ASYMPTOTIC BEHAVIORS OF RANDOM WALKS ON COUNTABLE GROUPS

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ABSTRACT

In this note we survey some topics in random walks on countable groups. The main focus is on quantitative estimates for random walk characteristics on amenable groups, in connections to geometric and algebraic properties of the underlying groups.

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1. INTRODUCTION: A BRIEF REVIEW OF SOME HISTORY

Random walks on general countable groups were introduced by H. Kesten in his thesis titled *Symmetric Random Walks on Groups* [63]. Let Γ be a countable group and let μ be a probability measure on Γ. We say that μ is symmetric if $\mu(g) = \mu(g^{-1})$ for all $g \in \Gamma$, and μ is nondegenerate if the support of μ generates Γ as a semigroup. Consider a random walk on Γ in which every step consists of right multiplication by a random group element distributed according to μ. In other words, take a sequence of independent random variables $(Y_n)_{n=1}^{\infty}$ on Γ distributed as μ. Let $W_0 = \mathrm{id}_\Gamma$, $W_n = Y_1 \cdots Y_n$. We refer to the process $(W_n)_{n=0}^{\infty}$ as a μ-random walk on Γ. The distribution of W_n is the nth convolution power $\mu^{(n)}$.

When Γ is generated by a finite subset $S \subset \Gamma$, consider the Cayley graph of (Γ, S), which is a graph with vertex set Γ and edge set $\{(g, gs) : g \in \Gamma, s \in S\}$. The word length $|g|_S$ of a group element is the smallest integer $n \geq 0$ for which there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$. A random walk with step distribution μ supported on $S \cup S^{-1}$ can be visualized as a random nearest neighbor exploration process on the Cayley graph.

The questions considered in [62,63] regard the relation between the spectrum of the associated linear operator P_μ on $\ell^2(\Gamma)$ and the structure of the group Γ, where $P_\mu f(x) = \sum_{y \in \Gamma} f(xy)\mu(y)$. The original definition of amenability, introduced by von Neumann to explain the Hausdorff–Banach–Tarski paradox, says that Γ is amenable if there is a Γ-invariant mean on $\ell^\infty(\Gamma)$. Kesten's characterization of amenability [62] states that when μ is a nondegenerate symmetric probability measure on Γ, the spectral radius $\lambda(\Gamma, \mu) = \lim_{n\to\infty} \mu^{(2n)}(\mathrm{id})^{1/2n}$ is 1 if and only if Γ is amenable. When Γ is amenable, one might further ask what is the behavior of the spectral distribution of P_μ near 1, or of the decay of return probability $\mu^{(2n)}(\mathrm{id})$ when n goes to infinity. Since the pioneering work of Varopoulos, such questions are studied using both analytic and geometric tools. In particular, there are close relations between the behavior of the heat semigroup $(P_\mu^n)_{n=0}^{\infty}$ on $\ell^2(G)$ and geometric properties captured through Sobolev-type inequalities, see the survey [98] and the monograph [99].

A real-valued function f on Γ is called μ-harmonic if it satisfies the mean value property that $f(x) = \sum_{y\in\Gamma} f(xy)\mu(y)$ for all $x \in \Gamma$. When all bounded μ-harmonic functions are constant, we say (Γ, μ) has the *Liouville property*. A theory for the non-Liouville case, in the more general context of locally compact second countable (abbreviated as *lcsc*) groups is initiated by Furstenberg [37,39,40], where a measure-theoretical object called *Poisson boundary* (also called Poisson–Furstenberg boundary) was introduced to represent the space of bounded μ-harmonic functions. In particular, the Poisson boundary of (G, μ) is trivial if and only if (G, μ) has the Liouville property. Note that the measure-theoretical Poisson–Furstenberg boundary is different from the topological Furstenberg boundary which is also introduced in [37].

Entropy is a crucial quantity in the study of Poisson boundaries. Let $(W_n)_{n=0}^{\infty}$ be a μ-random walk on a countable group Γ. Denote by $H_\mu(n)$ the (Shannon) entropy of W_n,

$$H_\mu(n) = H(W_n) = -\sum_{g\in\Gamma} \mu^{(n)}(g) \log \mu^{(n)}(g). \tag{1.1}$$

The limit $\mathbf{h}_\mu = \lim_{n\to\infty} H_\mu(n)/n$ is called the (Avez) asymptotic entropy of the μ-random walk. The celebrated *entropy criterion for the Liouville property*, due to Avez [7], Derriennic [28] and Kaimanovich–Vershik [60], states that if μ has finite entropy, then the asymptotic entropy is 0 if and only if (Γ, μ) is Liouville.

Suppose now Γ is generated by a finite subset $S \subset \Gamma$. When the measure μ has finite first moment with respect to the word norm $|\cdot|_S$, that is, $\sum |g|_S \mu(g) < \infty$, we may consider the speed function (also called rate of escape or drift), defined as

$$L_\mu(n) = \mathbb{E}\big[|W_n|_S\big] = \sum_{g\in\Gamma} |g|_S \mu^{(n)}(g). \tag{1.2}$$

The limit $\mathbf{l}_\mu = \lim_{n\to\infty} L_\mu(n)/n$ is called the asymptotic speed/drift. By Kingman's subadditive ergodic theorem, we have $|W_n|_S/n \to \mathbf{l}_\mu$ when $n \to \infty$ almost surely.

By the so called "fundamental inequality" that $\mathbf{h}_\mu \leq v\mathbf{l}_\mu$ (see, e.g., [19]), where v is the asymptotic volume growth rate of (Γ, S), we have that $\mathbf{l}_\mu = 0$ implies $\mathbf{h}_\mu = 0$. A theorem of Karlsson and Ledrappier [61] combined with the entropy criterion imply the following speed criterion: for a nondegenerate *centered* step distribution μ on Γ with finite first moment, the asymptotic speed $\mathbf{l}_\mu = 0$ if and only if (Γ, μ) is Liouville. In the special case where μ is a nondegenerate symmetric probability measure with finite support, the speed criterion is proved earlier in [96] by showing a general off-diagonal estimate for transition probabilities $P^n(x, y)$, where P is a reversible Markov operator on a countable state space. A generalization and improvement of this estimate is given in [22] (with a simpler proof) and is now called the Varopoulos–Carne inequality. Applied to a μ-random walk on Γ, the inequality reads: let $S = \operatorname{supp}\mu$ and take the word distance on the Cayley graph (Γ, S), then

$$P_\mu^n(x, y) \leq 2e^{-d_S(x,y)^2/2n}. \tag{1.3}$$

When (Γ, μ) is not Liouville, the Poisson boundary identification problem asks if one can find an "explicit" Γ-space X, a σ-algebra $\mathcal{B}$ on X and a probability measure ν on $\mathcal{B}$, such that $(X, \mathcal{B}, \nu)$ is isomorphic to the Poisson boundary of (Γ, μ) via a Γ-equivariant measurable isomorphism. The entropy criterions of Kaimanovich [57] provide powerful tools for the identification problem. As remarked in [59], a majority of known examples of nontrivial boundary behaviors of random walks on countable groups fall into one of the following two classes:

(i) Convergence of random walk sample paths to some suitable geometric boundary, in the presence of hyperbolicity or nonpositive curvature.

(ii) Pointwise stabilization of some notion of "configurations" along random walk sample paths.

Based on the limit behavior of the random walks observed, one can define a space $(X, \mathcal{B}, \nu)$ which is a quotient of the Poisson boundary (such a space is called a μ-boundary). Roughly speaking, the entropy criterion says that the candidate space $(X, \mathcal{B}, \nu)$ is isomorphic to the Poisson boundary if the random walk $(W_n)_{n=0}^{\infty}$ conditioned on its "limit" in X, has asymptotic entropy 0 almost surely (see [57, SECTION 4] for a precise statement). The ray and strip criterions in [57] provide more checkable sufficient conditions to ensure the conditional asymptotic entropy given $(X, \mathcal{B}, \nu)$ is 0 almost surely.

