WEIGHTED FOURIER EXTENSION ESTIMATES AND APPLICATIONS

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ABSTRACT

We describe some recent results on weighted Fourier extension estimates and their applications in PDEs and geometric measure theory.

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The goal of this paper is to go over some recent work by the author and her collaborators on Schrödinger maximal estimates, weighted Fourier extension estimates, and Falconer distance set problem. The study of Schrödinger maximal estimates arises from the pointwise convergence problem of the solution to the Schrödinger equation raised by Carleson in the late 1970s. Weighted Fourier extension estimates are closely related to the classical Fourier restriction problem raised by Stein and have various applications in PDEs and geometric measure theory. Falconer distance set problem was introduced by Falconer in the early 1980s and remains to be a difficult and wide open question in geometric measure theory. Two major special cases of the general weighted Fourier extension estimates apply to Schrödinger maximal estimates and Falconer distance set problem.

1. SCHRÖDINGER MAXIMAL ESTIMATES

The solution to the free Schrödinger equation

$$\begin{cases} iu_t - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

is

$$e^{it\Delta} f(x) = (2\pi)^{-n} \int e^{i(x\cdot\xi + t|\xi|^2)} \hat{f}(\xi)\, d\xi.$$

It is not hard to show that the solution $e^{it\Delta} f(x)$ converges to the initial data f in L^2 as the time t tends to 0. However, the problem of pointwise convergence is much harder. About 40 years ago, Carleson [10] proposed the question of identifying the optimal exponent s for which $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ almost everywhere whenever f lies in the Sobolev space $H^s(\mathbb{R}^n)$. He proved himself the convergence for $s \geq 1/4$ when $n = 1$. Dahlberg and Kenig [12] then showed that this result is sharp. The higher-dimensional case has since been studied intensely [4–6, 9, 11, 13, 15, 16, 20, 30–32, 36, 38, 39, 41, 42]. Recently, Bourgain [6] gave counterexamples showing that $s \geq \frac{n}{2(n+1)}$ is necessary for the pointwise convergence to hold. In collaboration with Guth and Li, and then with Zhang, we proved that Bourgain's bound is also sufficient (up to endpoint).

Theorem 1.1 ($n = 2$, Du, Guth, and Li [15]; $n \geq 3$, Du and Zhang [20]). *Let $n \geq 2$. For every $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2(n+1)}$, $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ almost everywhere.*

The pointwise convergence problem can be approached via a standard smooth approximation argument. Indeed, the convergence holds uniformly for Schwartz functions because of their rapid decay nature. Since the space of Schwartz functions is dense in the Sobolev space, to prove that $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ holds a.e. for any f in $H^s(\mathbb{R}^n)$, it is enough to show that the associated maximal function $\sup_{0<t\leq 1} |e^{it\Delta} f(x)|$ is bounded from $H^s(\mathbb{R}^n)$ to $L^p(B^n(c,1))$ for some $p \geq 1$ and any unit ball $B^n(c,1)$ in $\mathbb{R}^n$. Such estimates are called the *Schrödinger maximal estimates*. More precisely, results in Theorem 1.1 are consequences of the following two theorems on Schrödinger maximal estimates.

Theorem 1.2 (Du, Guth, and Li [15]). *For any $s > \frac{1}{3}$, the following bound holds: for any function $f \in H^s(\mathbb{R}^2)$,*

$$\Big\| \sup_{0<t\le 1} |e^{it\Delta} f| \Big\|_{L^3(B^2(0,1))} \le C_s \|f\|_{H^s(\mathbb{R}^2)}.$$

Theorem 1.3 (Du and Zhang [20]). *Let $n \ge 3$. For any $s > \frac{n}{2(n+1)}$, the following bound holds: for any function $f \in H^s(\mathbb{R}^n)$,*

$$\Big\| \sup_{0<t\le 1} |e^{it\Delta} f| \Big\|_{L^2(B^n(0,1))} \le C_s \|f\|_{H^s(\mathbb{R}^n)}.$$

