ON A CURIOUS PROBLEM AND WHAT IT LEAD TO

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ABSTRACT

This note tells a story of an open problem on the asymptotic behavior of the minimal number of generators of groups that motivated several of my research directions.

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In this note I will attempt to tell the story of an open problem on the minimal number of generators of groups that I am interested in for a long time and that motivated several of my research directions, sometimes in surprising ways. As stories go, the focus on the protagonist (the rank gradient problem) tends to do injustice to the other characters. This means that for some of the connected math described, even major results will get suppressed. I will also attempt to build up the character subjectively and from its birth, so not every result will be stated in its strongest form immediately, like Pallas Athene jumping out of her father's head in her full strength. In any case, the note circles around unsolved problems, which is the very opposite of this image. I will try to mitigate the damages with side stories and remarks. Finally, I believe that the truth, even mathematical, is inherently subjective and is born from a dialogue of people arriving from infinitely far. I attempt to tell this story from my own perspective but this should not be taken as a suggestion on my role in the projects I describe.

For a discrete group Γ, let $d(\Gamma)$ denote the minimal number of generators of Γ, that is, the minimal size of a subset S of Γ that generates Γ. We will also call this the *rank* of Γ, although the word "rank" is used by a lot of other notions already. We are interested in the case when Γ is also *residually finite*, that is, the intersection of its subgroups of finite index is the trivial subgroup. A rich source of residually finite groups is finitely generated matrix groups.

The rank is a rather mysterious invariant, already for finite groups. While natural (geometric) generating sets give suggestions for the rank, other generating sets may beat them to the punch. The finite symmetric group $\mathrm{Sym}(n)$ can be generated by all transpositions (i, j) and also by the neighboring transpositions $(i, i+1)$. But it can also be generated by just 2 elements. On this track, we know that every finite simple group can be generated by at most 2 elements, but as of now, this only follows from the classification of finite simple groups.

When dealing with infinite groups, the picture does not get clearer, either. Virtually the only general way to bound the rank from below is to use the first homology, and when this does not help, one has to play it by the ears. A beautiful exception is the Grushko–Neumann theorem. The rank is not only hard to control for abstract groups. When Γ arises as the fundamental group of a nice manifold, say, one would expect that a minimal generating set, as a family of loops, will carry some geometric meaning. While there are examples when this is indeed the case, in general this is too much to hope for.

When an invariant of a residually finite group is rather unruly, one can attempt to stabilize it by looking at its growth over its subgroups of finite index and hope that this will give a more robust invariant. The biggest success story here is L^2 cohomology, or, more generally, spectral theory and representation theory, as we will discuss later on. On the geometric side, this means that instead of the defining manifold or complex M of Γ, we look at the family of finite sheeted coverings of M and try to build a geometric understanding of asymptotic homotopy on these spaces.

Rank gradient. By the Nielsen–Schreier theorem, when Γ is a free group, and H is a finite index subgroup of Γ, we have $d(H) - 1 = (d(\Gamma) - 1)|\Gamma : H|$. In other words, the number

$$r(\Gamma, H) = \frac{d(H) - 1}{|\Gamma : H|}$$

is constant $d(\Gamma) - 1$ when Γ is free, hence for an arbitrary Γ, we have the inequality $r(\Gamma, H) \leq d(\Gamma) - 1$. A little exercise then shows that for $K \leq H \leq \Gamma$, we have $r(\Gamma, K) \leq r(\Gamma, H)$. This implies that for a *chain* of finite index subgroups $\Gamma = H_0 \geq H_1 \geq \cdots$, the limit

$$\mathrm{RG}\big(\Gamma, (H_n)\big) = \lim_{n \to \infty} r(\Gamma, H_n)$$

will exist. We call this the *rank gradient* of Γ with respect to the chain (H_n). The notion comes from Marc Lackenby [32], see also an early profinite version in [36]. One can also define it to an arbitrary subset $\{H_n\}$ of finite index subgroups of Γ as

$$\mathrm{RG}\big(\Gamma, \{H_n\}\big) = \inf_n r(\Gamma, H_n).$$

Many years ago, we started to study this notion by ourselves with Nik Nikolov, proved some initial results using elementary group theory, and soon realized that we do not know a convincing example for when the rank gradient in fact depends on the chain. A nonconvincing example comes from $\Gamma = F_2 \times F_2$: normal chains in Γ with trivial intersection have rank gradient zero, but chains that only walk down on one of the factors have positive rank gradient. We could not find an example, however, when the H_n are normal subgroups of finite index and their intersection is trivial. We still cannot.