Prototype examples of (i) are random walks on nonelementary Gromov word hyperbolic groups, where the geometric boundary is the visual boundary. In this case for a step distribution μ of finite entropy and finite log-moment, the visual boundary equipped with the hitting distribution is identified as the Poisson boundary of the μ-random walk in [57]. This type of geometric boundary identification holds for a wide class of groups acting on hyperbolic spaces (not necessarily proper or locally compact), see the work of Maher and Tiozzo [76] and references therein.

Prototype examples of (ii) are random walks on the so called lamplighter groups $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d = (\oplus_{\mathbb{Z}^d} \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}^d$. Symmetric random walks on $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$, $d \geq 3$, are considered in [60] as first examples of random walks on amenable groups with nontrivial Poisson boundary. A crucial observation there is that for some suitably chosen step distribution μ, when the projected random walk on $\mathbb{Z}^d$ is transient, the configuration in $\oplus_{\mathbb{Z}^d} \mathbb{Z}/2\mathbb{Z}$ stabilizes pointwise along the random walk trajectory almost surely. The question whether the μ-boundary from pointwise stabilization is the full Poisson boundary remained open, until resolved positively in the work of Erschler [33] for $d \geq 5$ and in Lyons and Peres [73] for all $d \geq 3$.

In the rest of this note we survey in more details some aspects around asymptotic behaviors of random walks. Because of limitations of space and the author's knowledge, this survey is not intended to be comprehensive; rather only a small selection of topics are discussed.

2. A FEW THEMES

Here are some loosely phrased questions that have emerged from the study of random walks on groups:

(1) Can some random walk behaviors (for classes of random walks on a group, say finite range nondegenerate symmetric random walks, random walks satisfying some moment condition, etc.) be deemed *group invariant*?

(2) What properties of the group can be characterized in terms of random walk behaviors?

(3) Can random walk behavior be used to understand groups and their actions?

In each of these directions of research, many natural questions remain open.

2.1. Stability problems

Question 1 is often casted as *stability* problems. It is interesting both from the point of view of understanding random walk behaviors, and searching for group invariants arising from stochastic processes on them. Regarding the behavior of return probabilities, the following stability theorem is established by Pittet and Saloff-Coste [87] using comparison of Dirichlet form techniques (for the definition of Dirichlet form in this context see Section 3.1). Given two nonincreasing functions $f, g : \mathbb{N} \to \mathbb{R}$, we say they are equivalent if there is a constant $C \geq 1$ such that $g(Cx)/C \leq f(x) \leq Cg(x/C)$. We say a probability measure μ on Γ has finite α-moment (with respect to the word distance) if $\sum_{g\in\Gamma} |g|_S^\alpha \mu(g) < \infty$.

Theorem 2.1 ([87]). *The equivalence class of the decay function $n \mapsto \mu^{(2n)}(\mathrm{id})$, where μ is the step distribution of a nondegenerate symmetric random walk of finite second moment on Γ, is a quasiisometry invariant.*

The finite second moment condition in the theorem is necessary. For example, on $\mathbb{Z}$ consider an α-stable-like measure $\mu_\alpha(x) = \frac{c_\alpha}{(1+|x|)^{\alpha+1}}$, where $\alpha \in (0, 2)$ and c_α is the normalizing constant so that μ_α has total mass 1. Then the decay function behaves like $\mu_\alpha^{(2n)}(0) \simeq n^{-\frac{1}{\alpha}}$, which is not equivalent to the decay function of symmetric simple random walk on $\mathbb{Z}$.

In [16], Bendikov and Saloff-Coste considered the question of fastest decay of return probability under a given moment condition. Given $\alpha \in (0, 2)$ and a constant $C > 0$, let $\mathcal{S}_{\Gamma,\alpha}$ be the set of all symmetric probability measures μ on Γ such that $\sum_{g\in\Gamma} |g|_S^\alpha \mu(g) \leq C$. Consider the following function of fastest decay under α-moment condition:

$$\Phi_{\Gamma,\alpha} : n \mapsto \inf\{\mu^{(2n)}(\mathrm{id}),\ \mu \in \mathcal{S}_{\Gamma,\alpha}\}.$$

It turns out to be convenient to consider the version defined using weak α-moment as well. For some specific classes of groups the behavior of $\Phi_{\Gamma,\alpha}$ can be understood rather well, see [16, 90]. However, the question whether the equivalence class of the function $\Phi_{\Gamma,\alpha}$ is a quasiisometry invariant remains open.

In contrast to the decay of return probabilities, where tools such as comparison of Dirichlet forms are available, for entropy and speed functions stability is a well-known open problem for general amenable groups. For instance, fix the group Γ, it is an important open question whether the behavior of the entropy function $n \mapsto H_\mu(n)$ or the speed function $n \mapsto \mathbb{E}[|X_n|_S]$, is stable among nondegenerate, symmetric, finitely supported step distributions on Γ. Note that it is known that the Liouville property is not stable under quasiisometry for general graphs, see [17, 75].

2.2. Characterizations in terms of random walks

Kesten's characterization of amenability cited earlier can be viewed as a first result in the direction of Question 2. Kesten asked in [64] for a characterization of recurrent groups. What are the finitely generated groups that can carry a nondegenerate recurrent symmetric random walk? This problem was settled by Varopoulos in the 1980s, invoking Gromov's polynomial growth theorem.

Theorem 2.2 ([97]). *Suppose* Γ *is a finitely generated group and there exists a nondegenerate symmetric probability measure* μ *on* Γ *such that the* μ*-random walk is recurrent. Then* Γ *is a finite extension of the trivial group* $\{\mathrm{id}\}$*,* $\mathbb{Z}$*, or* $\mathbb{Z}^2$.

The growth function is an important geometric invariant of a group. Let Γ be a finitely generated group and S be a finite generating set of Γ. The growth function $v_{\Gamma,S}(n)$ counts the number of elements with word length $|\gamma|_S \leq n$, that is,

$$v_{\Gamma,S}(n) = \left|\left\{\gamma \in \Gamma : |\gamma|_S \leq n\right\}\right|.$$

A finitely generated group Γ is of *polynomial growth* if there exists $0 < D < \infty$ and a constant $C > 0$ such that $v_{\Gamma,S}(n) \leq Cn^D$. It is of *exponential growth* if there exist $\lambda > 1$ and $c > 0$ such that $v_{\Gamma,S}(n) \geq c\lambda^n$. If $v_{\Gamma,S}(n)$ is subexponential, but Γ is not of polynomial growth, we say Γ is a group of *intermediate growth*. By Gromov's theorem [46], any group of polynomial growth is virtually nilpotent. For some classes of groups, the growth is either polynomial or exponential (for example, for solvable groups by results of Milnor and Wolf [79, 101], and for linear groups as it follows from the Tits alternative [93]).

The first examples of groups of intermediate growth are constructed by Grigorchuk in [43], answering a question of Milnor. These groups are indexed by infinite strings ω in $\{0, 1, 2\}^\infty$: for any such ω, four automorphisms $a, b_\omega, c_\omega, d_\omega$ of the rooted binary tree are associated. The group G_ω is generated by $S = \{a, b_\omega, c_\omega, d_\omega\}$. By [43], if ω is eventually constant then G_ω is virtually abelian (hence of polynomial growth); otherwise G_ω is of intermediate growth. The group G_ω is periodic (also called torsion) if and only if ω contains all three letters $0, 1, 2$ infinitely often. A special example of such a string is $(012)^\infty$; the corresponding group is called the *first Grigorchuk group*, which was introduced and shown to be an infinite torsion group Grigorchuk in [45]. First examples of simple groups of intermediate growth are constructed in the recent work of Nekrashevych [83, 84].

A key point in Varopoulos' proof is that a volume growth lower bound (geometric property of the underlying group) implies an upper bound for the decay of return probabilities (analytic property of the heat semigroup). More precisely, suppose that there are constants $c, d > 0$ such that the volume growth of the group Γ satisfies $v_{\Gamma,S}(n) \geq cn^d$ for all n, then there is a constant $c_1 > 0$ such that $\mu^{(2n)}(\mathrm{id}) \leq c_1 n^{-\frac{d}{2}}$. Since, by Gromov's theorem [46] and its version in van den Dries and Wilkie [94], a group of weak polynomial growth is virtually nilpotent, this estimate leads to a proof of Theorem 2.2.

