ON SOME PROPERTIES OF SPARSE SETS: A SURVEY

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ABSTRACT

Sparse sets are, by definition, sets that are small, either in cardinality, measure, dimension, or density. Curves, surfaces, and other submanifolds are standard examples of sparse sets in Euclidean space. However, many sparse sets naturally occurring in ergodic and geometric measure theory, such as Cantor-like sets or self-similar fractals, lack the regularity of the aforementioned objects. Despite this deficiency, many sparse sets are rich in arithmetic, geometric, and analytic properties that can be viewed as working substitutes for smoothness. This has led to a vibrant line of inquiry into the governing principles behind certain phenomena that are typically associated with submanifolds and that have the potential for ubiquity in far more general contexts. Structural and analytical properties of sparse sets, whether discrete or continuous, lie at the center of many problems in harmonic analysis, fractal geometry, combinatorics, and number theory. This is a survey of a few such problems that the author has worked on.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 42-02; Secondary 28-02, 28A80, 58C40, 05D10, 26A24

KEYWORDS

Hausdorff and Fourier dimension, differentiation theorems, growth of Laplace-Beltrami eigenfunctions on manifolds, Euclidean Ramsey theory

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 4, pp. 3224–3248
DOI 10.4171/ICM2022/153

Published by EMS Press

1. INTRODUCTION

Many problems in harmonic analysis and geometric measure theory, including restriction, Bochner–Riesz, Kakeya, Falconer conjectures, maximal operator bounds, and oscillatory integral estimates, are at heart questions involving size and structural properties of sparse sets. In classical formulations of the problems, these sparse sets are often lower-dimensional surfaces in Euclidean space, such as lines, curves, other submanifolds, zero sets of polynomials or real-analytic functions. For example, the famous Kakeya conjecture aims to quantify the size of a possibly small (i.e., Lebesgue-null) set that contain lines in every direction; intersection properties of lines or thin tubes (which are thickened versions of lines) are essential to its analysis. The restriction problem is a statement about the Lebesgue integrability of the Fourier transform of certain classes of functions in $\mathbb{R}^d$ after being restricted to a sphere; the latter hypersurface provides the geometric basis for this problem. Decay rates of multidimensional oscillatory integrals and integral operators are tied to critical points of their phases; if the phase function is polynomial or analytic in nature, the set of critical points is a semialgebraic set whose structure determines the asymptotic behavior of the integral. These avenues of research naturally use well-developed analytic and geometric notions, such as smoothness or curvature, of the underlying surfaces. A more recent research direction in harmonic analysis has been devoted to investigating extensions of classical results that are normally associated to surfaces and manifolds to the context of more general sets. More precisely, do similar results continue to hold for arbitrary Euclidean sets, possibly of fractional dimension, where tools like smoothness or curvature are unavailable, or have to be replaced by appropriate generalizations? To what extent are such results transferable between the discrete and continuum settings, and which features are unique? This in turn has led to a deeper investigation of the structure and content of sparse sets.

This article, arranged in three distinct and notationally self-contained parts, gives an overview of part of the author's work to date in this area, undertaken with many collaborators over the last decade and a half. The common theme is the study of sparse sets. Each section contains a brief motivation for the problems considered, and a statement of the main results. Proofs are delegated to the publications where the results appear. An in-depth discussion of the surrounding literature had to be scaled back due to space constraints, but the bibliography contains some important landmarks in the subject, as well as more exhaustive surveys.

2. MAXIMAL AVERAGES AND DIFFERENTIATION

2.1. Motivation of the problem

There is a vast literature on maximal and averaging operators over families of lower-dimensional submanifolds of $\mathbb{R}^d$. The aim is to quantify the behavior of such operators; for instance, what are the Lebesgue mapping properties of the maximal operator? What is the Lebesgue or Sobolev regularity of the averaging operator? For concreteness, we will focus here on the case of maximal operators over rescaled copies of a single submanifold. Assuming that the submanifold in question is sufficiently smooth, an important issue turns out to be its curvature. Roughly speaking, curved submanifolds admit nontrivial maximal estimates, whereas flat submanifolds do not. A fundamental and representative positive result is the *spherical maximal theorem*, due to Stein [90] for $d \geq 3$ and Bourgain [9] for $d = 2$. Recall that the spherical maximal operator in $\mathbb{R}^d$ is defined to be

$$\mathfrak{M}_{\mathbb{S}^{d-1}} f(x) := \sup_{r>0} \int_{\mathbb{S}^{d-1}} \left|f(x+ry)\right| d\sigma(y), \tag{2.1}$$

where σ is the normalized Lebesgue measure on the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$.

Theorem 2.1 (Stein [90], Bourgain [9]). *For any $d \geq 2$, the maximal operator $\mathfrak{M}_{\mathbb{S}^{d-1}}$ is bounded on $L^p(\mathbb{R}^d)$ for $p > \frac{d}{d-1}$, and this range of p is optimal.*

It is well known that Theorem 2.1 fails in all dimensions $d \geq 2$ if the sphere $\mathbb{S}^{d-1}$ is replaced by a polygonal line or the surface of a polytope. These geometric objects, while still piecewise smooth, do not have any curvature. In intermediate cases such as conical surfaces, which are flat along their generating rays but curved in other directions, maximal estimates may still be available but with weaker exponents. Many results of this type are known under varying smoothness and curvature conditions. We refer the reader to [19, 45, 46, 66–68, 88, 89, 91, 92] for an introduction to this prolific area of research and further references.

Stein's proof of the spherical maximal theorem for $d \geq 3$ exploits curvature only through the decay of the Fourier transform of the surface measure on the manifold, as do many of the other results just mentioned. Let us recall that for any finite measure μ on $\mathbb{R}^d$, its Fourier transform is defined as

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} d\mu(x). \tag{2.2}$$

In the case of the sphere, the Fourier transform decays like $|\xi|^{-\frac{d-1}{2}}$ at infinity; similar estimates hold for other convex hypersurfaces of codimension 1 with nonvanishing Gaussian curvature. The decay estimates are weaker for manifolds with flat directions, which in turn result in the restricted range of exponents in maximal and averaging estimates mentioned above.

The connection between the Fourier decay of a measure and the maximal average associated with it is exemplified in the following classical result of Rubio de Francia [73]. Given a measure μ, let us define its corresponding maximal operator

$$\mathfrak{M}_\mu f(x) := \sup_{r>0} \int \left|f(x+ry)\right| d\mu(y), \tag{2.3}$$

which is the supremal average of $|f(x+\cdot)|$ over all possible dilates of μ. If μ is the surface measure of $\mathbb{S}^{d-1}$, then $\mathfrak{M}_\mu$ coincides with the spherical maximal operator defined in (2.1).

Theorem 2.2 (Rubio de Francia [73]). *Suppose that μ is a compactly supported Borel measure on $\mathbb{R}^d$, $d \geq 1$, such that*

$$\left|\widehat{\mu}(\xi)\right| \leq C\left(1+|\xi|\right)^{-a} \tag{2.4}$$

for some $a > \frac{1}{2}$. Then the maximal operator $\mathfrak{M}_\mu$, defined as in (2.3) is bounded on $L^p(\mathbb{R}^d)$ for $p > (2a+1)/(2a)$.

Theorem 2.2 implies Theorem 2.1 for $d \geq 3$, since then the surface measure σ on the sphere obeys the above assumption with $a = \frac{d-1}{2} > \frac{1}{2}$. However, this Fourier decay is insufficient for $d = 2$. In other words, Theorem 2.2 fails to capture the circular maximal estimate in $\mathbb{R}^2$, for which $a = \frac{1}{2}$ just misses the stated range. Instead, Bourgain's proof of the circular maximal theorem for $d = 2$ relies more directly on the geometry involved. The relevant geometric information concerns intersections of pairs of δ-thickened circles, in other words, annuli of width δ. In an arrangement of many such annuli, most pairwise intersections are of the order δ^2, which is smaller by an order of magnitude than each annulus itself. While larger intersections are possible, Bourgain's proof shows that they do not occur frequently. An alternative proof of the circular maximal theorem that exploits finer properties of Fourier integral operators rather than Fourier decay can be found in [65].

2.2. Main results

The above class of results does not offer an easy extension to dimension one. Indeed, it is not clear a priori what a one-dimensional theory might look like, given that the real line has no nontrivial lower-dimensional submanifolds. However, given any $\varepsilon > 0$, there are many singular measures on $\mathbb{R}$ supported on sets of Hausdorff dimension $1 - \varepsilon$. Viewing ε as an analogue of "codimension," it is natural to ask whether by imposing additional structure on these sets that would assume the role of curvature, one might obtain L^p estimates similar to those in Theorem 2.1 for the associated maximal operators and for a range $p > p_\varepsilon$, where $p_\varepsilon \searrow 1$ as $\varepsilon \to 0$. Theorem 2.3 below, joint with Izabella Łaba, provides an affirmative answer to this question. Theorem 2.3 may be interpreted as the limiting situation as $\varepsilon \to 0$ (compare with Theorem 2.1 as $d \to \infty$) where the maximal range $(1, \infty]$ of p is achieved for a single set S of zero Lebesgue measure.

Theorem 2.3 ([59]). *For every $0 \leq \varepsilon < \frac{1}{3}$, there exists a probability measure $\mu = \mu(\varepsilon)$ supported on a Lebesgue-null set S of Hausdorff dimension $1 - \varepsilon$ such that $\mathfrak{M}_\mu$ is bounded on $L^p(\mathbb{R})$ for all $p > \frac{1+\varepsilon}{1-\varepsilon}$.*

The result above is one of many similar ones involving the operator on restricted sets and restricted scales. The interested reader is referred to [59] for other analogous statements concerning $\mathfrak{M}_\mu$ and its variants. As a consequence of Theorem 2.3, we obtain a differentiation theorem for μ that answers a question of Aversa and Preiss [2, 3, 69].