Problem 1 (Rank gradient). *Let Γ be finitely generated and let (H_n) and (K_n) be normal chains in Γ with trivial intersection. Does*

$$\mathrm{RG}\big(\Gamma, (H_n)\big) = \mathrm{RG}\big(\Gamma, (K_n)\big)?$$

What do you do when you encounter an elusive but attractive invariant? 1. Prove that it vanishes in some natural cases; 2. Try to look for translations or analogues in other fields and try to make mathematical energy flow through; 3. Connect it to some other, maybe tamer invariants; 4. Extend the notion wildly and see what happens. In what follows, I will describe some attempts of these points and where they lead.

The cost correspondence. A good starting exercise for the reader is to prove by hand that if Γ has a central element of infinite order, then the rank gradient vanishes for any normal chain with trivial intersection. After proving some starting results like this with Nik Nikolov on rank gradient, we managed to connect the rank gradient to cost.

The notion of cost was introduced by Gilbert Levitt [34] and most of the subsequent, deep work on it was done by Damien Gaboriau [24]. I will not define the notion here, just state that every probability measure preserving (p.m.p.) action of a countable group Γ has a cost, which is a real number between 1 and $d(\Gamma)$. A major question on cost is the following [24].

Problem 2 (Fixed price). *Let Γ be a countable group. Does every free p.m.p. action of Γ have the same cost?*

In his hallmark result [24], Damien Gaboriau showed that this is true for free groups. Since the cost only depends on the equivalence relation spanned by the action, it immediately follows that the free groups F_2 and F_3 are not orbit equivalent, a well-known open problem at the time.

In [9] we established the following correspondence. For a chain (Γ_n) in Γ, one can associate its coset tree $T(\Gamma, (\Gamma_n))$ as follows. The vertex set of T is the union of cosets Γ/Γ_n and the edges are defined by immediate inclusion of cosets. The group Γ acts on T by automorphisms and this action extends to the boundary ∂T of T as a continuous action. There is a natural measure on the boundary (the product measure on infinite walks) and the action preserves this measure.

Theorem 3 (Cost correspondence). *Let Γ be finitely generated and let (Γ_n) be a normal chain in Γ with trivial intersection. Then*

$$\mathrm{RG}\big(\Gamma, (\Gamma_n)\big) = \mathrm{c}\big(\Gamma, (\Gamma_n)\big) - 1,$$

where $\mathrm{c}(\Gamma, (\Gamma_n))$ *denotes the cost of* Γ *acting on* $\partial T(\Gamma, (\Gamma_n))$.

So, for chains, the rank gradient problem is a special case of the fixed price problem for profinite actions.

The existing cost theory immediately gave new vanishing results on rank gradient for a large class of groups, including amenable groups and more importantly, the so called *right angled* groups. These are groups that admit a list of generators of infinite order such that neighboring generators commute. It is an important class as it contains many nonuniform lattices, like $\mathrm{SL}(3, \mathbb{Z})$.

Looking at the cost literature, we also realized that a seemingly innocent result on cost would actually positively solve the RG problem. The question is whether the cost minus 1 is multiplicative for finite-index subrelations, just like rank minus 1 is for free groups. The result was announced to be solved at the time with versions of a preprint circulating but by now the community agrees that it should be considered unsolved. I still think that this could lead to a fruitful attack on either fixed price, or the rank gradient problem.