We now discuss some results that characterize properties of the underlying group Γ in terms of boundary behaviors of random walks. In [37] Furstenberg proved that if Γ is nonamenable, then for any nondegenerate step distribution μ on Γ, the Poisson boundary of (Γ, μ) is nontrivial; and the converse to this statement was conjectured to be true as well. This conjecture was proved independently by Kaimanovich and Vershik [60], Rosenblatt [88]:

Theorem 2.3 ([60, 88]). *A countable group* Γ *is amenable if and only if there is a nondegenerate symmetric random walk on* Γ *with trivial Poisson boundary.*

It is classical that for any step distribution on a virtually nilpotent group, the associated Poisson boundary is trivial, see [30]. In [60] it is conjectured that on any group of exponential growth, there exists a symmetric step distribution (of infinite support in general) with nontrivial Poisson boundary. The first result on nontrivial boundary behavior of random walks on intermediate growth groups is due to Erschler [32]. Note that by the entropy criterion for the Liouville property, any finite range random walk on a group of intermediate growth has trivial Poisson boundary. Thus to observe nontrivial boundary behavior, it is necessary to take random walk step distributions with infinite support.

Theorem 2.4 ([32]). *Let $\Gamma = G_\omega$ be a Grigorchuk group with $\omega \in \{0,1\}^\infty$, where ω contains infinite numbers of* 0 *and* 1. *Then Γ admits a symmetric measure μ of finite entropy such that the Poisson boundary of (Γ, μ) is nontrivial.*

We mention that the nontrivial limit behavior observed in Theorem 2.4 is of type (ii) as described in the Introduction.

The longstanding problem which groups admit random walks with nontrivial Poisson boundary is completely settled in the recent work of Frisch, Hartman, Tamuz, and Vahidi Ferdowsi [36]. Recall that Γ has the infinite conjugacy class property (ICC) if each of its nontrivial elements has an infinite conjugacy class. Note that in some works the definition of ICC also requires the group to be nontrivial. For a finitely generated group Γ, having no ICC quotient except the trivial one $\{\mathrm{id}\}$ is equivalent to being virtually nilpotent.

Theorem 2.5 ([36]). *Let Γ be a countable group. The following are equivalent:*

(i) *Γ has a quotient group $\bar{\Gamma}$ such that $\bar{\Gamma}$ is a nontrivial* ICC *group.*

(ii) *There is a probability measure μ on Γ with nontrivial Poisson boundary.*

The direction (ii) implies (i) in the statement is known from the earlier work of Jaworski in [52]. The direction (i) implies (ii) is proved in [36] by a novel construction of step distributions with nontrivial Poisson boundary, directly using the ICC property. Moreover, the measure μ can be taken to be a symmetric measure with finite Shannon entropy whose support generates Γ. Theorem 2.5 solves the aforementioned conjecture of Kaimanovich and Vershik positively; and moreover brings in the key insight that the algebraic condition of having a nontrivial ICC quotient plays a crucial role. We mention that the similar problem for nondiscrete locally compact groups remains open: for instance, how to characterize a locally compact group G which admits a step distribution μ with nontrivial Poisson boundary, where μ is absolutely continuously with respect to the Haar measure of G. For this formulation, an answer is known for *connected* compactly generated lcsc groups by [50]: the characterization is that G is not of polynomial growth. The question is open for totally disconnected locally compact groups. One may also drop the constraint on μ and formulate the problem for any step distribution on the group G. Dropping the assumption on μ may change possible boundary behaviors even on polynomial growth groups, see [51, 53].

2.3. Random walks as tools

Now we turn to Question 3. Random walks provide a natural tool to study stationary measures. Consider an action of Γ on a compact space X by homeomorphims and let μ be a probability measure on Γ. Denote by $P(X)$ the space of probability measures on the Borel σ-field of X. Then by a standard compact argument, there always exists a μ-stationary measure $\nu \in P(X)$, that is, ν satisfies $\sum_{g\in\Gamma} g.\nu\mu(g) = \nu$ where $g.\nu(A) = \nu(g^{-1}.A)$. One fundamental observation in [37] is that the martingale convergence theorem implies that almost surely, along the μ-random walk $(W_n)_{n=0}^{\infty}$, the sequence of measures $(W_n.\nu)$ converges in the weak topology of $P(X)$. In particular, the limit measures give rise to a Γ-equivariant map from the Poisson boundary of (Γ, μ) to $P(X)$. This map is sometimes referred to as the affine boundary map associated to $\Gamma \curvearrowright (X, \nu)$. Ideas of using Poisson boundaries to answer algebraic questions appear in the work of Furstenberg on lattice envelopes [38]. Furstenberg's ideas inspire the use of boundary theory in later works on rigidity phenomena, which we do not touch on here.

Next we discuss some works in which random walks are used as tools to prove amenability of groups. It is shown in von Neumann's work that finite groups and abelian groups are amenable and that the class of amenable groups (AG) is closed under four standard operations: taking (i) subgroups, (ii) quotients, (iii) group extensions, and (iv) direct unions. The term *elementary amenable* is coined by Day: denote by EG be the smallest class of groups which contains all finite groups and all abelian groups and is closed under operations (i)–(iv).

A finitely generated group Γ of subexponential growth is amenable: subexponential growth implies that there exists a subsequence of balls $B(\mathrm{id}, r_i)$ forming a Følner sequence. Chou shows in [23] that a finitely generated group in EG is either virtually nilpotent or of exponential growth; a torsion group in EG is locally finite; and a finitely generated simple group in EG is finite. Thus Grigorchuk groups of intermediate growth are in AG but not EG. As in [27], denote by SG the closure of all groups of subexponential growth under (i)–(iv). It is clear that $\mathrm{E}G \subseteq SG \subseteq \mathrm{A}G$.

A first example to separate SG and AG is shown in Bartholdi and Virág [14]. The example is called the Basilica group, which was first studied in [44]. The Basilica group B is a two-generated group acting on the rooted binary tree. One key idea in [14] is certain random walk on B enjoys self-similar properties compatible with the wreath recursions down the tree. The self-similarity allows one to efficiently use the recursion on the tree to study behaviors of the random walk. In [14] it is shown that the return probability of such a random walk on B decays subexponentially, thus B is amenable by Kesten's criterion. The idea of self-similar random walks is later extended to larger classes of groups acting on trees in [2, 13, 20, 58]. It is later understood that for proving amenability, one crucial property is that the induced random walk on certain orbital Schreier graph is recurrent. For instance, amenability of B can be shown in the unified framework of "extensive amenability," which emerges from the seminal work [54] and is developed in [55, 56].

For the rest of this subsection we focus on the relation of random walks with nontrivial Poisson boundary to volume growth of the group. The most direct way to obtain a growth lower bound is to exhibit distinct elements within a given radius. For instance, if Γ contains two elements a, b such that they generate a free semigroup, then thIS semigroup provides 2^n distinct elements within radius $n \max\{|a|_S, |b|_S\}$. In general, it can be rather challenging to find explicit elements within a given distance, see [11,71] on the first Grigorchuk group. As a consequence of the entropy criterion and Shannon's theorem, random walks with nontrivial Poisson boundary on Γ can provide lower bounds on volume growth of Γ. Heuristically, instead of exhibiting many distinct elements in a ball, one constructs random walks with positive asymptotic entropy, which indirectly imply there must be sufficiently many points in balls. The quantitative relation between the tail decay of the step distribution μ and growth of Γ can be made precise:

Lemma 2.6. *Suppose Γ admits a μ-random walk with nontrivial Poisson boundary, where the probability measure μ has finite entropy $H(\mu) < \infty$ and finite α-moment, for some $\alpha \in (0, 1]$. Then there is a constant $c > 0$ such that the volume function satisfies*

$$v_{\Gamma,S}(r) \geq \exp(cn^{\alpha}).$$

See [35, LEMMA 2.1] for a more general statement. To avoid possible periodicity issues, we may always assume that $\mu(\{\mathrm{id}_\Gamma\}) > 0$: changing μ to a convex combination of μ and the δ-mass at id does not change the space of harmonic functions. To make use of Lemma 2.6, one first constructs a random walk μ with finite entropy, which is designed to guarantee that there exists a tail event A of the μ-random walk whose probability is not in $\{0, 1\}$. Then we have:

Observation of one nontrivial tail event for μ-random walk

$$\Updownarrow$$

Poisson boundary of (G, μ) is nontrivial

$$\Downarrow$$

Volume lower bound from the moment condition satisfied by μ.