Theorem 2.4. *For $0 \leq \varepsilon < \frac{1}{3}$, let $\mu = \mu(\varepsilon)$ be the measure specified in Theorem 2.3. Then for every $f \in L^p(\mathbb{R})$ with $p \in ((1+\varepsilon)/(1-\varepsilon), \infty)$, we have*

$$\lim_{r \to 0} \left| \int f(x + ry)\, d\mu(y) - f(x) \right| = 0 \quad \text{for a.e. } x \in \mathbb{R}. \tag{2.5}$$

Thus for $\varepsilon = 0$, the measure $\mu = \mu(0)$ is supported on a full-dimensional set, is singular with respect to Lebesgue, and yet differentiates L^p in the sense of (2.5), as does the Lebesgue measure. Further, the maximal operator $\mathfrak{M}_\mu$ is bounded on the same Lebesgue spaces L^p (namely, $p \in (1, \infty)$) as the one-dimensional Hardy–Littlewood maximal function. However, unlike the Lebesgue measure, the measure $\mu = \mu(0)$ fails to differentiate L^1, as shown by Preiss [59, SECTION 8].

The measures $\mu = \mu(\varepsilon)$ in our results are constructed by randomizing a Cantor-type iteration. More precisely, we describe a random mechanism for building the nested Cantor iterates S_k as a union of finitely many intervals. The measure μ is then shown to be the weak-$*$ limit of the natural probability measures $1_{S_k}/|S_k|$, which are supported on $S = \bigcap_k S_k$.

It turns out that the proof of Theorem 2.3 does not use any Fourier decay conditions. Instead, the proof relies on geometric arguments akin to those in Bourgain's proof of the circular maximal theorem. The right substitute for Fourier decay turns out to be a correlation condition between affine copies of the sets S_k, providing the needed bound on the size of multiple intersections analogous to those arising in Bourgain's argument. Readers familiar with the proof of Theorem 2.1 for $d = 2$ or other similar results will recognize the correlation condition as a bound on the integrand (interpreted as the correlation function) in the expression for the L^n-norm of the dual linearized and discretized maximal operator, for large integer values of n. The proof attempts to minimize this integrand whenever possible through randomization arguments.

The threshold exponent $p_0 = (1+\varepsilon)/(1-\varepsilon)$ is suboptimal in general. Shmerkin and Suomala [82] have improved the range of p for random measures μ associated to an Ahlfors-regular variant of fractal percolation. Another improvement in a different direction is due to Łaba [57], who has obtained slightly weaker estimates for $\mathfrak{M}_\mu$, but for a much larger class of measures μ; in particular, her results apply to measures μ with self-similar supports of arbitrarily small Hausdorff dimension and no Fourier decay. Determining the Lebesgue boundedness of $\mathfrak{M}_\mu$ where μ is the Cantor–Lebesgue measure on the standard middle-third Cantor set remains an open problem.

3. SPARSE RESTRICTION OF LAPLACE–BELTRAMI EIGENFUNCTIONS

The study of eigenfunctions of Laplacians lies at the interface of several areas of mathematics, including analysis, geometry, mathematical physics, and number theory. These special functions arise in physics and in partial differential equations as modes of periodic vibration of drums and membranes. In quantum mechanics, they represent the stationary energy states of a free quantum particle on a Riemannian manifold.

Let (M, g) denote a compact, connected, n-dimensional Riemannian manifold without boundary, and $-\Delta_g$ the positive Laplace–Beltrami operator on M. It is well known [87, CHAPTER 3] that the spectrum of this operator is nonnegative and discrete. Let us denote its eigenvalues by $\{\lambda_j^2 : j \geq 0\}$, and the corresponding eigenspaces by $\mathbb{E}_j$. Without loss of generality, the positive square roots of the distinct eigenvalues can be arranged in increasing order, with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \lambda_j < \cdots \to \infty.$$

It is a standard fact [87, CHAPTER 3] that each $\mathbb{E}_j$ is finite-dimensional. Further, the space $L^2(M, dV_g)$ (consisting of functions on M that are square-integrable with respect to the canonical volume measure dV_g) admits an orthogonal decomposition in terms of $\mathbb{E}_j$:

$$L^2(M, dV_g) = \bigoplus_{j=0}^{\infty} \mathbb{E}_j.$$

One of the fundamental questions surrounding Laplace–Beltrami eigenfunctions targets their concentration phenomena, via high-energy asymptotics or high-frequency behavior. There are many avenues for this study, as exemplified in [1, 11, 13, 20, 32, 41, 61, 74, 79, 83, 99, 100]. One such approach involves studying the growth of the L^p-norms of these eigenfunctions as the eigenvalue goes to infinity. My joint work with Suresh Eswarathasan [27], the main focus of this section, lies in this category. Specifically, we describe the $L^2(M) \to L^p(\Gamma)$ mapping property of a certain spectral projector (according to the spectral decomposition above), where Γ is a Lebesgue-null subset of M. In particular, Γ does not enjoy any smooth structure, a point of departure from prior work where this feature was heavily exploited. We begin by reviewing the current research landscape that will help place the main results in context.

3.1. Motivation of the problem

The Weyl law in spectral theory provides an L^∞-bound on eigenfunctions on M [43]. The first results that establish L^p eigenfunction bounds for $p < \infty$ are due to Sogge [86].

Theorem 3.1 ([86]). *Given any manifold M as above and $p \in [2, \infty]$, there exists a constant $C = C(M, p) > 0$ such that the following inequality holds for all $\lambda \geq 1$:*

$$\|\varphi_\lambda\|_{L^p(M)} \leq C(1+\lambda)^{\delta(n,p)} \|\varphi_\lambda\|_{L^2(M)}, \quad \text{with} \tag{3.1}$$

$$\delta(n,p) = \begin{cases} \dfrac{n-1}{4} - \dfrac{n-1}{2p}, & \text{if } 2 \leq p \leq \dfrac{2(n+1)}{n-1}, \\ \dfrac{n-1}{2} - \dfrac{n}{p}, & \text{if } \dfrac{2(n+1)}{n-1} \leq p \leq \infty \end{cases}. \tag{3.2}$$

Here φ_λ is any eigenfunction of $-\Delta_g$ corresponding to the eigenvalue λ^2. The bound is sharp for the n-dimensional unit sphere $M = \mathbb{S}^n$, equipped with the surface measure.

Historically, an important motivation and source of inspiration for this line of investigation has been the Fourier restriction problem, which explores the behavior of the Fourier transform when restricted to curved surfaces in Euclidean spaces. In fact, the Stein–Tomas L^2-restriction theorem [96], originating in Euclidean harmonic analysis, was a key ingredient in an early proof of Theorem 3.1 for the sphere. Indeed, Theorem 3.1 may be viewed as a form of discrete restriction on M where the frequencies are given by the spectrum of the manifold, see, for example, [85]. Conversely, it is possible to recover the L^2-restriction theorem for the sphere from a spectral projection theorem such as Theorem 3.1 applied to the n-dimensional flat torus. The lecture notes of Yung [98, SECTION 2] contain a discussion of these implications.

Theorem 3.1 permits a number of independent proofs. For an argument that involves well-known oscillatory integral estimates of Hörmander applied to the smooth spectral projector (denoted $\rho(\sqrt{-\Delta_g} - \lambda)$), we refer the reader to the treatise [87]. The semiclassical approach of Koch, Tataru, and Zworski [54] has also yielded many powerful applications.

Finer information on eigenfunction growth may be obtained through L^p-bounds on φ_λ when restricted to smooth submanifolds of M. One expects φ_λ to assume large values on small sets. Thus its L^p-norm on a Lebesgue-null set such as a submanifold, if meaningful, is typically expected to be larger in comparison with the L^p-norm taken over the entire manifold M, as given by Theorem 3.1. The first step in this direction is due to Reznikov [70], who studied eigenfunction restriction phenomena on hyperbolic surfaces via representation-theoretic tools. The most general results to date on restricted norms of Laplace eigenfunctions are by Burq, Gérard, and Tzvetkov [14], and independently by Hu [44]. The work of Tacy [95] has extended these results to the setting of a semiclassical pseudodifferential operator (not merely the Laplacian) on a Riemannian manifold, while removing logarithmic losses at a critical threshold. Another particular endpoint result is due to Chen and Sogge [18]. We have summarized below the currently known best eigenfunction restriction estimates for a general manifold, combined from this body of work and for easy referencing later.

Theorem 3.2 ([14, 44, 95]). *Let $\Sigma \subset M$ be a smooth d-dimensional submanifold of M, equipped with the canonical measure $d\sigma$ that is naturally obtained from the metric g. Then for each $p \in [2, \infty]$, there exists a constant $C = C(M, \Sigma, p) > 0$ such that for any $\lambda \geq 1$ and any Laplace eigenfunction φ_λ associated with the eigenvalue λ^2, the following estimate holds:*

$$\|\varphi_\lambda\|_{L^p(\Sigma, d\sigma)} \leq C(1+\lambda)^{\delta(n,d,p)} \|\varphi_\lambda\|_{L^2(M, dV_g)}. \tag{3.3}$$

The exponent $\delta(n, d, p)$ admits a multipart description. Specifically,

$$\delta(n, n-1, p) = \begin{cases} \frac{n-1}{4} - \frac{n-2}{2p}, & \text{for } 2 \leq p \leq \frac{2n}{n-1}, \\ \frac{n-1}{2} - \frac{n-1}{p}, & \text{for } \frac{2n}{n-1} \leq p \leq \infty. \end{cases} \tag{3.4}$$

For $d \neq n-1$,

$$\delta(n, d, p) = \frac{n-1}{2} - \frac{d}{p}, \quad \text{for } 2 \leq p \leq \infty \text{ and } (d, p) \neq (n-2, 2). \tag{3.5}$$

For $(d, p) = (n-2, 2)$, the exponent $\delta(n, d, p)$ is still given by (3.5); however, there is an additional logarithmic factor $\log^{1/2}(\lambda)$ appearing in the right-hand side of inequality (3.3).

The proofs in [14] and [18] use a delicate analysis of oscillatory representations of the smoothed spectral projector $\chi(\sqrt{-\Delta_g} - \lambda)$ restricted to submanifolds Σ, combined with refined estimates influenced by the considered geometry. Alternatively, [44] uses general mapping properties for Fourier integral operators with prescribed degenerate canonical relations to obtain bounds for the oscillatory integral operators in question. There are several recurrent features in these proofs; namely, stationary phase methods, arguments involving integration by parts and operator-theoretic convolution inequalities. This methodology heavily relies on the fact that the underlying measures are induced by Lebesgue, which in turn is a consequence of M and Σ being smooth manifolds. We wanted to explore the accessibility of this machinery in the absence of smoothness, and to find working substitutes when such methods are unavailable.