The rank vs. Heegaard genus problem. In our project with Nik Nikolov, we studied Marc Lackenby's work and its topological motivations [32, 33] and got aware that using his results on Heegaard genus and expansion [31], proving the vanishing of rank gradient for $\Gamma = \mathrm{SL}(2, \mathbb{Z}[i])$ would solve a famous old problem in 3-manifold theory. The problem that is still open is whether for finite volume 3-manifolds, the ratio of the Heegaard genus and the rank can get arbitrarily large. Note that for hyperbolic manifolds, it was also open for a long time whether the rank can even *differ* from the Heegaard genus. Now this is solved by Tao Li [35]. The deal for the ratio is that by [31] the Heegaard genus grows linearly for any chain of subgroups with property (τ) and it is easy to produce a normal chain in Γ with vanishing rank gradient, since it is virtually a finitely generated free by cyclic group. So if the rank gradient is independent of the chain, then a chain of principal congruence subgroups in Γ will have property (τ) and hence positive Heegaard genus growth but vanishing rank gradient, which makes the ratio of the two invariants go to infinity. On the other hand, if the rank gradient may depend on the chain, then the fixed price problem is solved negatively.

That is, we managed to show that at least one of these well-studied problems have a negative solution, but we still do not know which one(s). This is certainly a good joke, but a word of caution is due here. It could very well be that eventually *both* problems have a negative solution but for entirely different reasons, and then our bridge between them will not prove to be useful, as no one walks through it.

Homology growth and Lück approximation. The first rational homology b_Q of a group is a trivial lower bound for its rank, in fact, it is the only general lower bound people use. As a consequence, the growth of homology satisfies

$$\lim_{n\to\infty} \sup \frac{b_Q(\Gamma_n)}{|\Gamma : \Gamma_n|} \leq \mathrm{RG}\big(\Gamma, (\Gamma_n)\big).$$

The good news here is that when Γ is finitely presented, a famous theorem of Wolfgang Lück [37] implies that the limit of the left-hand side exists and is independent of the chain.

Theorem 4 (Lück approximation). *Let Γ be a finitely presented group and let (Γ_n) be a chain of normal subgroups of finite index in Γ with trivial intersection. Then we have*

$$\lim_{n\to\infty} \frac{b_Q(\Gamma_n)}{|\Gamma : \Gamma_n|} = \beta^1(\Gamma),$$

where $\beta^1(\Gamma)$ is the first L^2 Betti number of Γ.

The L^2 story that starts here is quite extensive and beautiful (see [37] and around), but for us what matters now is that for finitely presented groups we have

$$\mathrm{RG}\big(\Gamma, (H_n)\big) \geq \beta^1(\Gamma).$$

We still do not know an example where there is a proper inequality here. Note that for a classical group theorist, it sounds quite weird that the abelianization of Γ_n should asymptotically control its rank! For instance, this suggests that if the Γ_n are perfect groups, then by some miracle we should be able to generate them by fewer elements than the trivial bound.

When turning to the measured setting and use the cost correspondence, this takes the form of a question already asked in the initial paper of Gaboriau [24]. He shows that for every free action of Γ the cost of the action minus 1 is at least $\beta^1(\Gamma)$ and no one knows an example when they are not equal.

I will not state the full generality of the Lück approximation result, but need to make some side comments here. First, the proof is really about spectral convergence. It is easy to see that for normal chains with trivial intersection, the eigenvalue distribution of any locally defined operator on the finite quotient will weakly converge to the spectral measure of the same operator in the limit. The gist of Lück approximation is to show that the measure of the set $\{0\}$ will also converge. This is a tightness result that does not follow from weak convergence in general. The result was later generalized by Andreas Thom [41] for arbitrary real values instead of 0.

Graph sequences, combinatorial cost, and the Farber condition. In our project leading to [9], we also studied what happens with arbitrary instead of normal subgroups or when we ease up on having trivial intersection. These were not arbitrary questions. First, we hoped to find counterexamples easier in this bigger class. Second, in a lot of cases when one is interested in the asymptotic behavior of an invariant on a family of finite index subgroups, they do not form a chain and are not normal. For instance, in number theory we often care about the congruence subgroups $\Gamma_0(N)$: they are not normal and do not form a chain. These questions has lead to a connection to graph limit theory.