Comparing the two Schrödinger maximal estimates above, we note that one can derive Theorem 1.3 in the case $n = 2$ from Theorem 1.2 using Hölder's inequality. Despite the fact that Theorem 1.3 in the cases $n = 1, 2$ can recover the almost sharp results of pointwise convergence problem, the sharp estimates for the L^2-norm of the Schrödinger maximal function are not as strong as the previous sharp estimates for the L^p-norm ($p = 4$ when $n = 1$ is due to Kenig, Ponce, and Vega [29], and $p = 3$ when $n = 2$ is due to Du, Guth, and Li [15]). Based on these results, it is natural to ask the following:

Question 1.4. Consider the Schrödinger maximal estimates of the form

$$\Big\| \sup_{0<t\le 1} |e^{it\Delta} f| \Big\|_{L^p(B^n(0,1))} \le C_s \|f\|_{H^s(\mathbb{R}^n)}. \tag{1.1}$$

(1) Let $n \ge 3$. Determine the optimal $p = p(n)$ for which (1.1) holds for any $s > \frac{n}{2(n+1)}$.

(2) Let $n \ge 3$ and fix $p > 2$. Identify the optimal range of $s = s(n, p)$ for which (1.1) holds.

Via a localization argument, Littlewood–Paley decomposition, and parabolic rescaling, the above question can be reduced to the problem of identifying the sharp exponent $\boldsymbol{\gamma(n, p)}$, which is the optimal γ such that

$$\Big\| \sup_{0<t\le R} |e^{it\Delta} f| \Big\|_{L^p(B^n(0,R))} \lessapprox R^{\gamma} \|f\|_{L^2}, \quad \forall f : \operatorname{supp} \hat{f} \subset B^n(0,1). \tag{1.2}$$

Here $A \lessapprox B$ means $A \le C_\varepsilon R^\varepsilon B$ for any $\varepsilon > 0$, $R \ge 1$. The known results [6,12,15,20,29] can be summarized as

$$\gamma(n, p) = \max\left\{ n\left(\frac{1}{p} - \frac{n}{2(n+1)}\right), 0 \right\} \tag{1.3}$$

for any $p \ge 1$ when $n = 1, 2$, and $1 \le p \le 2$ when $n \ge 3$.

It remains an interesting problem to determine $\gamma(n, p)$ for $p > 2$ when $n \ge 3$. It seemed possible that (1.3) should hold for any $p \ge 1$ and $n \ge 1$. However, we disproved this for a certain range of p when $n \ge 3$ by examining Bourgain's example [6] in all intermediate dimensions:

Theorem 1.5 (Du et al. [19]). *Let $n \ge 3$ and $p > 2$. Then*

$$\gamma(n, p) \ge \max_{m \in \mathbb{Z}, 1 \le m \le n} \left[\frac{n+m}{2} \left(\frac{1}{p} - \frac{1}{2} \right) + \frac{m}{2(m+1)} \right].$$

Let us look at two special cases of Theorem 1.5: by a direct calculation,

- if $\gamma_{n,p} = n(\frac{1}{p} - \frac{n}{2(n+1)})$, then $p \le p_0(n) := 2 + \frac{4}{(n-1)(n+2)}$;
- if $\gamma_{n,p} = 0$, then $p \ge p_1(n) := \max_{m\in\mathbb{Z},1\le m\le n}[2 + \frac{4}{n-1+m+n/m}]$.

Note that $p_0(n) < \frac{2(n+1)}{n} < p_1(n)$ when $n \ge 3$. Therefore, (1.3) fails for $p_0(n) < p < p_1(n)$ when $n \ge 3$.