There is a common theme in Theorems 3.1 and 3.2 above; namely, the left-hand side of both inequalities (3.1) and (3.3) involves the L^p-norm of an eigenfunction φ_λ but over different submanifolds of M (including M itself), and with respect to natural measures on these submanifolds. An interesting feature of the exponents $\delta(n, p)$ and $\delta(n, d, p)$ is that for large p, they are both of the form $(n-1)/2 - \alpha/p$, where

$$\alpha = \text{dimension of the space on which the } L^p\text{-norm of } \varphi_\lambda \text{ is measured}$$
$$= \begin{cases} \dim(M) = n & \text{in Theorem 3.1,} \\ \dim(\Sigma) = d & \text{in Theorem 3.2.} \end{cases} \tag{3.6}$$

In view of this commonality in (3.2), (3.4), and (3.5), we pose the following question:

- *Given an arbitrary Borel set $\Gamma \subseteq \Sigma$, does there exist a measure μ supported on Σ with respect to which we can estimate the growth of the eigenfunctions φ_λ?*

The nontrivial situation arises when Γ is Lebesgue-null, i.e., μ is singular with respect to the canonical measure on Σ. The optimal scenario would be to obtain bounds that reflect the dimensionality of the set Γ in the same way that Theorems 3.1 and 3.2 do. We answer this by presenting the main results of our article [27].

3.2. Main results

Given a compact n-dimensional Riemannian manifold (M, g), let $\Sigma \subseteq M$ be a smooth embedded submanifold of dimension $1 \le d \le n$, equipped with the restricted Riemannian metric naturally endowed by g. Let $(U, \mathfrak{u})$ be a local coordinate chart on Σ, where $U \subseteq \mathbb{R}^d$ is an open set containing $[0,1]^d$ and $\mathfrak{u} : U \to \mathfrak{u}(U) \hookrightarrow \Sigma$ is a smooth embedding. Given any $\varepsilon \in [0, 1)$, let $E \subseteq [0,1]^d$ be an arbitrary Borel set of Hausdorff dimension $\dim_{\mathbb{H}}(E) = d(1-\varepsilon)$. We refer the reader to the classical textbook of Mattila [64, CHAPTER 4] for the definitions and properties of Hausdorff dimension of sets in Euclidean spaces. The Borel set $E \subseteq [0,1]^d$ generates a corresponding Borel set $\Gamma = \Gamma[E]$ in Σ by setting $\Gamma := \mathfrak{u}(E)$. Conversely, every Borel subset Γ in $\mathfrak{u}([0,1]^d) \subseteq \Sigma$ can be identified with a set $E = \mathfrak{u}^{-1}(\Gamma) \subseteq [0,1]^d$. Similarly, any measure ν supported on Γ corresponds with a measure $\mu = \mathfrak{u}^*\nu$ on E via the pull-back $\mathfrak{u}^*$, i.e.,

$$\nu(A) := \mu\big(\mathfrak{u}^{-1}(A)\big) \quad \text{for all Borel sets } A \subseteq \Sigma. \tag{3.7}$$

The converse is also true; any Borel measure μ generates another measure ν on Γ through its push-forward, given by the same relation (3.7). Since $\mathfrak{u}$ is a diffeomorphism, and thus bi-Lipschitz, it preserves Hausdorff dimension [28, COROLLARY 2.4]; hence $\dim_{\mathbb{H}}(\Gamma) = \dim_{\mathbb{H}}(E) = d(1-\varepsilon)$. Let us define our critical exponent

$$p_0 = p_0(n, d, \varepsilon) := \frac{4d(1-\varepsilon)}{n-1}. \tag{3.8}$$

Our result below, representative of the class of results presented in [27], specifies a family of restricted eigenfunction estimates for every such set Γ.

Theorem 3.3 ([27]). *Let M, Σ, Γ and p_0 be as above. Then for every $\kappa > 0$ sufficiently small, there exists a probability measure $\nu^{(\kappa)}$ supported on Γ such that for all $\lambda \geq 1$ and all $p \in [2, \infty]$, we have the eigenfunction estimate*

$$\|\varphi_\lambda\|_{L^p(\Gamma,\nu^{(\kappa)})} \leq C_{\kappa,p}(1+\lambda)^{\theta_p}\Theta(\lambda;\kappa,p)\|\varphi_\lambda\|_{L^2(M,dV_g)}. \tag{3.9}$$

Here φ_λ denotes any L^2-eigenfunction associated with the eigenvalue λ^2 for the Laplace–Beltrami operator $-\Delta_g$ on M. For $p_0 > 2$, the exponent θ_p is given by

$$\theta_p = \theta_p(n,d,\varepsilon) := \begin{cases} \frac{n-1}{4} & \text{if } 2 \leq p \leq p_0 - \frac{4\kappa}{n-1}, \\ \frac{n-1}{2} - \frac{d(1-\varepsilon)}{p} & \text{if } p \geq p_0 - \frac{4\kappa}{n-1}. \end{cases} \tag{3.10}$$

The function Θ represents a power loss of κ/p beyond the critical threshold:

$$\Theta(\lambda;\kappa,p) := \begin{cases} 1 & \text{if } 2 \leq p \leq p_0 - \frac{4\kappa}{n-1}, \\ (1+\lambda)^{\frac{\kappa}{p}}(\log\lambda)^{\frac{1}{p}} & \text{if } p = p_0 - \frac{4\kappa}{n-1}, \\ (1+\lambda)^{\frac{\kappa}{p}} & \text{if } p > p_0 - \frac{4\kappa}{n-1}. \end{cases}$$

For $p_0 \leq 2$ and $2 \leq p \leq \infty$, we set

$$\theta_p = \theta_p(n,d,\varepsilon) := \frac{n-1}{2} - \frac{d(1-\varepsilon)}{p} \quad \text{and} \quad \Theta(\lambda;\kappa,p) = (1+\lambda)^{\frac{\kappa}{p}}.$$

The positive constant $C_{\kappa,p}$ in (3.9) may depend on $n, d, \varepsilon, \kappa$, and p, but is independent of λ. The probability measure $\nu^{(\kappa)}$ in (3.9), given by Frostman's lemma, obeys the following volume growth condition: there exists $C_\kappa > 0$ such that for all $x \in \Sigma$ and $r > 0$,

$$\nu^{(\kappa)}\big(B_g(x;r)\big) \leq C_\kappa r^{d(1-\varepsilon)-\kappa}, \tag{3.11}$$

where $B_g(x;r) \subseteq M$ denotes the Riemannian ball centered at x of radius r.

The estimate (3.9) is sharp for $p \geq \max(2, p_0)$, except possibly for the infinitesimal blow-up factor of $(1+\lambda)^{\kappa/p}$. More precisely, for every $\varepsilon \in [0,1)$, $d \leq n$, and $p \geq \max(2, p_0)$, the bound in (3.9) is realized, ignoring subpolynomial losses, for certain sets of dimension $d(1-\varepsilon)$ in $M = \mathbb{S}^n$. The estimate is not sharp for $2 \leq p < p_0$, when $p_0 > 2$. This is an artifact of our proof strategy.

For an arbitrary Borel set Γ, the information available about measures supported on it is limited. As a result, the measure $\nu^{(\kappa)}$ that realizes (3.9) varies with κ in general. Thus we are able to prove (3.9) only for all $\kappa > 0$ and not for $\kappa = 0$. On the other hand, if $\Gamma = M$ or if Γ is a submanifold of M, it follows from [14, 44, 87, 95] that there is a natural Lebesgue-induced measure on Γ for which (3.9) does hold with $\kappa = 0$. We show in [27] is that such an improvement holds in a generic sense. In Theorem 1.7 and Corollary 1.8 of [27], we provide a large class of sparse subsets Γ that are not submanifolds, each supporting *a single probability measure ν that obeys* (3.11) *for all $\kappa > 0$*, even though $C_\kappa \nearrow \infty$ as $\kappa \searrow 0$. For this measure ν, we show that a stronger version of (3.9) holds, with $\kappa = 0$. However,

Θ is then replaced by a function of slow growth in the range $p \geq \max(2, p_0)$. A precise functional form for Θ that quantifies the infinitesimal blowup is provided.

For $p \geq \max(2, p_0)$, the exponent θ_p in Theorem 3.3 is of the same form alluded to in (3.6), namely $\theta_p = (n-1)/2 - \alpha/p$ with $\alpha = d(1-\varepsilon) = \dim_{\mathbb{H}}(\Gamma)$. Thus our result may be viewed as a natural interpolation between the global estimates in [86] and the smooth restriction estimates in [14], bridging the estimates across a family of arbitrary Borel sets with continuously varying Hausdorff dimensions.

To the best of our knowledge, Theorem 3.3 is the first result of its kind in several distinct categories. First, it offers eigenfunction bounds restricted to *any Borel subset of positive Hausdorff dimension*, for every manifold M and every smooth submanifold Σ therein. Second, even for integers m, our result produces new sets of dimension m, for example, with $(n, d, \varepsilon) = (2, 2, 1/2)$, that are not necessarily contained in any m-dimensional submanifold, and yet capture the same eigenfunction growth bounds as smooth submanifolds of the same dimension, up to subpolynomial losses. Third, when $\varepsilon = 0$, our result provides examples of singular measures supported on submanifolds with respect to which the eigenfunctions obey the same L^p growth bounds, up to any prescribed κ-loss, as with the induced Lebesgue measure on the same submanifold.

The work of Burq, Gérard, and Tzvetkov [14, THEOREM 2] shows that when $n = 2$, $d = 1$ and Γ is a curve of nonvanishing geodesic curvature, Theorem 3.2 admits a significant improvement; namely the growth exponent $\delta(2, 1, p)$ can be replaced by the smaller exponent $\tilde{\delta}(2, 1, p) = 1/3 - 1/(3p)$ in the range $2 \leq p \leq 4$. The correct analogue of the nonvanishing curvature condition for an arbitrary sparse set Γ that would lead to similar improvements for Theorem 3.3 is as yet unknown.

