CONVEX GEOMETRY AND ITS CONNECTIONS TO HARMONIC ANALYSIS, FUNCTIONAL ANALYSIS AND PROBABILITY THEORY

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ABSTRACT

Convex geometry and analysis have connections to many areas of the mathematical sciences: PDEs, discrete geometry, optimization, theoretical computer science, and mathematical economics. No article could even scratch the surface of all of these. Instead, we shall begin by describing how the development of the subject was influenced over the last 50 years by two other fields, harmonic and functional analysis, and then discuss the subtle and still somewhat mysterious way in which convex domains exhibit properties that we normally expect to see within probability theory.

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INTRODUCTION

The task I was set in this article was to discuss convex geometry and analysis and their connections to other fields. As pointed out in the abstract, it would be impossible to write a readable article that even began to exhaust such a broad remit. Naturally, I have opted to explain the connections between convex geometry and the areas that I am most familiar with and, in order to make the article accessible to as large an audience as possible, I have included a few pages of introduction describing the classical theory. The three main sections cover the subject's connections with harmonic analysis, functional analysis, and probability theory, respectively. Some of the material goes back several decades but helps to provide a context for the more recent material: again the aim is to make the article widely accessible to nonspecialists.

It is natural to begin a discussion of convex geometry with the isoperimetric inequality: the statement that if you wish to enclose the largest volume with a given surface area, the optimal shape is a Euclidean ball. Equivalently, if a set $K \subset \mathbf{R}^d$ is measurable then

$$d v_d^{1/d} |K|^{(d-1)/d} \leq |\partial K|$$ [1]

where v_d is the volume of the Euclidean ball of radius 1. Formally, this is not a statement involving convexity since it applies to all sufficiently nice sets, but it is clear that in spirit we are talking about convex sets. It may not be any easier to give a *formal* proof that the optimizers are convex than to prove the full inequality; but it is *intuitively* clear that if your set has gaps, then you can push bits of it together so as to decrease its surface without changing its volume.

Isoperimetric principles of one sort or another appear all over mathematics: in particular, their generalizations in the form of large deviation inequalities play a crucial role in probability theory. Section 3 of this article discusses the influence of functional analysis on convex geometry, and in this section we shall describe how deviation principles are used to prove one of the most celebrated results in convex geometry, Dvoretzky's Theorem, which guarantees that all convex bodies have almost ellipsoidal sections of quite high dimension. The section will also discuss the reverse Santaló inequality of Bourgain and Milman and how this grew out of the interaction between functional analysis and geometry.

Quite early in the 20th century it was realized that the isoperimetric inequality can be extended from sets to functions to give the Gagliardo–Nirenberg–Sobolev inequality. If a measurable $f : \mathbf{R}^d \to \mathbf{R}$ has a gradient almost everywhere then

$$d v_d^{1/d} \left(\int_{\mathbf{R}^d} |f|^{d/(d-1)} \right)^{(d-1)/d} \leq \int_{\mathbf{R}^d} \|\nabla f\|_2.$$

It is in this spirit that we shall look at links between convex geometry and harmonic analysis in Section 2 of the article. We shall discuss a convolution inequality of Brascamp and Lieb that belongs firmly in harmonic analysis but which dovetails perfectly with a geometric

1 Throughout this article we shall use the modulus sign $|\cdot|$ to denote the volume measure of the appropriate dimension.

principle of Fritz John to prove the reverse form of the isoperimetric inequality found by the present author. We shall also discuss the beautiful monotone transportation map of Brenier and how Barthe used it to prove the Brascamp–Lieb inequality. Finally, we shall briefly discuss the quite extensive recent work on the stability of the isoperimetric inequality, in particular in the work of Fusco, Figalli, Jerison, Maggi, and Pratelli.

The final section, Section 4 of the article, focusses on a remarkable correspondence between convex geometry and probability. In linking geometry and harmonic analysis, we shall frequently switch between a convex domain and its indicator function. If the domain has volume 1 then its indicator is automatically the density of a random vector in $\mathbf{R}^d$. So at a trivial level there is obviously a reinterpretation of geometry[2] in terms of probability. But probability theory is much more than just analysis with total measure 1. Central to it is the concept of independence and a wealth of related ideas: filtrations, conditional expectations, and so on. Over the last three decades, it has become increasingly clear that the uniform measure on a convex domain exhibits properties that we would expect from the joint law of *independent* random variables: for example, the central limit theorem for convex domains that was conjectured by the present author and proved by Klartag. The background to these developments was a collection of conjectures made in the late 1980s and early 1990s, and on which quite a lot of progress has been made in the last 20 years. One of the motivations for these conjectures is their relationship to a lovely problem in theoretical computer science: the difficulty of computing volumes of convex sets. So we shall mention the algorithms of Dyer, Frieze, Kannan, Lovász, Simonovits, Applegate, Vempala, and Lee, which depend upon the rate at which Markov chains diffuse inside a convex body. This section also includes Paouris' decay estimate for the Euclidean norm on a convex set, the stochastic localization technique of Eldan and a very recent development by Chen. Being more recent, this material has not yet been highly digested, and so this section is much less polished than the earlier ones. Section 1 of the article will recall some standard facts from convex geometry that we shall refer to throughout the article.

Since this article cannot touch on all of the many areas in which convex analysis appears, we shall say nothing about the combinatorial theory of polytopes and its relation to the topology of complex varieties and very little about the huge field of optimization. An excellent starting point on polytopes is the article by Henk, Richter-Gebert, and Ziegler [67]. We shall also not mention the relationship between polyhedra and lattice points described in loving detail in the book by Barvinok [16]. If my selection of topics has a unifying theme, it is (as by now the reader will have guessed) the isoperimetric inequality.

1. THE FUNDAMENTALS OF CONVEX GEOMETRY

The aim of this section is to describe some of the most basic ideas in convex geometry. The list is far from exhaustive: the topics are selected mainly so that I can refer to them in the subsequent sections of the article.

2 Or at least the kind of geometry we are talking about.

1.1. The Brunn–Minkowski inequality

The basic Sobolev inequality mentioned in the introduction is one way to generalize the isoperimetric inequality, but there is another rather different generalization, which constitutes the most fundamental relation between volume and the linear structure of space. For a set A with a nice enough boundary, we can compute its surface area by considering the volume of its neighborhoods

$$A_\varepsilon = \{x : \|x - y\| \le \varepsilon, \text{ for some } y \in A\}.$$

We then compute the surface area as the "derivative" of volume

$$|\partial A| = \lim_{\varepsilon \to 0} \frac{|A_\varepsilon| - |A|}{\varepsilon}.$$

The ε-neighborhood can be described in a different way as

$$A + B(\varepsilon) = \{x + u : x \in A, u \in B(\varepsilon)\}$$

where $B(\varepsilon)$ is the Euclidean ball of radius ε. Therefore the isoperimetric inequality will follow from a sufficiently strong estimate from below for the volume of the sumset $A + B(\varepsilon)$. The Brunn–Minkowski inequality provides such an estimate for the sum of any two (let us say compact) sets.

Theorem 1 (Brunn–Minkowski). *Suppose A and B are compact sets in $\mathbf{R}^d$. Then*

$$|A + B|^{1/d} \ge |A|^{1/d} + |B|^{1/d}.$$

The inequality can be reformulated in terms of convex combinations of sets (rather than sums). For compact sets A and B in $\mathbf{R}^d$ and $\lambda \in (0,1)$,

$$\left|(1-\lambda)A + \lambda B\right|^{1/d} \ge (1-\lambda)|A|^{1/d} + \lambda|B|^{1/d}$$

and, by using the arithmetic/geometric mean inequality, we can deduce a multiplicative version, which has a number of advantages,

$$\left|(1-\lambda)A + \lambda B\right| \ge |A|^{1-\lambda}|B|^{\lambda}. \tag{1.1}$$

Among other things, this formulation has a natural generalization to functions that was found by Prékopa and Leindler (see, for example, [101]) and which can be very easily proved by induction on the dimension d. (This induction argument seems to have appeared first in [30].) The original inequality does not lend itself to such a proof because in order to deduce the d-dimensional result for (indicator functions of) sets, you need to apply the $(d-1)$-dimensional result for more general functions.

Theorem 2 (Prékopa–Leindler). *If $f, g, m : \mathbf{R}^d \to [0,\infty)$ are measurable, $\lambda \in (0,1)$, and, for each x and y,*

$$m\big((1-\lambda)x + \lambda y\big) \ge f(x)^{1-\lambda} g(y)^{\lambda},$$

then

$$\int m \ge \left(\int f\right)^{1-\lambda} \left(\int g\right)^{\lambda}.$$

A function $f : \mathbf{R}^d \to [0, \infty)$ is called *logarithmically concave* if its logarithm is concave (with the usual convention regarding $-\infty$). Equivalently, f is logarithmically concave if it satisfies

$$f\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} f(y)^{\lambda}$$

for all $x, y \in \mathbf{R}^d$ and $\lambda \in [0, 1]$. The Prékopa–Leindler inequality ensures that if f is such a function then its marginals are too, and from this it follows that convolutions of logarithmically concave functions are also logarithmically concave. The class of such functions thus constitutes a natural extension of the class of indicator functions of convex sets, but which is closed under the most common operations applied to densities in probability theory.

