CATEGORIFICATION AND APPLICATIONS

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ABSTRACT

We survey some new developments for categorification of quantum groups and their applications in representation theory.

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1. INTRODUCTION

Categorification, in the broad sense, refers to the realization of a mathematical object as the Grothendieck group of certain category. Lie theory is a domain with many interesting examples of categorifications. An important family among them consists of categorification of quantum groups and their representations.

Let $\mathfrak{g}$ be a Kac–Moody Lie algebra, $\mathbf{U}_v(\mathfrak{g})$ be its quantized enveloping algebra, and $\mathbf{U}_v^+$ be the positive part. The first categorification of $\mathbf{U}_v^+$ was constructed by Lusztig [41, 42] using perverse sheaves on the moduli stack of quiver representations. It reveals some deep structures of $\mathbf{U}_v^+$, including the existence of a remarkable basis called the canonical basis. It is defined as the classes of intersection complexes in the Grothendieck group. Kashiwara [29] also gave a construction of this basis using a different method.

A (naive) categorical $\mathfrak{g}$-action on an exact category $\mathcal{C}$ consists of pairs of exact adjoint endofunctors $\mathcal{E}_i, \mathcal{F}_i$ on $\mathcal{C}$ such that the Grothendieck group of $\mathcal{C}$ is a $\mathfrak{g}$-representation with the Chevalley generators in $\mathfrak{g}$ acting by the operators induced by $\mathcal{E}_i$, $\mathcal{F}_i$. A famous example is the categorical action of the affine Lie algebra $\widehat{\mathfrak{sl}}_p$ on the category of representations over a field of characteristic p of the symmetric groups of all ranks. The functors $\mathcal{E}_i$, $\mathcal{F}_i$ are given by some i-restriction and i-induction functors. Categorifical actions also had remarkable applications to representation theory of affine Hecke algebras and their cyclotomic quotients. Ariki [1] proved that the category of modules over affine Hecke algebras of type A at an eth root of unity categorifies the positive half of $\widehat{\mathfrak{sl}}_e$, and the category of modules over cyclotomic Hecke algebras categorifies an integrable irreducible $\widehat{\mathfrak{sl}}_e$-representation, with the classes of simple modules corresponding to the dual canonical basis. This result confirms a conjecture by Lascoux–Leclerc–Thibon [39] and provides character formulae for simple modules over cyclotomic Hecke algebras.

In 2008, a seminal work of Chuang–Rouquier [17] brought some new perspectives on categorifications. They introduced an enhanced notion of $\mathfrak{sl}_2$-categorical action, whose input requires not only exact adjoint endofunctors $\mathcal{E}, \mathcal{F}$ as above, but also some natural transformations $x \in \mathrm{End}(\mathcal{E})$, $\tau \in \mathrm{End}(\mathcal{E}^2)$ satisfying the defining relations for nil-affine Hecke algebras. They showed that many previously known examples of categorifications in Lie theory can be enhanced. The enhancement has two important advantages among others. First, it guarantees that the categorification of a simple integrable $\mathfrak{sl}_2$-representation is unique. Second, it provides derived self-equivalences between different blocks, categorifying the Weyl group action on the underlying representation. As an application, Broué's abelian defect group conjecture for symmetric groups was proved.

To extend this powerful theory from $\mathfrak{sl}_2$ to arbitrary Kac–Moody Lie algebra $\mathfrak{g}$, one needs correct substitutions for the nil-affine Hecke algebras. This is provided by a new family of $\mathbb{Z}$-graded algebras introduced by Khovanov–Lauda [33] and independently by Rouquier [53], called quiver Hecke algebras (also known as KLR algebras). The category of graded projective modules over these algebras gives a categorification of $\mathbf{U}_v^+$ in purely algebraic terms. Moreover, by work of Rouquier [55] and Varagnolo–Vasserot [65], it is equivalent to Lusztig's categorification. Integrable simple $\mathfrak{g}$-representations also admit categorifications by representations of cyclotomic quotients of quiver Hecke algebras by Kang–Kashiwara [25]. Moreover, Rouquier [53] proved the unicity of categorifications for these simple representations.

Quiver Hecke algebras have captured a tremendous amount of interest in the last decade and many important progress have been made. We do not intend to give a complete survey of the subject. Instead, we will focus on some new equivalences of categorifications and some recent applications to representation theory. Here is an outline of this report.

In Section 2 we review Lusztig's categorification of $\mathbf{U}_v^+$, quiver Hecke algebra, its realization as extension algebra of some ℓ-adic complexes, the theory of standard modules for quiver Hecke algebras and their relations to PBW basis. We also mention the monoidal categorification of the quantum cluster algebra structure on the quantum coordinate ring $\mathbf{A}_v^+$ by Kang–Kashiwara–Kim–Oh [27].

In Section 3 we discuss categorifications of quantized loop algebras. By the theory of K-theoretical Hall algebras, there is a natural categorification of the positive part of quantized loop algebras in terms of coherent sheaves on the cotangent dg-stacks of quiver representations. For a quiver of finite type, since the loop algebra is part of the corresponding affine Lie algebra, it can also be categorified using representations of quiver Hecke algebras. It is natural to ask whether these two categorifications are equivalent. An equivalence of this kind was given for $\mathfrak{sl}_2$ in [61]. It also gives an interesting comparison between two monoidal categorifications of the quantum cluster algebra structure on a quantum unipotent coordinate ring for $\widehat{\mathfrak{sl}}_2$, one in terms of quiver Hecke algebras as mentioned above and the other in terms of perverse coherent sheaves on affine Grassmannians, constructed by Cautis–Williams [15].

In Section 4 we discuss some recent applications of categorical actions to representation theory, including those of rational double affine Hecke algebras and those of finite reductive groups of classical types. We also discuss how to use categorical actions to construct representations of current algebras on the center of the underlying categories. As an application, we obtain an isomorphism between the center of cyclotomic quiver Hecke algebras and the singular cohomology of Nakajima quiver varieties in finite types. In a parallel context, we get an explicit computation of the cohomology of Gieseker moduli spaces.

2. QUIVER HECKE ALGEBRAS

2.1. Notation

For an exact (resp. triangulated) category $\mathcal{C}$, its Grothendieck group $[\mathcal{C}]$ is the quotient of the free $\mathbb{Z}$-module spanned by the isomorphism classes of objects in $\mathcal{C}$ by the relations $[M] = [M'] + [M'']$ whenever there is a short exact sequence $M' \to M \to M''$(resp. distinguished triangle). Abelian categories are naturally exact categories. Additive categories can be viewed as exact categories with short exact sequences being the split ones.

An exact category $\mathcal{C}$ is graded if it is equipped with an exact autoequivalence $\langle 1 \rangle : \mathcal{C} \to \mathcal{C}$. For such a category, the Grothendieck group $[\mathcal{C}]$ is a $\mathbb{Z}[v^{\pm 1}]$-module with $v[M] = [M\langle 1 \rangle]$. Here v is a formal variable. In particular, for an object $M \in \mathcal{C}$ and

$a(v) = \sum_r a_r v^r \in \mathbb{N}[v^{\pm 1}]$, we write $a(v)M = \oplus_r M\langle r\rangle^{\oplus a_r}$. We set

$$\mathrm{Hom}^{\bullet}_{\mathcal{C}}(-,-) = \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(-,-\langle r\rangle).$$

A standard example of graded category is the category of graded vector spaces over a field $\mathbb{k}$ with $(M\langle 1\rangle)_n = M_{n+1}$. Its Grothendieck group is isomorphic to $\mathbb{Z}[v^{\pm 1}]$, with the class of $M = \bigoplus_{n\in\mathbb{Z}} M_n$ mapping to its graded dimension $\mathrm{gdim}(M) = \sum_{n\in\mathbb{Z}}(\dim_{\mathbb{k}} M_n)v^n$.

An exact category $\mathcal{C}$ is monoidal if it is equipped with an exact bifunctor $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ satisfying certain associativity constraints. Such a bifunctor yields a ring structure on the abelian group $[\mathcal{C}]$. If $\mathcal{C}$ is graded monoidal, then $[\mathcal{C}]$ becomes a $\mathbb{Z}[v^{\pm 1}]$-algebra.

Let $\mathbb{k}$ be a field. For a graded $\mathbb{k}$-algebra A, let $\mathrm{Mod}(A)$ be the category of graded A-modules. Let $\mathrm{mod}(A)$, $\mathrm{proj}(A)$, $\mathrm{fmod}(A)$ be respectively the full subcategories consisting of finitely generated graded A-modules, finitely generated graded projective A-modules, and graded A-modules which are finite dimensional over $\mathbb{k}$.

2.2. The quantized enveloping algebra

Let I be a finite set. Fix a Cartan datum with a symmetric generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$, a set of simple roots $\{\alpha_i \mid i\in I\}$, a weight lattice P, and a symmetric bilinear form $P\times P\to\mathbb{Q}$, $(\lambda,\mu)\mapsto\lambda\cdot\mu$ such that $\alpha_i\cdot\alpha_j = a_{ij}$ and $\omega_i\cdot\alpha_j = \delta_{ij}$, where $\{\omega_i \mid i\in I\}$ are the fundamental weights. Let $\mathfrak{g} = \mathfrak{g}_A$ be the associated Kac–Moody Lie algebra. Let $Q = \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ be the root lattice. Set $Q^+ = \bigoplus_{i\in I}\mathbb{N}\alpha_i$ and $P^+ = \bigoplus_{i\in I}\mathbb{N}\omega_i$. Let Φ be the set of roots, and Φ^+ the set of positive roots. Let $\mathfrak{n}\subset\mathfrak{g}$ be the Lie subalgebra spanned by positive root spaces.