To establish (1.2), it is helpful to consider a more general setting, in which it is convenient to use induction on scales. By formalizing the being locally constant property, one can treat $|e^{it\Delta} f(x)|$ essentially as a constant on each unit cube, and therefore the left-hand side of (1.2) is equivalent to $\|e^{it\Delta} f\|_{L^p(X)}$, where X is a union of lattice unit cubes in $B^{n+1}(0, R)$ such that each lattice vertical thin tube of dimensions $1 \times \cdots \times 1 \times R$ contains exactly one unit cube from X. In particular, the set X satisfies the condition that $|X \cap B_r| \lesssim r^n$ for any ball B_r of radius $r \ge 1$. Based on this observation, we are led to consider a slight generalization of (1.2):

Question 1.6. Let $\mathcal{X}_{n+1,R}$ denote the collection of subsets X such that each X in $\mathcal{X}_{n+1,R}$ is a union of lattice unit cubes in $B^{n+1}(0, R)$ satisfying $|X \cap B_r| \le r^n$ for any ball B_r of radius $r \ge 1$.

Let $n \ge 3$ and fix $p > 2$. Determine the sharp exponent $\tilde{\boldsymbol{\gamma}}(\boldsymbol{n}, \boldsymbol{p})$, which is the optimal γ for which the following holds:

$$\left\|e^{it\Delta} f\right\|_{L^p(X)} \lessapprox R^{\gamma}\|f\|_2, \quad \forall X \in \mathcal{X}_{n+1,R},\ \forall f : \operatorname{supp} \hat{f} \subset B^n(0, 1). \tag{1.4}$$

The argument from Du, Guth, and Li [15] can be adapted to establish $\tilde{\gamma}(n = 2, p = 3) = 0$, which in turn determines $\tilde{\gamma}(n = 2, p)$ for all $p \ge 1$:

$$\tilde{\gamma}(n = 2, p) = \begin{cases} 2(\frac{1}{p} - \frac{1}{3}), & 1 \le p \le 3, \\ 0, & p \ge 3. \end{cases}$$

For general $n \ge 3$, the fractal L^2 Fourier extension estimate from Du and Zhang [20] gives the sharp exponent $\tilde{\gamma}(n, p = 2) = \frac{n}{2(n+1)}$, which also determines $\tilde{\gamma}(n, p)$ for all $1 \le p \le 2$:

$$\tilde{\gamma}(n, p) = n\left(\frac{1}{p} - \frac{n}{2(n+1)}\right), \quad \forall n \ge 3, \forall 1 \le p \le 2.$$

The main ingredients in the work of [15] include the polynomial partitioning method adapted by Guth to Fourier restriction problem [25] and Bourgain–Demeter's l^2 decoupling theorem [7]. The method of polynomial partitioning identifies algebraic structures where the Schrödinger solutions are most concentrated, and reduces the original 3-dimensional problem to an essentially 2-dimensional one. The reduced problem can then be solved by a bilinear refined Strichartz estimate, which is derived via decoupling and induction on scales.

The proof of the fractal L^2 Fourier extension estimate in [20] uses a broad–narrow analysis [5, 7, 8, 26]. In the broad case, there are $n + 1$ transverse frequency caps in $B^n(0, 1)$ making significant contributions, and we can apply either Bennett–Carbery–Tao's multilinear restriction estimates [2] or multilinear refined Strichartz estimates of Du et al. [16]. In the narrow case, we invoke l^2 decoupling [7] in dimension n and use an induction on scales argument which is rooted in the proof of the refined Strichartz estimates [15].

It remains a wide open and challenging problem to determine $\tilde{\gamma}(n, p)$ for $n \geq 3$ and $p > 2$.

2. WEIGHTED FOURIER EXTENSION ESTIMATES

In this section, we discuss the general weighted Fourier extension estimates, which include the Schrödinger maximal estimate as a major special case. Besides their own independent interest, such general estimates have various applications in PDEs and geometric measure theory.

Definition 2.1. Let $0 < \alpha \leq d$.