There is a rather odd consequence (or variant) of the Brunn–Minkowski inequality found by Busemann in [33]. Suppose K is a symmetric convex body, or equivalently the unit ball of a norm on $\mathbf{R}^d$. For each unit vector θ, look at the intersection of K with the $(d-1)$-dimensional subspace $\theta^{\perp}$ orthogonal to θ. Then the function

$$\theta \mapsto \left|K \cap \theta^{\perp}\right|^{-1}$$

that measures the reciprocal of the $(d-1)$-dimensional volume of the intersection, extends to a norm on $\mathbf{R}^d$. So this is a precise way to say that if you pick two nearby sections of K then a section between them cannot have volume much smaller than they do. Busemann's Theorem has a simple, but surprisingly useful extension [9]:

Theorem 3 (Busemann–Ball). *Let $f : \mathbf{R}^d \to [0, \infty)$ be an even logarithmically concave function whose integral is finite and strictly positive. Then for each $p \geq 1$, the function*

$$x \mapsto \left(\int_0^\infty f(rx) r^{p-1}\, dr\right)^{-1/p}$$

defines a norm on $\mathbf{R}^d$.

By generating a norm (and hence a convex set) from a logarithmically concave function, the theorem automatically transfers information about convex sets to logarithmically concave functions. Working with functions provides the flexibility to take marginals and convolutions without much affecting what is true. Many of the well-known inequalities for convex sets have analogues for logarithmically concave functions that can be proved in similar ways.

1.2. Fritz John's Theorem

In a famous paper [69] from 1948 on optimization problems, Fritz John gave an example that turned out to be extremely prescient and which has become one of the standard tools in understanding convex domains. The theorem characterizes the ellipsoid of largest volume inside a convex domain in terms of the geometric structure of the contact points between the ellipsoid and the surface of the body. There are two versions, one for symmetric bodies and one for general ones. To get the feel of the theorem, we will just quote the simpler symmetric version. (Throughout the article we will often quote results just for symmetric sets or even functions. In all cases they hold without the symmetry assumption, but the statements are often more complicated and the proofs need no additional ideas.)

Theorem 4 (John). *Let K be a symmetric convex body in $\mathbf{R}^d$. Then K contains a unique ellipsoid of largest volume. This ellipsoid is the standard Euclidean ball B_2^d if and only if the ball is indeed included in K and there are unit vectors $u_1, u_2, \ldots, u_m$ on the surface of K and positive weights $c_1, c_2, \ldots, c_m$ for which*

$$\sum_1^m c_i \langle u_i, x\rangle^2 = \|x\|^2 \tag{1.2}$$

for every $x \in \mathbf{R}^d$.

The condition shows that the contact points behave somewhat like an orthogonal basis. That forces their directions to be distributed in a well-spread-out way: they cannot all lie too close to a subspace of dimension less than d. By applying the identity to an orthonormal basis and summing, we get that

$$\sum c_i = d. \tag{1.3}$$

Plainly, K is included in the set

$$\{x : |\langle u_i, x\rangle| \leq 1, \text{ for all } i\}$$

and hence for any $x \in K$,

$$\|x\|^2 \leq \sum_1^m c_i = d.$$

Consequently, K not only includes the ball of radius 1 but is included in the ball of radius $\sqrt{d}$. Thus John's Theorem provides a way to use a linear map to make a convex body "as round as possible": choose the largest ellipsoid inside K and map that to the standard Euclidean ball.

1.3. The Blaschke–Santaló inequality and symmetrization

A crucial role is played in functional analysis by duality. The theory extends from symmetric convex bodies, the unit balls of norms on $\mathbf{R}^d$, to more general convex sets. The polar of a body K is

$$K^\circ = \{y : \langle y, x\rangle \leq 1, \text{ for all } x \in K\}.$$

The fundamental fact here is the Blaschke–Santaló inequality [21] and [104].

Theorem 5 (Blaschke–Santaló). *If K is a symmetric convex body then the product of the volumes of K and its polar K° is no more than that for the Euclidean balls v_d^2.*

As it stands the statement cannot be true for arbitrary (nonsymmetric) sets because if the origin is not inside the set, the polar will be unbounded. However, there is an extension to general sets, in which one first shifts the set K to the optimal position in space before taking the polar K°.

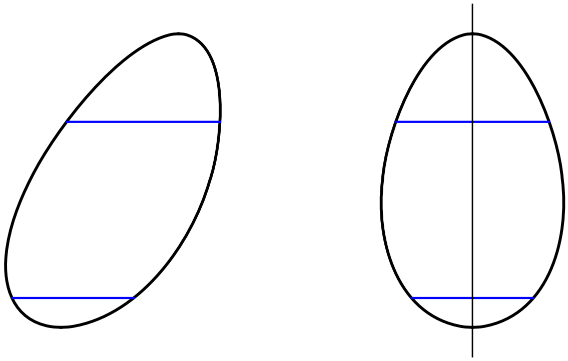

FIGURE 1
Symmetrizing a convex set

A classical and very lovely way to establish inequalities such as the isoperimetric inequality is by means of Steiner symmetrization. If U is a 1-codimensional subspace of $\mathbf{R}^d$ then we can *symmetrize* K with respect to U in the following way. For each line perpendicular to U, consider its intersection with K. Now shift that line segment so that it sits symmetrically either side of the subspace U; see Figure 1. Plainly, the new set has the same volume as K and Steiner showed that it has a smaller surface area. To establish the isoperimetric inequality, you need to show that, by repeatedly symmetrizing a set in different subspaces, you can (in the limit) turn it into a ball. This was done in a famous article by De Giorgi [41].

One can generalize this idea of symmetrization to subspaces of dimension other than $d-1$. If U is a subspace then we replace K by the set of all points of the form

$$u+(v-w)/2,$$

where u is a point in U, v and w are in the orthogonal complement $U^{\perp}$, and $u+v$ and $u+w$ are in K. It was found by Saint-Raymond [103] that, by using this type of symmetrization, one can give a proof of the Blaschke–Santaló inequality. (See also [9].) If you symmetrize K and its polar in orthogonal subspaces then the polar of the symmetrization contains the symmetrization of the polar. Therefore the product of the volumes goes up when you symmetrize. A lovely generalization of this argument to nonsymmetric sets was found by Meyer and Pajor [87].

1.4. Lévy's inequality

Isoperimetric inequalities hold in many manifolds although there are not too many examples where the exact optimizers are known. One case in which the optimum *is* known is that of the sphere S^{d-1} in Euclidean d-dimensional space. We use the rotation-invariant probability measure σ_{d-1} on the sphere to measure "volume." Lévy proved that among compact subsets of the sphere with a given measure, those with the smallest boundary are the spherical caps; see Figure 2.

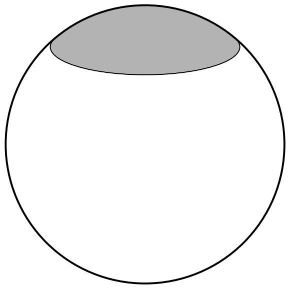

FIGURE 2
A spherical cap

The inequality can be extended. For each $\varepsilon > 0$, the subset of S^{d-1} of a given measure whose ε-neighborhood has smallest measure is a spherical cap. It was shown by Benyamini that this can be proved by a kind of symmetrization argument: the so-called 2-point symmetrization first introduced by Wolontis [110]. Benyamini's argument is included in the article [51]. The process is this. You start with a subset of the sphere. Choose a direction (let us say downwards) and, for each line in that direction that meets the sphere, ask whether the two points where it meets the sphere lie in the set. If both do, neither do, or just the bottom one does, then leave them alone. But if just the top one belongs to the set, then move it to the bottom. You thus compress the set as much as possible into the southern hemisphere. So this is actually a "compression" argument rather than a "symmetrization" argument, but they are clearly very similar in spirit.

Lévy's inequality implies a deviation estimate on the sphere something like the following:

Theorem 6. *Suppose $A \subset S^{d-1}$ and*

$$\sigma_{d-1}(A) \geq 1/2.$$

Then its ε-neighborhood has probability at least

$$\sigma_{d-1}(A_\varepsilon) \geq (1 - 2e^{-d\varepsilon^2/2}).$$

1.5. Differentiability

A classic and much loved text on convex analysis is that by Rockafellar [102]. His book was written with optimization in mind and so proceeds in a very different direction from this article, but we shall want one famous fact from that source. It contrasts appealingly with the warning we impress upon our students in their second or third analysis courses, namely that convergence of functions does not imply convergence of their derivatives.

Theorem 7. *If $\phi : \mathbf{R}^d \to \mathbf{R}$ is convex then f has a gradient almost everywhere. Indeed, the gradient exists outside a set of Hausdorff dimension at most $d-1$. If ϕ_k are convex functions* *converging locally uniformly to ϕ then wherever the derivatives exist we also have*

$$\nabla \phi_k \to \nabla \phi.$$

In fact, much more is true. There are a number of ways to make sense of the idea that convex functions are *twice* differentiable almost everywhere. It cannot be true in the classical sense because the function might fail to be differentiable on a dense set. However, if we are content to ask just for a second-order Taylor expansion instead of the existence of a classical second derivative then the Busemann–Feller–Alexandrov Theorem [1, 34] guarantees its existence almost everywhere. As you would expect, this Hessian of a convex function will be a positive semidefinite map almost everywhere. An excellent account of the various forms of twice differentiability, and several new arguments are contained in the article by Bianchi, Colesanti, and Pucci [20].

2. CONNECTIONS WITH HARMONIC ANALYSIS

The aim of this section is to describe a number of geometric principles that have been established by using very precise inequalities from harmonic analysis and how these methods then fed back into the study of the original inequalities. An important role is played here by the monotone transport of Brenier which became a powerful tool in PDEs and a subject of considerable attention in the late 1990s and early part of this century. A good reference is the monograph of Villani [109].