For $n\in\mathbb{N}$, $1\leqslant l\leqslant n$, define the following quantum integers in $\mathbb{Z}[v^{\pm 1}]$:

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}},\quad [n]^! = \prod_{r=1}^{n}[r],\quad \begin{bmatrix} n \\ l \end{bmatrix} = \frac{[n]^!}{[l]^![n-l]^!}.$$

The positive part ${}'\mathbf{U}_v^+$ of the quantized enveloping algebra is the unital $\mathbb{Q}(v)$-algebra generated by E_i for $i\in I$, subject to defining relations

$$\sum_{s+r=1-a_{ij}} (-1)^s E_i^{(s)} E_j E_i^{(r)} = 0,\quad \text{for } i\neq j\in I,$$

where $E_i^{(n)} = E_i^n/[n]^!$. It is a Q^+-graded algebra with $\deg(E_i) = \alpha_i$. There is a coproduct

$$\Delta : {}'\mathbf{U}_v^+ \to {}'\mathbf{U}_v^+ \otimes {}'\mathbf{U}_v^+,\quad E_i\mapsto E_i\otimes 1 + 1\otimes E_i,\quad \forall i\in I,$$

such that ${}'\mathbf{U}_v^+$ is a twisted[1] bialgebra. There is a unique nondegenerate symmetric bilinear form $(\cdot,\cdot)_v$ on ${}'\mathbf{U}_v^+$ determined by $(1,1)_v = 1$, $(E_i,E_j)_v = \delta_{i,j}/(1-v^2)$, and $(ab,c)_v = (a\otimes b,\Delta(c))_v$ for $a,b,c\in{}'\mathbf{U}_v^+$, where $(a\otimes b, c\otimes d)_v = (a,c)_v(b,d)_v$ on ${}'\mathbf{U}_v^+\otimes{}'\mathbf{U}_v^+$.

1 Here and below, twisted means the multiplication on ${}'\mathbf{U}_v^+\otimes{}'\mathbf{U}_v^+$ is $(a\otimes b)(c\otimes d) = v^{-\deg(b)\cdot\deg(c)} ac\otimes bd$ for homogeneous elements $a,b,c,d\in{}'\mathbf{U}_v^+$.

Let $\mathbf{U}_v^+$ be the $\mathbb{Z}[v^{\pm 1}]$-subalgebra of $'\mathbf{U}_v^+$ generated by $E_i^{(n)}$ for $i \in I$, $n \geqslant 1$. It is a $\mathbb{Z}[v^{\pm 1}]$-form for $'\mathbf{U}_v^+$, and its specialization at $v = 1$ is an integral form for the universal enveloping algebra of $\mathfrak{n}$. The dual form

$$\mathbf{A}_v^+ = \{x \in {}'\mathbf{U}_v^+ \mid (x, y)_v \in \mathbb{Z}[v^{\pm 1}],\ \forall y \in \mathbf{U}_v^+\}$$

is called the *quantum unipotent coordinate ring*. Its specialization at $v = 1$ is an integral form for the coordinate ring of the unipotent group associated with $\mathfrak{n}$.

2.3. Quivers and Ringel's Hall algebra

A quiver $\Gamma = (I, H)$ is an oriented graph with vertices set I and arrows set H. For $h : i \to j \in H$, we write $h' = i$, $h'' = j$. Let h_{ij} be the total number of arrows from i to j. Assume that Γ has no edge loops, then it determines a symmetric generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ given by $a_{ii} = 2$ and $a_{ij} = -h_{ij} - h_{ji}$ for $i \neq j$. Write $\mathfrak{g}_\Gamma = \mathfrak{g}_A$. We say Γ is of finite type if $\mathfrak{g}_\Gamma$ is finite dimensional.

Fix a field $\mathbb{F}$. A Γ-representation over $\mathbb{F}$ is a pair (x, V), where $V = \bigoplus_{i \in I} V_i$ is an I-graded $\mathbb{F}$-vector space, and $x = (x_h)_{h \in H}$ is a collection of linear maps $x_h : V_{h'} \to V_{h''}$. Its dimension is the element $\sum_{i \in I} \dim_{\mathbb{F}}(V_i)\alpha_i$ in Q^+. A morphism of representations $(x, V) \to (x', V')$ is a family of linear maps $a_i : V_i \to V_i'$ such that $a_{h''} x_h = x_h' a_{h'}$ for all $h \in H$. Fix $\beta = \sum_{i \in I} d_i \alpha_i \in Q^+$, the space of Γ-representations of dimension β is

$$X_\beta = \bigoplus_{h \in H} \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{d_{h'}}, \mathbb{F}^{d_{h''}}).$$

The group $G_\beta = \prod_{i \in I} \mathrm{GL}_{d_i}$ acts on X_β by $(gx)_h = g_{h''} x_h g_{h'}^{-1}$ for all $h \in H$, $g = (g_i) \in G_\beta$. Two representations are isomorphic if and only if they are in the same G_β-orbits. So the quotient stack $\mathcal{X}_\beta = [X_\beta / G_\beta]$ parametrizes the isomorphism classes of Γ-representations. For each $i \in I$, there is a unique simple Γ-representation S_i of dimension α_i with $x = 0$. All simple Γ-representations are of this form. For a sequence $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$ of elements in Q^+ whose entries sum up to β, let $\tilde{\mathcal{X}}_\nu$ be the moduli stack of flags of Γ-representations $\phi_\bullet = (\phi_1 \subset \cdots \subset \phi_m)$ such that $\phi_r / \phi_{r-1} \in \mathcal{X}_{\nu_r}$ for $1 \leqslant r \leqslant m$. We have a proper morphism

$$f_\nu : \tilde{\mathcal{X}}_\nu \to \mathcal{X}_\beta, \quad \phi_\bullet \mapsto \phi_m. \tag{2.1}$$

Let $\mathcal{X}_\beta^{\mathrm{nil}}$ be the union of the image of f_ν for all possible ν. Representations in $\mathcal{X}_\beta^{\mathrm{nil}}$ are called nilpotent. If Γ has no oriented cycles, then $\mathcal{X}_\beta = \mathcal{X}_\beta^{\mathrm{nil}}$ for all β.

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field. The space $\mathbb{Q}[\mathcal{X}_\beta^{\mathrm{nil}}]$ of $\mathbb{Q}$-valued functions on the finite set $\mathcal{X}_\beta^{\mathrm{nil}}$ is spanned by the characteristic functions 1_ϕ for $\phi \in \mathcal{X}_\beta^{\mathrm{nil}}$. The Hall algebra $\mathbf{H}(\Gamma, q) = \bigoplus_{\beta \in Q^+} \mathbb{Q}[\mathcal{X}_\beta^{\mathrm{nil}}]$ is an associative algebra with the multiplication given by

$$1_{\phi_1} * 1_{\phi_2} = \sum_{\phi \in \mathcal{X}_{\beta+\gamma}} c_\nu \left| f_\nu^{-1}(\phi) \cap p_\nu^{-1}(\phi_1, \phi_2) \right| 1_\phi,$$

for $\phi_1 \in \mathcal{X}_\beta$, $\phi_2 \in \mathcal{X}_\gamma$. Here $\nu = (\beta, \gamma)$, c_ν is some structural constant and p_ν is the morphism

$$p_\nu : \tilde{\mathcal{X}}_\nu \to \mathcal{X}_\beta \times \mathcal{X}_\gamma, \quad (\phi' \subset \phi) \mapsto (\phi', \phi/\phi'). \tag{2.2}$$

Theorem 2.1 ([52]). *The assignments $E_i \mapsto 1_{S_i}$ for $i \in I$ defines a Q^+-graded algebra embedding*

$$'\mathbf{U}_v^+|_{v=q^{-1/2}} \hookrightarrow \mathbf{H}(\Gamma, q).$$

It is an isomorphism when Γ is a quiver of finite type.

We recall some properties of $\mathcal{X}_\beta$ in two special examples which will be used later.

Example 2.2. Assume Γ is a quiver of finite type. Let $\mathbb{F} = \overline{\mathbb{F}}_q$. By a theorem of Gabriel, there is a bijection between Φ^+ and the set of isomorphism classes of indecomposable Γ-representations, sending α to M_α. There exists a total order on Φ^+ such that $\alpha' \prec \alpha$ if $\mathrm{Hom}_\Gamma(M_\alpha, M_{\alpha'}) = 0$. Fix such an order, then G_β-orbits on X_β are in bijection with descending sequences of elements in Φ^+ whose entries sum up to β. Such sequences are called *Kostant partitions*. We denote by Π_β the set of all Kostant partitions of β.

Example 2.3. Let $\mathbb{F} = \overline{\mathbb{F}}_q$. Consider the quiver $\hat{\Gamma} = (0 \rightrightarrows 1)$ associated with the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. The path algebra of $\hat{\Gamma}$ is isomorphic to the self-extension algebra of the tilting bundle $\mathscr{T} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)$ on the projective line $\mathbb{P}^1$. Thus there is an equivalence of derived categories

$$\mathrm{Ext}^\bullet(\mathscr{T}, -) : D^b\,\mathrm{Coh}(\mathbb{P}^1) \simeq D^b\,\mathrm{Rep}(\hat{\Gamma}).$$

It sends the line bundle $\mathscr{O}_{\mathbb{P}^1}(k)$ for $k \geqslant 0$ to the indecomposable preprojective $\hat{\Gamma}$-representation of dimension $\alpha_1 + k\delta$. Since any vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles, the isomorphism classes of rank r vector bundles are in bijection with the decreasing sequences in $\mathbb{Z}^r$. For $\beta = r\alpha_1 + n\delta$, let

$$\Lambda_\beta = \{(\lambda_1 \geqslant \cdots \geqslant \lambda_r) \in \mathbb{N}^r \mid \lambda_1 + \cdots + \lambda_r = n\}. \tag{2.3}$$

Let Bun_β^+ be the stack of vector bundles parametrized by Λ_β. Let $\mathcal{Y}_\beta$ be the open substack of $\mathcal{X}_\beta$ consisting of preprojective representations. Then the derived equivalence above yields an isomorphism of stacks $\mathrm{Bun}_\beta^+ \simeq \mathcal{Y}_\beta$.