The random walks with nontrivial Poisson boundary on the Grigorchuk group $G_{(01)^\infty}$ constructed by Erschler in Theorem 2.4 yield lower bounds which match rather tightly with upper bounds. More precisely, by [32, THEOREMS 2 AND 3], the growth function of $G = G_{(01)^\infty}$ satisfies

$$\exp\left(\frac{n}{\log^{2+\epsilon}(n)}\right) \lesssim v_{G,S}(n) \lesssim \exp\left(\frac{n}{\log^{1-\epsilon}(n)}\right), \quad \text{for any } \epsilon > 0.$$

The construction in [32] uses the fact that $G_{(01)^\infty}$ contains an infinite dihedral group. For the Grigorchuk group $G_{(012)^\infty}$, which is a torsion group by [45], random walks with nontrivial Poisson boundary and tight control over the tail decay are constructed in [35].

Theorem 2.7 ([35]). *Let $\alpha_0 = \frac{\log 2}{\log \lambda_0} \approx 0.7674$, where λ_0 is the positive root of the polynomial $X^3 - X^2 - 2X - 4$. For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ and a nondegenerate symmetric probability measure μ on $G = G_{(012)^\infty}$ of finite entropy and nontrivial Poisson boundary, where the tail decay of μ satisfies that for all $r \geq 1$,*

$$\mu\big(\{g : l_S(g) \geq r\}\big) \leq C_\epsilon r^{-\alpha_0+\epsilon}.$$

As a consequence, for any $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that for all $n \geq 1$,

$$v_{G,S}(n) \geq \exp(c_\epsilon n^{\alpha_0 - \epsilon}).$$

The volume lower bound in Theorem 2.7 matches up in exponent with the growth upper bound in [10]. In particular, combined with the upper bound we conclude that the volume exponent of the first Grigorchuk group $G = G_{(012)^\infty}$ exists and is equal to α_0, that is,

$$\lim_{n\to\infty} \frac{\log\log v_{G,S}(n)}{\log n} = \alpha_0.$$

A detailed sketch of the construction of measure μ stated in Theorem 2.7 can be found in [35, INTRODUCTION].

The method also applies to other period strings ω in $\{0, 1, 2\}^\infty$, which contains all three symbols infinitely often, to show the corresponding Grigorchuk group G_ω has a volume exponent $\alpha_\omega \in (0, 1)$, see [35, THEOREM B]. When ω is not periodic, it is shown in [43] that for some choices of ω, the growth of G_ω exhibits oscillating behavior: limit of $\log\log v_{G_\omega,S}(n)/\log n$ does not exist. Indeed, the family $\{G_\omega\}$ provides a continuum of mutually nonequivalent growth functions. This statement is shown in [43] with the introduction of the space of marked groups, see more discussion in Section 3.2. In [35, THEOREMS C AND 8.5] it is shown that for a large collection of nonperiodic ω, the oscillating growth function of G_ω can be estimated with good precision. In particular, combined with the upper bounds from [12], the estimates show that given any $\alpha \leq \beta$ in the interval $[\alpha_0, 1]$, where α_0 is the growth exponent of $G_{(012)^\infty}$, there is an $\omega \in \{0, 1, 2\}^\infty$ with

$$\liminf_{n\to\infty} \frac{\log\log v_{G_\omega,S}(n)}{\log n} = \alpha \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log\log v_{G_\omega,S}(n)}{\log n} = \beta.$$

We mention that it is an open problem whether one can find an example of intermediate growth group whose lower growth exponent is strictly less than α_0. Such questions are related to Grigorchuk's gap conjecture, see [42].

3. QUANTITATIVE BEHAVIOR OF RANDOM WALK CHARACTERISTICS

In this section we focus on quantitative estimates for random walks on groups. Here are some of the key interrelated aspects:

(i) What are the *spectral properties* of the convolution operator $f \mapsto f * \mu$ when μ is a symmetric probability measure on Γ? What is the behavior of the probability of return of a symmetric random walk driven by step distribution μ?

(ii) What is the asymptotic *entropic behavior*, that is, the behavior of $n \mapsto H_\mu(n)$ as n tends to infinity? Here $H_\mu(n)$ is the Shannon entropy of W_n as in (1.1).

(iii) What is the *escape behavior* of transient random walks captured in terms of some given distance function on the group, say, in the form of average displacement as in (1.2) or more refined descriptions?

(iv) What is the structure of sets μ-harmonic functions (bounded, positive, of polynomial growth, of a given growth type, slow or fast, etc.)?

In this section we mainly focus on topics around (i)–(iii); unbounded harmonic functions are discussed in the next section.

3.1. Isoperimetric profiles

We first introduce some notations for (i). Let R_μ be the right convolution operator $\ell^2(\Gamma) \to \ell^2(\Gamma)$ defined as $R_\mu(f)(x) = (f * \mu)(x) = \sum_{g\in\Gamma} f(xg^{-1})\mu(g)$. The return probability to identity at time n is given by $\mu^{(n)}(\mathrm{id}) = \langle R_\mu^n \delta_{\mathrm{id}}, \delta_{\mathrm{id}}\rangle_{\ell^2(\Gamma)}$. When μ is symmetric, R_μ is a self-adjoint operator. Denote by $E_\lambda^{R_\mu} = \mathbf{1}_{(-\infty,\lambda]}(R_\mu)$ the spectral projections of R_μ. The spectral measure of R_μ is given by

$$N_{R_\mu}\big((-\infty,\lambda]\big) = \big\langle E_\lambda^{R_\mu}(\delta_{\mathrm{id}}), \delta_{\mathrm{id}}\big\rangle_{\ell^2(\Gamma)}.$$

The relation between return probabilities and the spectral measure is expressed through the transform

$$\mu^{(2n)}(\mathrm{id}) = \int_{\mathbb{R}} \lambda^{2n} dN_{R_\mu}(\lambda) = \int_{-1}^{1} \lambda^{2n} dN_{R_\mu}(\lambda).$$

The last equality is because R_μ is a Markov operator. When Γ is an infinite amenable group, one may draw information on the behavior of the spectral measure near 1 from the decay of return probabilities, using Tauber–Karamata theorems for Laplace transforms.

The decay of return probability $\mu^{(2n)}(\mathrm{id})$ is closely related to isoperimetric profiles of R_μ, which in the discrete setting can be introduced for more general reversible Markov operators. Let V be a countable set, typically the vertex set of a graph, and let $P : V \times V \to [0,1]$ be the transition probabilities of a reversible Markov chain on V. Denote by π a reversing measure for P, that is, $\pi(x)P(x,y) = \pi(y)P(y,x)$ for all $x, y \in V$. Consider the associated Dirichlet form

$$\mathcal{E}_P(f_1, f_2) = \frac{1}{2}\sum_{x,y}\big(f_1(y) - f_1(x)\big)\big(f_2(y) - f_2(x)\big)\pi(x)P(x,y),$$

which is a bilinear form on $\mathrm{Dom}(\mathcal{E}_P) = \{f \in L^2(V,\pi) : \mathcal{E}_P(f,f) < \infty\}$. The L^2-*isoperimetric profile* of P, also called the *spectral profile*, is defined as

$$\Lambda_{2,P} : \mathbb{R}_+ \to [0,1], \qquad v \mapsto \inf\big\{\lambda_P(\Omega) : \Omega \subseteq V, \pi(\Omega) \le v\big\},$$

where $\lambda_P(\Omega)$ is the lowest eigenvalue of the Laplacian operator $I - P$ with Dirichlet boundary condition in Ω,

$$\lambda_P(\Omega) = \inf\big\{\mathcal{E}_P(f,f) : \mathrm{supp}\,(f) \subseteq \Omega, \|f\|_{L^2(V,\pi)} = 1\big\}.$$

The L^1-isoperimetric profile is defined analogously. Using an appropriate coarea formula, $\Lambda_{1,P}$ can equivalently be defined more geometrically as

$$\Lambda_{1,P}(v) = \inf\left\{\frac{\sum_{x,y\in V} \mathbf{1}_\Omega(x)\mathbf{1}_{V\setminus\Omega}(y)\pi(x)P(x,y)}{\pi(\Omega)} : \pi(\Omega) \le v\right\},$$

where the quantity $\sum_{x,y\in V} \mathbf{1}_\Omega(x)\mathbf{1}_{V\setminus\Omega}(y)\pi(x)P(x,y)$ measures the size of the boundary of Ω with respect to P.