4. CONFIGURATIONS IN SPARSE SETS

Another related field of research, at the interface of harmonic analysis, geometric measure theory, and fractal geometry, is the study of patterns or configurations in sparse sets. Questions here are typically of the following type: *Under what conditions must a small set contain a given pattern? Can it contain many? Can a large set avoid specified patterns? How can one quantify the patterns contained in a set?* Stated in this level of generality, these questions lack precision, both in the quantification of size and in the specification of patterns. “Large” could be interpreted in the context of cardinality, Lebesgue measure, asymptotic or Banach density, Hausdorff, Minkowski or Fourier dimension. “Patterns” could be geometric in nature, for example, arithmetic or geometric progressions, equilateral triangles, parallelograms; alternatively, they could be algebraic, such as solutions of certain equations. This line of investigation has a particularly rich history in number theory and additive combinatorics where the ambient space is often the space of integers, or subsets thereof. It has expanded into an active research area in the continuum setting within the last two decades. While the questions often look similar in the discrete and continuous regimes, the answers are sometimes very different.

4.1. Existence and avoidance of linear patterns

Can large sets avoid many patterns? Regardless of the many possible variants of such a question, it would seem that a natural answer would be "no," with any reasonable definition. Indeed, there is a large body of work that supports this intuition; see [5, 15, 35–37, 42, 58].

However, there are also many results in the literature that challenge this intuition, especially when slight variations in the notions of size lead to very different conclusions regarding the existence of patterns. For example, in the discrete setting, a classical result of Behrend [4] and Salem and Spencer [78] says that for any $\varepsilon > 0$ and all sufficiently large positive integers M, there exists a set $X_M \subseteq [M] := \{0, 1, 2, \dots, M-1\}$ such that $\#(X_M) > M^{1-\varepsilon}$ and X_M contains no nontrivial three-term arithmetic progression. This is in sharp contrast with the celebrated results of Roth [71, 72] and Szemerédi [93, 94], which state that for any $k \geq 3$ and any $c > 0$, there exists $M_0 \geq 1$ such that for $M \geq M_0$, any set $X_M \subseteq [M]$ obeying $\#(X_M) \geq cM$ contains a nontrivial k-term arithmetic progression. The work of Ruzsa [75, 76] on subsets of integers avoiding nontrivial solutions of linear equations has been particularly influential in subsequent research in additive combinatorics.

Similar results exist in the continuum as well. For instance, one can deduce from the Lebesgue density theorem that any set in $\mathbb{R}$ with a Lebesgue density point contains a nontrivial affine copy of any finite configuration. This conclusion applies therefore to any set of positive Lebesgue measure. On the other hand, Keleti [52] constructs a compact subset $E \subseteq [0, 1]$ with Hausdorff dimension 1 but Lebesgue measure zero such that there does not exist any nontrivial solution of $x - y = z - w$, with $x < y \leq z < w$ and $(x, y, z, w) \in E^4$. In particular, E avoids all three-term arithmetic progressions. Subsequent results [23, 30, 40, 52, 53, 62, 63] have explored the issue of avoidance further, providing examples of sets of large Hausdorff dimension that omit increasingly general families of algebraic and geometric patterns. Let us recall from [64, THEOREM 8.8] or [28, SECTION 4.1] that the *Hausdorff dimension* $\dim_{\mathbb{H}}(A)$ of a Borel set $A \subseteq \mathbb{R}^n$ is the supremum of exponents $\alpha > 0$ with the following property: there exists a probability measure μ supported on A such that for some positive, finite constant C_1,

$$\mu\big(B(x, r)\big) \leq C_1 r^{\alpha} \quad \text{for all } x \in \mathbb{R}^n,\ r > 0. \tag{4.1}$$

These results suggest a general rule of thumb: large Hausdorff dimension is usually not enough to ensure that a set contains a specified family of patterns.

On the other hand, the situation is expected to be different for sets A of large *Fourier dimension*. The Fourier dimension $\dim_{\mathbb{F}}(A)$ of a Borel set $A \subseteq \mathbb{R}^n$ is defined as the supremum of exponents $\beta \leq n$ obeying the following condition: there exist a probability measure μ supported on A and a positive finite constant C_2 such that

$$\big|\widehat{\mu}(\xi)\big| \leq C_2\big(1 + |\xi|\big)^{-\beta/2} \quad \text{for all } \xi \in \mathbb{R}^n, \quad \text{where } \widehat{\mu}(\xi) := \int e^{-ix\xi} d\mu(x). \tag{4.2}$$

Frostman's lemma [64, P. 168] states that $\dim_{\mathbb{F}}(A) \leq \dim_{\mathbb{H}}(A)$ for any Borel set A. This inequality implies that sets of large Fourier dimension form a smaller subclass within the class of sets of large Hausdorff dimension. It gives rise to the intuition that such sets are more likely to enjoy additional properties rooted in the Fourier decay of the supporting measures; in particular, they could possibly contain a richer class of patterns. A Borel set whose Fourier dimension equals its Hausdorff dimension is called a *Salem set*. One therefore hopes that a Salem set of large dimension may contain patterns that a non-Salem set of the same Hausdorff dimension does not. While this naive expectation turns out to be false in general (more on this in Section 4.2), there is a core of truth in this heuristic principle. In joint work with Yiyu Liang [60], we have made this precise. Our results in this direction form the main content of this subsection and the next.

The intuition that large Salem sets are richer in structure than their non-Salem counterparts of the same dimension is perhaps also due to the known examples of such sets. Salem sets are ubiquitous among random sets. Many random constructions yield sets that are, on the one hand, often (almost surely) Salem and, on the other hand, embody verifiable algebraic or geometric structure. The first such random construction is due to Salem himself [77]; many subsequent random constructions have appeared in [7,8,16,17,25,49,50,58,81]. Deterministic examples of Salem sets are comparatively fewer [38,39,47,48,51], but they arise naturally in number theory [6,12,24] and are rich in arithmetic patterns as well. The work of Körner [55,56], which explicitly addresses the relation between the rate of decay of the Fourier transform of a measure and possible algebraic relations within its support, is perhaps closest to the main theme of this section.

In the results to be stated shortly, we will provide a quantitative formulation of the heuristic principle that Salem sets possess richer structure, in the specific context of translation-invariant linear patterns. More precisely, we will be concerned with algebraic patterns that occur as a nontrivial zero of some function in the class

$$\mathscr{F} = \mathscr{F}(\mathbb{N}) := \bigcup_{v=2}^{\infty} \mathscr{F}_v(\mathbb{N}), \quad \text{where} \tag{4.3}$$

$$\mathscr{F}_v(\mathbb{N}) := \left\{ f(x_0, \dots, x_v) := m_0 x_0 - \sum_{i=1}^{v} m_i x_i \;\middle|\; \begin{array}{l} m_0, \dots, m_v \in \mathbb{N}, \quad m_0 = \sum_{i=1}^{v} m_i, \\ \gcd(m_0, m_1, \dots, m_v) = 1 \end{array} \right\}. \tag{4.4}$$

Here $v \in \mathbb{N} \setminus \{1\}$ and $\mathbb{N} := \{1, 2, \dots\}$.

Definition 4.1. Let us briefly review the patterns whose existence or avoidance we will explore in this section.

- Given $f \in \mathscr{F}_v(\mathbb{N})$, a vector $x = (x_0, x_1, \dots, x_v) \in \mathbb{R}^{v+1}$ is said to be a *zero of* f if it obeys the equation $f(x_0, \dots, x_v) = 0$. Such a vector x will also be referred to as a *solution* of the equation $f(x_0, \dots, x_v) = 0$.
- A zero $x = (x_0, \dots, x_v) \in \mathbb{R}^{v+1}$ of a function $f \in \mathscr{F}_v(\mathbb{N})$ is said to be *nontrivial* if the entries of x are all distinct. All other zeros of f are called *trivial*. The terms "trivial" and "nontrivial" apply with the same definition to solutions of equations of the form $f = 0$ as well.

- Given a set $E \subseteq \mathbb{R}$, we say that E *contains a nontrivial zero of* $f \in \mathscr{F}_v(\mathbb{N})$ if there exists $x = (x_0, x_1, \ldots, x_v) \in E^{v+1}$ with all distinct entries such that $f(x) = 0$. If no such $x \in E^{v+1}$ exists, we say that E *avoids all nontrivial zeros of* f.

- A set $E \subseteq \mathbb{R}$ is said to contain *a nontrivial translation-invariant rational linear pattern* if it contains a nontrivial zero of some $f \in \mathscr{F}$.

A three-term arithmetic progression (x_0, x_1, x_2) with nonzero common difference is a simple example of a nontrivial translation-invariant rational linear pattern, since it is a nontrivial zero of the function $f(x_0, x_1, x_2) = 2x_0 - (x_1 + x_2)$. If a vector

$$x = (x_0, \ldots, x_v) \in \mathbb{R}^{v+1}$$

is a trivial zero of some $f \in \mathcal{F}_v(\mathbb{N})$ with $v \geq 3$ but has at least two distinct entries, then the vector $y = (y_0, \ldots, y_{v'})$ consisting of the distinct entries of x provides a nontrivial zero of some $g \in \mathcal{F}_{v'}(\mathbb{N})$, $v' < v$.

In the following subsection, we provide answers to variants of the following question: *given $\mathscr{F}^* \subseteq \mathscr{F}(\mathbb{N})$, how large a set $E \subset \mathbb{R}$, in the sense of Fourier dimension, can one construct that avoids all the nontrivial zeros of all $f \in \mathscr{F}^*$? Alternatively, are sets of large enough Fourier dimension guaranteed to contain a nontrivial zero of some $f \in \mathscr{F}^*$?* The requirement $m_0 = \sum_{i=1}^{v} m_i$ in $\mathcal{F}_v(\mathbb{N})$ is designed to avoid trivial answers; without this assumption, one can always find an avoiding interval (of positive Lebesgue measure) centered around 1.