2.1. The reverse isoperimetric inequality

In 1937 Behrend [18] asked a rather natural question about reversing the isoperimetric inequality. If a set looks like a scattering of dust it can have huge surface area but very small volume. Even if the set is convex there is no upper bound for the surface in terms of the volume, because the set could be a pancake. Behrend's question was this: suppose you are allowed to apply a linear map to your convex set which preserves the volume but makes the surface area as small as possible. For which convex set is the minimal surface area largest? The natural conjecture is that in each dimension, the simplex is the solution to this max–min problem.

In 1961 Petty [96] found a characterization of the optimal affine image for each convex body.

Theorem 8 (Petty). *A convex body $K \subset \mathbf{R}^d$ has the least surface area among all its affine images of the same volume, if and only if for every $x \in \mathbf{R}^d$,*

$$\frac{d}{|\partial K|} \int_{\partial K} \langle n, x \rangle^2 = \|x\|^2, \tag{2.1}$$

where the integral is taken with respect to the area measure on the boundary of K and n is the unit normal at each point of the boundary.

The condition is clearly very similar to the Fritz John condition in Theorem 4 and, in fact, there are a number of results with the same general "shape"; see [61]. In spite of this attractive characterization, Behrend's question was not answered until 1990 [11], and it turned out that the affine image which was best adapted to solving the problem was actually the one characterized by John rather than the optimal one for surface area. It is an elementary exercise to check that if a convex $K \subset \mathbf{R}^d$ includes the Euclidean ball of radius 1, then $|\partial K| \le d|K|$ with equality if K is a polytope whose facets all touch the ball (and in many other cases). Since there is equality for a regular solid simplex, we can prove the reverse isoperimetric inequality by showing that the regular simplex has the largest volume among all bodies whose ellipsoid of maximal volume is the Euclidean unit ball. (Among symmetric convex bodies, the cube has the largest volume ratio; the proof is similar but a bit simpler.)

Theorem 9 (Ball). *Suppose K is a convex set in $\mathbf{R}^d$, the ellipsoid of largest volume in K is the Euclidean ball $B(1)$, and T is a regular simplex with the same maximal ellipsoid. Then*

$$|K| \le |T|,$$

and consequently,

$$\frac{|\partial K|}{|K|^{(d-1)/d}} \le \frac{|\partial T|}{|T|^{(d-1)/d}}.$$

As mentioned in the introduction, the proof of this theorem depended upon a convolution inequality of Brascamp and Lieb. The famous inequality of Young for convolutions states that if $f, g, h : \mathbf{R} \to [0, \infty)$ are measurable and $1/p + 1/q + 1/r = 2$ then

$$\int_{\mathbf{R}} f * g * h \le \|f\|_p \|g\|_q \|h\|_r.$$

The inequality holds on any locally compact group using integrals with respect to Haar measure. On compact groups where constant functions belong to all L_p-spaces, the inequality is sharp, but for the real line it is not. The sharp version was found for certain exponents by Beckner [17] and in full generality by Brascamp and Lieb [29]. The extremal functions for the inequality are Gaussian densities rather than constant functions.

The key to proving the reverse isoperimetric inequality, Theorem 9, was to recognize that the Brascamp–Lieb inequality dovetails perfectly with Fritz John's Theorem. The appropriate formulation is this.

Theorem 10 (Brascamp–Lieb). *Suppose that unit vectors (u_i) in $\mathbf{R}^d$ and weights (c_i) satisfy the John condition*

$$\sum_1^m c_i \langle u_i, x\rangle^2 = \|x\|_2^2$$

for all $x \in \mathbf{R}^d$. Then if (f_i) are nonnegative measurable functions on $\mathbf{R}$,

$$\int_{\mathbf{R}^d} \prod_1^m f_i\big(\langle u_i, x\rangle\big)^{c_i} \le \prod_1^m \left(\int_{\mathbf{R}} f_i\right)^{c_i}.$$

Some feel for the inequality can be gained by observing that there is obviously equality if the (f_i) are identical Gaussian densities. If $f_i(t) = e^{-t^2}$ for each i then

$$\prod f_i\big(\langle u_i, x\rangle\big)^{c_i} = \exp\Big(-\sum c_i \langle u_i, x\rangle^2\Big) = e^{-\|x\|_2^2}$$

by the Fritz John condition (1.2). The integral of this function is the same as the product

$$\prod\left(\int_{\mathbf{R}} f_i\right)^{c_i} = \left(\int e^{-t^2}\right)^d$$

by equation (1.3). There is something pleasingly counterintuitive about the fact that we prove an inequality for which the simplex is extremal by using a result from harmonic analysis in which Gaussian densities are extremal. The resolution of the paradox lies in the fact that the Brascamp–Lieb inequality is sharp (whatever the f_i) if the vectors u_i form an orthonormal basis.

It is natural to conjecture an extension of the Brascamp–Lieb inequality in which one replaces the rank-one projections $x \mapsto \langle u_i, x\rangle u_i$ by orthogonal projections of higher rank. This generalized inequality was proved by Lieb in a later article **[81]**.

2.2. Monotone transport

A few years after the proof of the reverse isoperimetric inequality, Barthe **[15]** gave an elegant new proof of the generalized Brascamp–Lieb inequality using the optimal transportation map discovered by Brenier. A map $T : \mathbf{R}^d \to \mathbf{R}^d$ transports a probability measure μ on $\mathbf{R}^d$ to a probability measure ν if, for each measurable set $A \subset \mathbf{R}^d$, we have

$$\mu\big(T^{-1}(A)\big) = \nu(A). \tag{2.2}$$

Among all maps that do so, one can ask for the one that minimizes the total cost

$$\int_{\mathbf{R}^d} c(x, Tx)\, d\mu(x)$$

for some cost function c. (So $c(x, y)$ is the cost of moving unit measure from x to y.) Brenier **[32]** realized that for one very specific cost function, $c(x, y) = \|x - y\|^2$, the square of the Euclidean distance, the optimal map exists under only very weak hypotheses about the measures and has a very special form: it is the gradient of a convex function. (This map is called a monotone transport map by analogy with the 1-dimensional case in which the derivative of a convex function is monotone.) A version for still more general measures was found by McCann **[86]**. For our purposes, the following theorem, which has a very simple proof **[12]**, gives a good enough picture:

Theorem 11 (Brenier–McCann). *If μ and ν are probability measures on $\mathbf{R}^d$, ν has compact support and μ assigns no mass to any set of Hausdorff dimension $d-1$, then there is a convex function $\phi : \mathbf{R}^d \to \mathbf{R}$, so that $T = \nabla\phi$ transports μ to ν.*

The hypothesis on μ corresponds precisely to the conclusion of the differentiability Theorem 7 since we need ϕ to be differentiable almost everywhere with respect to μ.

Barthe's proof of the Brascamp–Lieb inequality involves transporting the given densities to Gaussian densities and checking that the integral of the product increases as a result. The latter depends crucially upon the fact that the transportation map is the gradient of a convex function and hence that its Hessian is positive semidefinite symmetric. So the argument constitutes a kind of symmetrization technique in which we do not know exactly what map we are using but we do have an inequality for its Hessian. We introduced monotone transport because of Barthe's proof of the Brascamp–Lieb inequality but it has an obvious alternative point of contact with convex analysis: it involves the gradient of a convex function. The contact is actually much closer. If $K \subset \mathbf{R}^d$ is a convex body then, by Theorem 7, it has a well-defined outward unit normal almost everywhere on its surface. By the divergence theorem, the integral of this normal over the surface is zero. The Gauss map which takes a point on the surface to the normal at that point, transports the surface area measure to a measure ν on the sphere satisfying

$$\int_{S^{d-1}} \theta \, d\nu(\theta) = 0. \tag{2.3}$$

A beautiful classical theorem of Minkowski goes in the other direction, just like Theorem 11.

Theorem 12 (Minkowski existence theorem). *Suppose that ν is a finite measure on S^{d-1} for which equation (2.3) holds and whose support spans $\mathbf{R}^d$. Then there is a convex body K whose Gauss map transports its surface measure to ν.*

This theorem may look a bit different from Theorem 11 because in this case we appear to build the surface measure at the same time as the convex set instead of being given both measures to begin with. But in reality we are effectively given the surface measure, namely it is $(d-1)$-dimensional Hausdorff measure.

Another elegant approach to the Brascamp–Lieb inequality, this one using heat flow methods, was found by Carlen, Lieb, and Loss [37] and Bennett, Carbery, Christ, and Tao [19]. In the latter article the formulation of the inequality that matches John's Theorem is called the "geometric form" of the inequality. Following Barthe's argument, monotone transportation was also used to give very elegant proofs of a number of other geometric inequalities (with best possible constants). For the purposes of this article, the obvious paper to mention is that of Cordero-Erausquin, Nazaret, and Villani [40] where they study the Sobolev and Gagliardo–Nirenberg inequalities. The original proofs of the best constants were found by Aubin [6] and Talenti [108] and by Del Pino and Dolbeault [42].