2.4. Lusztig's categorification

Based on Ringel's construction and Grothendieck's sheaf–function correspondence, Lusztig constructed the following categorification of $\mathbf{U}_v^+$. Let $\mathbb{F} = \overline{\mathbb{F}}_q$ and $\Bbbk = \overline{\mathbb{Q}}_\ell$. Let $D_c^b(\mathcal{X}_\beta)$ be the bounded derived category of constructible ℓ-adic sheaves on $\mathcal{X}_\beta$. The category

$$D_c^b(\mathcal{X}) = \bigoplus_\beta D_c^b(\mathcal{X}_\beta)$$

admits a monoidal structure given by the convolution product

$$\mathscr{F}_1 * \mathscr{F}_2 = f_{\nu *} p_\nu^*(\mathscr{F}_1 \boxtimes \mathscr{F}_2)[-\beta \cdot \gamma], \quad \text{for } \mathscr{F}_1 \in D_c^b(\mathcal{X}_\beta),\ \mathscr{F}_2 \in D_c^b(\mathcal{X}_\gamma),$$

where $\nu = (\beta, \gamma)$ and p_ν, f_ν are the morphisms defined in (2.1) and (2.2). For $i \in I$, let $\mathscr{L}_i = \Bbbk_{\mathcal{X}_{\alpha_i}}[-1]$ be the (shifted) constant sheaf on $\mathcal{X}_{\alpha_i}$. For $\nu = (\nu_1, \ldots, \nu_n) \in I^n$, we have

$$\mathscr{L}_{\nu_1} * \cdots * \mathscr{L}_{\nu_n} \simeq f_{\nu *}(\Bbbk_{\tilde{\mathcal{X}}_\nu}[\dim(\tilde{\mathcal{X}}_\nu)]).$$

Denote this complex by $\mathscr{L}_\nu$. Note that by the decomposition theorem, it is a direct sum of intersection complexes on $\mathcal{X}_\beta$ up to some shifts. Let I^β be the subset of I^n consisting of sequences ν whose entries sum up to β. Set $\mathscr{L}_\beta = \bigoplus_{\nu \in I^\beta} \mathscr{L}_\nu$.

Lusztig defined a full additive subcategory $\mathcal{U}_\beta^+$ of $D_c^b(\mathcal{X}_\beta)$ which is generated by the indecomposable summands of $\mathscr{L}_\beta$ and closed under the shifts by [1]. The sum $\mathcal{U}^+ = \bigoplus_{\beta \in Q^+} \mathcal{U}_\beta^+$ is stable under convolution. Hence it is a graded monoidal subcategory of $D_c^b(\mathcal{X})$. We have an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras

$$[\mathcal{U}^+] \simeq \mathbf{U}_v^+$$

with $[\mathscr{L}_i] \mapsto E_i$ for $i \in I$ and $[1] \mapsto v$.

A basis of $[\mathcal{U}^+]$ as a $\mathbb{Z}[v^{\pm 1}]$-module is given by the isomorphism classes of indecomposable objects in $\mathcal{U}^+$ modulo shifts. Let $\mathbf{B}_\beta$ be the set of isomorphism classes of intersection complexes which appear as direct summands of $\mathscr{L}_\beta$ up to some shift. Then $\mathbf{B} = \bigsqcup_{\beta \in Q^+} \mathbf{B}_\beta$ is a basis of $[\mathcal{U}^+]$, called the *canonical basis*.

2.5. Quiver Hecke algebras

Let $\Bbbk$ be a field. For $i, j \in I$, set $Q_{ij}(u, v) = (-1)^{h_{ij}}(u - v)^{-a_{ij}}$ for $i \neq j$ and $Q_{ii} = 0$. For $\beta \in Q^+$ of height n, the symmetric group S_n acts on the set I^n by permutation. The subset I^β is stable under this action. Write $s_k = (k, k+1) \in S_n$ for $1 \leqslant k \leqslant n-1$.

Definition 2.4 ([33, 53]). For $\beta \in Q^+$ of height n, the quiver Hecke algebra R_β is the unital $\Bbbk$-algebra generated by $x_1, \dots, x_n$, $\tau_1, \dots, \tau_{n-1}$ and e_ν for $\nu \in I^\beta$, subject to the following defining relations:

$$x_k x_l = x_l x_k, \quad x_k e_\nu = e_\nu x_k, \quad e_\nu e_{\nu'} = \delta_{\nu,\nu'} e_\nu, \quad \sum_{\nu \in I^\beta} e_\nu = 1,$$

$$\tau_l e_\nu = e_{s_l(\nu)} \tau_l, \quad \tau_k \tau_l = \tau_l \tau_k \quad \text{if } |k-l| > 1, \quad \tau_k^2 e_\nu = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e_\nu,$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k) e_\nu = \delta_{\nu_k, \nu_{k+2}} \frac{Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_{k+2},\nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e_\nu,$$

$$(\tau_k x_l - x_{s_k(l)} \tau_k) e_\nu = \begin{cases} -e_\nu, & \text{if } l = k, \nu_k = \nu_{k+1}, \\ e_\nu, & \text{if } l = k+1, \nu_k = \nu_{k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

It admits a $\mathbb{Z}$-grading with $\deg(e_\nu) = 0$, $\deg(x_k) = 2$, and $\deg(\tau_l e_\nu) = -a_{\nu_l, \nu_{l+1}}$.

Remark 2.5. Quiver Hecke algebras are also defined for symmetrizable Cartan datum and for more general choices of parameters Q_{ij}. These generalizations are important as they provide new categorification results beyond those with geometrical origins. But to simplify exposition, we will not discuss them in this survey.

Example 2.6. For any $i \in I$, the algebra $R_{n\alpha_i}$ is isomorphic to the nil-affine Hecke algebra NH_n. Consider the polynomial ring $\mathrm{Pol}_n = \Bbbk[x_1, \dots, x_n]$. Let Z_n be the subring of symmetric polynomials. Recall that Pol_n is a free graded Z_n-module of rank $v^{\frac{n(n-1)}{2}}[n]^!$. We have an algebra isomorphism $\rho : \mathrm{NH}_n \to \mathrm{End}_{Z_n}(\mathrm{Pol}_n)$ such that $\rho(x_k)$ is the multiplication by x_k and $\rho(\tau_l) = \frac{s_l - 1}{x_l - x_{l+1}}$ is the Demazure operator. So NH_n is a matrix algebra over Z_n. It has a unique indecomposable self-dual grade projective module $P_n = v^{-\frac{n(n-1)}{2}} \mathrm{Pol}_n$, and $\mathrm{NH}_n \simeq [n]^! P_n$ as graded NH_n-modules.

Example 2.7. Let $\mathbf{H}_{n,q}$ be the affine Hecke algebra of type A. It is generated by $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$, $T_1, \ldots, T_{n-1}$ subject to usual defining relations. Assume $q \neq 1$. Fix a finite subset I in $\Bbbk^\times$, define a quiver Γ_q with vertices set I and arrows $i \to qi$. Its connected components are either of type A or of affine type A. Let $\mathrm{Mod}(\mathbf{H}_{n,q})_I$ be the category of $\mathbf{H}_{n,q}$-modules over which $X_1, \ldots, X_n$ acts locally finitely with eigenvalues in I. Brundan–Kleshchev [12] and Rouquier [53] proved that $\mathrm{Mod}(\mathbf{H}_{n,q})_I$ is equivalent to the category of R_n-modules over which $x_1, \ldots, x_n$ act nilpotently, where $R_n = \bigoplus_{|\beta|=n} R_\beta$ is the quiver Hecke algebra for Γ_q.

For $\beta, \gamma \in Q^+$, the element $e_{\beta,\gamma} = \sum_{\nu \in I^\beta, \nu' \in I^\gamma} e_{\nu\nu'}$ is an idempotent in $R_{\beta+\gamma}$. There is a natural algebra embedding $R_\beta \otimes_\Bbbk R_\gamma \to e_{\beta,\gamma} R_{\beta+\gamma} e_{\beta,\gamma}$. For $M \in \mathrm{Mod}(R_\beta)$, $N \in \mathrm{Mod}(R_\gamma)$, the induction

$$M \circ N = R_{\beta+\gamma} e_{\beta,\gamma} \otimes_{R_\beta \otimes_\Bbbk R_\gamma} (M \otimes_\Bbbk N).$$

yields a monoidal structure on $\mathrm{Mod}(R) = \bigoplus_{\beta \in Q^+} \mathrm{Mod}(R_\beta)$. The restriction $\mathrm{Res}_{\beta,\gamma}(-) = e_{\beta,\gamma}(-)$ is right adjoint to the induction. Both functors are exact and preserve the subcategories $\mathrm{mod}(R)$, $\mathrm{proj}(R)$, and $\mathrm{fmod}(R)$. So the Grothendieck groups of these categories become twisted bialgebras.

There are also duality functors $\circledast$ and $\sharp$ on $\mathrm{fmod}(R_\beta)$ and $\mathrm{proj}(R_\beta)$, given respectively by $M^\circledast = \mathrm{Hom}_\Bbbk(M, \Bbbk)$, $P^\sharp = \mathrm{Hom}_{R_\beta}(P, R_\beta)$, both viewed as left R_β-modules via the unique antiinvolution on R_β fixing the generators. They induce involutions on $[\mathrm{fmod}(R)]$ and on $[\mathrm{proj}(R)]$ such that $v \mapsto v^{-1}$.