(1) We say that μ is an *α-dimensional measure* in $B^d(0,1)$ if it is a positive Borel measure, supported in the unit ball $B^d(0,1)$, that satisfies

$$c_\alpha(\mu) := \sup_{x \in \mathbb{R}^d, r > 0} \frac{\mu(B(x,r))}{r^\alpha} < \infty.$$

(2) We say that H is an *α-dimensional weight* in $\mathbb{R}^d$ if it is a nonnegative measurable function on $\mathbb{R}^d$ that satisfies

$$\int_{B(x,r)} H(x)\,dx \leq r^\alpha, \quad \forall x \in \mathbb{R}^d,\ \forall r \geq 1.$$

(3) Let $\mathcal{X}_{d,\alpha,R}$ denote the collection of subsets X such that each X in $\mathcal{X}_{d,\alpha,R}$ is a union of lattice unit cubes in $B^d(0,R)$ satisfying

$$\left|X \cap B(x,r)\right| \leq r^\alpha, \quad \forall x \in \mathbb{R}^d,\ \forall r \geq 1.$$

Let S denote either the unit sphere $\mathbb{S}^{d-1}$ or the truncated paraboloid $\mathbb{P}^{d-1}$. Let $d\sigma$ be the induced Lebesgue measure on S. Consider the Fourier extension operator

$$Ef(x) = E_S f(x) = \int_S e^{i\omega\cdot x} f(\omega)\,d\sigma(\omega).$$

Note that $E_{\mathbb{P}^{d-1}} f$ corresponds to free Schrödinger solutions. We are interested in the following *weighted Fourier extension estimates*:

Question 2.2. Determine the sharp exponent $\boldsymbol{\gamma(d, \alpha, p)}$, which is the optimal γ for which the following two equivalent estimates hold:

(1) For any α-dimensional weight H in $\mathbb{R}^d$ and any function $f \in L^2(S, d\sigma)$,

$$\|Ef\|_{L^p(B^d(0,R);H dx)} \lessapprox R^\gamma \|f\|_2; \tag{2.1}$$

(2) For any subset $X \in \mathcal{X}_{d,\alpha,R}$ and any function $f \in L^2(S, d\sigma)$,

$$\|Ef\|_{L^p(X)} \lessapprox R^\gamma \|f\|_2. \tag{2.2}$$

To see the equivalence of the two estimates (2.1) and (2.2), one direction is easy: given $X \in \mathcal{X}_{d,\alpha,R}$, the characteristic function of X is an α-dimensional weight in $\mathbb{R}^d$; the other direction was proved in Du and Zhang [20] where the being locally constant property and dyadic pigeonhole argument play key roles. The advantage of the expression (2.2) is that it allows us to take into account geometric structures more directly.

The case $\alpha = n = d - 1$ of (2.2) is a generalization of Schrödinger maximal estimates as described in Section 1. Estimates (2.1) for general α are related to the study of spherical average Fourier decay rates of fractal measures [3, 17, 20–23, 33–35, 40, 43]. For α around $\frac{d}{2}$, estimates (2.1) have drawn particular interest because of their application to Falconer's distance set problem [17, 20–24, 43]. The techniques in the proof of Theorems 1.2 and 1.3 are used to establish the following results:

Theorem 2.3. *Let H be an α-dimensional weight in $\mathbb{R}^d$.*

(1) (Du et al. [17]). *For $\frac{3}{2} < \alpha \le 2$,*

$$\|Ef\|_{L^3(B^3(0,R);Hdx)} \lessapprox \|f\|_2, \quad \forall f \in L^2(S, d\sigma). \tag{2.3}$$

(2) (Du and Zhang [20]). *For $d \ge 3$ and $\frac{d}{2} < \alpha < d$,*

$$\|Ef\|_{L^2(B^d(0,R);Hdx)} \lessapprox R^{\frac{\alpha}{2d}} \|f\|_2, \quad \forall f \in L^2(S, d\sigma). \tag{2.4}$$

In particular, (2.3) gives that $\gamma(d = 3, \alpha = 2, p = 3) = 0$, which in turn determines the exact $\gamma(d = 3, \alpha = 2, p)$ for all $p \ge 1$. For $\frac{3}{2} < \alpha < 2$, it is unknown, but expected, that $\gamma(d = 3, \alpha, p) = 0$ for some p smaller than 3.