2.3. Projections and surface area

If K is a symmetric convex body in $\mathbf{R}^d$ and for each unit vector θ we consider the $(d-1)$-dimensional volume $|P_\theta(K)|$ of the projection of K onto the orthogonal complement of the span of θ, then it is easy to see that the map

$$\theta \mapsto \left|P_\theta(K)\right|$$

extends to a norm on $\mathbf{R}^d$. The unit ball of this norm has volume

$$V_K = dv_d \int_{S^{d-1}} \frac{1}{|P_\theta K|^d} \, d\sigma_{d-1}$$

and, unlike the surface area, this quantity is unchanged if we apply a linear map of determinant 1 to K. On the other hand, the surface area of K is (apart from the obvious constant) the average of the volumes of the projections

$$|\partial K| = \frac{dv_d}{v_{d-1}} \int_{S^{d-1}} |P_\theta K| \, d\sigma_{d-1}.$$

So it is natural to ask whether there is a strong form of the isoperimetric inequality guaranteeing that V_K is minimized over bodies of a given volume by the Euclidean ball. This is, indeed, the case and was proved by Petty [97]. Petty's projection inequality has been considerably generalized in the work of Lutwak, Yang, and Zhang [83] and Haberl and Schuster [65].

The corresponding reverse question was solved by Zhang [111] who proved that V_K is *maximized* over bodies of a given volume by the simplex. Whereas Petty's Theorem strengthens the isoperimetric inequality, Zhang's Theorem does not follow from Theorem 9 with the correct constant because for the simplex the volume $|P_\theta K|$ is not constant as a function of θ, and so there is strict inequality in the Hölder inequality that one would wish to invoke.

2.4. Stability

Whenever one has an important inequality like the isoperimetric inequality, it is natural to ask about its stability. If a set has small surface area, must it resemble a Euclidean ball? Again, strictly speaking, this is not necessarily a problem about convex sets: the question might well make sense for other sets, as long as we specify carefully what we mean by "resemble." The most famous classical result in this direction is that of Bonnesan [24] from 1924 which is, indeed, specific to convex sets. He proved that for a convex region C in the plane with area A and perimeter P, there are concentric discs D_1 and D_2 with radii r_1 and r_2 for which $D_1 \subset C \subset D_2$ and with

$$P^2 \geq 4\pi\big(A + (r_2 - r_1)^2\big).$$

Thus if C almost satisfies the isoperimetric inequality $P^2 \geq 4\pi A$ with equality, its boundary can be sandwiched between two very similar circles. This result was not extended to higher dimensions until 1989 when Fuglede [55] showed that for a convex body K in $\mathbf{R}^d$, the Hausdorff distance of K from the closest Euclidean ball can be estimated by a certain power of the gap between $|\partial K|$ and the surface area of a ball of the same volume. This sort of conclusion is clearly impossible without the convexity assumption since a general set could have a tiny piece far from the rest of it, which contributes very little to either the volume or the surface area.

However, if we choose to measure the distance of a set from a ball by the volume of their symmetric difference then we can drop the convexity. In a couple of articles, in particular [66], Hall proved the following:

Theorem 13 (Hall). *For each d, there is a constant $C(d)$ so that if A is a measurable set in $\mathbf{R}^d$ then there is a Euclidean ball B of the same volume as K for which*

$$|A \triangle B|^4 \leq C(d)\big(|\partial A| - |\partial B|\big).$$

Hall conjectured that the exponent 4 in this theorem was not optimal and that 2 was the correct exponent. This was proved by Fusco, Maggi, and Pratelli in [57]. Normally, the only reasonable way to prove a stability estimate for a sharp inequality is to take a proof of the inequality and to "watch carefully" what it does, so as to track how much the quantities on each side change as you run through the proof. (Since the stability result implies the original inequality, your argument had better give a proof of the original, so you might as well start with such a proof.) Each of the stability results mentioned so far, tracks the Steiner symmetrization proof of the isoperimetric inequality which was mentioned in Section 1.3.

Knöthe [76] found a different approach to the Brunn–Minkowski inequality and (a fortiori) to the isoperimetric inequality. He used a measure transport map satisfying an equation like (2.2), but whose derivative at each point is upper triangular rather than positive semidefinite symmetric. His argument works just as well with the Brenier map except that to make it rigorous you need some regularity for the map, and this is more difficult to establish in the case of monotone transport. (The main reference here is the subtle regularity theory of Cafarelli [36] for the Monge–Ampére equation.) In their article [50], Figalli, Maggi, and Pratelli obtain stability results by tracking the transportation proof of the isoperimetric inequality instead of the symmetrization proof. The main thrust of their article is that the latter method works for "anisotropic" isoperimetric inequalities. It was pointed out by Gromov [92] that Knöthe's argument works even when you measure surface area in a direction-dependent (anisotropic) way, whereas the symmetrization argument cannot possibly work because the extremal cases are no longer Euclidean balls.

As a consequence of the Brunn–Minkowski inequality, we know that the map

$$\varepsilon \mapsto |A + B(\varepsilon)|^{1/d}$$

is concave in ε. If we control the surface area of A then we control the derivative of the map at 0 and hence its value at each ε. Therefore one can strengthen the stability estimates for the isoperimetric inequality by showing that A must look like a ball under the weaker assumption that the volume $|A + B(\varepsilon)|^{1/d}$ is not too large. This was done by Figalli and Jerison [49]. There are two very comprehensive surveys of all of these stability results, namely by Fusco [56] and Maggi [84]. To some extent, the existence of these surveys, by authors heavily involved in the developments, has prompted me to give relatively brief descriptions.

3. APPLICATIONS OF FUNCTIONAL ANALYSIS

In the 1970s and 1980s researchers in geometric functional analysis began to focus on quantitative problems in finite-dimensional normed spaces rather than problems in infinite dimensions that were, at least in spirit, qualitative. The work led to many new results in convex geometry and, perhaps most strikingly, the reverse Santaló inequality of Bourgain and Milman. These developments really began with Dvoretzky's Theorem about a decade earlier.

3.1. Dvoretzky's Theorem

We shall say that a symmetric convex body K is t-equivalent to an ellipsoid if for some ellipsoid $\mathcal{E}$,

$$\mathcal{E} \subset K \subset t\mathcal{E}.$$

This is the same as saying that the normed space with unit ball K is isomorphic to Euclidean space with a constant of isomorphism t. Dvoretzky [44] answered a question of Grothendieck by proving that high-dimensional convex bodies have fairly high-dimensional slices that are almost indistinguishable from ellipsoids.

Theorem 14 (Dvoretzky). *For each positive integer k and each $\varepsilon > 0$, there is an integer d so that any d-dimensional symmetric convex body has a k-dimensional slice that is $(1+\varepsilon)$-equivalent to an ellipsoid.*

This theorem was one of the earliest triumphs of the probabilistic method: the use of probability theory to construct (or demonstrate the existence of) mathematical objects with special properties. About 12 years later, Milman [89] found a different proof which gives the optimal dependence of k on d. (The fact that it *is* optimal is shown, for example, by the cube.)

Theorem 15 (Milman). *For each $\varepsilon > 0$, there is a constant $c(\varepsilon) > 0$ so that any d-dimensional symmetric convex body has a k-dimensional slice that is $(1+\varepsilon)$-equivalent to an ellipsoid with*

$$k \geq c(\varepsilon) \log d.$$

Milman's proof is now the most familiar. It proceeds in several steps. Assume (by applying a linear transformation) that the Euclidean unit ball is the ellipsoid of largest volume inside the convex body K. Then if $\|\cdot\|_K$ is the norm whose unit ball is K, we have that $\|x\|_K \leq \|x\|_2$ for every x in $\mathbf{R}^d$ giving a bound on the Lipschitz constant of $\|\cdot\|_K$ as a function on the Euclidean sphere. Using the first step, you check that if $\|x\|_K$ is roughly constant on a reasonably dense finite subset of the sphere in some k-dimensional subspace, then it will be roughly constant on the whole sphere in that subspace. A simple argument shows that the sphere in $\mathbf{R}^k$ has a fairly dense subset with only about 4^k points. Now using the Lipschitz property again and Levy's isoperimetric inequality, Theorem 6, you show that $\|\cdot\|_K$ is roughly constant on a huge part of the Euclidean sphere in $\mathbf{R}^d$. Now choose a space at random from among all k-dimensional subspaces and conclude that with high probability the norm, restricted to 4^k points in this subspace, will be almost constant.

The proof seems complete, but a moment's thought shows that there is a point glossed over. We transformed K to make it as round as possible in the hope that there would then be large sets on the sphere where $\|\cdot\|_K$ is almost constant. However, so far we have only used the fact that K includes the ball of radius 1. That would still be true if K were a huge set with no similarity to a ball whatsoever. To make the details of the argument work, we need to know that the average of the norm over the Euclidean sphere is not too small (using the fact that the Euclidean ball is the ellipsoid of *maximal* volume). Thus we have a final step using a result of Dvoretzky and Rogers which shows that if the ellipsoid of maximal volume inside K is the Euclidean unit ball then for some c independent of d,

$$\int_{S^{d-1}} \|\theta\|_K \, d\sigma_{d-1}(\theta) \geq c\sqrt{\log d}.$$

Dvoretzky's original argument is more complicated. Dvoretzky introduced the first and last steps but did not apply the discretization. Instead, he showed "directly" that the norm is almost constant on a k-dimensional subspace. As a result, instead of considering neighborhoods of substantial subsets of the sphere, he was forced to consider neighborhoods of sets that meet a substantial proportion of the k-dimensional subspaces. Milman's method threw into sharper relief the idea that on a space like the sphere, which satisfies a deviation principle, any Lipschitz function will be almost equal to its average on a huge part of the space. This viewpoint led to a series of important results in the 1980s which will be discussed in Section 3.3, but in the next short subsection we shall say a bit more about Euclidean slices.

3.2. Sections of ℓ_p balls

In 1974 Kašin [71] showed that the finite-dimensional L_1 spaces, ℓ_1^d, have Euclidean subspaces of much higher dimension than is guaranteed by Dvoretzky's Theorem.