Between the L^1- and L^2-isoperimetric profiles, we have the following inequality, often referred to as Cheeger's inequality (see, e.g., [68]):

$$\frac{1}{2}\Lambda_{1,P}^2 \leq \Lambda_{2,P} \leq \Lambda_{1,P}. \tag{3.1}$$

The nonobvious direction $\Lambda_{2,P} \geq \frac{1}{2}\Lambda_{1,P}^2$ is useful for transferring L^1-expansion inequalities to spectral profile lower bounds. The Coulhon–Saloff-Coste inequality [26], which implies, for example,

$$\Lambda_{1,R_\mathbf{u}}\big(2v_{\Gamma,S}(r)\big) \geq \frac{1}{2r},$$

where $\mathbf{u}$ is the uniform measure on $S \cup S^{-1}$ and $v_{\Gamma,S}$ is the volume function function of (Γ, S), can be proved by an elementary mass displacement type argument. Sharp L^1-expansion inequalities on $\mathbb{Z}^d$ can be derived from Loomis–Whitney inequalities, see, e.g., [72, SECTION 6.6]; further connections between isoperimetric inequalities and entropy inequalities (and consequences such as Loomis–Whitney, Harper inequalities) are investigated in [48].

The use of Nash-type inequalities to estimate return probabilities in the discrete setting was introduced in [95]. It turns out Nash inequalities are equivalent to Faber–Krahn-type inequalities, where the latter is of the form $\Lambda_{2,P}(v) \geq f(v)$ for some positive function f. In fact, in very general settings, it is known that various forms of functional inequalities are equivalent, see [8] and references therein. Comparison of forms, considered in [29] for random walks on finite groups, is a useful tool to deduce isoperimetric inequalities for a Markov operator P of interest from known results on other Markov operators.

Through a series of works by Coulhon and Grigor'yan [24, 25], it is shown that under some mild conditions, the asymptotic decay of return probability $\sup_{x\in V} P^{2n}(x,x)$ and the L^2-isoperimetric profile of P determine each other. More precisely, suppose that $\pi^* = \inf_{x\in V}\pi(x) > 0$. Let $\gamma(t)$ be the function defined by the equation

$$t = \int_{\pi_*}^{\psi(t)} \frac{dv}{\Lambda_{2,P}(v)v}, \tag{3.2}$$

then under a mild regularity assumption, the return probabilities satisfy

$$\sup_{x\in V} \frac{P^{2n}(x,x)}{\pi(x)} \simeq \frac{1}{\psi(2n)}. \tag{3.3}$$

For the Markov operator R_μ, where μ is a probability measure on a countable amenable group, a precise formula relating the behavior of the spectral measure N_{R_μ} near 1 and Λ_{R_μ} near infinity is obtained in [15], under the assumption that $\Lambda_{R_\mu} \circ \exp$ is doubling near infinity.

For classes of groups where explicit estimates of return probabilities and isoperimetric profiles are known, the read may consult [15, TABLE 1] and pointers to references there. In addition to the Table, for free solvable groups see [89], and for discrete subgroups of upper triangular matrices over a local field see [92].

A consequence of (3.2) and (3.3) is the following isoperimetry test for transience. Suppose we have a Faber–Krahn inequality for an irreducible reversible Markov operator P of the form $\Lambda_{2,P}(s) \geq f(s)$, $s \in [1,\infty)$, where f is a continuous positive decreasing function on $[1,\infty)$. If

$$\int_1^\infty \frac{ds}{s^2 f(s)} < \infty, \tag{3.4}$$

then the Markov chain with transition operator P is transient, see **[41, THEOREM 6.12]**. Recall the type (ii) boundary behavior described in the Introduction, which often relies on the transience of induced random walks on certain orbits. In particular, isoperimetry of induced random walks on Schreier graphs play an important role in the construction of random walks with nontrivial Poisson boundary in Theorem 2.7.

3.2. The space of marked groups and realization problems

The speed function realization problem, which is often attributed to Vershik, asks what kind of functions can be realized as the speed function of a simple random walk on some finitely generated group. By realizing a given function f as a speed function, we mean finding (Γ,μ) such that the speed of the μ-random walk satisfies $f(n)/C \leq L_\mu(n) \leq Cf(n)$ for some constant $C \geq 1$. Similar realization problems can be posed for other random walk characteristics such as the entropy function $H_\mu(n)$; and for geometric invariants such as the growth function $v_{\Gamma,S}$.

A complete solution to a realization problem consists of two parts. The first part identifies constraints on the functions; the second part shows that all functions under the necessary constraints can be realized. Consider the speed function $L_\mu(n)$, where μ is a nondegenerate symmetric probability measure of finite support on Γ. Then the triangle inequality for the norm $|\cdot|_S$ and translation invariance imply that $L_\mu(n)$ is a subadditive function, $L_\mu(n+m) \leq L_\mu(n) + L_\mu(m)$. Another constraint is known as the "universal diffusive lower bound", which is proved in Lee and Peres **[69]** building on an earlier idea of Erschler: there is a universal constant $c > 0$ such that for any infinite amenable group Γ equipped with a finite generating set S, for any symmetric probability measure μ on Γ whose support contains S, we have

$$L_\mu(n) \geq c\sqrt{p_* n}, \quad \text{where } p_* = \min_{\gamma \in S} \mu(\gamma).$$

For more discussion on the connection of such a general bound to harmonic embeddings into a Hilbert space, see Section 4. The diffusive lower bound is achieved, for example, by simple random walk on $\mathbb{Z}$: take $\Gamma = \mathbb{Z}$ and $\mu(\pm 1) = \frac{1}{2}$. Given these constraints, the speed function realization problem asks what functions between $\sqrt{n}$ and n can be realized as speed function of finite range symmetric random walks on groups.

Theorem 3.1 (**[21]**). *There exists a universal constant $C > 1$ such that the following holds. For any function $f : [1,\infty) \to [1,\infty)$ such that $f(1) = 1$ and $x/f(x)$ is nondecreasing, there exist a group Δ equipped with a finite generating set T and a nondegenerate symmetric probability measure μ on Δ of finite support such that*

- *the speed and entropy functions satisfy* $L_\mu(n) \simeq_C H_\mu(n) \simeq_C \sqrt{n} f(\sqrt{n})$;
- *the* L^p*-isoperimetric profile satisfies* $\Lambda_{p,R_\mu}(v) \simeq_C \left(\frac{f(\log(e+v))}{\log(e+v)}\right)^p$ *for any* $p \in [1,2]$;
- *the return probability satisfies* $-\log(\mu^{(2n)}(\mathrm{id})) \simeq_C w(n)$, *where* $w(n)$ *is determined by* $n = \int_1^{w(n)} \left(\frac{s}{f(s)}\right)^2 ds$.