Due to a recent example in Du [14], (2.4) is sharp when $d - 1 \le \alpha < d$, and it determines the exact value of $\gamma(d, \alpha, p)$ for all $d - 1 \le \alpha < d$ and $1 \le p \le 2$. For $\alpha < d - 1$, it is expected that there is still room to improve the estimate (2.4). The key feature of the examples from [14] is that for $\alpha \in [m, m + 1]$ the corresponding examples are concentrated around hyperplanes of dimension m or $m + 1$. This explains in some way why (2.4) only gives sharp results for large α. When working towards answering Question 2.2 for small α, we need to explore new methods which can reduce the original question that in a much lower dimension.

Theorem 2.3 gives certain state-of-the-art results for several problems in PDEs and geometric measure theory, including size of the divergence set of Schrödinger solutions, Fourier decay rates of fractal measures, and Falconer distance set problem.

2.1. Divergence set of Schrödinger solutions

A natural refinement of Carleson's problem was initiated by Sjögren and Sjölin [38]: determine the size of the divergence set, in particular, consider

$$\alpha_n(s) := \sup_{f \in H^s(\mathbb{R}^n)} \dim\Big\{x \in \mathbb{R}^n : \lim_{t \to 0} e^{it\Delta} f(x) \ne f(x)\Big\},$$

where dim denotes the Hausdorff dimension. It is known that

$$\alpha_n(s) = \begin{cases} n, & \text{for } s \le \frac{n}{2(n+1)} \quad \text{(Bourgain [6], Lucà and Rogers [32])}, \\ n - 2s, & \text{for } \frac{n}{4} \le s \le \frac{n}{2} \quad \text{(Žubrinić [44], Barceló et al. [1])}. \end{cases}$$

An open problem is to determine $\alpha_n(s)$ for $n \geq 2$ and $\frac{n}{2(n+1)} < s < \frac{n}{4}$. For α in this range, the best known lower bounds (examples) are due to Lucà and Rogers [31, 32], and the best known upper bounds follow from Theorem 2.3:

$$\alpha_n(s) \leq n + 1 - \frac{2(n+1)s}{n} \quad \text{(Du and Zhang [20])}.$$

An improvement of estimate (2.4) will give a better upper bound of $\alpha_n(s)$. More precisely, if

$$\|Ef\|_{L^2(B^{n+1}(0,R);Hdx)} \lessapprox R^\gamma \|f\|_2, \quad \forall \alpha\text{-dimensional weight } H,\ \forall f \in L^2(\mathbb{P}^n, d\sigma),$$

then we have $\alpha_n(s) \leq \alpha_0$, where α_0 is the root for α to the equation $\gamma + \frac{n-\alpha}{2} = s$ [33].

2.2. Fourier decay rates of fractal measures

Let $\beta_d(\alpha, S)$ denote the average Fourier decay rate of fractal measures, which is defined as the supremum of the numbers β for which

$$\|\widehat{\mu}(R\cdot)\|^2_{L^2(S,d\sigma)} \lesssim c_\alpha(\mu)\|\mu\| R^{-\beta}, \tag{2.5}$$

whenever $R > 1$ and μ is an α-dimensional measure in $B^d(0,1)$. The problem of identifying the value of $\beta_d(\alpha, \mathbb{S}^{d-1})$ was proposed by Mattila [35].

