LATTICE SUBGROUPS ACTING ON MANIFOLDS

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ABSTRACT

We discuss recent progress in understanding rigidity properties of smooth actions of higher-rank lattices. We primarily discuss questions of existence in low dimensions (Zimmer's conjecture), classification in the smallest possible dimension, and further classification assuming dynamical properties of the action. Two common themes arise in the proofs: (1) dynamical properties of the lattice action are mimicked by certain measures on an induced G-space; (2) such measures often exhibit additional rigidity properties. Throughout, we state some open problems and possible directions for future research.

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1. INTRODUCTION: LATTICES, GROUP ACTIONS, AND RIGIDITY

1.1. Rigidity of linear representations

For $n \geq 2$, consider the group $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ of $n \times n$ integer matrices with determinant 1 or a more general lattice subgroup of $\mathrm{SL}(n, \mathbb{R})$. There is a stark distinction between the case $n = 2$ and $n \geq 3$; in particular, relative to various group- and representation-theoretic properties, the group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is rather "flexible" whereas the group $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ is very "rigid" whenever $n \geq 3$.

Indeed, when $n = 2$, linear representations $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ are very flexible. In contrast, when $n \geq 3$, linear representations $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ exhibit many well-known rigidity properties; we highlight local rigidity of the inclusion $\iota\colon \Gamma \to \mathrm{SL}(n, \mathbb{R})$ [62, 64], local rigidity of general representations $\pi\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ [52, 59, 65], and Mostow's strong rigidity [50, 53, 56]. The principle result that includes those above is Margulis' superrigidity theorem. Roughly, Margulis' theorem states (when $n \geq 3$) that any representation $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ coincides—up to a compact error—with the restriction of a continuous representation $\pi\colon \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(d, \mathbb{R})$. Since representations of $\mathrm{SL}(n, \mathbb{R})$ are classified, this more-or-less classifies all representations of Γ.

1.2. The general setting

Throughout, G will be a connected noncompact semisimple Lie group. We will always assume the Lie algebra of G is simple and say that G is a simple Lie group. Throughout, we will typically assume that G has higher real rank. (The Lie algebra $\mathfrak{g}$ of G admits an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k}\mathfrak{a}\mathfrak{n}$. The *real rank* of $\mathfrak{g}$ is $\dim(\mathfrak{a})$ and G is *higher rank* if it has real rank at least 2.) At times we may also assume G has finite center though that is not technically necessary for most results.

Such groups G admit biinvariant Haar measures. A *lattice* in G is a discrete subgroup Γ of G such that the coset space G/Γ has finite volume. A lattice Γ is *cocompact* if, in addition, the quotient G/Γ is compact; otherwise, Γ is *nonuniform*. When G is simple and is of higher real rank, we say a lattice Γ in G is a *higher-rank lattice*.

For simplicity of exposition, we formulate most results and conjectures in the case that $G = \mathrm{SL}(n, \mathbb{R})$ (though many results and conjectures hold for wider classes of groups). The real rank of $\mathrm{SL}(n, \mathbb{R})$ is $n - 1$ and thus we typically assume $n \geq 3$ to ensure we are in the higher-rank setting. The standard example of a lattice subgroup in $G = \mathrm{SL}(n, \mathbb{R})$ is the subgroup $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. The subgroup $\mathrm{SL}(n, \mathbb{Z})$ is nonuniform, though we note that $\mathrm{SL}(n, \mathbb{R})$ admits cocompact lattices.

1.3. Actions on manifolds and the Zimmer program

Beyond linear representations, we might replace the vector space $\mathbb{R}^d$ with a compact manifold M and replace the finite-dimensional Lie group $\mathrm{GL}(d, \mathbb{R})$ with $\mathrm{Diff}^r(M)$, the group of all C^r-diffeomorphisms[1] of M. A homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ then defines a

1 If $r \geq 1$ is not integral, we write $r = k + \beta$ where $k \in \mathbb{N}$ and $\beta \in (0, 1)$ and say that $f\colon M \to M$ is C^r if it is C^k and if the kth derivatives of f are β-Hölder continuous.

C^r *action* of Γ on M. If vol is a smooth volume form on M, we also consider $\mathrm{Diff}^r_{\mathrm{vol}}(M)$, the group of volume-preserving diffeomorphisms, and study volume-preserving actions $\alpha\colon \Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$.