When the function f *is sublinear, that is,* $\lim_{x\to\infty} f(x)/x = 0$, *the group* Δ *can be chosen to be elementary amenable with asymptotic dimension 1.*

Here the notation $f \simeq_C g$ means $g(x)/C \le f(x) \le Cg(x)$. In particular, the first item gives a satisfying answer to the speed function realization problem. The statement for speed function realization between $n^{3/4}$ and n^γ, $\gamma < 1$, is obtained earlier by Amir and Virag in [4]. The group Δ constructed in Theorem 3.1 is of exponential growth. It is an open problem for the entropy function, whether a nondegenerate symmetric random walk μ on a group Γ of exponential growth always satisfies $H_\mu(n) \gtrsim \sqrt{n}$.

As cited in Section 2.3, the space of marked groups is introduced by Grigorchuk in [43] to show that there are $2^{\aleph_0}$ groups with pairwise inequivalent growth functions. A k-marked group is a pair (Γ, T), where $T = (t_1, \dots, t_k)$ is an ordered k-tuple in Γ^k which generates Γ. Equivalently, let $\mathbf{F}_k$ be the rank k free group; (Γ, T) corresponds to the kernel of the homomorphism $\mathbf{F}_k \to \Gamma$ which sends the jth free generator of $\mathbf{F}_k$ to t_j, $1 \le j \le k$. Denote by $\mathcal{M}_k$ the space of k-marked groups. The product topology on $2^{\mathbf{F}_k}$ induces a topology on $\mathcal{M}_k$, via the identification described above. This topology on $\mathcal{M}_k$ is sometimes called the Cayley–Grigorchuk topology, as two marked groups are close if their labeled Cayley graphs agree on a large ball around the identity. Under this topology, $\mathcal{M}_k$ is a metrizable compact Hausdorff space.

The product operation in $\mathcal{M}_k$ is called a diagonal product: consider a collection of marked groups $((\Gamma_i, T_i))_{i\in I}$ their diagonal product, denoted by $\bigotimes_{i\in I}(\Gamma_i, T_i)$, is the quotient of $\mathbf{F}_k$ with kernel $\bigcap_{i\in I} \ker(\mathbf{F}_k \to \Gamma_i)$. In some situations it is possible to understand well the structure of a diagonal product. Consider a converging sequence of marked groups $((\Gamma_i, T_i))_{i=1}^\infty$ in $\mathcal{M}_k$ and denote by $(\Gamma_0, T_0) = \lim_{i\to\infty}(\Gamma_i, T_i)$. Then the limit (Γ_0, T_0) is a marked quotient of the diagonal product $\Delta = \bigotimes_{i=1}^\infty(\Gamma_i, T_i)$. When the sequence Γ_i consists of finite groups, Δ is a FC-central extension of Γ_0. The construction in [21] takes diagonal product of a sequence of marked groups which converges to a wreath product of the form $\Gamma_0 = (A \times B) \wr \mathbb{Z}$ where A, B are finite groups. The sequence is chosen so that one can understand what elements in $\ker(\Delta \to \Gamma_0)$ are, and, moreover, explicitly estimate the word length of such elements with respect to the marking on Δ. The flexibility in the construction allows proving Theorem 3.1.

3.3. Relations between random walk characteristics

The three random walk characteristic functions, namely the decay of return probabilities, the entropy function, and the speed function post constraints on each other.

3.3.1. Between speed and entropy

As a consequence of the Varopoulos–Carne inequality and the fundamental inequality mentioned earlier, for a symmetric probability measure μ on G with finite support, entropy and speed satisfy

$$\frac{1}{n}\left(\frac{1}{4}L_{\mu}(n)\right)^2 - 1 \le H_{\mu}(n) \le (v+\varepsilon)L_{\mu}(n) + \log n + C, \tag{3.5}$$

where v is the exponential volume growth rate of $(G, \operatorname{supp}\mu)$, $C > 0$ is an absolute constant, see [4, 31]. For example, if we know $H_{\mu}(n) \simeq n^{\theta}$ (that is, the entropy exponent is θ), then the speed function is constrained by $n^{\theta} \lesssim L_{\mu}(n) \lesssim n^{(1+\theta)/2}$.

In [1], the joint realization problem of speed and entropy is considered. The construction in [21] with a sequence of expanders as input and the one with finite dihedral groups allows showing that for any entropy exponent $\theta \in [1/2, 1]$, all speed exponents allowed by the constraint (3.5) can be realized. That is, for any $\theta \in [\frac{1}{2}, 1]$ and $\gamma \in [\frac{1}{2}, 1]$ satisfying $\theta \le \gamma \le \frac{1}{2}(\theta + 1)$, there exists a finitely generated group G and a symmetric probability measure μ of finite support on G, such that the random walk on G with step distribution μ has entropy exponent θ and speed exponent γ, see [21, COROLLARY 1.3, PROPOSITION 3.17]. The case where both exponents θ, γ belong to $[\frac{3}{4}, 1]$ was treated by Amir [1].

3.3.2. Between return probabilities and entropy

Let μ be a symmetric probability measure of finite entropy on a group G. In [86, 90], the following connection between return probability and entropy is shown. Let μ be a symmetric probability measure of finite entropy on Γ. Then:

- if $-\log \mu^{(2n)}(\mathrm{id})/n^{1/2} \to 0$ as $n \to \infty$, then the pair (G, μ) has the Liouville property;
- furthermore, if $-\log \mu^{(2n)}(\mathrm{id}) \lesssim n^{\beta}$ where $\beta \in (0, 1/2)$, then the entropy function satisfies

$$H_{\mu}(n) \lesssim n^{\frac{\beta}{1-\beta}},$$

see [86, THEOREMS 1.1 AND 3.2]. The sharpness of this bound, which turns a return probability lower estimate into an entropy upper estimate, is demonstrated on a family of groups called bubble groups, which are considered in [67].

If instead of slow decay of return probabilities, one has estimates on the spectral profiles of balls, $\lambda_{R_{\mu}}(B(\mathrm{id}, r)) \lesssim r^{-\theta}$, then, by [86, THEOREM 1.6], the α-moment of displacement of the μ-random walk $(W_n)_{n=1}^{\infty}$ satisfies

$$\mathbb{E}\Big[\max_{1\le k\le n} |W_k|_S^{\alpha}\Big] \le C n^{\alpha/\theta}, \quad \text{for any } \alpha \in (0, \theta).$$

3.4. Connection to metric embeddings

The study of embeddings of finitely generated groups (viewed as a metric space with word distance on its Cayley graph) into Hilbert space was initiated by Gromov [47]. In the seminal work [102], G. Yu proved that groups that admit coarse embeddings into Hilbert space satisfy the coarse Baum–Connes conjecture.

Distortion of embeddings of finite metric spaces has been extensively studied in the theory of Banach spaces. Similar to the notion of distortion, Guentner and Kaminker [49] introduce a natural quasiisometry invariant that characterizes how close to bi-Lipschitz can an embedding of an infinite group into a Banach space be. Let Γ be a group generated by a finite set S and equipped with the associated left-invariant word metric d_S. For a Banach space X let $\alpha^*_X(\Gamma)$ be the supremum over all $\alpha \geq 0$ such that there exists a Lipschitz mapping $f : \Gamma \to X$ and $c > 0$ such that for all $x, y \in \Gamma$ we have $\|f(x) - f(y)\|_X \geq c d_S(x, y)^\alpha$. Similarly, one can define the equivariant compression exponent $\alpha^\#_X(\Gamma)$ by restricting to equivariant Lipschitz maps, namely $\|f(gx) - f(gy)\|_X = \|f(x) - f(y)\|_X$ for all $g, x, y \in G$. When the target space is the classical Lebesgue space $L_p([0,1])$, we write $\alpha^*_p(\Gamma)$ and $\alpha^\#_p(\Gamma)$ for the compression exponents.