4.2. Main results

In [58, THEOREM 1.2], we showed, in joint work with Izabella Łaba, that if a compact set $A \subseteq [0, 1]$ supports a probability measure μ obeying a ball condition of the type (4.1) and a Fourier decay condition of the type (4.2), then A contains a nontrivial three-term arithmetic progression, provided (a) $\beta > 2/3$, (b) the constants C_1 and C_2 are appropriately controlled, and (c) the exponent α is sufficiently close to 1, depending on C_1, C_2, and β. The article [58, SECTION 7] also contains a large class of examples of Salem sets that verify the hypotheses of [58, THEOREM 1.2]. This leads to a natural question whether the technical growth conditions (b) on C_1, C_2 are truly necessary, and whether progressions exist in any set of large enough Fourier dimension. This naive expectation is, however, false. Shmerkin [80, THEOREMS A AND B] has recently proved the existence of a compact full-dimensional Salem set contained in [0, 1] that avoids all nontrivial arithmetic progressions. The existence of such a Salem set seems, at first glance, to contradict the conventional belief that such sets should enjoy richer structure.

4.2.1. Rational linear patterns

Our next three results show that even though a Salem set of large dimension can avoid a specific linear pattern (or even finitely many) given by $\mathcal{F}$, it cannot avoid all of them.

Theorem 4.2 ([60]). *Given $v \in \mathbb{N}$, $v \geq 2$, let $E \subseteq [0, 1]$ be a closed set satisfying $\dim_{\mathbb{F}}(E) > \frac{2}{v+1}$, i.e., there exist some $\beta > \frac{1}{v+1}$, a probability measure μ supported on E,* *and some positive constant C such that*

$$\left|\hat{\mu}(\xi)\right| \leq C\left(1+|\xi|\right)^{-\beta}. \tag{4.5}$$

Then E contains a nontrivial zero of some $f \in \mathscr{F}_v(\mathbb{N})$ defined in (4.4). *In other words, there exists $\{m_0, \dots, m_v\} \subseteq \mathbb{N}$ satisfying $m_0 = \sum_{i=1}^{v} m_i$, such that E contains a nontrivial solution of the equation*

$$\sum_{i=1}^{v} m_i x_i = m_0 x_0. \tag{4.6}$$

Corollary 4.3 ([60]). *Let $E \subseteq [0,1]$ be a closed set of positive Fourier dimension. Then E contains a nontrivial translation-invariant rational linear pattern, in the sense of Definition* 4.1.

We compare Theorem 4.2 with earlier results of Körner [55, 56]. For instance, in [56, LEMMA 2.3] he shows that if E is a subset of the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with $\dim_{\mathbb{F}}(E) > \frac{2}{(v+1)}$, then there exist integers $m_0, m_1, \dots, m_v \in \mathbb{Z}$, not all zero, and distinct points $x_0, x_1, \dots, x_v \in E$ such that

$$m_0 x_0 = m_1 x_1 + \cdots + m_v x_v \pmod 1. \tag{4.7}$$

A priori, one does not know the number of integers m_j that are zero in the above equation, the signs of the nonzero integers m_j and whether the equation is translation-invariant. On the other hand, the linear equations stemming from $\mathscr{F}_v(\mathbb{N})$ and underlying Theorem 4.2 are exact (not modulo integers), and the coefficients $m_0, \dots, m_v$ are all positive with the further constraint $m_0 = m_1 + \cdots + m_v$. Körner [56, THEOREM 2.4] also constructs a set $E \subseteq \mathbb{T}$ of Fourier dimension $1/v$ with the following property: there does not exist any nonzero vector $(m_0, \dots, m_v) \in \mathbb{Z}^{v+1}$ for which the equation (4.7) admits a nontrivial solution consisting of distinct points $x_0, x_1, \dots, x_v \in E$. Körner's construction, based on a Baire category argument, is nonexplicit. We ask the interested reader to compare Körner's construction of an avoiding set with the avoidance results in this paper (Theorems 4.5, 4.6, and 4.10). The sets that we construct avoid more restricted classes of equations but are of larger Fourier dimension.

As another point of contrast, we mention a construction of Keleti [53] that provides, for any countable set $T \subseteq (0,1)$, a subset $E \subseteq [0,1]$ of Hausdorff dimension 1 that does not contain any triple of distinct points $\{x, y, z\}$ such that $tx + (1-t)y = z$ for any $t \in T$. Choosing $v = 2$ and $T = \mathbb{Q} \cap (0,1)$, the set of rationals in $(0,1)$, we observe that $\dim_{\mathbb{F}}$ in Theorem 4.2 cannot be replaced by $\dim_{\mathbb{H}}$. Generalizing Keleti's result, Mathé [63] proves the existence of a rationally independent set in $\mathbb{R}$ of full Hausdorff dimension. We recall that a set $E \subseteq \mathbb{R}$ is *rationally independent* if for any integer $v \geq 2$ and any choice of distinct points $x_1, x_2, \dots, x_v \in E$,

$$\sum_{j=1}^{v} a_j x_j = 0 \quad \text{with } \{a_1, \dots, a_v\} \subseteq \mathbb{Z} \quad \text{implies} \quad a_1 = a_2 = \cdots = a_v = 0.$$

Theorem 4.2 implies that such sets cannot be Salem. Indeed, any set $E \subseteq \mathbb{R}$ of positive Fourier dimension will support a probability measure μ that satisfies (4.5) for some $v \in \mathbb{N}$

and some $\beta > 1/(v+1)$. By Theorem 4.2, it will contain a rationally dependent $(v+1)$-tuple of distinct points that obeys a relation of the form (4.6).

Corollary 4.4 ([60]). *There can be no rationally independent set in $\mathbb{R}$ of positive Fourier dimension.*

However, it is possible for a large Salem set to avoid nontrivial zeros of any *finite* sub-collection of $\mathscr{F}$, as our next result illustrates.

Theorem 4.5 ([60]). *Let $\mathscr{F}$ be as in (4.3). Given any finite collection $\mathscr{G} \subseteq \mathscr{F}$, there exists a set $E \subseteq [0,1]$ with $\dim_{\mathbb{F}} E = 1$ such that E contains no nontrivial zero of any $f \in \mathscr{G}$.*

Corollary 4.4 and Theorem 4.5 lead to a natural question: *does there exist a full-dimensional Salem set that avoids the nontrivial zeros of some countably infinite sub-collection of $\mathscr{F}$?* We answer this question in the affirmative; see Theorem 4.10. At the moment, we do not know how to characterize such subcollections.

Our next result attempts to strike a balance of a different sort between Theorems 4.2 and 4.5. While Theorem 4.2 dictates that a Salem set of large Fourier dimension must necessarily contain a nontrivial zero $(x_0, x_1, \dots, x_v)$ of some function $f \in \mathcal{F}_v$ and some $v \geq 2$, it a priori does not specify the diameter or spread of such a solution,

$$\operatorname{diam}(x_0, \dots, x_v) = \max\bigl\{|x_i - x_j|;\ i, j \in \{0, 1, \dots, v\}, i \neq j\bigr\},$$

which could in principle be very small; in other words, the nontrivial solution could be "almost trivial." We now show that it is possible to construct a full-dimensional Salem set that prohibits, in a quantifiable way, nontrivial zeros from being almost trivial.

Theorem 4.6 ([60]). *There exists a set $E \subseteq [0,1]$, $\dim_{\mathbb{F}} E = 1$ with the following property. For every $v \geq 2$ and every $f \in \mathcal{F}_v(\mathbb{N})$ defined as in (4.4), there exists $\kappa > 0$ such that whenever there exists a $(v+1)$-tuple $(x_0, x_1, \dots, x_v) \in E^{v+1}$ with*

$$\operatorname{diam}(x_0, x_1, \dots, x_v) < \kappa \quad \text{and} \quad f(x_0, x_1, \dots, x_v) = 0, \tag{4.8}$$

we have that $x_0 = x_1 = \cdots = x_v$. In particular, a nontrivial zero of f in E, if it exists, would obey $\operatorname{diam}(x_0, \dots, x_v) \geq \kappa$.

In addition, the constant $\kappa = \kappa_N$ can be chosen uniformly for all $f \in \mathscr{F}$ whose coefficients are bounded by N.

4.2.2. General linear patterns

The statements of Theorems 4.2 and 4.5 lead to an interesting possibility. Let $\mathscr{F}_v(\mathbb{R}_+)$ denote the class of translation-invariant linear functions in $(v+1)$ variables with real positive coefficients. Then $\mathscr{F}_v(\mathbb{R}_+)$ can be identified with the $(v-1)$-dimensional set

$$\mathscr{T}_v = \bigl\{\mathbf{t} \in (0,1)^{v-1} : t_1 + t_2 + \cdots + t_{v-1} < 1\bigr\}, \tag{4.9}$$

which is a half-space of $\mathbb{R}^{v-1}$ restricted to the open unit cube, via the map

$$\mathbf{t} = (t_1, \dots, t_{v-1}) \in \mathscr{T} \mapsto f_{\mathbf{t}} \in \mathscr{F}_v(\mathbb{R}_+), \quad \text{where}$$

$$f_{\mathbf{t}}(x) = x_0 - (t_1 x_1 + t_2 x_2 + \cdots + t_v x_v), \quad t_v = 1 - \sum_{i=1}^{v-1} t_i.$$

Under this map, the class $\mathscr{F}_v(\mathbb{N})$ is identified with the positive rationals in $\mathscr{T}_v$, and hence is of Hausdorff dimension zero. On the other hand, $\mathscr{F}_v(\mathbb{R}_+)$ is of positive $(v-1)$-dimensional Lebesgue measure. One is then led to ask: *Given a collection $\overline{\mathscr{F}} \subseteq \mathscr{F}_v(\mathbb{R}_+)$ that is of positive Lebesgue measure or large Hausdorff dimension under this identification, does there exist a set $E \subseteq \mathbb{R}$ of large Fourier dimension that avoids all nontrivial zeros of $\overline{\mathscr{F}}$?* In our next two theorems, we answer this question in the affirmative, in the special case of trivariate equations, where $v = 2$ and $\overline{\mathscr{F}}$ can be viewed as a subset of $(0, 1)$. In Theorem 4.7, the class $\overline{\mathscr{F}}$ is identified with a union of intervals, in Theorem 4.8 with collections of badly approximable numbers.