In dimension two, the exact decay rates are known:

$$\beta_2(\alpha, S) = \begin{cases} \alpha, & \alpha \in (0, 1/2] \quad \text{(Mattila [34])}, \\ 1/2, & \alpha \in [1/2, 1] \quad \text{(Mattila [34])}, \\ \alpha/2, & \alpha \in [1, 2] \quad \text{(Wolff [43])}. \end{cases}$$

In higher dimensions, it is known that $\beta_d(\alpha, S) = \alpha$ in the range $\alpha \in (0, \frac{d-1}{2}]$, and

$$\beta_d(\alpha, \mathbb{P}^{d-1}) = \frac{(d-1)\alpha}{d} \quad \text{for } d \geq 3 \text{ and } d-1 \leq \alpha < d \quad \text{(Du and Zhang [20], Du [14])}.$$

In other cases, $\beta_d(\alpha, S)$ is still a mystery. The current best lower bounds are

$$\beta_d(\alpha, S) \geq \begin{cases} \alpha, & \alpha \in (0, \frac{d-1}{2}] \quad \text{(Mattila [34])}, \\ \frac{d-1}{2}, & \alpha \in [\frac{d-1}{2}, \frac{d}{2}] \quad \text{(Mattila [34])}, \\ \frac{(d-1)\alpha}{d}, & \alpha \in [\frac{d}{2}, d] \quad \text{(Du et al. [17, } d=3\text{], Du and Zhang [20, } d \geq 4\text{])}. \end{cases}$$

For upper bounds, the author's recent work [14] includes a summary. New upper bounds are obtained in [14] for $\beta_d(\alpha, \mathbb{S}^{d-1})$ with $d \geq 4$, $\alpha > \frac{d}{2}$, and for $\beta_d(\alpha, \mathbb{P}^{d-1})$ with $d \geq 3$, $\alpha > \frac{d-1}{2}$.

By a duality argument and Hölder's inequality, the weighted Fourier extension estimates (2.1) and $\beta_d(\alpha, S)$ are related as follows: if

$$\|Ef\|_{L^p(B^d(0,R);Hdx)} \lessapprox R^\gamma \|f\|_2, \quad \forall \alpha\text{-dimensional weight } H,\ \forall f \in L^2(S, d\sigma),$$

then $\beta_d(\alpha, S) \geq 2(\frac{\alpha}{p} - \gamma)$. Therefore, in order to determine the exact $\beta_d(\alpha, S)$, it is of particular interest to study Question 2.2 for fractional dimension $\alpha \in (\frac{d-1}{2}, d-1)$.

2.3. Falconer's distance set problem

The Falconer distance set conjecture, which is a famously difficult problem in geometric measure theory, is a continuous version of the celebrated Erdős distinct distance conjecture whose two-dimensional case was resolved by Guth and Katz [28]. The study of the Falconer problem is naturally related to Fourier restriction theory, projection theory of fractal measures, and incidence geometry. It has attracted a great amount of attention over the decades and has seen some very recent breakthroughs. See [17,18,20,27] and the references therein for more details.

Let $E \subset \mathbb{R}^d$ be a compact set. Its *distance set* $\Delta(E)$ is defined by $\Delta(E) := \{|x-y| : x, y \in E\}$.

Conjecture 2.4 (Falconer [24]). *Let $d \geq 2$ and $E \subset \mathbb{R}^d$ be a compact set. Then*

$$\dim(E) > \frac{d}{2} \Rightarrow \left|\Delta(E)\right| > 0.$$

Here $|\cdot|$ denotes the Lebesgue measure and $\dim(\cdot)$ is the Hausdorff dimension.