Theorem 2.8 ([33]). *There are unique isomorphisms of twisted $\mathbb{Z}[v^{\pm 1}]$-bialgebras*

$$\begin{aligned} \big[\mathrm{proj}(R)\big] &\simeq \mathbf{U}_v^+, \quad [R_{\alpha_i}] \mapsto E_i, \quad i \in I, \\ \big[\mathrm{fmod}(R)\big] &\simeq \mathbf{A}_v^+, \end{aligned}$$

such that $([P],[N])_v = \mathrm{gdim}\,\mathrm{Hom}_R(P^\sharp, N)$ *for* $P \in \mathrm{proj}(R)$ *and* $N \in \mathrm{fmod}(R)$.

The following theorem shows that quiver Hecke algebras provide a purely algebraic description of Lusztig's category $\mathcal{U}^+$.

Theorem 2.9 ([55, 65]). *There is an isomorphism of graded algebras*

$$R_\beta \simeq \mathrm{Ext}^\bullet_{D^b_c(\mathcal{X}_\beta)}(\mathscr{L}_\beta, \mathscr{L}_\beta)^{\mathrm{op}}. \tag{2.4}$$

The functor $\oplus_\beta \mathrm{Ext}^\bullet_{D^b_c(\mathcal{X}_\beta)}(\mathscr{L}_\beta, -)$ *yields an equivalence of graded monoidal categories*

$$\mathcal{U}^+ \simeq \mathrm{proj}(R),$$

which sends $\mathbf{B}$ *to the classes of indecomposable self-dual projective modules.*

Remark 2.10. This theorem was extended to the setting $\mathbb{F} = \mathbb{C}$ and $\Bbbk$ a field of positive characteristic by Maksimau [44].

Remark 2.11. Kang–Kashiwara–Park [28] introduced quiver Hecke algebras for quivers with edge loops and established the analog of isomorphism (2.4) in this setting.

2.6. Representations of geometric extension algebras

Consider an arbitrary algebraic variety X over $\mathbb{F} = \overline{\mathbb{F}}_q$ equipped with an action of a reductive group G and a G-equivariant proper morphism $f : \tilde{X} \to X$ from a smooth variety $\tilde{X}$. Assume that X, G, f are defined over $\mathbb{F}_q$. Let $\mathcal{X} = [X/G]$ and $\tilde{\mathcal{X}} = [\tilde{X}/G]$. Let $\Bbbk = \overline{\mathbb{Q}}_\ell$. The push-forward of constant sheaf $\mathscr{L}_f = f_* \Bbbk_{\tilde{X}}[\dim(\tilde{X})]$ is a self-dual semisimple complex in $D^b_c(\mathcal{X})$. The main player of this section is the Yoneda algebra

$$A_f = \mathrm{Ext}^\bullet_{D^b_c(\mathcal{X})}(\mathscr{L}_f, \mathscr{L}_f)^{\mathrm{op}}.$$

Many interesting algebras in representation theory arise in this way. We have just seen that quiver Hecke algebras are of this kind. Historically, Lusztig first gave such a realization for degenerate affine Hecke algebras using the Springer resolutions. There are many other examples, including Schur algebras, degenerate double affine Hecke algebras, and the mathematical definition of Coulomb branch by Braverman–Finkelberg–Nakajima.

In [32], Kato studied homological properties of A_f. Assume G acts on X with finitely many orbits $\{O_\lambda\}_{\lambda \in \Xi}$, and every point in X has a connected stabilizer in G. Then each orbit supports a unique G-equivariant simple perverse sheaf IC_λ given by the intermediate extension of $\Bbbk_{O_\lambda}[\dim O_\lambda]$ to X. By the decomposition theorem, we have

$$\mathscr{L}_f = \bigoplus_{\lambda \in \Xi_f} \mathrm{IC}_\lambda \otimes L_\lambda,$$

where L_λ are self-dual graded vector spaces and $\Xi_f = \{\lambda \in \Xi \mid L_\lambda \neq 0\}$. The set $\{L_\lambda\}_{\lambda \in \Xi_f}$ is a complete collection of nonisomorphic self-dual simple graded A_f-modules. For each λ, the A_f-module $P_\lambda = \mathrm{Ext}^\bullet_{D^b_c(\mathcal{X})}(\mathscr{L}_f, \mathrm{IC}_\lambda)$ is a projective cover of L_λ. Let $j_\lambda : O_\lambda \to X$ be the natural embedding, define the standard module as

$$\Delta_\lambda = \mathrm{Ext}^\bullet_{D^b_c(\mathcal{X})}\Big(\mathscr{L}_f, j_{\lambda *}\big(\Bbbk_{O_\lambda}[\dim O_\lambda]\big)\Big). \tag{2.5}$$

We equip Ξ with the partial order $\prec$ given by the closure relations on the orbits.

Theorem 2.12 ([32]). *Assume that $\Xi_f = \Xi$ and that*

(1) *the algebra A_f is pure of weight zero,*

(2) *the complex IC_λ is pointwise pure for every $\lambda \in \Xi$.*

Then the category $(\mathrm{mod}(A_f), \{\Delta_\lambda\}_{\lambda \in \Xi}, \prec)$ is a polynomial highest weight category.

The notion of polynomial highest weight category was introduced by Kleshchev [36]. Being of *polynomial highest weight* means that in $\mathrm{mod}(A_f)$ the projective module P_λ is filtered by standard modules Δ_μ with $\mu \succeq \lambda$ and Δ_λ appears only once as a quotient, and that $\mathrm{End}^\bullet_{A_f}(\Delta_\lambda)$ is a polynomial ring over which Δ_λ is a finitely generated free module, for all $\lambda \in \Xi$. In this case, the algebra A_f has finite global dimension and the following generalized BGG reciprocity holds:

$$[P_\lambda : \Delta_\mu]_v = [\overline{\Delta}_\mu : L_\lambda]_v, \tag{2.6}$$

where $\overline{\Delta}_\mu = \Delta_\mu \otimes_{\mathrm{End}^\bullet_{A_f}(\Delta_\mu)} \Bbbk$ is called a *proper standard module*, and $[- : -]_v$ stands for the graded multiplicities.

Remark 2.13. A version of Theorem 2.12 for $\mathbb{F} = \mathbb{C}$ and $\Bbbk$ a field of positive characteristic was established by McNamara [48], where purity conditions are replaced by parity conditions.

The theorem is applicable to quiver Hecke algebras when Γ is of finite type. Thus $\mathrm{mod}(R_\beta)$ is a polynomial highest weight category in these cases. In fact, for any quiver, the stabilizer of a point in X_β is connected, and the purity assumption (1) is satisfied. However, the finite type assumption is crucial to guarantee that $\Xi_f = \Xi$ is finite. The purity assumption (2) is proved by Lusztig [42] for finite type quiver, but unknown in general. In the affine case, one needs to modify $\mathrm{mod}(R_\beta)$ to get a similar result. Here is an example.

Example 2.14 ([61]). Let $\hat{\Gamma}$ be the Kronecker quiver. Let $\mathcal{Y}_\beta$ be the open substack of preprojective representations in $\mathcal{X}_\beta$ defined in Example 2.3. Recall that the points in $\mathcal{Y}_\beta$ are indexed by the finite set Λ_β. Let $j_\beta : \mathcal{Y}_\beta \to \mathcal{X}_\beta$ be the natural embedding. Set

$$S_\beta = \mathrm{Ext}^\bullet_{D^b_c(\mathcal{Y}_\beta)}\big(j^*_\beta(\mathscr{L}_\beta), j^*_\beta(\mathscr{L}_\beta)\big)^{\mathrm{op}}. \tag{2.7}$$

Then the category $\mathrm{mod}(S_\beta)$ is polynomial highest weight. In this case, the purity assumption (2) is proved using the isomorphism $\mathcal{Y}_\beta \simeq \mathrm{Bun}^+_\beta$ and the fact that Bun^+_β admits an affine paving.

2.7. Standard modules and PBW bases

There is also an algebraic approach for standard modules over quiver Hecke algebras, which works for symmetrizable generalized Cartan matrices as well.

Assume that $\mathfrak{g}_\Gamma$ is of finite type. A convex order on Φ^+ is a total order $\prec$ such that if $\alpha \preceq \beta$ and $\alpha + \beta$ is a root, then $\alpha \preceq \alpha + \beta \preceq \beta$. For each positive root α, and any $n \in \mathbb{N}$, a finitely generated $R_{n\alpha}$-module L is called *semicuspidal* if $\mathrm{Res}_{\lambda,\mu}(L) \neq 0$ implies λ is a sum of roots $\preceq \alpha$ and μ is a sum of roots $\succeq \alpha$. Semicuspidal modules form an abelian subcategory in $\mathrm{mod}(R_{n\alpha})$ which is equivalent to $\mathrm{mod}(\mathrm{NH}_n)$ in Example 2.6. In particular, it contains a unique self-dual simple module $L_{n\alpha}$. Let $\Delta_{n\alpha}$ be its projective cover inside this subcategory of semicuspidal modules. Then $\Delta^{\circ n}_\alpha = [n]^! \Delta_{n\alpha}$ and $L_{n\alpha} = v^{\frac{n(n-1)}{2}} L^{\circ n}_\alpha$. Recall that a Kostant partition π is of the form $(\beta_1^{m_1}, \dots, \beta_k^{m_k})$ with $\beta_1 \succ \dots \succ \beta_k$. Set

$$\Delta_\pi = \Delta_{m_1\beta_1} \circ \dots \circ \Delta_{m_k\beta_k}, \quad \overline{\Delta}_\pi = L_{m_1\beta_1} \circ \dots \circ L_{m_k\beta_k}.$$

Let Π_β be the set of Kostant partitions of β, and equip it with the bilexicographic order.