Theorem 16 (Kašin). *For each d, there is a subspace of ℓ_1^d of dimension at least $d/2$ which is 32-isomorphic to a Euclidean space.*

He used the fact that the unit ball of $B_1^d = \{(x_i)_1^d : \sum |x_i| \leq 1\}$ contains a Euclidean ball of radius $1/\sqrt{d}$ whose volume is as large as $1/2^d$ times the volume of B_1^d. This remarkable fact, that the unit ball of ℓ_1^d has almost spherical slices of dimension proportional to d, was reproved in [51] using Milman's approach to Dvoretzky's Theorem.

A familiar phenomenon in functional analysis is that the L_p spaces for $p < 2$ behave very differently from those for $p > 2$: several important identities in Hilbert space become inequalities in L_p, in one direction for $p < 2$ but in the other direction for $p > 2$. Kašin's argument can be used to show that for $p < 2$ the space ℓ_p^d contains subspaces of proportional dimension that are almost Euclidean. However, for $p > 2$ the correct dependence is $d^{2/p}$ as shown in [51]. This fact was the starting point for Bourgain's remarkable solution to the Λ_p problem on subspaces of L_p spanned by trigonometric characters [26].

3.3. The reverse Santaló inequality

In 1939 Mahler asked a very natural question, prompted by applications in the geometry of numbers. We already saw that the product of the volumes of a symmetric convex body and its polar cannot be more than for the Euclidean ball v_d^2. Mahler asked whether the minimum occurs for the pair consisting of the cube and the so-called cross-polytope, the unit balls of ℓ_∞^d and ℓ_1^d, respectively, for which the product is $4^d/d!$. He also asked whether the minimum over all (not necessarily symmetric) bodies occurs for the simplex. The precise questions are still open but, for example, the products v_d^2 and $4^d/d!$ have a ratio of about $(\pi/2)^d$, so for most purposes it is enough to have an estimate

$$|K|.|K^\circ| \geq c^d v_d^2$$

for some positive constant c. Such an estimate was proved in a well-known article of Bourgain and Milman [28].

Theorem 17 (Bourgain–Milman). *There is a constant $c > 0$ so that if K is a symmetric convex body and K° is its polar then*

$$|K|.|K^\circ| \geq \left(\frac{c}{d}\right)^d.$$

The assumption of symmetry was removed in subsequent works, but the ideas involved in proving the more general statement do not really add anything to the original.

The original proof of the theorem used a subtle, but rather technical, estimate of Milman's [90] (which he called the lower M^*-estimate), together with the theory of type and cotype developed principally by Kwapień, Maurey, and Pisier; see in particular [85, 98]. A crucial result of the latter is Pisier's estimate for the norm of the Rademacher projection on a finite-dimensional space [99]. If K is a convex body that is t-equivalent to an ellipsoid in the sense of Section 3.1 then there is a linear image $\tilde{K}$ so that the norms whose unit balls are $\tilde{K}$ and its polar satisfy

$$\int_{S^{d-1}} \|\theta\|_{\tilde{K}} \, d\sigma_{d-1}(\theta) \int_{S^{d-1}} \|\phi\|_{\tilde{K}^\circ} \, d\sigma_{d-1}(\phi) \leq C(1 + \log t) \tag{3.1}$$

for some constant C. As alluded to in Section 2.3, an upper estimate for

$$\int_{S^{d-1}} \|\theta\|_{\tilde{K}} \, d\sigma_{d-1}(\theta)$$

yields a lower estimate for the volume of K because of Hölder's inequality. So Pisier's result gives an estimate

$$|K|.|K^\circ| \geq \left(\frac{c}{d \log d}\right)^d$$

which is much stronger than the estimate that follows from John's Theorem, but contains an extra $\log d$ that is not present in the Bourgain–Milman Theorem. Milman's lower M^*-estimate demonstrates the existence of a high-dimensional subspace on which $\|\cdot\|_{\tilde{K}}$ is controlled in terms of the quantity

$$\int_{S^{d-1}} \|\phi\|_{\tilde{K}^\circ} \, d\sigma_{d-1}(\phi).$$

This and the very weak (logarithmic) dependence of the integral on the distance of a normed space from Euclidean made possible an iterative argument: apply a linear map that makes the integral small, find a subspace much closer to Euclidean and repeat. Shortly after Theorem 17, Milman proved another result in the same spirit: his reverse Brunn–Minkowski inequality [91]. The reverse Santaló and reverse Brunn–Minkowski inequalities are explained at length in the books [5, 100].

In the years following the Bourgain–Milman Theorem, there have been a number of other proofs using very different methods. Kuperberg [77] found one that uses topology, and Nazarov [93] presented a method using complex analysis. Quite recently an "elementary" proof was found by Giannopoulos, Paouris, and Vritsiou [60] which is very much in the spirit of convex geometry. This will be discussed in Section 4.6 below.

4. THE PROBABILISTIC PICTURE

The aim of this last section is to explain the somewhat cryptic assertion made in the the introduction that convex domains exhibit many of the properties we expect of the joint densities of independent random variables. In the previous section it was explained how the probabilistic method appears in the proofs of subtle geometric results. But here we are after a more intimate connection between geometry and probability. Instead of probability being a tool for proving geometric facts, we want to see classical probability theory actually mimicked by the geometry.

4.1. The cube and the Gaussian isoperimetric inequality

To begin gently, observe that the indicator function of the cube $[-1/2, 1/2]^n$ is exactly the joint density of n independent random variables, each one uniformly distributed on the interval $[-1/2, 1/2]$. If our convex set happened to be a rectangle then it would be the joint density of independent random variables, but we have to choose the coordinate system carefully. If a long thin rectangle is not aligned with the coordinates then its coordinates are highly dependent; see Figure 3.

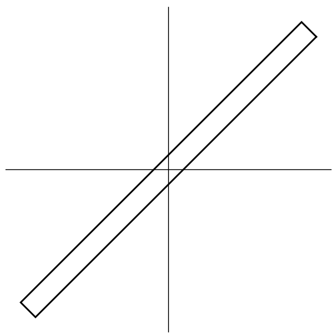

FIGURE 3
A rectangle misaligned

For a general convex domain, there is no natural choice of a coordinate system, so, in order to witness its similarity to a joint density, we must first transform the domain in such a way that the choice of coordinate system does not really matter. If (X_i) are not just independent but also *identically distributed* then all marginals of the joint distribution have the same variance: the vector $(X_1, \ldots, X_d)$ has the property that for any unit vector (θ_i),

$$\mathrm{E}\Big(\sum \theta_i X_i\Big)^2 = \mathrm{E}X_1^2.$$

So, given a symmetric convex domain, we start by applying the linear map which makes its inertia tensor a multiple of the identity. We call a symmetric domain K *isotropic* if

$$\int_K \langle u, x\rangle^2 \, du = L^2 \|x\|_2^2 \tag{4.1}$$

for some L and every vector x. Note that this condition again resembles the conditions of John (1.2) and Petty (2.1). (If the domain is not symmetric, we also shift it so that $\int_K x = 0$.) For an isotropic body, it makes sense to ask whether its indicator looks like the joint density of IID random variables.

The cube is the only example of a convex domain which *exactly* corresponds to IID random variables, but the model we want to keep in mind is that of the standard Gaussian density on $\mathbf{R}^d$, namely

$$x \mapsto g(x) = \frac{1}{(\sqrt{2\pi})^d} \exp\bigl(-\|x\|_2^2/2\bigr).$$

Although this is not the density of a convex set, it is logarithmically concave and is the joint density of independent 1-dimensional Gaussians. There is an isoperimetric principle for the Gaussian density established independently by Borell [25] and by Sudakov and Tsirel'son [106]. We write

$$\gamma(A) = \int_A g$$

for the Gaussian measure of a measurable set A in $\mathbf{R}^d$.

Theorem 18 (Borell–Sudakov–Tsirel'son). *Suppose A is a measurable subset of $\mathbf{R}^d$ and H is a half-space with the same Gaussian measure, $\gamma(A) = \gamma(H)$. Then for each $\varepsilon > 0$ the ε-neighborhoods of these sets satisfy*

$$\gamma(A_\varepsilon) \geq \gamma(H_\varepsilon).$$

Both articles establish the theorem by using the isoperimetric inequality on the sphere and a limiting process. There are other, direct, proofs; a particularly elegant one was found by Bobkov [22]. From this isoperimetric principle, we can immediately obtain a deviation estimate: if $\gamma(A) = 1/2$ then the t-neighborhood of A has large measure:

$$\gamma(A_t) \geq 1 - 1/2e^{-t^2/2}.$$

It follows from this, or can easily be checked by simple calculation, that most of the mass of the Gaussian density in $\mathbf{R}^d$ lies in a spherical shell of constant thickness (the constant being independent of dimension) and radius about $\sqrt{d}$. In other words, most of the mass lies in a shell much thinner than its radius.