Different methods have been invented to lower the dimensional threshold. To name a few landmarks: in 1985, Falconer [24] showed that $|\Delta(E)| > 0$ if $\dim(E) > \frac{d}{2} + \frac{1}{2}$. This dimensional threshold has since been lowered gradually. It was further lowered by Wolff [43] to $\frac{4}{3}$ in the case $d = 2$, and by Erdoğan [22] to $\frac{d}{2} + \frac{1}{3}$ when $d \geq 3$. These records were recently broken with the following state-of-the-art thresholds:

$$\begin{cases} \frac{5}{4}, & d = 2 \quad \text{(Guth et al. [27])}, \\ \frac{9}{5}, & d = 3 \quad \text{(Du et al. [17])}, \\ \frac{d^2}{2d-1} = \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4}, & d \geq 3 \text{ and } d \text{ is odd} \quad \text{(Du and Zhang [20])}, \\ \frac{d}{2} + \frac{1}{4}, & d \geq 4 \text{ and } d \text{ is even} \quad \text{(Du et al. [18])}. \end{cases}$$

By a classical analytic approach of Mattila [34], we can approach Falconer's problem via Fourier decay rates of fractal measures and thus via weighted Fourier extension estimates. This is the route taken in many prior works, including [17,20,22,43]. More precisely, if

$$\|Ef\|_{L^p(B^d(0,R);Hdx)} \lessapprox R^\gamma \|f\|_2, \quad \forall \alpha\text{-dimensional weight } H, \ \forall f \in L^2(S, d\sigma),$$

then $|\Delta(E)| > 0$ if $\dim(E) > \alpha_0$, where α_0 is the root for α to the equation $\alpha = d - 2(\frac{\alpha}{p} - \gamma)$.

In a recent breakthrough by Guth et al. [27], they studied the two-dimensional Falconer problem, and developed a new method that modifies the original Mattila's approach. Their argument consists primarily of two steps. First, prune the natural Frostman measure μ on E by removing "bad" wave packets at different scales, and show that the error introduced in the pruning process can be controlled. Second, apply a refined decoupling inequality to estimate some L^2 quantity involving the pruned good measure.

The above arguments do not readily extend to higher dimensions. In [27], to verify that the pruned measure is close enough to the original Frostman measure, one applies a radial projection theorem of Orponen [37] that assumes the measure has dimension $\alpha > d - 1$. However, when $d \geq 3$, this condition fails to hold if α is close enough to $\frac{d}{2}$.

In a recent work Du et al. [18], we overcame this difficulty by introducing another ingredient into the process: orthogonal projections of the original measure. Combining orthogonal projections and Orponen's radial projection theorem, we were able to remove certain *bad* part from the original measure and approach Falconer's distance set problem via the following:

Question 2.5. Prove weighted L^2 Fourier extension estimates for *good* functions:

$$\|Ef\|_{L^2(B^d(0,R);Hdx)} \lessapprox R^{\gamma}\|f\|_2, \quad \forall\alpha\text{-dimensional weight } H, \ \forall \text{ good } f \in L^2(S, d\sigma).$$

By the techniques from [18], we can define good functions as follows: we say $f \in L^2(S, d\sigma)$ is *good* if in its wave packet decomposition $f = \sum_{T\in\mathbb{T}} f_T$ (here for each wave packet f_T, Ef_T is essentially supported on a tube T of dimensions $R^{1/2} \times \cdots \times R^{1/2} \times R$, and f_T is supported on a cap $\theta = \theta(T) \subset S$ of radius $R^{-1/2}$), for each tube $T \in \mathbb{T}$ with $f_T \neq 0$,

$$\int_T H(x)\,dx \lessapprox \begin{cases} R^{\alpha}R^{-\frac{d}{4}}, & d \text{ is even,} \\ R^{\alpha}R^{-\frac{d-1}{4}}, & d \text{ is odd.} \end{cases} \tag{2.6}$$

Note that since H is α-dimensional, we have that the total weight H on $B^d(0, R)$ is $\leq R^{\alpha}$. Condition (2.6) says that a function f is good if the weight H on each relative tube from the wave packet decomposition of f is just a small proportion of the total weight. Roughly speaking, all the relative tubes are *light*. To further improve the current results for Falconer's distance set problem, one may explore other tools from geometric measure theory which could help removing more *bad* parts from the original measure and so the functions under consideration are *good* at various scales in contrast with (2.6).

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