Theorem 4.7 ([60]). *Let us fix any $p \in \mathbb{N}$ with $p \geq 2$ and any $\alpha \in (0, 1)$. Then there exist some $\kappa = \kappa(p, \alpha) > 0$ and $E \subseteq [0, 1]$ with $\dim_{\mathbb{F}}(E) \geq \alpha$ such that E contains no nontrivial solution of*

$$tx + (1-t)y = z \quad \textit{for all } t \in \bigcup_{q=1}^{p-1} \left(\frac{q}{p} - \kappa, \frac{q}{p} + \kappa \right).$$

For fixed constants $0 < \tau, c \leq 1$, let us define the collection $\mathscr{E}_{c,\tau}$ of badly approximable numbers as follows:

$$\mathscr{E}_{c,\tau} := \left\{ t \in (0,1) : \left| t - \frac{q}{p} \right| > \frac{c}{p^{1+\tau}}, \text{ for all } \frac{q}{p} \in \mathbb{Q}, p \in \mathbb{N}, q \in \mathbb{Z}, \gcd(p,q) = 1 \right\}. \tag{4.10}$$

Sets of this type have applications in number theory, and their sizes have been widely studied. For example, if $\tau = 1$, then the Hausdorff dimension of $\mathscr{E}_{c,\tau}$ is of the order of $1 - O_c(1)$ as $c \to 0$. We refer the reader to [84, THEOREM 1.3] and the bibliography in this article for a survey of such results.

Theorem 4.8 ([60]). *For every $\varepsilon_0 \in (0, \frac{1}{2})$, there exists a set $E \subseteq [0, 1]$ with $\dim_{\mathbb{F}}(E) = \frac{1}{1+\tau}$ such that E contains no nontrivial solution of*

$$tx + (1-t)y = z, \quad \textit{for any } t \in \mathscr{E}_{c,\tau} \cap (\varepsilon_0, 1 - \varepsilon_0).$$

The combined strategies of Theorems 4.7 and 4.8 imply the following corollary.

Corollary 4.9 ([60]). *Let us fix $0 < \tau$, $c \leq 1$, $\varepsilon_0 \in (0, \frac{1}{2})$, and $p \in \mathbb{N} \setminus \{1\}$. Then for all sufficiently large $M \in \mathbb{N}$ and $\kappa = \frac{1}{2pM}$, there exists $E \subseteq [0, 1]$ with $\dim_{\mathbb{F}} E = \frac{1}{1+\tau}$ such that E contains no nontrivial solution of*

$$tx + (1-t)y = z, \quad \textit{for any } t \in \left[C \cap (\varepsilon_0, 1 - \varepsilon_0) \right] \cup \left[\bigcup_{q=1}^{p-1} \left(\frac{q}{p} - \kappa, \frac{q}{p} + \kappa \right) \right].$$

The sets of forbidden coefficients t in Theorems 4.7 and 4.8 are large, as a consequence of which the avoiding sets we obtain are not of full dimension. *Is it possible to* *construct a full-dimensional Salem set for which the set of forbidden coefficients is still quantifiably large?* Our next result provides an affirmative answer to this question, while also addressing the question posed after Theorem 4.5.

Theorem 4.10 ([60]). *There exists an infinite set* $\mathfrak{C} \subseteq (0,1)$ *and* $E \subseteq [0,1]$ *with* $\dim_{\mathbb{F}} E = 1$ *such that* E *contains no nontrivial solution of*

$$tx + (1-t)y = z \quad \textit{for any } t \in \mathfrak{C}. \tag{4.11}$$

The set $\mathfrak{C}$ *contains infinitely many rationals and uncountably many irrationals.*

It is natural to ask whether there exists a version of Shmerkin's theorem [80] or Theorem 4.5 for a finite but arbitrary collection of equations in $\mathscr{F}_v(\mathbb{R}_+)$; for instance, does there exist a full-dimensional Salem set E that contains no nontrivial solution of $tx + (1-t)y = z$, for any prespecified irrational $t \in (0,1)$? We are currently unable to provide an answer to this question. Also, the proof techniques of this paper are not immediately generalizable to other types of translation invariant equations, for example, when

$$\sum_{i=1}^{m} s_i x_i = \sum_{j=1}^{n} t_j y_j \quad \text{with} \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} t_j = 1, \quad 0 < s_i, t_j < 1 \text{ and } m, n \geq 2,$$

or when the equation is nonlinear, say $x_3 - x_1 = (x_2 - x_1)^2$. We are pursuing these directions in ongoing work.

4.3. A Roth-type result for dense Euclidean sets

A main objective of Ramsey theory is the study of geometric configurations in large, but otherwise arbitrary sets. A typical problem in this area reads as follows: given a set S, a family $\mathcal{F}$ of subsets of S and a positive integer r, is it true that any r-coloring of S yields some monochromatic configuration from $\mathcal{F}$? More precisely, for any partition of $S = S_1 \cup \cdots \cup S_r$ into r disjoint subsets, does there exist $i \in \{1, 2, \ldots, r\}$ and $F \in \mathcal{F}$ such that $F \subseteq S_i$? In discrete (respectively Euclidean) Ramsey theory, S is generally $\mathbb{Z}^d$ (respectively $\mathbb{R}^d$), and sets in $\mathcal{F}$ are geometric in nature. For example, if X is a fixed finite subset of $\mathbb{R}^d$, such as a collection of equally spaced collinear points or vertices of an isosceles right triangle, then $\mathcal{F} = \mathcal{F}(X)$ could be the collection of all isometric copies or all homothetic copies of X in S. A coloring theorem refers to a choice of S and $\mathcal{F}$ for which the answer to the above-mentioned question is yes. Such theorems are often consequences of sharper, more quantitative statements known as density theorems. A fundamental result with $S = \mathbb{N} = \{1, 2, \ldots\}$ is Szemerédi's theorem [93] (already mentioned in Section 4.1), which states that if $E \subseteq \mathbb{N}$ has positive upper density, i.e.,

$$\limsup_{N \to \infty} \frac{|E \cap \{1, \ldots, N\}|}{N} > 0,$$

then E contains a k-term arithmetic progression for every k. This in particular implies van der Waerden's theorem [34, 97], which asserts that given $r \geq 1$, any r-coloring of $\mathbb{N}$ must produce a k-term monochromatic progression, i.e., a homothetic copy of $\{1, 2, \ldots, k\}$. This subsection is devoted to joint work with Brian Cook and Akos Magyar [22], where we are concerned with certain density theorems in Ramsey theory over $\mathbb{R}^d$.

4.3.1. Motivation of the problem

A basic and representative result of the type we are interested in states that, with $d \geq 2$, a set $A \subseteq \mathbb{R}^d$ of positive upper Banach density contains all large distances, i.e., for every sufficiently large $\lambda \geq \lambda_0(A)$ there are points $x, x+y \in A$ such that $\|y\|_2 = \lambda$. Recall that the positive upper Banach density of A is defined as

$$\bar{\delta}(A) := \lim_{N \to \infty} \sup_{x \in \mathbb{R}^d} \frac{|A \cap (x + [0, N]^d)|}{N^d}.$$

The quoted result was obtained independently, along with various generalizations, by a number of authors, for example, Furstenberg, Katznelson, and Weiss [31], Falconer and Marstrand [29], and Bourgain [10].

To paraphrase, the above result shows that for any two-point configuration $X \in \mathbb{R}^d$ we are guaranteed the existence, up to congruence, of *all sufficiently large* dilates of X inside of A. The term configuration simply refers to a finite point set. From this point of view, it is a natural question to ask is whether similar statements exist that involve configurations with a greater number of points. If one looks for *some* (rather than every) sufficiently large dilate of a given configuration, such results are well known in the discrete regime of the integer lattice, under suitable assumptions of largeness on the underlying set. These results can often be translated to existence of configurations in the Euclidean setting as well. For instance, Roth's theorem [71] in the integers states that a subset of $\mathbb{Z}$ of positive upper density contains a three-term arithmetic progression $\{x, x+y, x+2y\}$, and it easily implies that a measurable set $A \subseteq \mathbb{R}$ of positive upper density contains a three-term progression whose gap size can be arbitrarily large. Results ensuring *all sufficiently large* dilates of a configuration in a set of positive Banach density are stronger, and their proofs typically more difficult. Bourgain [10] shows that if $X = \{x_1, \ldots, x_k\}$ is any nondegenerate k-point simplex in $\mathbb{R}^d$, $d \geq k \geq 2$ (i.e., if $\{x_2 - x_1, \ldots, x_k - x_1\}$ spans a $(k-1)$-dimensional space), then any subset of $\mathbb{R}^d$ of positive upper Banach density contains a congruent copy of λX for *all sufficiently large* λ.

On the other hand, a simple example given in [10] shows that there is a set $A \subseteq \mathbb{R}^d$ in any dimension $d \geq 1$, such that the gap lengths of all 3-progressions in A do not contain all sufficiently large numbers. In other words, the result of [10] is false for the degenerate configuration $X = \{0, e_1, 2e_1\}$, where e_1 is the canonical unit vector in the x_1-direction. More precisely, the counterexample provided in [10] is the set A of points $x \in \mathbb{R}^d$ such that $|\|x\|_2^2 - m| \leq \frac{1}{10}$ for some $m \in \mathbb{N}$. The parallelogram identity

$$2\|y\|_2^2 = \|x\|_2^2 + \|x+2y\|_2^2 - 2\|x+y\|_2^2$$

then dictates that $|\|y\|_2^2 - \frac{\ell}{2}| \leq \frac{4}{10}$ (for some $\ell \in \mathbb{N}$) for any progression $\{x, x+y, x+2y\} \subseteq A$. Thus the squares of the gap lengths are restricted to lie close to the half-integers, and therefore cannot realize all sufficiently large numbers.

The counterexample above has an interesting connection with a result in Euclidean Ramsey theory due to Erdős et al. [26]. Let us recall [33] that a finite point set X is said to be *Ramsey* if for every $r \geq 1$, there exists $d = d(X, r)$ such that any r-coloring of $\mathbb{R}^d$ contains a congruent copy of X. A result in [26] states that every Ramsey configuration X is spherical, i.e., the points in X lie on an Euclidean sphere. (The converse statement is currently an open conjecture due to Graham [33]). Since a set of three collinear points is nonspherical, it is natural to ask whether Bourgain-type counterexamples exist for any nonspherical X. This question was posed by Furstenberg and answered in the affirmative by Graham [33].