For $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, actions $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ on manifolds are again quite flexible. However, by analogy with rigidity properties of linear representations, we might ask if (possibly volume-preserving) actions of higher-rank lattices exhibit rigidity properties analogous to those that hold for linear representations. To motivate statements, it is useful to recall some standard low-dimensional, algebraically defined actions by lattices in $\mathrm{SL}(n,\mathbb{R})$.

(1) *Affine actions on tori.* Consider the case that Γ has finite index in $\mathrm{SL}(n,\mathbb{Z})$. We obtain an action $\alpha\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^n)$ on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ given by $\alpha(\gamma)(x+\mathbb{Z}^n) = \gamma\cdot x + \mathbb{Z}^n$ for every matrix $\gamma \in \Gamma \subset \mathrm{SL}(n,\mathbb{Z})$. Since there exists $\gamma \in \mathrm{SL}(n,\mathbb{Z})$ with all eigenvalues outside of the unit circle, this gives an example of an affine Anosov action (see Definition 4.3). Observe that these actions preserve the Haar measure on $\mathbb{T}^d$.

(2) *Projective actions.* Given any lattice subgroup $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$, the linear action of Γ on $\mathbb{R}^n$ induces an action on the space of rays (or lines) in $\mathbb{R}^n$ through the origin. We thus obtain an action of Γ on the $(n-1)$-dimensional sphere S^{n-1} (or $\mathbb{R}P^{n-1}$). The subgroup $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ also acts on Grassmanians of higher-dimensional planes in $\mathbb{R}^n$ and on spaces of flags in $\mathbb{R}^n$. These actions are all left actions of Γ on G/Q for some parabolic subgroup $Q \subset G = \mathrm{SL}(n,\mathbb{R})$. We remark that these actions admit no Γ-invariant probability measure.

(3) *Isometric actions.* Certain cocompact lattices $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ admit representations $\pi\colon \Gamma \to \mathrm{SU}(n)$ with infinite image (see discussion in [69, SECTIONS 6.7, 6.8, WARNING 16.4.3]). The representation π then induces an isometric action of Γ on the $(2n-2)$-dimensional space $M = \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1)\times \mathrm{U}(n-1))$.

In the early 1980s, Zimmer established a superrigidity theorem for linear cocycles over ergodic, measure-preserving actions of higher-rank Lie groups and their lattices (see [72]). The cocycle superrigidity theorem, its corollaries, and contemporaneous results of Zimmer's (see [72–77]) led Zimmer to formulate several conjectures and questions concerning (C^∞, volume-preserving) actions of higher-rank simple Lie groups and their lattices. These questions, conjectures, and more recent extensions are usually referred to as the *Zimmer program*. Roughly, the Zimmer program aims to establish analogues of rigidity results for linear representations in the setting of smooth actions on compact manifolds. See, for instance, [24] for an overview and statements of many conjectures in this area.

2. LOW DIMENSIONS AND ZIMMER'S CONJECTURE

We present some motivation, state a contemporary version of *Zimmer's conjecture*, and outline recent progress in the area. See also the article by D. Fisher in the same proceedings for related discussion.

2.1. Motivation and Zimmer's conjecture

For $n \geq 3$, let Γ be a lattice subgroup of $\mathrm{SL}(n,\mathbb{R})$. Recall the action of Γ on S^{n-1} and, assuming Γ is commensurable with $\mathrm{SL}(n,\mathbb{Z})$, the affine action of Γ on $\mathbb{T}^n$ discussed in Section 1.3. Zimmer's conjecture asserts that these represent the minimal dimensions in which nontrivial actions of such Γ could occur. To be precise, note that if $\Gamma' \subset \Gamma$ is a finite-index normal subgroup, the finite quotient group $F = \Gamma/\Gamma'$ may act on manifolds of arbitrary dimension. This induces an action of Γ that should be considered rather trivial. Assuming $\dim(M)$ is sufficiently small, Zimmer's conjecture states all actions of Γ factor through the action of a finite group.

Conjecture 2.1 (Zimmer's conjecture for lattices in $\mathrm{SL}(n,\mathbb{R})$). *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice subgroup. Let M be a compact manifold.*

(1) *If $\dim(M) < n-1$, then any homomorphism $\Gamma \to \mathrm{Diff}(M)$ has finite image.*

(2) *In addition, if vol is a volume form on M and if $\dim(M) = n-1$, then any homomorphism $\Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$ has finite image.*

A motivation (by analogy) for this conjecture is the following corollary of Margulis' superrigidity theorem: *Let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$. For any $d < n$, the image of any representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ is finite.* Indeed, using that there are no nontrivial representations $\pi\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$, the image $\rho(\Gamma)$ is contained in a compact subgroup K of $\mathrm{GL}(d,\mathbb{R})$; Margulis further studies representations into compact Lie groups and shows the Lie algebra of K contains only copies of $\mathfrak{su}(n)$, the compact real form of $\mathfrak{sl}(n,\mathbb{R})$. A dimension count implies the Lie algebra of K vanishes and thus K is finite.