What connects two different chains of normal subgroups in Γ with trivial intersection? The best answer I know is that they are locally indistinguishable. That is, from every vertex, the corresponding Schreier graphs locally look more and more like the infinite Cayley graph of Γ. When you only ask that the same holds for *most* vertices, you get the notion of Benjamini–Schramm convergence [13].

In the homological direction, we found Michael Farber's extension of the Lück approximation theorem [22] and the subsequent work of Nicolas Bergeron and Damien Gaboriau [14] on when and how such an extension may fail. It is clear that the Farber condition is equivalent to asking that the action of Γ on the boundary of the coset tree is essentially free. We called these chains *Farber chains*.

For a fixed generating set S of Γ, one can visualize the chain (Γ_n) by looking at the sequence of finite Schreier graphs $(\mathrm{Sch}(\Gamma/\Gamma_n, S))$ and attempt to understand rank gradient using the asymptotic metric geometry of this graph sequence. Now we can state what the Farber condition is, in various ways. For a permutation action of Γ and $g \in \Gamma$, let $\mathrm{Fix}(g, \Gamma/\Gamma_n)$ denote the number of fixed points of g.

Proposition 5. *Let Γ be a group generated by the finite symmetric set S and let (Γ_n) be a sequence of subgroups in Γ. Then the following are equivalent:*

1. *For every $1 \neq g \in \Gamma$, we have*
$$\lim_{n\to\infty} \frac{\mathrm{Fix}(g, \Gamma/\Gamma_n)}{|\Gamma : \Gamma_n|} = 0 \quad \textit{(Farber condition);}$$
2. *A random conjugate of Γ_n as an invariant random subgroup weakly converges to the trivial one;*
3. $\mathrm{Sch}(\Gamma/\Gamma_n, S)$ *Benjamini–Schramm converges to* $\mathrm{Cay}(\Gamma, S)$*;*
4. *The coset actions of Γ on Γ/Γ_n form a sofic approximation of Γ.*

These equivalences are all easy once one learns the language. However, these forms are important to note, as they highlight the fields that got connected by this theory: in order, representation theory, ergodic theory, graph limits, and soficity.

An example for a Farber sequence is the above mentioned congruence subgroups $\Gamma_0(p)$ (p prime). Let me remark that the notion of Farber sequence can be naturally extended to a sequence of lattices in a fixed locally compact group. In [12] that some people call the 7 samurai paper, we prove that in a higher rank simple Lie group, *every* sequence of lattices with covolume tending to infinity is Farber! We use invariant random subgroups in the proof, a notion that was coined (but not invented) in [6]. There would be a lot to tell here, but this story is about rank gradient, so we stop at this point.

When looking at it as a graph theory problem, the rank gradient problem asks if the asymptotic rank is a *local invariant*, or whether it depends on global properties of $\mathrm{Sch}(\Gamma/\Gamma_n, S)$.

The idea of bringing graph limit theory to cost and L^2 theory is due to Gabor Elek [21], who in an early paper [20] defined the combinatorial cost of a graph sequence and proved the analogues of the known results on cost and homology growth in this setting. A quick definition of combinatorial cost is as follows. For a sequence of finite graphs, a *bi-Lipshitz rewiring* of the sequence is another graph sequence using the same vertex sets, such that there exists a constant L where the distance of every edge in one graph can be substituted by a path of length at most L in the other. The combinatorial cost is the infimum of edge densities that can be achieved by such a rewiring. Note that much later, two groups consisting of Alessandro Carderi, Damien Gaboriau, and Mikael de la Salle, and me and Laszlo Toth, respectively, independently showed that using an ultraproduct language [19] or local–global convergence [10], the combinatorial cost is, in fact, equal to the cost of a suitable limiting object. But by then, the damage was done and graph limit theory was affecting the field in various ways.