Theorem 2.15 ([13]). *Assume that $\mathfrak{g}_\Gamma$ is of finite type.*

(1) *The category* $\mathrm{mod}(R_\beta)$ *is polynomial highest weight with* $\{\Delta_\pi\}_{\pi\in\Pi_\beta}$ *being the standard modules, and* $\{\overline{\Delta}_\pi\}_{\pi\in\Pi_\beta}$ *being the proper standard modules.*

(2) *For each* π, *the module* $\overline{\Delta}_\pi$ *has a unique simple quotient* L_π. *The set* $\{L_\pi\}_{\pi\in\Pi_\beta}$ *is a complete collection of nonisomorphic self-dual simple graded* R_β*-modules.*

Note that if the convex order on Φ^+ satisfies the property given in Example 2.2, then the standard module Δ_π here coincide with the geometrical one in (2.5).

Remark 2.16. Part (2) gives a new parametrization of simple R_β-modules. It generalizes Zelevinsky's parametrization of simple modules for affine Hecke algebras of type A in terms of multi-segments.

For each choice of a reduced expression for the longest element $w_0 = s_{i_1}\cdots s_{i_N}$ in the Weyl group, we have a convex order $\alpha_1 \succ \cdots \succ \alpha_N$ on Φ^+ with $\alpha_k = s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k})$. Lusztig [43] defined the PBW basis for $\mathbf{U}_v^+$ as follows. The root vectors are $E_{\alpha_k} := T_{i_1}\cdots T_{i_{k-1}}(E_{i_k})$ in $\mathbf{U}_v^+$, where T_i are certain braid group operators. The dual root vectors are $E_\alpha^* = (1-v^2)E_\alpha \in \mathbf{A}_v^+$. The PBW basis is $\{E_\pi = E_{\beta_1}^{(m_1)}\cdots E_{\beta_k}^{(m_k)} \mid \pi \in \Pi\}$, and the dual PBW basis for $\mathbf{A}_v^+$ is $\{E_\pi^* = v^{s_\pi}E_{\beta_1}^{*m_1}\cdots E_{\beta_k}^{*m_k} \mid \pi\in\Pi\}$, where $s_\pi = \sum_{r=1}^k m_r(m_r-1)/2$ and $\Pi = \bigsqcup_{\beta\in Q^+}\Pi_\beta$. Under the isomorphisms $[\mathrm{proj}(R)] \simeq \mathbf{U}_v^+$ and $[\mathrm{fmod}(R)] \simeq \mathbf{A}_v^+$, we have $[\Delta_\pi] = E_\pi$, $[\overline{\Delta}_\pi] = E_\pi^*$. In particular, the homological property (2.6) implies that the transfer matrix between the PBW basis and the canonical basis is unitriangular with off-diagonal entries belong to $v\mathbb{N}[v]$. This confirms a conjecture of Lusztig.

This theory has been extended to symmetric affine type by McNamara [49] and Kleshchev–Muth [37]. For a real positive root α, the category of semicuspidal $R_{n\alpha}$-modules is again equivalent to $\mathrm{mod}(\mathrm{NH}_n)$. The new ingredient is a classification of semicuspidal representations for the imaginary roots. Once these representations are constructed, one can proceed as above to define Δ_π, $\overline{\Delta}_\pi$ indexed by (generalized) Kostant partitions. They give categorifications for the PBW basis and the dual PBW basis defined by Beck [3]. There is a similar positivity on the coefficients of the transfer matrix.

An important difference in the affine case is that the category $\mathrm{mod}(R_\beta)$ is no more polynomial highest weight, and its global dimension may be infinite. However, it has an interesting monoidal subcategory $\mathcal{D}_\beta$ with nice properties. Namely, let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated with a finite Lie algebra $\mathfrak{g}$. Let $\hat{\Phi}^{++}$ be the subset of real roots $\alpha + k\delta$ such that $\alpha\in\Phi^+$, $k \geqslant 0$. We can choose a preorder on the set $\hat{\Phi}$ of affine roots such that $\hat{\Phi}^{++} \subset \hat{\Phi}_{\prec\delta}$. Let Π_β^+ be the set of Kostant partitions of β which are supported on $\hat{\Phi}^{++}$. Then the subcategory $\mathcal{D}_\beta$ of $\mathrm{mod}(R_\beta)$ generated by L_π for $\pi\in\Pi_\beta^+$ is a polynomial highest weight category. In particular, it has finite global dimension. So the category of projective objects in $\mathcal{D}_\beta$ and the derived category $D^b(\mathcal{D}_\beta)$ have the same Grothendieck group. Put

$\mathcal{D} = \bigoplus_{\beta \in Q^+} \mathcal{D}_\beta$. We have

$$[D^b(\mathcal{D})] \simeq \mathbf{U}_v(\mathfrak{n}[z]). \tag{2.8}$$

Example 2.17. For the Kronecker quiver in Example 2.3, the closure relation on the orbits in $\mathcal{Y}_\beta$ is compatible with the convex preorder $\alpha_0 \succ \alpha_0 + \delta \succ \cdots \succ \mathbb{Z}\delta \succ \cdots \succ \alpha_1 + \delta \succ \alpha_1$. We have an equivalence of categories $\mathcal{D}_\beta \simeq \operatorname{mod}(S_\beta)$, where S_β is the algebra in (2.7). We have seen in Example 2.14 that this category is polynomial highest weight. The algebraic standard modules again coincide with the geometric ones.

Remark 2.18. The algebra S_β is a semicuspidal algebra for the category $\mathcal{D}_\beta$. The semicuspidal algebra for the imaginary part was studied by Klechshev–Muth [37] and Maksimau–Minets [45].

2.8. Monoidal categorification of quantum cluster algebras

For a symmetric Kac–Moody algebra $\mathfrak{g}$ and any element w in its Weyl group, Geiß–Leclerc–Schröer [23] showed that the quantum unipotent coordinate ring $\mathbf{A}_v(\mathfrak{n}(w))$ is a quantum cluster algebra, where $\mathfrak{n}(w) = \bigoplus_{\alpha \in \Phi^+ \cap w^{-1}(\Phi^-)} \mathfrak{n}_\alpha$. A cluster algebra is a subring of the fraction field of a quantum torus, with some special elements called cluster variables, which are grouped into some overlapping subsets called clusters. The clusters are obtained from an initial one by a combinatorial procedure called mutations. A product of elements inside the same cluster is called a cluster monomial. It was conjectured that the cluster monomials in $\mathbf{A}_v(\mathfrak{n}(w))$ all belong to the dual canonical basis, see Kimura [35]. This conjecture was proved by Kang–Kashiwara–Kim–Oh [27] using a monoidal categorification of $\mathbf{A}_v(\mathfrak{n}(w))$ by modules over quiver Hecke algebras, see Kashiwara's ICM talk [30] for a nice survey on this subject.

A key ingredient in this construction is the study of products of real simple objects. A simple R_β-module L is called *real* if $L \circ L$ is simple. It was shown in [26, 27] that if either M or N is a real simple object, then $\operatorname{Hom}_R(M \circ N, N \circ M) = \Bbbk\mathbf{r}$, where $\mathbf{r} : M \circ N \to N \circ M$ is a nonzero map given by a construction called *renormalized r-matrix*. Moreover, the image of $\mathbf{r}$ is simple, isomorphic to the head of $M \circ N$, and the socle of $N \circ M$ (with grading ignored). In [27] it was shown that given the presence of a quantum cluster structure on the Grothendieck ring of a monoidal category, how renormalized r-matrices reduce the existence of iterated mutations to the existence of one step mutation.

Renormalized r-matrices naturally show up in other contexts, including finite-dimensional representations of quantum affine algebras (which was studied before quiver Hecke algebras), as well as in representations of p-adic groups, see, e.g., [38]. Recently, Cautis–Williams [15] constructed renormalized r-matrices for perverse coherent sheaves on affine Grassmannians, and used them to construct a monoidal categorification of a quantum coordinate ring for $\widehat{\mathfrak{sl}}_2$.

3. COHERENT CATEGORIFICATION OF QUANTIZED LOOP ALGEBRAS

3.1. K-theoretical Hall algebra

We have explained the categorifications of $\mathbf{U}_v^+$ by perverse sheaves on the stack of quiver representations, and its algebraic counterpart by modules over quiver Hecke algebras. Now, we discuss a categorification of the quantized loop algebra via coherent sheaves.

Let $\Gamma = (I, H)$ be a quiver, $\mathfrak{g} = \mathfrak{g}_\Gamma$ and let $\mathfrak{n}$ be the positive part in $\mathfrak{g}$. The loop algebra $\mathfrak{n}[z^{\pm 1}]$ (with z being a formal variable) is a Lie algebra with bracket $[xz^m, yz^n] = [x, y]z^{m+n}$ for $x, y \in \mathfrak{n}$.

Definition 3.1 (Drinfeld). The quantized enveloping algebra $'\tilde{\mathbf{U}}_v^+$ for $\mathfrak{n}[z^{\pm 1}]$ is the $\mathbb{Q}(v)$-algebra generated by $E_{i,n}$ with $i \in I, n \in \mathbb{Z}$, subject to the following defining relations:

(1) for $i, j \in I$, we have $(v^{a_{ij}} z - w)E_i(z)E_j(w) = (z - v^{a_{ij}} w)E_j(z)E_i(w)$,

(2) for $i \neq j$, put $l = 1 - a_{ij}$, we have the Serre relation

$$\mathrm{Sym}_z \sum_{r=0}^{l} (-1)^r \left[\begin{smallmatrix} l \\ r \end{smallmatrix}\right] E_i(z_1) \cdots E_i(z_r) E_j(w) E_i(z_{r+1}) \cdots E_i(z_l) = 0.$$

Here $z, w, z_1, \ldots, z_l$ are variables, $E_i(z) = \sum_{n \in \mathbb{Z}} E_{i,n} z^{-n}$, the operator Sym_z is averaging with respect to the commutator $[a, b]_z = ab - zba$.

Let $\tilde{\mathbf{U}}_v^+$ be the $\mathbb{Z}[v^{\pm 1}]$-subalgebra of $'\tilde{\mathbf{U}}^+$ generated by the quantum divided powers $E_{i,n}^{(r)}$ with $i \in I, n \in \mathbb{Z}, r \geqslant 1$. We explain now its relationship with K-theoretical Hall algebra.