As explained, the aim of this final section of the article is to discuss the extent to which convex domains exhibit features like those of the Gaussian density. To set the scene, let us remark that we already have a deviation inequality for the Euclidean ball: Theorem 6 works just as well for the solid ball as for the sphere. Moreover, Pisier noticed that we can get a similar inequality for the cube by transporting the Gaussian measure to Lebesgue measure on the cube. However, one cannot hope to obtain such a sub-Gaussian deviation principle for a general convex domain. The unit ball of ℓ_1^d has volume $2^d/d!$ so the scaled copy that has volume 1 is about d times as large. Its marginal in a coordinate direction decays like $(1 - x/d)^d$ and this is only subexponential, rather than sub-Gaussian. For a general convex domain, a subexponential deviation estimate can be provided by a Poincaré inequality which estimates the smallest nontrivial eigenvalue $\lambda_1(K)$ of the Neumann Laplacian on the domain. In an influential paper [38] Cheeger showed that (on any compact Riemannian manifold) this eigenvalue cannot be too small if there is an isoperimetric inequality for subsets in the manifold. On the other hand, Gromov and Milman [62] showed that a lower bound for λ_1 does imply a subexponential deviation estimate. The statement that $\lambda_1(K)$ is the first nontrivial eigenvalue can be written as an inequality: if $s : K \to \mathbf{R}$ is differentiable and perpendicular to the trivial constant eigenfunction

$$\int_K s = 0,$$

then

$$\lambda_1(K) \int_K s^2 \le \int_K \|\nabla s\|_2^2. \tag{4.2}$$

Observe that if K is isotropic with constant L as in equation (4.1), then for any unit vector e we have

$$\int_K \langle u, e\rangle^2 \, du = L^2.$$

So by taking s to be the function $s : u \mapsto \langle u, e\rangle$ whose gradient everywhere is the vector e of length 1, we conclude that $\lambda_1(K)$ cannot be larger than $1/L^2$.

The Laplacian with respect to Gaussian measure, namely the operator

$$s \mapsto -\frac{\nabla.(g\nabla s)}{g} = -\Delta s + \langle x, \nabla s\rangle,$$

satisfies a Poincaré inequality with constant 1: the linear functions $x \mapsto \langle u, x\rangle$ are the eigenfunctions of this operator that have the smallest nonzero eigenvalue. The conjectures discussed below are intended to capture the extent to which convex domains share with Gaussian densities the properties of having Gaussian marginals, satisfying a Poincaré inequality or concentrating mass in a thin shell. Before addressing the conjectures that frame our probabilistic picture of convex domains, it is helpful to discuss sections of convex bodies.

4.2. Sections of convex bodies

From now on we shall assume that K is a symmetric convex domain in $\mathbf{R}^d$ with volume 1 which is isotropic, that is,

$$\int_K \langle u, x\rangle^2 \, du = L^2 \|x\|_2^2 \tag{4.3}$$

for all $x \in \mathbf{R}^d$. As a consequence of the Brunn–Minkowski inequality, each marginal density of K is logarithmically concave, and that means we can relate its maximum value to its "variance" L^2. Hensley [68] pointed out that this implies that if H is a 1-codimensional subspace of $\mathbf{R}^d$ then the slice $H \cap K$ has volume between, say, $1/(4L)$ and $1/L$. The obvious question is "How big is L?". We can apply (4.3) to an orthonormal basis to get

$$\int_K \|u\|_2^2 = dL^2,$$

and so it is clear that L is *minimized* if K is a Euclidean ball. In this case L is approximately $1/\sqrt{2\pi e}$. So the question is how *large* L can be? It is tempting to think that a convex domain of volume 1 must obviously have some 1-codimensional slices as large as those of the Euclidean ball of the same volume and use Hensley's result to deduce that L must be at most a constant independent of dimension. In 1956 Busemann and Petty [35] asked a general version of this question: Is it true that if K and B are symmetric convex bodies and every 1-codimensional slice through the center of K has $(d-1)$-dimensional volume smaller than the corresponding slice of B, then K itself must have smaller d-dimensional volume than B? The answer is no, and the simplest counterexamples take B to be the Euclidean ball, so there is no hope of estimating L in this way. A negative answer in 12 dimensions was provided in 1975 by Larman and Rogers [78] with K a random perturbation of the Euclidean ball. Some years later I proved that each 1-codimensional slice of the cube has volume at most $\sqrt{2}$ and, as a result, if the dimension is at least 10, the unit cube has all its slices smaller than those of the Euclidean ball of volume 1; see [8] and [10].

The Busemann–Petty problem is now solved in all dimensions. Lutwak [82] showed that the problem can be reformulated in terms of intersection bodies (the unit balls of the norms generated by Busemann's Theorem 3) and using this Gardner [58] proved that the problem has a positive solution in 3 dimensions, while Zhang [112] proved it for 4 dimensions. For dimension 5 and above, the solution is negative, and a unified treatment of the problem can be given using ideas of Koldobsky; see the article by Gardner, Koldobsky, and Schlumprecht [59]. (In the special case in which K is the Euclidean ball, the answer to the Busemann–Petty question is yes in all dimensions: every convex body has a slice as *small* as those of the Euclidean ball of the same volume.) So there remains the question: Is there an upper bound, independent of dimension, for the variance of isotropic convex domains in Euclidean space? This will be the subject of the first conjecture in the next subsection.

4.3. The conjectures

The aim of this subsection is to describe three conjectures that have motivated much of the work in high-dimensional geometry over the last two decades, each of which describes a sense in which the indicators of convex domains look like the densities of independent random variables or, more specifically, like Gaussian densities.

Conjecture 19 (Bourgain's slicing conjecture). *There is a constant M independent of dimension so that if K is an isotropic symmetric convex domain of volume 1 in $\mathbf{R}^d$ then*

$$\int_K \|u\|_2^2 \le M^2 d.$$

While this conjecture is usually attributed to Bourgain, I personally do not think he actually believed it. The question of just how large the integral can be remains open but there has been dramatic recent progress that will be discussed in Section 4.6. Note that the conjecture is equivalent to the following tantalisingly simple statement: there is a constant $\delta > 0$ so that every convex body of volume 1 has a 1-codimensional slice of volume at least δ; hence the name of the conjecture. As explained in the previous section, you cannot hope to prove the conjecture by showing that every convex body of volume 1 has a slice as large as the Euclidean ball of the same volume. What is trivial is that every convex domain of volume 1 must have width at most that of the Euclidean ball of the same volume, in at least one direction, and this is about $\sqrt{d}$. By the Cavalieri principle, it must have a slice of volume at least $1/\sqrt{d}$. On the face of it, this conjecture appears to be saying much less than that a convex domain looks like a Gaussian density; this point will be taken up in Section 4.4 below.

The second conjecture was made by the present author in the mid-1990s (and later published in a joint article with Anttila and Perissinaki [3]) and also by Brehm and Voigt [31]. Roughly speaking, it says the following:

Conjecture 20 (The central limit problem). *Let K be an isotropic, convex domain of volume 1 in $\mathbf{R}^d$. Then in all but a small proportion of directions, the 1-dimensional marginals of K are approximately Gaussian.*

(The precise formulation stipulates that as $d \to \infty$ the proportion decreases to 0 and the distance of the marginals from Gaussian also decreases to 0.)

The third and final conjecture concerns the Poincaré constant for an isotropic convex domain. It has been known for a century that for a bounded connected domain $\Omega \subset \mathbf{R}^d$ there is a spectral gap for the Neumann Laplacian on Ω. The gap can be very small if Ω is a dumbbell-shaped domain because then s can be equal to -1 on one of the weights, 1 on the other, and only have a nonzero gradient on the narrow bar that joins the two weights. Even if Ω is convex, the constant can be large if the set is long and thin since then s can take large values at the two ends by changing only very slowly along the length of Ω. However, for an *isotropic* convex set K, there is a bound depending only upon dimension. As was remarked earlier the spectral gap cannot be more than $1/L^2$. In their article [70], Kannan, Lovász, and Simonovits conjectured that this is the correct order.

Conjecture 21 (Kannan–Lovász–Simonovits). *There is a constant C independent of dimension so that if K is an isotropic, convex domain of volume 1 then for any differentiable s on K with $\int_K s = 0$,*

$$\int_K s^2 \leq CL^2 \int_K \|\nabla s\|_2^2 \tag{4.4}$$

where L is the "slicing constant" of K, that is,

$$\int_K \langle u, x\rangle^2 \, du = L^2 \|x\|_2^2.$$

Thus the conjecture is that for convex domains, linear functions are approximately the worst for the spectral gap problem, just as they are for the Gaussian density.

In the same article Kannan, Lovász, and Simonovits gave a better bound for the spectral gap than the trivial one that can be deduced just from a bound on the diameter of K. To do so, they used a localization method for proving inequalities originally employed by Payne and Weinberger [95], which is roughly as follows. Among convex sets, the cones are "extremal" in the sense that they are only just convex. As you scan along a cone in $\mathbf{R}^d$ the $(d-1)$-dimensional volume of the slices is given by $x \mapsto l(x)^{d-1}$ where l is a real-valued linear function. You have a pair of functions f and g and you want to contradict the claim that they both have positive integral (thereby proving the inequality you want). Assume that they do. Choose a hyperplane that cuts space into two pieces on which the integrals of f are the same and pick the piece on which the integral of g is the larger. Thus you have found a smaller region on which both functions have positive integral. Keep doing this and take a "limit." It is possible to show that the limiting region is an infinitesimal truncated cone: formally, a line segment and a weight function of the form $l(x)^{d-1}$ for some linear function l. You now just have to prove the inequality you want in this 1-dimensional setting. The estimate given by Kannan, Lovász, and Simonovits is

$$\int_K s^2 \leq CdL^2 \int_K \|\nabla s\|_2^2$$

with an additional factor of the dimension d, instead of inequality (4.4). The same estimate was later established by Bobkov [23] using entropy arguments.

The next subsection explains the probabilistic picture of geometry that grew out of these conjectures and how this led to a study of the entropy of logarithmically concave random variables. The following subsections will then describe the state of play on each of the three conjectures. There are a number of books and survey articles on these topics, for example, [2,75].