Theorem 4.11 (Graham [33]). *Let X be a finite nonspherical set. Then for any $d \geq 2$, there exist a set $A \subseteq \mathbb{R}^d$ with $\bar{\delta}(A) > 0$ and a set $\Lambda \subset \mathbb{R}$ with $\underline{\delta}(\Lambda) > 0$ so that A contains no congruent copy of λX for any $\lambda \in \Lambda$.*

4.3.2. Main result

It is interesting to observe that while Bourgain's counterexample prevents an existence theorem for three term arithmetic progressions of all sufficiently large *Euclidean* gaps, it does not exclude the validity of such a result when the gaps are measured using some other metric on $\mathbb{R}^d$ that does not obey the parallelogram law. In [22] we prove that such results do indeed exist for the l^p metrics $\|y\|_p := (\sum_{i=1}^d |y_i|^p)^{1/p}$ for all $1 < p < \infty$, $p \neq 2$. In this sense, a counterexample as described above is more the exception rather than the rule.

Variations of our arguments also work for other metrics given by specific classes of positive homogeneous polynomials of degree at least 4 and those generated by symmetric convex bodies with special structure. Results of the first type were obtained in the finite field setting by Cook and Magyar [21]. Also, the arguments here can be applied to obtain similar results for certain other degenerate point configurations.

Theorem 4.12 ([22]). *Let $1 < p < \infty$, $p \neq 2$. Then there exists a constant $d_p \geq 2$ such that for $d \geq d_p$ the following holds. Any measurable set $A \subseteq \mathbb{R}^d$ of positive upper Banach density contains a three-term arithmetic progression $\{x, x+y, x+2y\} \subseteq A$ with gap $\|y\|_p = \lambda$ for all sufficiently large $\lambda \geq \lambda(A)$.*

The result is sharp in the range of p. Easy variants of the example in [10] show that Theorem 4.12 and in fact even the two-point results of [10,29,31] cannot be true for $p = 1$ and $p = \infty$. Indeed, if $A = \mathbb{Z}^d + \varepsilon_0[-1,1]^d$ for some small $\varepsilon_0 > 0$, then, on the one hand, A is of positive upper Banach density. On the other hand, if $x, x+y \in A$ for some $y \neq 0$, then both $\|y\|_\infty$ and $\|y\|_1$ are restricted to lie within distance $O(\varepsilon_0)$ from some positive integer.

Indeed, counterexamples similar to [10] and the above can be constructed for norms given by a symmetric, convex body, a nontrivial part of whose boundary is either flat or coincides with an l^2-sphere. An appropriate formulation of a positive result for a general norm, and indeed the measurement of failure of the parallelogram law for such norms, remains an interesting open question.

We do not know whether the p-dependence of the dimensional threshold d_p stated in the theorem is an artifact of our proof. In our analysis, d_p grows without bound as $p \nearrow \infty$, while other implicit constants involved in the proof blow up near $p = 1$ and $p = 2$. It would be of interest to determine whether Theorem 4.12 holds for all $d \geq 2$ for the specified values of p.

Since three distinct collinear points cannot lie on an l^p-sphere for any $p \in (1, \infty)$, Theorem 4.12 shows that a result of the type considered by Graham in [33] is in general false for an l^p-sphere if $p \neq 1, 2, \infty$. Thus any connection between Ramsey-like properties and the notion of sphericality appears to be a purely l^2 phenomenon.

ACKNOWLEDGMENTS

I am deeply grateful to all my collaborators, whose generous insights have shaped my work and mathematical tastes. Conversations with the undergraduate and graduate students and postdoctoral fellows at UBC have been a constant source of joy and inspiration that have propelled my research—my warm thanks to all of them. Parts of the work took place in workshops and conferences hosted by the Fields Institute (2008), Banff International Research Station (2010, 2019), Mathematical Sciences Research Institute (2017), and Park City Mathematics Institute (2018). The creative space and facilities enabled by these institutes are gratefully acknowledged.

FUNDING

The author's work described in this article was partially supported by three consecutive Discovery grants (2007–2022) from the Natural Sciences and Engineering Research Council of Canada (NSERC), a Ruth E. Michler Fellowship from the Association for Women in Mathematics and Cornell University (2015), a scholarship from the Peter Wall Institute of Advanced Study (2018–2019) and a Simons Fellowship (2019–2020).

REFERENCES

[1] N. Anantharaman, Entropy and the localization of eigenfunctions. *Ann. of Math. (2)* **168** (2008), no. 2, 435–475.

[2] V. Aversa and D. Preiss, Hearts density theorems. *Real Anal. Exchange* **13** (1987), no. 1, 28–32.

[3] V. Aversa and D. Preiss, *Sistemi di derivazione invarianti per affinita*. Preprint (unpublished), Complesso Universitario di Monte S. Angelo, Napoli, 1995.

[4] F. A. Behrend, On sets of integers which contain no three terms in arithmetical progression. *Proc. Nat. Acad. Sci.* **32** (1946), 331–332.

[5] M. Bennett, A. Iosevich, and K. Taylor, Finite chains inside thin subsets of $\mathbb{R}^d$. *Anal. PDE* **9** (2016), no. 3, 597–614.

[6] A. S. Besicovitch, Sets of fractional dimensions (IV): on rational approximation to real numbers. *J. Lond. Math. Soc.* **9** (1934), no. 2, 126–131.

[7] C. Bluhm, Random recursive construction of Salem sets. *Ark. Mat.* **34** (1996), 51–63.

[8] C. Bluhm, On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets. *Ark. Mat.* **36** (1998), no. 2, 307–316.

[9] J. Bourgain, Averages in the plane over convex curves and maximal operators. *J. Anal. Math.* **47** (1986), 69–85.

[10] J. Bourgain, A Szemerédi type theorem for sets of positive density in $\mathbb{R}^k$. *Israel J. Math.* **54** (1986), no. 3, 307–316.

[11] J. Bourgain and Z. Rudnick, Restriction of toral eigenfunctions to hypersurfaces and nodal sets. *Geom. Funct. Anal.* **22** (2012), 878–937.

[12] J. D. Bovey and M. M. Dodson, The Hausdorff dimension of systems of linear forms. *Acta Arith.* **45** (1986), no. 4, 337–358.

[13] N. Burq, P. Gérard, and N. Tzetkov, Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations. *Ann. Sci. Éc. Norm. Supér. (4)* **38** (2005), 255–301.

[14] N. Burq, P. Gérard, and N. Tzetkov, Restrictions of the Laplace–Beltrami eigenfunctions to submanifolds. *Duke Math. J.* **138** (2007), no. 3, 445–487.

[15] V. Chan, I. Łaba, and M. Pramanik, Point configurations in sparse sets. *J. Anal. Math.* **128** (2016), no. 1, 289–335.

[16] X. Chen, Sets of Salem type and sharpness of the L^2-Fourier restriction theorem. *Trans. Amer. Math. Soc.* **368** (2016), no. 3, 1959–1977.

[17] X. Chen and A. Seeger, Convolution powers of Salem measures with applications. *Canad. J. Math.* **69** (2017), no. 2, 284–320.

[18] X. Chen and C. Sogge, A few endpoint geodesic restriction estimates for eigenfunctions. *Comm. Math. Phys.* **329** (2014), 435–459.

[19] M. Christ, A. Nagel, E. M. Stein, and S. Wainger, Singular and maximal Radon transforms: analysis and geometry. *Ann. of Math.* **150** (1999), 489–577.

[20] Y. Colin de Verdière, Ergodicité et fonctions propres du Laplacien. *Comm. Math. Phys.* **102** (1985), 497–502.

[21] B. Cook and Á. Magyar, On restricted arithmetic progressions over finite fields. *Online J. Anal. Comb.* **7** (2012), 1–10.

[22] B. Cook, Á. Magyar, and M. Pramanik, A Roth-type theorem for dense subsets of $\mathbb{R}^d$. *Bull. Lond. Math. Soc.* **49** (2017), no. 4, 676–689.

[23] J. Denson, M. Pramanik, and J. Zahl, Large sets avoiding rough patterns. 2019, arXiv:1904.02337. Springer volume "Harmonic Analysis and Applications", edited by Michail Rassias.

[24] H. G. Eggleston, Sets of fractional dimensions which occur in some problems of number theory. *Proc. Lond. Math. Soc.* **54** (1952), no. 2, 42–93.

[25] F. Ekström, Fourier dimension of random images. *Ark. Mat.* **54** (2016), no. 2, 455–471.

[26] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. H. Spencer, and E. G. Straus, Euclidean Ramsey theorems. *J. Combin. Theory Ser. A* **14** (1973), 341–363.

[27] S. Eswarathasan and M. Pramanik, Restriction of Laplace–Beltrami eigenfunctions to arbitrary sets on manifolds. *Int. Math. Res. Not.* **rnaa167** (2020), DOI 10.1093/imrn/rnaa167.

[28] K. Falconer, *Fractal geometry: mathematical foundations and applications*. 1st edn., Wiley Brothers, 1990.

[29] K. Falconer and J. Marstrand, Plane sets with positive density at infinity contain all large distances. *Bull. Lond. Math. Soc.* **18** (1986), no. 5, 471–474.

[30] R. Fraser and M. Pramanik, Large sets avoiding patterns. *Anal. PDE* **11** (2018), no. 5, 1083–1111.

[31] H. Furstenberg, Y. Katznelson, and B. Weiss, Ergodic theory and configurations in sets of positive density. In *Mathematics of Ramsey theory*, pp. 184–198, Springer, Berlin–Heidelberg, 1990.

[32] P. Gérard and É. Leichtnam, Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.* **71** (1993), 559–607.

[33] R. L. Graham, Recent trends in Euclidean Ramsey theory. *Discrete Math.* **136** (1994), 119–127.

[34] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey theory*. 2nd edn., John Wiley, New York, 1990.

[35] A. Greenleaf and A. Iosevich, On triangles determined by subsets of the Euclidean plane, the associated bilinear operators and applications to discrete geometry. *Anal. PDE* **5** (2012), no. 2, 397–409.