Let $\bar{\Gamma}$ be the quiver obtained from Γ by adding an arrow $\bar{h} : h'' \to h'$ for each $h \in H$. For $\beta \in Q^+$, let $\bar{X}_\beta$ be the space of representations of $\bar{\Gamma}$ of dimension β. Then we have a natural isomorphism $\bar{X}_\beta \simeq T^* X_\beta$. The action of G_β on $\bar{X}_\beta$ is Hamiltonian with the moment map given by

$$\mu_\beta : \bar{X}_\beta \to \mathfrak{g}_\beta, \quad (x_h, x_{\bar{h}})_{h \in H} \mapsto \sum_{h \in H} [x_h, x_{\bar{h}}].$$

We impose $\mathbb{C}^\times$-actions on $\bar{X}_\beta$ and on $\mathfrak{g}_\beta$ by dilations of weight 1 and weight 2, respectively. Set $G_\beta^c = G_\beta \times \mathbb{C}^\times$. Then μ_β is G_β^c-equivariant. The cotangent dg-stack of $\mathcal{X}_\beta$ is the quotient stack

$$T^* \mathcal{X}_\beta = \left[\bar{X}_\beta \times^R_{\mathfrak{g}_\beta} \{0\} / G_\beta^c\right].$$

Here $\bar{X}_\beta \times^R_{\mathfrak{g}_\beta} \{0\}$ is the derived fiber of μ_β at zero. In concrete terms, $\bar{X}_\beta \times^R_{\mathfrak{g}_\beta} \{0\} = \mathrm{Spec}(\mathbf{A}_\beta)$, where $\mathbf{A}_\beta = S(\bar{X}_\beta) \otimes S(\mathfrak{g}_\beta[1]\langle 2 \rangle)$ is a graded dg-algebra with the differential given by the contraction by $\mu_\beta \in S^2(\bar{X}_\beta) \otimes \mathfrak{g}_\beta^* \langle -2 \rangle$. Here $\langle 1 \rangle$ is the degree shift for the internal grading induced by the $\mathbb{C}^\times$-action.

Let $D^b \mathrm{Coh}(T^* \mathcal{X}_\beta)$ be the derived category of coherent sheaves on the dg-stack $T^* \mathcal{X}_\beta$. Equivalently, it is the derived category of graded $\mathbf{A}_\beta \rtimes G_\beta$-modules whose cohomology is finitely generated over $H^0(\mathbf{A}_\beta)$. There is a convolution product on

$$D^b \mathrm{Coh}(T^* \mathcal{X}) = \bigoplus_{\beta \in Q^+} D^b \mathrm{Coh}(T^* \mathcal{X}_\beta),$$

so that it becomes a graded monoidal triangulated category. The $\mathbb{Z}[v^{\pm 1}]$-algebra $[D^b \operatorname{Coh}(T^*\mathcal{X})]$ is called the *K-theoretical Hall algebra*. We also consider a triangulated monoidal subcategory $D^b \operatorname{Coh}(T^*\mathcal{X})_{\mathrm{nil}}$ consisting of complexes with cohomology supported on a closed substack of nilpotent elements.

Example 3.2. Consider $\Gamma = \bullet$, the quiver for $\mathfrak{sl}_2$. Then $\mathfrak{n}[z^{\pm 1}] \simeq \mathbb{C}[z^{\pm 1}]$. Write $Q^+ = \mathbb{N}\alpha$. Since Γ has no arrow, we have $X_{r\alpha} = \bar{X}_{r\alpha} = \{0\}$ and $T^*\mathcal{X}_{r\alpha} = [\{0\} \times^R_{\mathfrak{gl}_r} \{0\}/\operatorname{GL}^c_r]$. Hence $\mathbf{A}_{r\alpha}$ is the exterior algebra $S(\mathfrak{gl}_r[1]\langle 2\rangle)$ with zero differential. By Koszul duality, we have

$$D^b \operatorname{Coh}(T^*\mathcal{X}_{r\alpha}) \simeq D^b \operatorname{Coh}\big([\mathfrak{gl}_r/\operatorname{GL}^c_r]\big).$$

For each irreducible GL_r-representation $V(\lambda)$ of highest weight λ, set $\mathcal{O}(\lambda)_{r\alpha} = \mathcal{O}_{\mathfrak{gl}_r} \otimes V(\lambda)$. Let $\omega_1, \dots, \omega_r$ be the fundamental weights. Then $\mathcal{O}(n\omega_r)_{r\alpha}$ with $r \geqslant 1$ and $n \in \mathbb{Z}$ generate $\bigoplus_{r\geqslant 0} D^b \operatorname{Coh}([\mathfrak{gl}_r/\operatorname{GL}^c_r])$ as a monoidal triangulated category. We have an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras

$$\bigoplus_{r\geqslant 0}\big[D^b \operatorname{Coh}\big([\mathfrak{gl}_r/\operatorname{GL}^c_r]\big)\big] \simeq \tilde{\mathbf{U}}^+_v, \quad \big[\mathcal{O}(n\omega_r)_{r\alpha}\big] \mapsto E^{(r)}_{\alpha,n}.$$

Now, assume that Γ is an arbitrary quiver with no edge loop. Then for each $i \in I$, we have $\mathcal{X}_{r\alpha_i} = [\{0\}/\operatorname{GL}_r]$ and the vector bundles $\mathcal{O}(n\omega_r)_{r\alpha_i} \in \operatorname{Coh}(T^*\mathcal{X}_{r\alpha_i})$ as defined above. We have the following theorem.

Theorem 3.3 ([66]). *There is a unique surjective $\mathbb{Z}[v^{\pm 1}]$-algebra homomorphism*

$$\phi : \tilde{\mathbf{U}}^+_v \to \big[D^b \operatorname{Coh}(T^*\mathcal{X})_{\mathrm{nil}}\big], \quad E^{(r)}_{i,n} \mapsto \big[\mathcal{O}(n\omega_r)_{r\alpha_i}\big].$$

Moreover, ϕ is an isomorphism if Γ is of finite or affine type except $A^{(1)}_1$. In particular, $D^b \operatorname{Coh}(T^*\mathcal{X})_{\mathrm{nil}}$ gives a categorification of $\tilde{\mathbf{U}}^+_v$.

Remark 3.4. K-theoretical Hall algebras are constructed more generally for quivers with potential by Padurariu [51] using a category of singularities. Conjecturally, they are isomorphic to the positive part of Okounkov–Smirnov quantum affine algebras.

3.2. Equivalence of constructible and coherent categorifications

If $\mathfrak{g}$ is of finite type, its affine Lie algebra $\hat{\mathfrak{g}}$ is a central extension of the loop algebra $\mathfrak{g}[z^{\pm 1}]$. The Kac–Moody positive part $\hat{\mathfrak{n}}$ and the loop algebra $\mathfrak{n}[z^{\pm 1}]$ shares a common Lie subalgebra, which is $\mathfrak{n}[z]$.

Recall that for quiver Hecke algebras R_β of type $\hat{\mathfrak{g}}$, we have introduced the category $\mathcal{D}$ which categorifies $\mathbf{U}_v(\mathfrak{n}[z])$, see (2.8). On the other side, $\tilde{\mathbf{U}}^+_v$ is categorified by coherent sheaves on $T^*\mathcal{X}$, and $\mathbf{U}_v(\mathfrak{n}[z])$ is the subalgebra generated by divided powers $E^{(r)}_{i,n}$ for $i \in I$, $n \geqslant 0$, $r \geqslant 1$. Let $D^b \operatorname{Coh}(T^*\mathcal{X})_+$ be the triangulated subcategory of $D^b \operatorname{Coh}(T^*\mathcal{X})$ generated by $\mathcal{O}(n\omega_r)_{r\alpha_i}$ for $i \in I$, $n \geqslant 0$, $r \geqslant 1$. It also categorifies $\mathbf{U}_v(\mathfrak{n}[z])$. It is natural to ask whether these two categorifications are equivalent.

Question 3.5. Is there an equivalence of triangulated graded monoidal categories

$$D^b(\mathcal{D}) \simeq D^b \operatorname{Coh}(T^*\mathcal{X})_+$$

which induces the identity on the Grothendieck group?

In [61], a version of such an equivalence was given for $\mathfrak{g} = \mathfrak{sl}_2$. On the quiver Hecke side, we consider the category $\mathcal{D}_\beta$ attached to the Kronecker quiver in Example 2.17. The simple objects in this category are parametrized by the finite set Λ_β in (2.3). For $\beta = r\alpha_1 + n\delta$, let $D^b \operatorname{Coh}([\mathfrak{gl}_r / \mathrm{GL}_r^c])_\beta$ be the triangulated subcategory of $D^b \operatorname{Coh}([\mathfrak{gl}_r / \mathrm{GL}_r^c])$ generated by $\mathcal{O}(\lambda)_{r\alpha}$ for $\lambda \in \Lambda_\beta$, see Example 3.2. We conjecture that in this case there is an equivalence of graded monoidal categories

$$D^b(\mathcal{D}_\beta) \simeq D^b \operatorname{Coh}\big([\mathfrak{gl}_r / \mathrm{GL}_r^c]\big)_\beta.$$

Note that both categories can be viewed as categories over $\mathfrak{gl}_r // \mathrm{GL}_r$. The fiber at zero on the coherent side is $D^b \operatorname{Coh}([\mathcal{N}_r / \mathrm{GL}_r])_\beta$, where $\mathcal{N}_r \subset \mathfrak{gl}_r$ is the nilpotent cone. It has a perverse coherent t-structure defined by Arinkin–Berzukavnikov [2], whose heart $\operatorname{PCoh}([\mathcal{N}_r / \mathrm{GL}_r^c])_\beta$ is the category of equivariant perverse coherent sheaves on $\mathcal{N}_r$. The fiber at zero on the quiver Hecke side is a subcategory $\mathcal{D}_\beta^\sharp$ of $\mathcal{D}_\beta$ with the same simple objects as in $\mathcal{D}_\beta$.