In the volume-preserving setting, Conjecture 2.1(2) is motivated by the following corollary of Zimmer's cocycle superrigidity theorem: *For $n \geq 3$, $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$, and $\dim(M) < n$, a volume-preserving action $\Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$ preserves a measurable Riemannian metric on TM.* If this metric were C^0, the image $\alpha(\Gamma)$ would be contained in the compact isometry group of this metric. A dimension count again yields finiteness. Thus, if $\dim(M)$ is sufficiently small, one might expect the image $\alpha(\Gamma)$ to be contained in a compact isometry group K of M. To extend the conjecture to other groups, to each simple, noncompact Lie group G we associate 3 positive integers $v(G)$, $n(G)$, and $d(G)$ defined, roughly, as follows:

(1) $v(G)$ is the minimal dimension of G/H as H varies over all proper closed subgroups $H \subset G$. (We remark that H is a parabolic subgroup in this case.)

(2) $n(G)$ is the minimal dimension of a nontrivial linear representation of (the Lie algebra of) G.

(3) $d(G) = v(G_{\mathrm{cmt}})$ is the minimal dimension of all nontrivial homogeneous spaces of the compact real form, G_{cmt}, of G.

We also define another number, $r(G)$, first defined in [8], which arises from certain dynamical arguments. A simpler definition of $r(G)$ is the following:

(4) $r(G) = v(G')$ where G' is a maximal $\mathbb{R}$-split subgroup of G (with the same reduced restricted root system as G).

We note that $n(G)$, $d(G)$, and $v(G)$ depend only on the Lie algebra $\mathfrak{g}$ of G; $r(G)$ depends only on the restricted root system of $\mathfrak{g}$. See Tables 1, 2, and 3, in Appendix A for computations of the numbers $v(G)$, $d(G)$, $n(G)$, and $r(G)$ for various classical groups. Given the integers $n(G)$, $d(G)$, and $v(G)$, we have the following general conjecture.

Conjecture 2.2 (Zimmer's conjecture; general). *Let $\Gamma \subset G$ be a lattice in a connected higher-rank simple Lie group G. Let M be a compact manifold and let* vol *be a volume form on M.*

(1) *If $\dim(M) < \min\{n(G), d(G), v(G)\}$ then any homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ has finite image.*

(2) *If $\dim(M) < \min\{n(G), d(G)\}$ then any homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$ has finite image.*

(3) *If $\dim(M) < \min\{v(G), n(G)\}$ then for any homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}(M)$, the image $\alpha(\Gamma)$ preserves a Riemannian metric.*

(4) *If $\dim(M) < n(G)$ then for any homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$, the image $\alpha(\Gamma)$ preserves a Riemannian metric.*

We are intentionally vague about the regularity of the action in the conjecture as it is unclear what the optimal regularity should be. In parts (3) and (4), the invariant Riemannian metric should be at least C^0. Most results discussed below require the action to be at least $C^{1+\text{Hölder}}$ though some results hold for C^1 or even C^0 actions. We note that part (3) of Conjecture 2.2 implies part (1) and part (4) implies part (2) by compactness of the isometry group of the invariant metric, superrigidity, and definition of $d(G)$.

Many prior results towards this conjecture focused on actions on the circle including [10, 32, 68] and for volume-preserving (and general measure-preserving) actions on surfaces including [29, 30, 55]. See also [31] and [22] for results on real-analytic actions and [11, 13, 14] for results on holomorphic and birational actions. There are also many results (including in the C^0 setting) for actions of specific lattices on manifolds with certain topology, where topological obstructions constrain the possible actions; a partial list of such results includes [2, 54, 66, 67, 70, 78].

2.2. Work of Brown, Fisher, and Hurtado

The series of papers [5–7] established Conjecture 2.1, Zimmer's conjecture, for C^r actions by lattices in $\mathrm{SL}(n, \mathbb{R})$.

Theorem 2.3 ([7]). *Conjecture 2.1 holds for C^r actions, $r > 1$.*

For actions by general higher-rank lattices, the same series of papers establishes the following which directly implies Theorem 2.3 (see Table 1 in Appendix A).