The idea of connecting the notion of Markov type, which is an important metric invariant introduced by K. Ball [9], to Banach compression exponent of infinite groups first appears in Austin, Naor, and Peres [6]. For wreath products, in [81] an explicit formula for the Hilbert compression exponent of $H \wr \mathbb{Z}$ is shown, assuming that the lamp group H satisfies $\alpha^\#_2(H) = \frac{1}{2\beta^*(H)}$, where $\beta^*(H)$ is the supremum of upper speed exponent of symmetric random walk of bounded step distribution on H. Further, in [82] which significantly extends the method in [6,81], the L^p-compression exponent of $\mathbb{Z} \wr \mathbb{Z}$ was determined for $p \geq 1$. In [21] it is shown that for any $p \in [1,2]$ and a finitely generated infinite group H, the equivariant L_p-compression exponent of the wreath product $H \wr \mathbb{Z}$ is

$$\alpha^\#_p(H \wr \mathbb{Z}) = \min\left\{\frac{\alpha^\#_p(H)}{\alpha^\#_p(H) + (1 - \frac{1}{p})}, \alpha^\#_p(H)\right\}.$$

When applying the Markov-type method, one has the flexibility of choosing which Markov chains to consider: for instance, α-stable like random walks in [82] and jumping processes confined on finite subsets of $H \wr \mathbb{Z}$ in [21].

It is known that distortion of metric embeddings can be captured by Poincaré inequalities of general forms. In particular, the Markov-type inequalities mentioned above can be viewed as a special form of Poincaré inequalities. Other types of obstructions to low distortion embeddings can be observed in the metric geometry of finitely generated groups. The construction of diagonal product Δ with infinite dihedral groups as input in [21] contains scaled ℓ^∞-cubes of growing sizes in Δ. Sharp estimates of distortion of embeddings of ℓ^∞-cubes into L^p-spaces are provided by the deep work of Mendel and Naor on metric cotype in [77]. Explicit evaluation of compression exponents of such diagonal products yields the following. With certain choice of parameters, such groups also provide the first examples where L_p-compression exponent, $p > 2$, is strictly larger than the Hilbert compression exponent. It might be interesting to investigate this collection of groups in the program on quasiisometric rigidity of solvable groups.

Theorem 3.2 ([21]). *For any $\frac{2}{3} \le \alpha \le 1$, there exists a 3-step solvable group Δ such that for any $p \in [1,2]$,*

$$\alpha_p^*(\Delta) = \alpha_p^\#(\Delta) = \alpha.$$

Further, there exists a 3-step solvable group Δ_1 such that for all $p \in (2,\infty)$,

$$\alpha_p^\#(\Delta_1) \ge \frac{3p-4}{4p-5} > \alpha_2^\#(\Delta_1) = \frac{2}{3}.$$

4. UNBOUNDED HARMONIC FUNCTIONS AND EQUIVARIANT EMBEDDINGS INTO HILBERT SPACES

Besides bounded harmonic functions one may consider other classes of harmonic functions and their relation to random walks. In this section we focus on harmonic functions of at most linear growth on amenable groups. Unlike the boundary theory associated with bounded harmonic functions, there is no systematic theory developed for this class of harmonic functions. Throughout this section, let Γ be a finitely generated group and take a symmetric probability measure μ of finite generating support on Γ.

Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ on a separable Hilbert space $\mathcal{H}$. A map $b : \Gamma \to \mathcal{H}$ is a 1-*cocycle* if $b(gh) = b(g) + \pi_g b(h)$ for all $g, h \in \Gamma$. Because of the cocycle equality, given a probability measure μ on G, b is μ-harmonic if $\sum_{s\in\Gamma} b(s)\mu(s) = 0$. A μ-harmonic 1-cocycle $b : G \to \mathcal{H}$ is also referred to as an *equivariant harmonic embedding* of G into $\mathcal{H}$.

As a special case of results in [80] (for the finitely presented case) and [66], a finitely generated group G does not have Kazhdan's Property (T) if and only if it admits a nonconstant equivariant μ-harmonic embedding into a Hilbert space. For an exposition of the proof in the setting of finitely generated groups, see [65, APPENDIX]. In the amenable case, nontrivial μ-harmonic embeddings can be constructed more explicitly by using μ-random walks, see, for example, [69, SECTION 3] and [34].

One may ask about properties of unitary representations associated with nonconstant μ-harmonic 1-cocycles. In [91] Shalom introduced the following notions in connection to the large-scale geometry of the groups. We say Γ has Property H_{FD} (H_{F}, or H_{T}, respectively) if for every nonconstant μ-harmonic 1-cocycle $b : G \to \mathcal{H}$, the associated representation π has a finite-dimensional (finite, or trivial, respectively) subrepresentation. We say π is *weakly mixing* if it does not admit any finite-dimensional subrepresentations. It is clear that Properties H_{FD}, H_{F}, and H_{T} are in increasing strength, while the sharpest one of them, H_{T}, implies that all μ-harmonic 1-cocyles are homomorphisms to $\mathcal{H}$.

4.1. Martingale small-ball probabilities

The existence of a nontrivial equivariant μ-harmonic embedding $b : \Gamma \to \mathcal{H}$ implies the a diffusive lower bound for speed of a μ-random walk on G, see [69]. In this subsection, we review bounds on small-ball probabilities of the martingale $b(W_t)$, which provide additional information about the behavior of the random walk. Note that from the cocycle equality, $\|b(gs) - b(g)\|_{\mathcal{H}} = \|b(s)\|_{\mathcal{H}}$, in particular the map b is Lipschitz.

Consider a martingale $(X_t)_{t=0}^{\infty}$ with respect to filtration $(\mathcal{F}_t)_{t=0}^{\infty}$ taking values in a Hilbert space $\mathcal{H}$. Under the assumption of bounded increments and that the conditional variances $\mathbb{E}[\|X_{t+1} - X_t\|^2 | \mathcal{F}_t]$ are constant, general small-ball probabilities estimates are proved independently in [70] and [5]. Applied to the martingale $(b(W_t))_{t=0}^{\infty}$, where $(W_t)_{t=0}^{\infty}$ is a μ-random walk on Γ, we have:

Theorem 4.1 ([5,70]). *For any nonconstant equivariant μ-harmonic embedding $b : G \to \mathcal{H}$, there is a constant $C > 0$, such that for all $t, r \geq 1$,*

$$\mathbb{P}\big(\|b(W_t)\| \leq r\big) \leq \frac{Cr}{\sqrt{t}}.$$

Note that this general bound cannot be improved, since, for instance, it is sharp for a simple random walk (W_t) on $\mathbb{Z}$ where $\mu(\pm 1) = 1/2$ and $b : \mathbb{Z} \to \mathbb{R}$ given by $b(z) = z$. Note that in this example the representation associated with b is trivial.

When $b : G \to \mathcal{H}$ is a nonconstant harmonic 1-cocycle with weakly mixing representation π, the martingale $X_t = b(W_t)$ satisfies an asymptotic orthogonality condition: for large k, the direction of the increment $X_{t+k} - X_t$ is almost orthogonal to X_t. More precisely, when the representation π is weakly mixing, there exists a sequence of nonincreasing constants $(\epsilon_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \epsilon_k = 0$ and for any $t, k \in \mathbb{N}$, the martingale $X_t = b(W_t)$ satisfies

$$\frac{1}{k}\mathbb{E}\bigg[\Big\langle \frac{X_t}{\|X_t\|}, X_{t+k} - X_t\Big\rangle^2 | \mathcal{F}_t\bigg] \leq \epsilon_k \quad \text{almost surely.} \tag{4.1}$$

This claim can be deduced directly from [85, LEMMA], which is a step in Ozawa's functional analytic proof of the Gromov polynomial growth theorem.

It turns out for $\mathcal{H}$-valued martingales with bounded increments and asymptotic orthogonality property (4.1), one can obtain superpolynomial decay bounds for small-ball probabilities, following a classical Foster–Lyapunov drift-type supermartingale argument. Applied to the martingale $(b(W_t))_{t=0}^{\infty}$, we have the following superpolynomial decay estimate for small-ball probabilities. If $b : G \to \mathcal{H}$ is a nontrivial harmonic 1-cocycle with weakly mixing representation π, then for any $\beta > 0$, there exists a constant $C > 0$ depending on β such that

$$\mathbb{P}\big(\|b(W_t)\| \leq r\big) \leq C\left(\frac{r}{\sqrt{t}}\right)^{\beta} \quad \text{for all } t, r \geq 1. \tag{4.2}$$

4.2. Some open problems

Since its introduction in [91], it is believed that for amenable groups, Property H_{FD} is a rather strong property only satisfied by certain "small" groups. This is reflected in the state that the only known examples of groups with Property H_{FD} are nilpotent groups, polycyclic groups, wreath product $F \wr \mathbb{Z}$ with finite F, and certain extensions of such groups. There are very limited known tools to establish that a given group has Property H_{FD}. If Γ embeds as a lattice in a nondiscrete locally compact group G, then one might use the representation theory of G: this approach is carried out in [91] to establish that polycyclic groups have Property H_{FD}. Probabilistic approaches using random walks, see [85], require strong coupling properties. For instance, it is still not known whether the wreath product $F \wr \mathbb{Z}^2$, where F is finite, has Property H_{FD}.