When you ease up on normality, the intersection of the subgroups really will not matter, even for chains, and it is the Farber condition that will really affect the behavior. By [24], every aperiodic p.m.p. action of an amenable group has vanishing cost, but the rank gradient correspondence only works for Farber chains. Indeed, it is not true that for an amenable group the rank gradient vanishes for any normal chain, an easy counterexample comes from the lamplighter group. However, Marc Lackenby [33] showed that for *finitely presented* amenable groups, the rank gradient vanishes for arbitrary normal chains, that is, trivial intersection is not needed there. By pushing his trichotomy theorem a bit further, together with Nik Nikolov and Andrei Jaikin-Zapirain, we managed to show that for finitely presented amenable groups, the rank gradient also vanishes for arbitrary chains [7]. This is one of the examples I know where a result on rank gradient does not seem to have an immediate cost counterpart.

Weak containment. Using the above observation on local behavior, with Gabor Elek we attempted to solve the rank gradient problem by showing that the Schreier graphs of any two normal chains in the same group can be asymptotically "massaged onto each other" by almost covering maps. We soon realized that what we look at is already investigated for p.m.p. actions by Alekos Kechris [29] under the name weak containment and is also strongly connected to the notion of local–global convergence introduced by Bela Bollobas and Oliver Riordan [16] and developed by Hamed Hatami, Laszlo Lovasz, and Balazs Szegedy [27]. Ironically, we found that, in fact, the opposite of what we attempted holds, and proved the following rigidity result in [4].

Theorem 6 (Weak containment rigidity). *If a strongly ergodic p.m.p. action of* Γ *weakly contains a finite action of* Γ *then it factors onto it. In particular, if two normal chains in* Γ *with property* (τ) *define weakly equivalent coset tree actions, then the two chains are refinements of each other.*

From the point of view of the rank gradient problem, this can be considered as a harsh no entry sign, but it does show that arithmetic lattices admit uncountably many weakly inequivalent p.m.p. actions.

Although weak containment has not yet been successful to prove new results on rank gradient, let me mention a quite elegant application by Alekos Kechris [30]. A result of Lewis Bowen [17] implies that for the free group F_n, its profinite completion weakly contains any free p.m.p. action of F_n. By the cost monotonicity result for weak containment [29], this implies that among free p.m.p. actions of F_n, the cost is *minimal* for the profinite completion. Now using the cost–rank-gradient correspondence above, one yields that the cost of any free action of F_n is at least n, hence F_n has fixed price n, giving an alternate proof for the famous starting result of Damien Gaboriau.

The graph limit language suggested another possible attempt at the rank gradient problem, using a factor of iid generating set for the Γ_n. For a Farber chain, the quotient Schreier graphs look like the infinite limiting Cayley graph from most points. So, one may apply the same rule using an iid seed, to get a cheap generating set, and since the seed was iid, the resulting cheap rewiring also works for any Farber sequence the same way. This attempt also did not work (yet) but it did eventually lead to the result with Benjy Weiss [11] that every free action of a countable group Γ weakly contains its iid actions, hence showing that iid actions of Γ have maximal cost.

Homology torsion growth. When trying to interpret the fixed price 1 result of Damien Gaboriau for right angled groups with Tsachik Gelander and Nik Nikolov in the finite setting, we realized that there is an interesting rewiring complexity notion hiding behind it and that the notion can be used to prove vanishing of the first homology torsion growth.

More precisely, when building the cheap rewiring on the finite level, using Damien Gaboriau's trick, not only the rewiring gets cheap, but at the same time its complexity also stays low. In particular, it gives cost $1+\varepsilon$ with a bi-Lipschitz constant that is polynomial in $1/\varepsilon$. We then realized that this is enough to prove not only the vanishing of rank gradient, but also the vanishing of the first homology torsion growth.