4.4. The probabilistic picture clarified

It was remarked in Section 1.1 that the class of logarithmically concave densities is an extension of the class of convex domains which has the virtue of being closed under the most common operations of probability theory. Facts about convex domains usually transfer to this larger class: for example, using Theorem 3 it is not hard to show that if M is a bound in the slicing conjecture for a given dimension then eM works for logarithmically concave densities in the same dimension (see [9]). When studying a convex domain, it makes sense to consider its indicator function which takes the value 1 on the domain. But when looking at more general densities, it is not natural to normalize by fixing the value at a point. In the context of probability theory, it is clearly much more natural to consider probability densities $f : \mathbf{R}^d \to \mathbf{R}$ for which the covariance matrix is the identity; so for all x,

$$\int_{\mathbf{R}^d} f(u)\langle u, x\rangle^2 = \|x\|_2^2.$$

Once you rescale in this way, two of the conjectures read differently. The central limit problem, of course, stays as it is: we still want marginals to look Gaussian, just with a different variance.

The KLS conjecture now becomes as simple as it could be, namely the slicing constant L simply disappears from the statement.

Conjecture 22 (KLS probabilistic version). *There is a constant C independent of dimension so that if $f : \mathbf{R}^d \to [0, \infty)$ is an even logarithmically concave probability density whose covariance matrix is the identity, then for any differentiable $s : \mathbf{R}^d \to \mathbf{R}$ with compact support and $\int sf = 0$,*

$$\int s^2 f \leq C \int \|\nabla s\|_2^2 f.$$

The slicing problem states that in passing from the convex body normalization to the probabilistic normalization, we haven't had to rescale too much. So it now specifies that the value $f(0)$ cannot be more than M^d for some constant M. However, the quantity $f(0)$ looks rather unnatural in the probabilistic setting: among other things, it is very unstable under probabilistic operations such as convolution. Fortunately, it can be replaced by a proxy that is much more appropriate. It is not difficult to check that for an even logarithmically concave density the entropy, $\mathrm{Ent}(f) = -\int_{\mathbf{R}^d} f \log f$, satisfies

$$-\log f(0) \leq \mathrm{Ent}(f) \leq -\log f(0) + cd$$

for some constant c. (Within a few days of my pointing this out, Fradelizi showed me a neat argument that gives the optimal constant, $c = 1$. He included it into an article some years later [54].) Therefore the slicing problem now reads

Conjecture 23 (Slicing, probabilistic version). *There is a constant C independent of dimension so that if $f : \mathbf{R}^d \to [0, \infty)$ is an even logarithmically concave probability density whose covariance matrix is the identity, then*

$$\mathrm{Ent} f \geq -Cd.$$

Among random vectors with a given covariance matrix, the Gaussian has the largest entropy. The gap between the entropy of a random vector on $\mathbf{R}^d$ with density f and the entropy of the Gaussian is a well-known and very natural measure of how far the random vector is from being Gaussian. So this version of the slicing problem shows clearly that it does *indeed* constitute a statement about logarithmically concave densities being similar to Gaussians.

This entropic formulation of the problem was the motivation behind a series of articles [4, 13] of Artstein, Barthe, Naor, and the author, which used a local version of the Brunn–Minkowski inequality to find a new formula for the entropy (or more precisely, the Fisher information) of a marginal distribution. This did not solve the slicing problem, but led us to (among other things) the solution of an old problem in information theory: Is the central limit theorem driven by an analogue of the second law of thermodynamics? Since the Gaussian has the largest entropy among random variables with a given variance, it makes sense to ask whether the central limit theorem can be "explained" by the fact that normalized sums of IID random variables have increasing entropy that drives them to the Gaussian.

Theorem 24 (Artstein–Ball–Barthe–Naor). *If (X_i) are IID square-integrable random variables then the entropies*

$$\mathrm{Ent}\left(\frac{1}{\sqrt{n}} \sum_1^n X_i\right)$$

increase with n.

This theorem constitutes an application of convex geometry to information theory rather than the other way around, but the machinery developed in [13] did have one surprising consequence for geometry. Following the work of Bakry and Emery [7], most studies of entropy consider the evolution of a random vector along the Ornstein–Uhlenbeck semigroup. This is a semigroup $\{P_t\}_{t\geq 0}$ of convolution operators on L_1 which can be defined as follows. If f is the density of a random vector X then $P_t f$ is the density of the random vector $X_t = \sqrt{e^{-2t}}X + \sqrt{1-e^{-2t}}G$ where G is a standard Gaussian independent of X. Thus the semigroup evolves the original random vector towards the Gaussian. The logarithmic Sobolev inequality of Gross [63] ensures that the rate of decrease of the entropy gap between the random vector X_t at time t and the Gaussian limit is at least a certain multiple of the gap:

$$2(\mathrm{Ent} G - \mathrm{Ent} X_t) \leq -\frac{\partial}{\partial t}(\mathrm{Ent} G - \mathrm{Ent} X_t).$$

As a result the entropy gap decays to zero at least as fast as the exponential e^{-2t}. One consequence of the methods found in [13] is that if we start with a random variable X with a logarithmically concave density $f = e^{-\phi}$ for which the Laplacian

$$s \mapsto -\frac{\nabla.(f\nabla s)}{f} = -\Delta s + \langle \nabla\phi, \nabla s\rangle$$

itself has a spectral gap, then the entropic convergence is enhanced. So if the KLS conjecture holds for a particular density f, we get more rapid convergence of the entropy gap to zero. With some care, this can be used to show that the initial entropy gap was not too large to start with and hence yield an estimate for the slicing constant for f. The argument appears in [14].

Theorem 25 (Ball–Nguyen). *Let $f : \mathbf{R}^d \to \mathbf{R}$ be an isotropic even logarithmically concave probability density satisfying a Poincaré inequality*

$$\int s^2 f \leq C \int \|\nabla s\|_2^2 f$$

for differentiable s with compact support satisfying $\int sf = 0$. Then the slicing constant of f is at most e^{16C}.

Some years later Eldan and Klartag [47] gave a much tighter estimate for the relationship between the spectral gap and the slicing bound, not for each convex domain but "globally." They showed that an estimate C in the KLS conjecture for *all* convex domains transfers to an estimate of some constant multiple of C in the slicing problem for all domains. The most interesting thing about the argument is that they apply the spectral gap property to a more natural function, the Euclidean norm, than in the case of the theorem for individual domains stated above. Their result has acquired new significance following a recent article of Chen [39] and will be taken up in Section 4.6. The last three sections deal with what is known on the three conjectures stated above.

4.5. The central limit problem

It was shown by Sudakov [105] and by Diaconis and Freedman [43] that an isotropic probability measure μ on high-dimensional space will have Gaussian marginals, in the sense of the central limit problem stated above, as long as the measure satisfies a thin shell estimate of the kind that Gaussian measure satisfies. If

$$\int_{\mathbf{R}^d} \langle u, x\rangle^2 \, d\mu(u) = L^2 \|x\|_2^2$$

for all x then the typical radius of the random vector with law μ is $L\sqrt{d}$. The thin shell condition states that most of the mass of μ lies in a shell of roughly this radius, whose thickness is significantly smaller, namely

$$\mu\big(\big|\|x\|_2 - L\sqrt{d}\big| > \varepsilon L\sqrt{d}\big) < \varepsilon.$$

A measure satisfying this will have Gaussian marginals in most directions up to an error of ε plus a term depending only on d (that is, $o(1)$ as $d \to \infty$). In the case of the uniform measure on a convex set, one can obtain essentially optimal estimates (in terms of ε and d) as in [3]. This thin shell condition is clearly implied by a Poincaré inequality for the measure, so the KLS conjecture is stronger than both the central limit conjecture and the slicing problem. The KLS conjecture would give the thin shell property with ε of the order of $1/\sqrt{d}$ which is the best one could possibly hope for. It was explained in Section 4.3 that Kannan, Lovász, and Simonovits had used localization to obtain a bound on the spectral gap for isotropic convex sets or logarithmically concave functions. That bound does not provide an estimate for ε that tends to 0 as the dimension grows.

The central limit problem was solved in 2006 by Klartag [73] and shortly afterwards a completely different proof was given by Fleury, Guédon, and Paouris [53]. The key idea in Klartag's article is to show that typical marginals of the body of fairly high dimension, say $\log d$, have almost exactly rotation-invariant densities, a kind of Dvoretzky Theorem for marginal densities instead of sections. (The possibility of such an approximate rotation invariance was suggested by Gromov during the 1980s.) For a rotation invariant density, the thin shell property is a 1-dimensional question that is easily solved. Then, since the 1-dimensional marginals of the original body are the 1-dimensional marginals of the $\log d$-dimensional ones, they must be almost Gaussian.

A year or so before the proof of the central limit theorem for convex domains, Paouris [94] proved an optimal decay estimate for the Euclidean norm, which is clearly related to the thin shell estimate but ultimately has rather different consequences.

Theorem 26 (Paouris). *Suppose K is an isotropic, symmetric convex body of volume 1 in $\mathbf{R}^d$ and*

$$\int_K \|x\|_2^2 = L^2 d.$$

Then the volume of the part of K where $\|x\|_2$ is significantly larger than $L\sqrt{d}$ decays as

$$\big|\{x \in K : \|x\|_2 \geq cL\sqrt{d}t\}\big| < e^{-\sqrt{d}t}$$

for some constant c independent of dimension and K, and for all $t > 1$.