Theorem 2.4 ([5] cocompact case; [7] nonuniform case). *Let $\Gamma \subset G$ be a lattice in a connected higher-rank simple Lie group G. Let M be a compact manifold and let $r > 1$.*

(1) *If $\dim(M) < r(G)$ then any homomorphism $\Gamma \to \mathrm{Diff}^r(M)$ has finite image.*

(2) *In addition, if vol is a volume form on M and if $\dim(M) = r(G)$ then any homomorphism $\Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$ has finite image.*

We outline the broad steps in the proof of Theorem 2.4. Readers interested in the case of actions by cocompact lattices in $\mathrm{SL}(n,\mathbb{R})$ may consult expository accounts in [3] and [12] for detailed proofs.

Step 1: subexponential growth. Fix a lattice subgroup Γ as in Theorem 2.4. We have that Γ is finitely generated. Given $\gamma \in \Gamma$, let $|\gamma| = |\gamma|_S$ denote the word-length of γ relative to some finite symmetric generating set S. Equip TM with a Riemannian metric.

Definition 2.5. An action $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ has *uniform subexponential growth of derivatives* if for every $\varepsilon > 0$ there exists $C = C_\varepsilon$ such that for every $\gamma \in \Gamma$,

$$\sup_{x \in M} \left\| D_x\alpha(\gamma) \right\| \le C e^{\varepsilon|\gamma|}.$$

The following is the primary technical result established in [5–7].

Theorem 2.6 ([5, THEOREM 2.3], [7, THEOREM C]). *Let Γ and M be as in Theorem 2.4. For $r > 1$, let $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ be an action. Suppose that either*

(1) $\dim(M) < r(G)$, *or*

(2) $\dim(M) \le r(G)$ *and α preserves a smooth volume.*

Then α has uniform subexponential growth of derivatives.

Step 2: strong property (T) and averaging Riemannian metrics. The lattices Γ in Theorem 2.4 are known to have strong property (T). Strong property (T) was introduced by V. Lafforgue in [45] and shown for cocompact lattices in higher-rank groups in [16, 45] and extended to nonuniform lattices by de la Salle in [15]. An action $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ induces an action on Riemannian metrics. If $r \ge 2$, one can average over elements of this action and apply strong property (T) to obtain the following.

Theorem 2.7 ([5, THEOREM 2.4]). *Let Γ be a finitely generated group and let M be a compact manifold. For $k \ge 2$, let $\alpha\colon \Gamma \to \mathrm{Diff}^k(M)$ be an action. If α has uniform subexponential growth of derivatives and if Γ has strong property (T) then $\alpha(\Gamma)$ preserves a Riemannian metric that is $C^{k-1-\delta}$ for all $\delta > 0$.*

For $C^{1+\text{Hölder}}$ actions, the proof can be adapted to establish an analogue of Theorem 2.7. For C^1 actions, an analogue of Theorem 2.7 is obtained in [4, PROPOSITION 5].

Step 3: Margulis superrigidity. From Steps 1 and 2, the image $\alpha(\Gamma)$ is contained in a compact group K. Finiteness then follows immediately from Margulis' superrigidity, and a dimension count since (one can check) $r(G) < d(G)$; Theorem 2.4 follows.

2.3. C^1 actions

To establish a C^1 version of Conjecture 2.2, an analogue of Theorem 2.7 is given by [4, PROPOSITION 5]; it remains to establish a C^1 analogue of Theorem 2.6. However, a crucial step in the proof of Theorem 2.6 uses Pesin theory and Ledrappier–Young theory, which requires the consideration of C^r actions for $r > 1$. Still, a partial analogue of Theorem 2.6 holds under stronger constraints on the dimension of M.

Theorem 2.8 ([4]). *Let $\Gamma \subset G$ be a lattice in a connected, simple, higher-rank Lie group G. Let M be a compact manifold.*

(1) *If $\dim(M) < \operatorname{rank}(G)$, then any homomorphism $\Gamma \to \operatorname{Diff}^1(M)$ has finite image.*

(2) *In addition, if vol is a volume form on M and if $\dim(M) \leq \operatorname{rank}(G)$, then any homomorphism $\Gamma \to \operatorname{Diff}^2_{\mathrm{vol}}(M)$ has finite image.*

We note that the dimension bounds in Theorem 2.8 only coincide with the dimensions in Conjecture 2.2 in the case $G = \mathrm{SL}(n, \mathbb{R})$.