For a finitely presented group, it is easy to show that the size of the torsion part of the abelianization of a subgroup is at most exponential in the index of the subgroup. Hence, the right growth notion to consider is

$$\mathrm{t}\big(\Gamma,(\Gamma_n)\big)=\lim_{n\to\infty}\frac{\log(\mathrm{tor}(\Gamma_n))}{|\Gamma:\Gamma_n|},$$

assuming this limit exists. Torsion homology growth is studied by various groups for various reasons, see [15] and references therein. In [5] we prove the following.

Theorem 7 (First torsion homology growth). *Let Γ be a right angled group and let (Γ_n) be a Farber sequence in Γ. Then*

$$\mathrm{t}\big(\Gamma, (\Gamma_n)\big) = 0.$$

In fact, the proof works for any Farber sequence where the above defined by-Lipschitz constant is subexponential in the index. This brought an interesting connection to the Bergeron–Venkatesh conjecture [15], that made me understand something about how unruly the notion of rank may be in reality.

A special case of the Bergeron–Venkatesh conjecture says that for a principal congruence chain in $\Gamma = \mathrm{SL}(2, \mathbb{Z}[i])$, the first torsion homology growth is a positive constant. If we believe this, and also that the rank gradient is zero for these chains (these are both tall orders of course), then the previous theorem implies that while these congruence subgroups may admit cheap generating sets, their complexity must be exponential in the error $1/\varepsilon$. That is, we may not be able to find them in a nice and geometric way as they hide deep in the cruel and dark embrace of algebra.

Vanishing theorems can be cool, but they tend to emit a somewhat pessimistic aura. After all, at the end, we reach zero. However, in the case of first homology torsion growth, currently no one can do better, as the following is still open.

Problem 8. *Is there a finitely presented group Γ and a Farber sequence (Γ_n) in Γ such that $\mathrm{t}(\Gamma, (\Gamma_n)) > 0$?*

While there are lower bounds for the torsion, currently they do not get this high.

It is a natural question whether there is a natural "higher rank" notion of being right angled so that the above vanishing theorem generalizes for higher homology torsion. Recently this was addressed in the paper [2], together with Nicolas Bergeron, Mikolaj Fraczyk, and Damien Gaboriau.

Uniform rank gradient and Poisson processes. As we discussed above, to solve the rank vs Heegaard genus problem, it would be enough to effectively estimate the rank of principal congruence subgroups of $\mathrm{SL}(2, \mathbb{Z}[i])$. Note that this approach is a blessing and a curse at the same time. Indeed, while the ambient group and its congruence subgroups seem very concrete, they are inherently number-theoretic which means that any attempt would also involve some possibly rather nontrivial number theory. In fact, the same could be said when we want to estimate the rank gradient of any discrete group, using geometric methods. Indeed, unless the group is of some quite special form, like close to free, right angled, or amenable, the geometry of its Cayley graphs seem quite complicated.

It turns out, however, that when we ask for much more, it immediately forces our hand in a good way and seems to give a much simpler image to deal with. Let G be a locally compact group, but for simplicity just concentrate on when G is a simple real Lie group. When Γ is a lattice in G, it is finitely generated, moreover, by the work of Tsachik Gelander [26], we have

$$d(\Gamma) \leq C \operatorname{vol}(G/\Gamma),$$

where C is an absolute constant. The notion of Farber sequences make perfect sense, using either Benjamini–Schramm convergence of the quotient spaces G/Γ (or X/Γ where X is the symmetric space for G) or invariant random subgroups.

Problem 9. *Let G be a semisimple Lie group and let (Γ_n) be a Farber sequence of lattices in G. Does*

$$\mathrm{RG}\big(G, (\Gamma_n)\big) = \lim \frac{d(\Gamma_n) - 1}{\mathrm{vol}(G/\Gamma_n)}$$

exist?

If it does, then the limit is independent of the sequence, since you can merge two Farber sequences and they stay Farber. This is one of the advantages over using chains of lattices. In fact, one can define Benjamini-Schramm convergence in the realm of Riemannian manifolds [3]. On this language one gets the Poisson point process on a symmetric space as the limit of independent random subsets of the finite volume manifolds.