Theorem 3.6 ([61]). *For $\beta = r\alpha_1 + n\delta$ with $r \geqslant 1$, there is an equivalence of graded triangulated categories*[2]

$$D^{\mathrm{perf}}(\mathcal{D}_\beta^\sharp) \simeq D^{\mathrm{perf}} \operatorname{Coh}\big([\mathcal{N}_r / \mathrm{GL}_r^c]\big)_\beta,$$

which induces an equivalence of graded abelian categories

$$\mathcal{D}_\beta^\sharp \simeq \operatorname{PCoh}\big([\mathcal{N}_r / \mathrm{GL}_r^c]\big)_\beta.$$

Further, this equivalence is compatible with the proper stratified structures on both sides.

The proof of this theorem uses a derived equivalence between $\mathcal{D}_\beta$ and the category of constructible sheaves on the stack of preprojective representations $\mathcal{Y}_\beta \simeq \mathrm{Bun}_\beta^+$ in Example 2.3, the derived geometric Satake equivalence between $D^b \operatorname{Coh}([\mathfrak{gl}_r / \mathrm{GL}_r^c])$ and the equivariant derived category of constructible sheaves on the affine Grassmannian for GL_r established by Bezrukavnikov–Finkelberg [8], and a version of Radon transform between Bun_β^+ and the affine Grassmannian.

Remark 3.7. This theorem has a similar flavor as the equivalence between two categorifications of affine Hecke algebras established by Bezrukavnikov [6].

Remark 3.8. For $w = (s_0 s_1)^N$ in the affine Weyl group of $\widehat{\mathfrak{sl}}_2$, by [27] the quantum cluster algebra $\mathbf{A}_v(w)$ has a monoidal categorification by a subcategory in $\mathcal{D}$. Cautis–Williams [15] constructed another monoidal categorification using equivariant perverse coherent sheaves on the affine Grassmannian for GL_N. The theorem above combined with a functor of Finkelberg–Fujita [22] yields a faithful functor between these two categorifications, which is expected to be an equivalence.

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2 Here "perf" refers to the subcategory of perfect complexes.

4. CATEGORICAL REPRESENTATIONS AND APPLICATIONS

4.1. Categorical representations

The categorified quantum group is a monoidal $\Bbbk$-linear 2-category $\mathcal{U}$ with objects being elements in the weight lattice P, the set of 1-morphisms generated by $\mathcal{E}_i$, $\mathcal{F}_i$ for $i \in I$, and 2-morphisms generated by $x \in \mathrm{End}(\mathcal{E}_i)$, $\tau \in \mathrm{End}(\mathcal{E}_i\mathcal{E}_j)$, $\eta_i : 1 \to \mathcal{F}_i\mathcal{E}_i$, $\varepsilon_i : \mathcal{E}_i\mathcal{F}_i \to 1$, subject to a list of relations. Khovanov–Lauda [34] and Rouquier [53] independently introduced a definition of $\mathcal{U}$, with different sets of generators and relations for 2-morphisms. Brundan [11] proved that they are equivalent.

A categorical $\mathcal{U}$-representation is a 2-functor from $\mathcal{U}$ to the 2-category of $\Bbbk$-linear categories. In concrete terms, it consists of a collection of $\Bbbk$-linear categories $\{\mathcal{C}_\mu\}_{\mu\in P}$ equipped with adjoint functors $\mathcal{E}_i : \mathcal{C}_\mu \to \mathcal{C}_{\mu+\alpha_i}$, $\mathcal{F}_i : \mathcal{C}_{\mu+\alpha_i} \to \mathcal{C}_\mu$, and natural transformations x, τ, ε_i, η_i satisfying the defining relations in $\mathcal{U}$. In this case, we also say that $\mathcal{C} = \bigoplus_{\mu\in P} \mathcal{C}_\mu$ carries a categorical $\mathfrak{g}$-action.

For $\mathfrak{g}$ of type A or affine type A, a *categorical* $\mathfrak{g}$*-action* on an abelian and Artinian category $\mathcal{C}$ is equivalent to the following data (see [53]):

- a decomposition $\mathcal{C} = \bigoplus_{\mu\in P} \mathcal{C}_\mu$,
- a pair of biadjoint endofunctors $\mathcal{E}$, $\mathcal{F}$ on $\mathcal{C}$,
- natural transformations $X \in \mathrm{End}(\mathcal{E})$, $T \in \mathrm{End}(\mathcal{E}^2)$,

such that X acts on $\mathcal{E}$, $\mathcal{F}$ with eigenvalues in I, the generalized eigenfunctors $\mathcal{E}_i$, $\mathcal{F}_i$ for $i \in I$ yield a $\mathfrak{g}$-action on the Grothendieck group $[\mathcal{C}]$ such that $[\mathcal{C}_\mu]$ is the μ-weight space, and X, T satisfy defining relations for affine Hecke algebras.

Many representation categories carrie such actions, including those of symmetric groups, cyclotomic Hecke algebras, the category $\mathcal{O}$ for $\mathfrak{gl}_n$, etc. By Chuang–Rouquier [17], the existence of such a categorical action implies that the categories $\mathcal{C}_\mu$ for μ lying in the same Weyl group orbit are derived equivalent. They also constructed a crystal structure on the set of simple objets in $\mathcal{C}$. In [58], it is proved that the classes of these simple objects form a perfect basis in the Grothendieck group, which has nice unicity properties.

4.2. Minimal categorification

Let $\mathfrak{g}$ be any symmetrizable Kac–Moody algebra. For a dominant weight $\lambda \in P^+$, the irreducible $\mathfrak{g}$-representation of highest weight λ has an integral form $\mathbf{V}_v(\lambda)$, which is a quotient of $\mathbf{U}_v^+$. Let $\mathbf{V}_v^*(\lambda)$ be the dual form.

Definition 4.1. The *cyclotomic quiver Hecke algebra* R_β^λ is the $\mathbb{Z}$-graded algebra defined as the quotient of R_β by the two-sided ideal generated by $\sum_{\nu\in I^\beta} x_1^{\lambda\cdot\alpha_{\nu_1}} e_\nu$.

Kang–Kashiwara [25] proved that $\mathbf{V}_v(\lambda)$ is categorified by $\mathcal{C}_\mu = \mathrm{proj}(R_\beta^\lambda)$ for $\mu = \lambda - \beta$, with $\mathcal{F}_i : \mathrm{proj}(R_\beta^\lambda) \to \mathrm{proj}(R_{\beta+\alpha_i}^\lambda)$ given by $R_{\beta+\alpha_i}^\lambda e_{\beta,i} \otimes_{R_\beta^\lambda} -$, and the adjoint functor given by $\mathcal{E}_i(-) = e_{\beta,i}(-)$ viewed as left R_β^λ-modules. The 2-morphisms x, τ are given by multiplying with the same named generators in R_β. The representation $\mathbf{V}_v^*(\lambda)$ is categorified by $\bigoplus_{\beta\in Q^+} \mathrm{fmod}(R^\lambda_\beta)$. This result generalizes Ariki's [1] categorification theorem for cyclotomic Hecke algebras. Rouquier [53] proved that the categorification of $\mathbf{V}_v(\lambda)$ by $\Bbbk$-linear additive categories is unique.

4.3. Applications to representations of rational double affine Hecke algebras

Cyclotomic rational double affine Hecke algebra (=CRDAHA) is a special family of symplectic reflection algebras introduced by Etingof–Ginzburg [21]. They are associated with complex reflection groups $G(l, 1, n)$ and some parameters. They have a category $\mathcal{O}$, which is a highest weight cover of the category of finitely generated modules over cyclotomic Hecke algebras. This representation category can be viewed as a generalization of the q-Schur algebra, and provides an important example of category $\mathcal{O}$ associated with quantization of symplectic resolutions. The Grothendieck group of these category $\mathcal{O}$ (summed over n) can be naturally identified with the *Fock space* $\mathbf{F}_v$ of level l. The latter is a combinatorial model which gives a concrete realization of integrable $\widehat{\mathfrak{sl}}_e$-representations. Rouquier [54] conjectured that the classes of simple modules in $\mathcal{O}$ correspond to the dual canonical basis in $\mathbf{F}_v$. This yields character formulae for these simple modules in terms of affine Kazhdan–Lusztig polynomials.

In [58], a categorical $\widehat{\mathfrak{sl}}_e$-action on $\mathcal{O}$ was constructed using the induction and restriction functors defined by Bezukavnikov–Etingof [7]. Varagnolo–Vasserot [64] constructed a categorical $\widehat{\mathfrak{sl}}_e$-action on an affine type A parabolic category $\mathcal{O}$, and conjectured it should be equivalent to that for CRDAHA. This conjecture was proved independently by Losev [40] and Rouquier–Shan–Varagnolo–Vasserot [56]. As a consequence, Rouquier's conjecture was confirmed. Further, by [59], the parabolic affine category $\mathcal{O}$ admits a Koszul grading. By the equivalence above, this transfers to a Koszul grading on the category $\mathcal{O}$ of CRDAHA. Moreover, its Koszul dual is the category $\mathcal{O}$ of another CRDAHA. This confirms a conjecture of Chuang–Miyachi [16]. The Koszul duality categorifies the level–rank duality on the Fock space.

On the Fock space, there is also an interesting Heisenberg algebra action. A categorification of this action was constructed in [62] and it was used to prove a conjecture of Etingof [20] on the number of finite dimensional representations of these CRDAHA.