Question 2.9. Let Γ be a lattice subgroup of $G = \mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}(n, n)$, or $\mathrm{SO}(n, n + 1)$. Do Theorem 2.6 and Conjecture 2.2(1)–(2) hold for C^1 actions of Γ?

2.4. C^0 actions and actions on the circle

Given the results of Theorem 2.4 and Theorem 2.8, it is natural to ask if any analogous results hold for actions by homeomorphisms. For actions of general higher-rank lattices, most results on C^0 actions have focused on (non-volume-preserving) actions on the circle or the interval. We mention in particular [68] where it is shown that actions of higher-$\mathbb{Q}$-rank groups Γ on the circle are finite. A recent breakthrough by B. Deroin and S. Hurtado [17] completely resolves the question of C^0 action on the circle (among many other results).

Theorem 2.10 (Corollary of [17, THEOREM 1.5]). *Let Γ be a lattice in higher-rank simple Lie group G. For every action $\alpha\colon \Gamma \to \operatorname{Homeo}(S^1)$, the image $\alpha(\Gamma)$ is finite.*

The proof of Theorem 2.10 follows somewhat the approach in the proof of Theorem 2.4 but, due to the lack of differentiability, new tools need to be developed. We mention only one novelty of working on the circle used in [17]: following [18], one may replace minimal C^0 actions with bi-Lipschitz actions.

2.5. Beyond $\mathbb{R}$-split groups

Theorem 2.4 only gives the optimal dimension bounds for Conjecture 2.2(3) (and thus Conjecture 2.2(1)) in the case of $\mathbb{R}$-split Lie groups; see Tables 1 in Appendix A. Further analysis of objects arising in the proof of Theorem 2.6 establishes the conjectured bounds in Conjecture 2.2(3) for some nonsplit groups. This was first shown for actions of lattices in $\mathrm{SL}(n,\mathbb{C})$ in [71]: *For $n \geq 3$, Conjecture 2.2(3) holds for C^r ($r > 1$) actions of cocompact lattices in* $\mathrm{SL}(n,\mathbb{C})$. The same holds for lattices in general complex simple groups. Beyond actions by lattices in complex Lie groups, one can establish Conjecture 2.2(3) for large parameter ranges of many nonsplit Lie groups. The following (nonexhaustive list) gives some ranges where such results can be shown.

Theorem 2.11 (J. An, A. Brown, and Z. Zhang; in preparation). *Conjecture 2.2(3) holds for C^r ($r > 1$) actions of lattices in the following Lie groups:*

(1) *all higher-rank simple complex Lie groups;*

(2) $\mathrm{SL}(n,\mathbb{H})$ *with $n \geq 9$;*

(3) $\mathrm{SO}^+(m,n)$ *with $2 \leq n < m \leq \frac{1}{2}(n^2 - n + 4)$;*

(4) $\mathrm{SU}(m,n)$ *with $6 \leq n \leq m \leq \frac{1}{4}(n^2 - 3n + 6)$;*

(5) $\mathrm{SO}^*(2n)$ *with $n \geq 30$.*

This naturally leads to the following question.

Question 2.12. Does Conjecture 2.2(3) hold for actions of lattices in all higher-rank simple Lie groups?

We also show some partial results towards Conjecture 2.2(4).

Theorem 2.13 (J. An, A. Brown, and Z. Zhang; in preparation). *Let Γ be a lattice in* $\mathrm{SL}(n,\mathbb{C})$ *for $n \geq 4$. If $\dim(M) \leq n(G) - 2$ then for any homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$ ($r > 1$), the image $\alpha(\Gamma)$ preserves a Riemannian metric.*

2.6. Dimension gaps between (3) and (4) of Conjecture 2.2

Theorem 2.4 implies all statements of Conjecture 2.2 for actions by lattices in $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{Sp}(n,\mathbb{R})$ since $r(G) = v(G) = n(G) - 1 < d(G)$. However, for the $\mathbb{R}$-split groups $G = \mathrm{SO}(n,n)$ and $G = \mathrm{SO}(n,n+1)$, we have

$$r(G) = v(G) = n(G) - 2 < d(G) = n(G) - 1 < n(G).$$

Thus, for these groups, Theorem 2.4 implies Conjecture 2.2(1)–(3) but does not imply Conjecture 2.2(4). This gap also arises for $\mathbb{R}$-split exceptional groups and many non-$\mathbb{R}$-split groups. For instance, Theorem 2.13 implies that volume-preserving actions of lattices in $\mathrm{SL}(n,\mathbb{C})$ (for $n \geq 4$) preserve a Riemannian metric if $\dim(M) \leq 2n - 2$; Conjecture 2.2(4) asserts the same should hold if $\dim(M) = 2n - 1 = n(G) - 1$.