In this subsection, we discuss two problems on general amenable groups, in both situations knowing that the group Γ does not have Property H_{F} implies positive answers. These problems provide motivations to understand better the class of groups with Property H_{F}.

4.2.1. Dimension of the space of linear growth harmonic functions

Problem 4.2 ([78]). Let $\mathrm{HF}_1(\Gamma, \mu)$ denote the space of μ-harmonic functions on Γ whose growth is bounded by a linear function. Is it true that $\mathrm{HF}_1(\Gamma, \mu)$ is finite dimensional if and only if Γ is of polynomial growth?

For more background and discussions around this question see [78, INTRODUCTION]. The "if" direction is known: it is a step in Kleiner's proof [65] of Gromov's polynomial growth theorem. Towards the "only if" direction, when Γ does not have Property H_{FD}, there exists an irreducible unitary representation $\pi : \Gamma \to \mathcal{H}$ which is weakly mixing and associated with a nonconstant μ-harmonic cocycle $b : \Gamma \to \mathcal{H}$. Consider the subspace of Lipschitz μ-harmonic functions $\{f_v\}_{v\in\mathcal{H}}$ given by $f_v(g) = \langle b(g), v\rangle$. One can check that since π is weakly mixing, the space $\{f_v\}_{v\in\mathcal{H}}$ is infinite dimensional. When Γ has Property H_{FD} but not H_{F}, then it virtually admits a solvable group of exponential growth as a quotient group, see [91, PROPOSITION 4.2.3]. In this case applying the results in [78] to a solvable quotient group then lifting back to Γ show that $\mathrm{HF}_1(\Gamma, \mu)$ is infinite dimensional. Therefore the problem remains open only for groups with Property H_{F}.

4.2.2. Occupation time of balls

For a transient μ-random walk $W = (W_n)_{n=1}^{\infty}$ on Γ, one can consider the occupation time of a finite set, that is, the total amount of time the random walk spent in the given set. Of particular interest is the occupation time of balls:

$$\mathcal{N}_W(r) := \left|\left\{n \in \mathbb{N} : W_n \in B_S(\mathrm{id}, r)\right\}\right| = \sum_{\gamma\in B_S(\mathrm{id},r)} G_\mu(\mathrm{id}, \gamma),$$

where $B_S(\mathrm{id}, r)$ is the set of vertices within graph distance r to the identity on the Cayley graph (Γ, S), and $G_\mu(x, y) = \sum_{n=0}^{\infty} \mu^{(n)}(x^{-1}y)$ is the Green function of the μ-random walk.

Problem 4.3 ([74]). If $(W_n)_{n=0}^{\infty}$ is a transient symmetric random walk on a finitely generated group G, then

$$\mathbb{E}\big(\mathcal{N}_W(r)\big) \lesssim r^2,$$

where $\mathcal{N}_W(r)$ is the occupation time of the ball $B_S(\mathrm{id}, r)$ as defined above.

The motivation for the conjectured quadratic bound is as follows. Let τ_r be the first exit time of the ball $B(\mathrm{id}, r)$ of a random walk starting at the identity. Take a equivariant μ-harmonic embedding $b : G \to \mathcal{H}$ normalized such that $\mathbb{E}\|b(W_1)\|^2 = 1$. Applying the optional stopping theorem to the martingale $\|b(W_t)\|^2 - t$, we deduce that $\mathbb{E}(\tau_r) \lesssim r^2$.

Heuristically, since the random walk is assumed to be transient, once it has left a ball $B_S(\mathrm{id}, Cr)$, where C is a large constant, the chances that it comes back to $B(\mathrm{id}, r)$ is small. Hence conjecturally, the expected occupation time of balls should admit a quadratic upper bound as well.

In [74], it is shown that in general, for any nondegenerate symmetric transient random walk $(W_n)_{n=0}^{\infty}$ on a finitely generated infinite group Γ, we have

$$\mathbb{E}\big(\mathcal{N}_W(r)\big) \lesssim r^2 \sqrt{\log v_{\Gamma,S}(r)}, \tag{4.3}$$

where $v_{\Gamma,S}(r)$ is the volume growth function of (Γ, S). In particular, on groups of exponential volume growth it yields the upper bound $r^{5/2}$. The bound $\mathbb{E}(\mathcal{N}_W(r)) \lesssim r^3$ on exponential growth groups is shown in [18] relying only on the Varopoulos bound that on such groups $\mu^{(2n)}(\mathrm{id}) \lesssim e^{-n^{1/3}}$.

We mention a connection of polynomial upper bounds on occupation times of balls to positive μ-harmonic functions. See [100, CHAPTER IV] for a treatment of the Martin boundary, which is a topological boundary representing positive harmonic functions. The bound $\mathbb{E}(\mathcal{N}_W(r)) \le Cr^D$ implies that the minimum of the Green function of the μ-random walk satisfies

$$\min_{\gamma \in B(\mathrm{id},r)} G_\mu(\mathrm{id}, \gamma) \le \frac{Cr^D}{|B(\mathrm{id}, r)|}.$$

In particular, when Γ is of exponential growth, by the classical bound $\mathbb{E}(\mathcal{N}_W(r)) \lesssim r^3$, the minimum of the Green function in $B(\mathrm{id}, r)$ decays exponentially in r. By translation invariance, we can write the Green function as a telescoping product

$$\frac{G_\mu(\mathrm{id}, y_1 y_2 \cdots y_n)}{G_\mu(\mathrm{id}, \mathrm{id})} = \prod_{i=0}^{n-1} \frac{G_\mu(y_i, y_{i+1} \cdots y_n)}{G_\mu(y_{i+1}, y_{i+1} \cdots y_n)}, \quad \text{where } y_0 = \mathrm{id},\ y_i \in \Gamma.$$

Then an argument by contradiction shows that exponential decay of $\min_{\gamma \in B(\mathrm{id},r)} G_\mu(\mathrm{id}, \gamma)$ in r implies that there exists $s \in S^2$ and a sequence $(\gamma_n)_{n=0}^{\infty}$ in Γ going to infinity such that $\lim_{n\to\infty} G_\mu(s, \gamma_n)/G_\mu(\mathrm{id}, \gamma_n) < 1$. This shows that the Martin kernel $K(\cdot, \xi)$ is not constant in the first coordinate for some point ξ in the Martin boundary; equivalently, there are nonconstant positive μ-harmonic functions on Γ. Thus it gives another proof (though similar in spirit) of the result in [3], for any nondegenerate symmetric probability measure μ on a group Γ of exponential growth.

For Problem 4.3, when when Γ does not have Property H_{FD}, we can apply the estimate (4.2) to a nonconstant μ-harmonic cocycle $b : \Gamma \to \mathcal{H}$ with weakly mixing π. Indeed, choose any $\beta > 2$, since b is C-Lipschitz, we have

$$\mathbb{E}\big(\mathcal{N}_W(r)\big) \le r^2 + \sum_{t=r^2}^{\infty} \mathbb{P}\big(\|b(W_t)\| \le Cr\big) \le r^2 + C' \sum_{t=r^2}^{\infty} \left(\frac{r}{\sqrt{t}}\right)^{\beta} \le C'' r^2.$$

When Γ has Property H_{FD} but not H_{F}, then one can verify the quadratic bound by directly examining the random walk on a virtual quotient which is solvable of exponential growth. Therefore the problem is open only for groups with Property H_{F} that are not virtually nilpotent.

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