4.4. Applications to representations of finite reductive groups

Categorical actions are also constructed on the category of unipotent representations of classical finite algebraic groups, over a field of characteristic ℓ different from the defining characteristic. For $G = \mathrm{GL}_n(\mathbb{F}_q)$, this was done by Chuang–Rouquier [17]. For finite unitary groups and finite classical groups of type B, C, it was constructed by Dudas–Varagnolo–Vasserot [18, 19]. In all these cases, the functors $\mathcal{E}$ and $\mathcal{F}$ are given by Harish-Chandra restriction and induction functors. The underlying Grothendieck group is a level one Fock space in the case of $\mathrm{GL}_n(\mathbb{F}_q)$, and some explicit level 2 Fock spaces for the other classical types. As a consequence, Broué's abelian defect group conjecture is proved for unipotent ℓ-blocks of these groups at linear prime ℓ.

4.5. Applications to the study of center and cohomology

Let $\mathcal{C}$ be a graded $\Bbbk$-linear category. Denote the identity functor by $1_{\mathcal{C}}$. The center of $\mathcal{C}$ is the graded $\Bbbk$-algebra $Z^\bullet(\mathcal{C}) = \mathrm{End}(1_{\mathcal{C}})$. Given a pair of biadjoint endofunctors $\mathcal{E}, \mathcal{F}$ and $x \in \mathrm{End}(\mathcal{E})$, Bernstein [5] introduced the following operator:

$$Z_{\mathcal{E}}(x) : Z^\bullet(\mathcal{C}) \to Z^\bullet(\mathcal{C}),$$
$$z \mapsto \left(1_{\mathcal{C}} \xrightarrow{\eta} \mathcal{F} 1_{\mathcal{C}} \mathcal{E} \xrightarrow{\mathcal{F} z x} \mathcal{F} 1_{\mathcal{C}} \mathcal{E} \xrightarrow{\varepsilon'} 1_{\mathcal{C}} \right),$$

where η and ε' are the unit and counit in the biadjunction.

When $\mathcal{C}$ carries a categorical $\mathfrak{g}$-action, it is equipped with a family of biadjoint functors $\mathcal{E}_i, \mathcal{F}_i$ and an endomorphism $x \in \mathrm{End}(\mathcal{E}_i) \simeq \mathrm{End}(\mathcal{F}_i)^{\mathrm{op}}$. So we get a family of operators $x^+_{i,r} = Z_{\mathcal{F}_i}(x^r)$, $x^-_{i,r} = Z_{\mathcal{E}_i}(x^r)$ for $i \in I, r \geqslant 0$. By Beliakova–Habiro–Lauda–Webster [4] and Shan–Varagnolo–Vasserot [60], these operators define an action of the current algebra $L\mathfrak{g}$ on $Z^\bullet(\mathcal{C})$. If $\mathfrak{g}$ is of type ADE, then $L\mathfrak{g} = \mathfrak{g}[z]$ and the operators $x^+_{i,r}, x^-_{i,r}$ correspond to $E_i \otimes z^r$, $F_i \otimes z^r$, respectively.

This construction applied to the minimal categorification in Section 4.2 allows to establish an isomorphism between the center of cyclotomic quiver Hecke algebras and the singular cohomology of quiver varieties. Quiver varieties are a family of complex symplectic varieties $\mathfrak{M}^\lambda_\beta$ introduced by Nakajima [50]. Here $\lambda \in P^+$, $\beta \in Q^+$. Nakajima defined a $\mathfrak{g}$-action on the sum over β of the middle cohomology of $\mathfrak{M}^\lambda_\beta$ with coefficient in $\Bbbk$. Varagnolo [63] extended this to an $L\mathfrak{g}$-action on the total cohomology $\oplus_\beta \mathrm{H}^\bullet(\mathfrak{M}^\lambda_\beta)$.

Theorem 4.2 ([4,60]). *Assume $\mathfrak{g}$ is of type ADE. Fix $\lambda \in P^+$. There is an isomorphism of $L\mathfrak{g}$-modules*

$$\bigoplus_{\beta \in Q^+} Z^\bullet(R^\lambda_\beta) \simeq \bigoplus_{\beta \in Q^+} \mathrm{H}^\bullet(\mathfrak{M}^\lambda_\beta), \tag{4.1}$$

which respects Q^+-grading and intertwines the product on the center and the cup product on the cohomology.

This isomorphism is canonical in the following sense. It is not hard to show that the center of R_β is canonically isomorphic to $\mathrm{H}^\bullet([\mathrm{pt}/G_\beta])$. The quotient map $R_\beta \to R^\lambda_\beta$ induces a map on the center $\kappa_a : Z^\bullet(R_\beta) \to Z^\bullet(R^\lambda_\beta)$, which may not be surjective in general. On the geometrical side, the quiver variety $\mathfrak{M}^\lambda_\beta$ admits an open embedding into $[\mathrm{pt}/G_\beta]$. The pull-back gives the so-called *Kirwan map* $\kappa_g : \mathrm{H}^\bullet([\mathrm{pt}/G_\beta]) \to \mathrm{H}^\bullet(\mathfrak{M}^\lambda_\beta)$. McGerty and Nevins [47] proved that κ_g is surjective for any quiver, including those with edge loops. The isomorphism (4.1) fits into the following diagram:

$$\begin{array}{ccc} Z^\bullet(R_\beta) & \xrightarrow{\sim} & \mathrm{H}^\bullet([\mathrm{pt}/G_\beta]) \\ \Big\downarrow{\scriptstyle \kappa_a} & & \Big\downarrow{\scriptstyle \kappa_g} \\ Z^\bullet(R^\lambda_\beta) & \xrightarrow{(4.1)} & \mathrm{H}^\bullet(\mathfrak{M}^\lambda_\beta). \end{array}$$

Remark 4.3. Quiver varieties carry a symplectic G_λ-action and an additional $\mathbb{C}^\times$-action rescaling the symplectic form. The theorem admits a G_λ-equivariant version by considering cyclotomic quiver Hecke algebras defined over the ring $H^\bullet_{G_\lambda}(\mathrm{pt})$. However, adding

$\mathbb{C}^\times$-equivariance on the geometrical side changes the $L\mathfrak{g}$-action to a Yangian action. It is not known how to realize the Yangian action on the center side.

Remark 4.4. There is also a similar isomorphism between the cocenter $\mathrm{Tr}^\bullet(R^\lambda_\beta)$ of R^λ_β and the Borel–Moore homology of a Langrangian subvariety in $\mathfrak{M}^\lambda_\beta$. Moreover, in [4] it is proved that for $\mathfrak{g}$ of type ADE, the cocenter of the 2-category $\mathcal{U}$ is an idempotent version of $L\mathfrak{g}$. For general type of Kac–Moody algebra $\mathfrak{g}$, it is proved in [60] that $\mathrm{Tr}^\bullet(R^\lambda) = \oplus_\beta \mathrm{Tr}^\bullet(R^\lambda_\beta)$ is always a cyclic $\mathfrak{g}[z]$-module.

There is an interesting variation of this result for the Jordan quiver $\bullet\circlearrowleft$.

In this case, the quiver variety $\mathfrak{M}^r_n$ is the Gieseker moduli space parametrizing framed rank r torsion-free sheaves on $\mathbb{P}^2$ with the second Chern class equal to n. It carries an action of $\mathrm{GL}_r \times \mathrm{GL}_2$, where GL_r acts on the framing and GL_2 acts on $\mathbb{P}^2$. Let $G_r = \mathrm{GL}_r \times \mathbb{C}^\times$, with $\mathbb{C}^\times = \mathrm{diag}(t, t^{-1})$ in GL_2. Let $\mathbf{k} = \mathrm{H}_{G_r}(\mathrm{pt}) = \Bbbk[\hbar][y_1, \ldots, y_r]^{\mathfrak{S}_r}$ and $\mathbf{k}'$ be its fraction field. Maulik–Okounkov [46] and Schiffmann–Vasserot [57] independently proved that for fixed r, there is an affine W-algebra action on the (localized) equivariant cohomology $\mathbb{M}_r = \oplus_n \mathrm{H}^\bullet_{G_r}(\mathfrak{M}^r_n) \otimes_{\mathbf{k}} \mathbf{k}'$, confirming a version of the AGT conjecture concerning pure $N = 2$ gauge theory for the group SU_r. The quiver Hecke algebra R_n associated with the Jordan quiver is the degenerate affine Hecke algebra over the ring $\mathbf{k}$. Define its cyclotomic quotient $R^r_n = R_n/(x_1 - y_1)\cdots(x_1 - y_r)$. The quotient map induces a morphism $\kappa^r_n : Z^\bullet(R_n) \to Z^\bullet(R^r_n)$, which is only surjective after localization.

Theorem 4.5 ([60]). *Fix $r \geqslant 1$. There is an action of the affine W-algebra on*

$$\bigoplus_{n\in\mathbb{N}} Z^\bullet(R^r_n) \otimes_{\mathbf{k}} \mathbf{k}'$$

constructed using Bernstein operators. The module obtained is isomorphic to $\mathbb{M}_r$. Moreover, there is a ring isomorphism

$$\mathrm{im}(\kappa^r_n) \simeq \mathrm{H}^\bullet_{G_r}(\mathfrak{M}^r_n), \quad \forall n \in \mathbb{N}.$$

In particular, since the ring $\mathrm{im}(\kappa^r_n)$ has a presentation by generators and relations given by Brundan [10], this theorem gives an explicit description for the ring structure on $\mathrm{H}^\bullet_{G_r}(\mathfrak{M}^r_n)$. It also generalizes the results of Göttsche–Soergel [24] and Vasserot [67] for Hilbert scheme of n points on $\mathbb{C}^2$, which is $\mathfrak{M}^1_n$.

Remark 4.6. A similar description for the equivariant cohomology of Calogero–Moser spaces was established in [9].

Remark 4.7. S. Cautis, A. Lauda, A. Licata, and J. Sussan [14] showed that the cocenter of Khovanov's Heisenberg category is a quotient of the W-algebra above.

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