F))$ between nonstandard spaces of Schwartz functions (or measures) on a reductive group over a local field, adapted to a representation ρ of its dual group, and its dual. These spaces and operators would generalize Fourier transform on the space $\mathcal{S}(\mathrm{Mat}_n(F))$ of Godement–Jacquet theory, for the case $\rho =$ the standard representation of $\check{G} = \mathrm{GL}_n$, and would similarly give rise to the local functional equation for the L-functions associated to ρ. They were introduced by Braverman and Kazhdan [15,17], and advanced in the work of Ngô [63].

- The descent of such transforms to spaces of test measures for the Arthur–Selberg trace formula, or for the Kuznetsov formula. In the latter case, those would be operators
$$\mathcal{H}_\rho : \mathcal{S}_\rho(\mathcal{Y}) \to \mathcal{S}_{\rho^*}(\mathcal{Y})$$
between enlarged spaces of measures for the Kuznetsov formula, such as those encountered in Theorems 5.3.5 and 5.4.3.

The two notions are closely related, but it is the latter that we would like to focus on here. It is natural to ask the question of whether one can describe $\mathcal{H}_\rho$ explicitly, and prove a Poisson summation formula, globally, in the sense that the diagram

$$\begin{array}{ccc} \mathcal{S}_\rho(\mathcal{Y}(\mathbb{A})) & \xrightarrow{\mathcal{H}_\rho} & \mathcal{S}_{\rho^*}(\mathcal{Y}(\mathbb{A})) \\ & {\scriptstyle \mathrm{RTF}_{\mathcal{Y}}} \searrow \quad \swarrow {\scriptstyle \mathrm{RTF}_{\mathcal{Y}}} & \\ & \mathbb{C} & \end{array} \tag{6.2}$$

should commute. This would lead to an independent proof of the functional equation of the pertinent L-functions.

Such Hankel transforms have been described by Jacquet [40] for $\rho =$ the standard representation of GL_n (the paper [35] is closely related), and by me [76] for $\rho =$ the symmetric square representation of GL_2. It would be interesting to examine if these formulas admit an interpretation in terms of quantization, like the transfer operators in this paper.

It seems counter to the strategy of the Langlands program to seek such a proof of the functional equation, independent of functoriality. On the other hand, the similarity between diagrams (6.1) and (6.2) is enticing. More fundamentally, trace formulas with nonstandard test functions, such as those in Langlands' original "beyond endoscopy" proposal, or the Kuznetsov formula appearing in (6.1), have the L-functions embedded into them. Obtaining the spectral decomposition of those formulas will likely require more than "brute force" analytic number theory, and a Poisson summation formula of the form (6.2) could help resolve the problem. This idea was successfully employed in [73] for a new proof of Waldspurger's formula for toric periods in PGL_2 via a nonstandard comparison of the form (6.1). Therefore, it might be that, in the "beyond endoscopy" program, functoriality and the functional equation of L-functions should be studied hand-in-hand.

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