[36] A. Greenleaf, A. Iosevich, B. Liu, and E. Palsson, A group-theoretic viewpoint on Erdős–Falconer problems and the Mattila integral. *Rev. Mat. Iberoam.* **31** (2015), no. 3, 799–810.

[37] A. Greenleaf, A. Iosevich, and M. Pramanik, On necklaces inside thin subsets of $\mathbb{R}^d$. *Math. Res. Lett.* **24** (2017), no. 2, 347–362.

[38] K. Hambrook, Explicit Salem sets in $\mathbb{R}^2$. *Adv. Math.* **311** (2017), 634–648.

[39] K. Hambrook, Explicit Salem sets and applications to metrical Diophantine approximation. *Trans. Amer. Math. Soc.* **371** (2019), no. 6, 4353–4376.

[40] V. Harangi, T. Keleti, G. Kiss, P. Maga, A. Mathé, P. Mattila, and B. Strenner, How large dimension guarantees a given angle? *Monatsh. Math.* **171** (2013), no. 2, 169–187.

[41] B. Helffer, A. Martinez, and D. Robert, Ergodicité et limite semi-classique. *Comm. Math. Phys.* **109** (1987), 313–326.

[42] K. Henriot, I. Łaba, and M. Pramanik, On polynomial configurations in fractal sets. *Anal. PDE* **9** (2016), no. 5, 1153–1184.

[43] L. Hörmander, Spectral function of an elliptic operator. *Acta Math.* **121** (1968), 193–218.

[44] R. Hu, L^p norm estimates of eigenfunctions restricted to submanifolds. *Forum Math.* **21** (2009), 1021–1052.

[45] A. Iosevich and E. Sawyer, Maximal averages over surfaces. *Adv. Math.* **132** (1997), 46–119.

[46] A. Iosevich and E. Sawyer, Three problems motivated by the average decay of the Fourier transform. In *Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001)*, pp. 205–215, Contemp. Math. 320, Amer. Math. Soc., Providence, RI, 2003.

[47] V. Jarník, Diophantischen Approximationen und Hausdorffsches Mass. *Mat. Sb.* **36** (1929), 371–382.

[48] V. Jarník, Über die simultanen diophantischen Approximationen (German). *Math. Z.* **33** (1931), no. 1, 505–543.

[49] J. P. Kahane, Images browniennes des ensembles parfaits. *C. R. Acad. Sci. Paris, Sér. A–B* **263** (1966), A613–A615.

[50] J. P. Kahane, Images d'ensembles parfaits par des séries de Fourier gaussiennes. *C. R. Acad. Sci. Paris, Sér. A–B* **263** (1966), A678–A681.

[51] R. Kaufman, On the theorem of Jarník and Besicovitch. *Acta Arith.* **39** (1981), 265–267.

[52] T. Keleti, A 1-dimensional subset of the reals that intersects each of its translates in at most a single point. *Real Anal. Exchange* **24** (1998/1999), no. 2, 843–844.

[53] T. Keleti, Construction of one-dimensional subsets of the reals not containing similar copies of given patterns. *Anal. PDE* **1** (2008), no. 1, 29–33.

[54] H. Koch, D. Tataru, and M. Zworski, Semiclassical L^p estimates. *Ann. Henri Poincaré* **5** (2007), 885–916.

[55] T. Körner, Measure on independent sets, a quantitative version of a theorem of Rudin. *Proc. Amer. Math. Soc.* **135** (2007), no. 12, 3823–3832.

[56] T. Körner, Fourier transforms of measures and algebraic relations on their support. *Ann. Inst. Fourier (Grenoble)* **59** (2009), no. 4, 1291–1319.

[57] I. Łaba, Maximal operators and decoupling for $\Lambda(p)$ Cantor measures. 2018, arXiv:1808.05657.

[58] I. Łaba and M. Pramanik, Arithmetic progressions in sets of fractional dimension. *Geom. Funct. Anal.* **19** (2009), no. 2, 429–456.

[59] I. Łaba and M. Pramanik, Maximal operators and differentiation theorems for sparse sets. *Duke Math. J.* **158** (2011), no. 3, 347–410.

[60] Y. Liang and M. Pramanik, Fourier analysis and avoidance of linear patterns. 2020, arXiv:2006.10941.

[61] E. Lindenstrauss, Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2)* **163** (2006), 165–219.

[62] P. Maga, Full dimensional sets without given patterns. *Real Anal. Exchange* **36** (2010/2011), 79–90.

[63] A. Máthé, Sets of large dimension not containing polynomial configurations. *Adv. Math.* **316** (2017), 691–709.

[64] P. Mattila, Geometry of sets and measures in Euclidean space, Cambridge Stud. Adv. Math., Cambridge Univ. Press, Cambridge, 1995.

[65] G. Mockenhaupt, A. Seeger, and C. Sogge, Wave front sets, local smoothing and Bourgain's circular maximal theorem. *Ann. of Math.* **136** (1992), 207–218.

[66] A. Nagel, A. Seeger, and S. Wainger, Averages over convex hypersurfaces. *Amer. J. Math.* **115** (1993), 903–927.

[67] D. H. Phong and E. M. Stein, Hilbert integrals, singular integrals, and Radon transforms I. *Acta Math.* **157** (1986), 99–157.

[68] D. H. Phong and E. M. Stein, Hilbert integrals, singular integrals, and Radon transforms II. *Invent. Math.* **86** (1986), 75–113.

[69] D. Preiss, Lectures given in Ravello, 1985. Unpublished.

[70] A. Reznikov, Norms of geodesic restrictions for eigenfunctions on hyperbolic surfaces and representation theory. Unpublished update to a work from 2005. 2010, arXiv:math/0403437v3.

[71] K. Roth, On certain sets of integers. *J. Lond. Math. Soc.* **28** (1953), 104–109.

[72] K. Roth, Irregularities of sequences relative to arithmetic progressions, IV. *Period. Math. Hungar.* **2** (1972), 301–306.

[73] J. L. Rubio de Francia, Maximal functions and Fourier transforms. *Duke Math. J.* **53** (1986), 395–404.

[74] Z. Rudnick and P. Sarnak, The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.* **161** (1994), no. 1, 195–213.

[75] I. Ruzsa, Solving a linear equation in a set of integers I. *Acta Arith.* **65** (1993), no. 3, 259–282.

[76] I. Ruzsa, Solving a linear equation in a set of integers II. *Acta Arith.* **72** (1995), no. 4, 385–397.

[77] R. Salem, On singular monotonic functions whose spectrum has a given Hausdorff dimension. *Ark. Mat.* **1** (1951), no. 4, 353–365.

[78] R. Salem and D. C. Spencer, On sets of integers which contain no three terms in arithmetical progression. *Proc. Natl. Acad. Sci. USA* **28** (1942), 561–563.

[79] P. Sarnak, Arithmetic quantum chaos. In *The Schur lectures (Tel Aviv, 1992)*, pp. 183–236, Israel Math. Conf. Proc. 8, Bar-Ilan Univ., Ramat Gan, Israel, 1995.

[80] P. Shmerkin, Salem sets with no arithmetic progressions. *Int. Math. Res. Not.* **7** (2017), 1929–1941.

[81] P. Shmerkin and V. Suomala, Spatially independent martingales, intersections, and applications. *Mem. Amer. Math. Soc.* **251** (2018), no. 1195, v+102 pp.

[82] P. Shmerkin and V. Suomala, New bounds on Cantor maximal operators. 2021, arXiv:2106.14818.

[83] A. I. Shnirelman, Ergodic properties of eigenfunctions. *Uspekhi Mat. Nauk* **29** (1974), no. 6, 181–182 (in Russian).

[84] D. Simmons, A Hausdorff measure version of the Jarník–Schmidt theorem in Diophantine approximation. *Math. Proc. Cambridge Philos. Soc.* **164** (2018), no. 3, 413–459.

[85] C. Sogge, Oscillatory integrals and spherical harmonics. *Duke Math. J.* **53** (1986), 43–65.

[86] C. Sogge, Concerning the L^p norms of spectral clusters of second-order elliptic operators on compact manifolds. *J. Funct. Anal.* **77** (1988), 123–138.

[87] C. Sogge, *Fourier integrals in classical analysis*. Cambridge Tracts in Math., Cambridge Univ. Press, Cambridge, 1993.

[88] C. Sogge and E. M. Stein, Averages of functions over hypersurfaces in $\mathbb{R}^n$. *Invent. Math.* **82** (1985), 543–556.

[89] C. Sogge and E. M. Stein, Averages over hypersurfaces II. *Invent. Math.* **86** (1986), 233–242.

[90] E. M. Stein, Maximal functions: spherical means. *Proc. Natl. Acad. Sci. USA* **73** (1976), 2174–2175.

[91] E. M. Stein, *Harmonic analysis*. Princeton Univ. Press, Princeton, 1993.

[92] E. M. Stein and S. Wainger, Problems in harmonic analysis related to curvature. *Bull. Amer. Math. Soc.* **84** (1978), 1239–1295.

[93] E. Szemerédi, On sets of integers containing no 4 elements in arithmetic progression. *Acta Math. Sci. Hung.* **20** (1969), 89–104.

[94] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression. *Acta Arith. Collection of articles in memory of Juriĭ Vladimirovič Linnik* **27** (2015), 199–245.

[95] M. Tacy, Semiclassical L^p estimates of quasimodes on submanifolds. *Comm. Partial Differential Equations* **35** (2010), no. 8, 1538–1562.

[96] P. Tomas, A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc.* **81** (1975), 477–478.

[97] B. L. van der Waerden, Beweis einer Baudetschen Vermutung. *Nieuw Arch. Wiskd.* **15** (1927), 212–216.

[98] P. Yung, *Spectral projection theorems on compact manifolds*. Lecture notes, available under "Notes/Slides" at https://maths-people.anu.edu.au/~plyung/Sogge.pdf.

[99] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.* **55** (1987), 919–941.

[100] S. Zelditch and M. Zworski, Ergodicity of eigenfunctions for ergodic billiards. *Comm. Math. Phys.* **175**, 673–682 (199.

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