To address this problem, with Sam Mellick [8], we recently introduced a cost theory for point processes of locally compact groups. Note that Alessandro Carderi has already introduced the cost of p.m.p. actions of locally compact groups in his nice paper [18] and used an ultraproduct language to prove that the maximal cost of a p.m.p. action of G dominates the rank gradient, at least for uniformly discrete Farber sequences of lattices. Our approach of using point processes allows us to remove his uniform discreteness assumption and answer his question whether $G \times \mathbb{Z}$ has fixed price 1.

In the paper [8] we prove that the Poisson processes have maximal cost among free point processes and that this number dominates the rank gradient of any Farber sequence in G. This is an analogue of my theorem with Benjy Weiss for discrete groups [11], as Poisson processes are arguably the substitutes of iid actions in the locally compact setting. In particular, if the cost of the Poisson process is 1, then any free point process has cost 1 and the rank gradient vanishes for any Farber sequence of lattices.

Instead of G, one can again consider Poisson processes on its symmetric space X, as they have the same cost. In particular, to settle the rank vs Heegaard genus problem, it would be enough to show that Poisson processes on the hyperbolic space H^3 have cost 1.

Problem 10 (Poisson cost). *Does the Poisson process on H^3 of intensity* 1 *have cost* 1*?*

This seems to be a much simpler and more direct geometric-stochastic question than estimating the rank of congruence subgroups directly. On the other hand, if this cost happens to be greater than 1, this would not tell anything about the rank gradient, but it would still imply the existence of a countable equivalence relation whose cost does not equal to its first L^2 Betti number, answering a question of Damien Gaboriau (see [25] on L^2 numbers of countable equivalence relations).

Apart from the case when X is the upper half-plane, as of now, nothing is known for semisimple Lie groups. The reasonable conjecture is that for every other G, the cost of the Poisson processes should vanish. When we look at homological counterparts, we still get a nontrivial task. For rational homology growth, the 7 samurai project [12] and [1] settled most of the questions. In the direction of mod p homology growth, Mikolaj Fraczyk [23] proved in a beautiful paper that when G has higher rank and property (T), then the first mod 2 homology growth vanishes for arbitrary Farber sequences of lattices. In fact, he showed that every homology class admits a cycle of length that is sublinear in the volume. The difficulty is clearly shown in the fact that for odd primes, these are still open, although it would follow from the vanishing of the cost of Poisson processes.

Further questions on rank gradient. Kazhdan's property (T) is a strong property that can be used in manifold ways, in particular, it implies that the first L^2 Betti number vanishes. So, it makes sense to ask the following.

Problem 11 (Kazhdan groups). *Does the rank gradient vanish for finitely presented, residually finite groups with property (T)?*

In other words, does it vanish for every normal chain with trivial intersection? When switching to the ergodic side, this asks whether property (T) groups have fixed price 1. This is also open, however, Tom Hutchcroft and Gabor Pete recently showed in a recent, very nice paper [28] that such groups always admit an action with cost 1, that is, the infimal cost of Γ is 1. It would be natural to use [11] here, but the processes they ingeniously generate are not factor of iid, so their result does not establish fixed price 1, and also does not seem to settle the rank gradient problem for these groups. Nevertheless, it is still tempting to try to adapt their method somehow in the finite setting and yield a vanishing result on rank gradient.

In another direction, it would be interesting to say something meaningful on groups with positive rank gradient. Marc Lackenby's trichotomy theorem [32] gives some restrictions and also his theorem that finitely presented groups with positive p-gradient are large. On the other hand, if we omit the finite presentation condition, we will have some positive rank gradient monsters lurking around, as Denis Osin [39] and Jan-Christoph Schlage-Puchta [40] showed. A specific question due to Nik Nikolov is as follows.

Problem 12. *Can a group satisfying a nontrivial identity have a positive rank gradient?*

Nik Nikolov recently managed to show that in this case, the profinite gradient does vanish [38].

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