The restriction that t should be larger than 1 means that the theorem does not yield a concentration of volume in a shell but it gives an excellent decay rate for large radii. The proof of this theorem depends upon a delicate analysis of the integrals

$$\int_{S^{d-1}} \|\theta\|_K^q \, d\sigma_{d-1}(\theta)$$

of powers of the norm corresponding to K. This in turn depends upon a study of the norm restricted to subspaces in the same spirit as the Bourgain–Milman Theorem, the new ingredient here being that a crucial role is played by subspaces of dimension $\sqrt{d}$. The optimality (apart from the value of c) is shown by sets like the unit ball of ℓ_1^d as remarked earlier.

The proof of the central limit problem given by Fleury, Guédon, and Paouris uses a variant of the methods of Theorem 26. Each of these two articles gives an estimate for the thin shell problem with ε only logarithmic in the dimension. Klartag [74] quickly gave a power-type estimate and this was improved by Fleury [52], and again by Guédon and E. Milman [64] by combining the techniques from [53,73]. Then in [79] Lee and Vempala showed how to use the stochastic localization method of Eldan to estimate the thin shell bound. At the time this gave a bound of the form $\varepsilon = 1/d^{1/4}$, but the recent work of Chen reduces this almost to $1/\sqrt{d}$.

4.6. The slicing conjecture

Bourgain not only posed the slicing problem but gave the first significant estimate. As remarked above, there is a trivial bound of $\sqrt{d}$ for slicing constants in dimension d. In [27] Bourgain improved this dramatically to $d^{1/4} \log d$.

Theorem 27 (Bourgain). *For some M independent of dimension, if K is an isotropic symmetric convex body of volume* 1 *then*

$$\frac{1}{d} \int_K \|x\|_2^2 \leq M(d^{1/4} \log d)^2.$$

Bourgain's argument used much of the functional-analytic machinery that had just become available: Pisier's estimate (3.1), Talagrand's majorizing measure theorem [107], and an interpolation argument using the Brunn–Minkowski inequality.

In the late 1980s when the problem was first discussed, there was considerable interest in the question of whether the slicing constant is an isomorphic invariant for normed spaces. Suppose we have two t-equivalent norms on $\mathbf{R}^d$ with unit balls J and K which are isotropic and have volume 1. If the slicing constant of K is L, must it be that the constant for J is at most some fixed function of t times as large as L? Klartag [72] showed some 15 years later that, in this form at least, the question is a bit of a red herring.

Theorem 28 (Klartag). *If K is a convex body and $\varepsilon > 0$, there is another body T which is $(1+\varepsilon)$-equivalent to K and whose slicing constant is at most $C/\sqrt{\varepsilon}$.*

So every convex body is quite similar to one with a bounded slicing constant. Thus the only way that the slicing constant *can* be an isomorphic invariant is that it is essentially the same for all convex bodies. Klartag pointed out that by combining this with Paouris' estimate, Theorem 26, one can eliminate the $\log d$ factor in Bourgain's Theorem to give an estimate for slicing constants of $d^{1/4}$.

Klartag's argument is very surprising. He considers the following logarithmically concave function:

$$y \mapsto e^{\langle x,y\rangle}$$

restricted to K, for different choices of x. Using the theory of monotone transport discussed in Section 2.2, he shows that for an appropriate x, the slicing constant for this function can be bounded in terms of the volume product $|K|.|K^\circ|$ and then invokes a nonsymmetric version of Theorem 3 to create a convex set from the logarithmically concave function. To get Theorem 28, he used the reverse Santaló inequality (Theorem 17). Some years later Giannopoulos, Paouris, and Vritsiou realized that the final step could be avoided in a rather dramatic way. It is possible to estimate the volume product of a body in terms of its slicing constant. In itself that is not very surprising: the slicing conjecture is a strong statement, and already in [9] there was a very simple proof that the slicing conjecture implies Milman's reverse Brunn–Minkowski inequality. But the key point here is that the powers of the volume product in Klartag's Theorem and in the reverse direction are different, and this means that they can be combined to yield a proof of the reverse Santaló inequality. That in turn can be fed back into Klartag's Theorem. So there is now an "elementary" approach to Theorem 17.

It was remarked in Section 4.4 that the KLS conjecture implies the slicing conjecture and that Eldan and Klartag [47] had sharpened the dependence. Their argument applies the spectral gap property to the function $\|x\|_2$ and so what they actually prove is that estimates in the slicing conjecture can be deduced from estimates for the thin shell bound discussed in the previous subsection. Their original argument involved the construction of a Riemannian metric related to a convex body which seems to have little or nothing to do with the other ideas discussed in this article, although one can see a link to the proof of Theorem 28. Subsequently, the machinery of stochastic localization developed by Eldan, which will be discussed in the next section, was used by Eldan and Lehec [48] to give an alternative proof. The best estimates currently known in the slicing problem have just been improved dramatically as will be explained at the end of the next subsection.

4.7. The KLS conjecture

It was remarked in Section 4.1 that for a convex set, a bound on the spectral gap is equivalent to an isoperimetric inequality for subsets, that is, a bound on the Cheeger constant. The Cheeger constant for a domain K is the minimum over all subsets A of K of the ratio

$$\frac{|\partial A|}{|A|.|K-A|}$$

where the "surface area" $|\partial A|$ of A includes only the part of the surface inside K. So the constant tells you how small can be the area of a (curved) cut that divides the set into roughly equal pieces. In the paper [88] of E. Milman, there is a detailed explanation of the relationship between the Cheeger constant, the spectral gap, and (on the face of it) much weaker notions for convex domains.

The interest of computer scientists in the spectral gap problem for convex domains arose because of the problem of effective computation of volume for convex sets. To compute volume deterministically, even to within a factor that is exponential in the dimension, is computationally hard. However, in [45] Dyer, Frieze, and Kannan found a randomized algorithm which involved running a random walk inside the set: sampling from a set is more or less equivalent to computing its volume. There has been a succession of improvements in the run time of the algorithm by Lovász, Simonovits, Applegate, Vempala, Lee, and the original authors over a period of 30 years. A very helpful survey of the history is provided by Lee and Vempala [80]. Some of these improvements involve choosing enhanced random walks, but others depend upon getting better estimates for the geometry of the domains being sampled. In order to sample effectively, you want the random walk to mix throughout the domain quickly and the barrier to that happening will be a bottleneck: a way of cutting the domain into substantial pieces with a cut whose area is small. If your domain has this dumbbell shape then a random walk can get trapped in one of the weights. This is exactly the Cheeger constant problem. So better estimates in the KLS conjecture immediately imply better run times in the algorithm.

A very new approach to the problem was found by Eldan, [46]. Instead of *convolving* with a Gaussian as in the Ornstein–Uhlenbeck process above, Eldan's method can be thought of as the apparently simpler one of multiplying by a Gaussian density. But it is a random density and the aim is to show that the *typical* product behaves as you would like. If you multiply by a Gaussian with large variance you do not really change the logarithmically concave density: if you multiply by a Gaussian with small variance you get essentially a Gaussian, for which you know everything. The problem is to keep track of how quantities change in going from one to the other. The key is to effect the multiplications by a stochastic process, what Eldan called stochastic localization. The process is governed by a stochastic differential equation, but Eldan explains that this can be thought of in the following way. At each infinitesimal step, you multiply the density by a linear function whose gradient has a random direction. A linear function such as $x \mapsto 1 - x_1$ puts greater weight on one half of the density, thus mimicking the localization technique described in Section 4.3. A familiar fact in analysis is that the product of two "complementary" linear functions

$$(1 - x_1)(1 + x_1) = 1 - x_1^2$$

gives you a hump which is the first step towards a Gaussian.

When he introduced the method, Eldan used it to show that the thin shell property for a logarithmically concave density implies the KLS conjecture up to a factor that is only a power of $\log d$. As explained above, Lee and Vempala modified the technique to prove an estimate $\varepsilon = 1/d^{1/4}$ for the thin shell problem and hence by [47] a bound of $d^{1/4}$ for the slicing conjecture, the same as Klartag's. At that point it was tempting to wonder whether this might be the correct order of the worst slicing constant on the grounds that two (or even three) completely different methods gave (essentially) the same bound. However, in a recent remarkable breakthrough, Chen [39] found a way to use stochastic localization to get a bound for the KLS conjecture, the thin shell problem and the slicing problem, which is $O(d^{\alpha})$ for every $\alpha > 0$.

Theorem 29 (Eldan–Chen). *For every $\alpha > 0$, there is a constant $C(\alpha)$ so that for every symmetric, isotropic convex domain $K \subset \mathbf{R}^d$ of volume 1 and every differentiable $s : K \to \mathbf{R}$ with $\int_K s = 0$, we have*

$$\int_K s^2 \leq C(\alpha) d^\alpha \int_K \|\nabla s\|_2^2.$$

4.8. Conclusion

I introduced the final section by suggesting that convex bodies mimic classical probability theory. But with hindsight one should perhaps see the situation differently. Independence and convexity each, in their different ways, force a measure to be "genuinely" high-dimensional, as opposed to being a low-dimensional measure that accidentally lies in a high-dimensional space. What makes a measure roughly Gaussian is the high-dimensionality. How is it that the extra freedom in high dimensions creates what appears to be more order and predictability instead of less? I admit to being biased but surely a clue is given by Theorem 24: the disorder that comes from high-dimensionality is the sort of disorder that is found in physical systems, namely the disorder of high entropy. And increased entropy presents to low-dimensional human eyes as uniformity and regularity.

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