Question 2.14. Does Conjecture 2.2(4) hold for lattices Γ in $\mathrm{SO}(n,n)$, $\mathrm{SO}(n,n+1)$, or $\mathrm{SL}(n,\mathbb{C})$, $n \geq 3$? Specifically, if $\dim(M) < n(G)$, does every volume-preserving action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty_{\mathrm{vol}}(M)$ preserve a (C^0 or C^∞) Riemannian metric?

We note that every lattice Γ in $G = \mathrm{SO}(n,n)$ and $G = \mathrm{SO}(n,n+1)$ admits a nonisometric action on a compact manifold of dimension $n(G) - 1$. Indeed, there is a parabolic subgroup $Q \subset G$ with codimension $v(G) = n(G) - 2$; the left action of Γ on G/Q is nonisometric. Taking $M = (G/Q) \times S^1$, let Γ act on M naturally on the left in the first coordinate and as the identity in the second coordinate. We note that this action does not preserve any volume form on M so does not contradict Question 2.14. As a first step towards Question 2.14, we might rule out related constructions that would yield counterexamples as in the following.

Problem 2.15. Let Γ be a lattice in $G = \mathrm{SO}(n,n)$ or $G = \mathrm{SO}(n,n+1)$. Show there is no volume-preserving action of Γ on $M = (G/Q) \times S^1$ with infinite image.

As a first step towards solving Problem 2.15, one might restrict to actions that factor onto the projective action on G/Q. In a related direction, we also pose the following.

Question 2.16. For $n \geq 3$, let Γ be a lattice in $G = \mathrm{SO}(n,n)$ or $\mathrm{SO}(n,n+1)$. Suppose that $\dim(M) = n(G) - 1$ and that $\alpha \colon \Gamma \to \mathrm{Diff}^\infty(M)$ is an action that does not preserve any (C^0 or C^∞) Riemannian metric. Is there either (1) an invariant embedded G/Q in M on which the dynamics restricts to the standard action or (2) an invariant open subset $U \subset M$ restricted to which the dynamics factors onto the standard action on G/Q?

3. CLASSIFICATION IN LOWEST DIMENSIONS AND RIGIDITY OF PROJECTIVE ACTIONS

For $n \geq 3$, Theorem 2.3 implies that actions by lattices in $\mathrm{SL}(n,\mathbb{R})$ are finite when $\dim(M) < n - 1$. When $\dim(M) = n - 1$, recall the natural action of Γ on S^{n-1} or $\mathbb{R}P^{n-1}$. In work in progress, we show these to be the only actions with infinite image.

Theorem 3.1 (A. Brown, F. Rodriguez Hertz, Z. Wang; in preparation). *For $n \geq 3$, let Γ be a lattice subgroup of $\mathrm{SL}(n,\mathbb{R})$. Let M be a connected compact manifold of dimension $n - 1$. Fix $r > 1$ and let $\alpha \colon \Gamma \to \mathrm{Diff}^r(M)$ be an action with infinite image $\alpha(\Gamma)$. Then*

(1) *there is a C^r-diffeomorphism h between M and either S^{n-1} or $\mathbb{R}P^{n-1}$ such that*

(2) *for all $x \in M$ and $\gamma \in \Gamma$, $h(\alpha(\gamma)(x)) = \gamma \cdot h(x)$ where the right-hand side denotes the standard projective action of Γ on S^{n-1} or $\mathbb{R}P^{n-1}$.*

The techniques used to prove Theorem 3.1 also give local rigidity of higher-dimensional projective actions, extending the results of **[38]** and **[44, THEOREM 17]**.

Theorem 3.2. *For $n \geq 3$, let $\mathcal{F}$ be a flag manifold (of flags in $\mathbb{R}^n$) and let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice subgroup. Then the standard action $\rho \colon \Gamma \to \mathrm{Diff}(\mathcal{F})$ is $C^{\infty,1,\infty}$-locally rigid.*

Theorem 3.2 says for any action $\alpha \colon \Gamma \to \mathrm{Diff}^\infty(\mathcal{F})$ sufficiently C^1 close to the standard projective action ρ, there exists a C^∞ diffeomorphism $h \colon \mathcal{F} \to \mathcal{F}$ such that $h \circ$

$\alpha(\gamma) = \rho(\gamma) \circ h$ for all $\gamma \in \Gamma$. The above results lead to the following question classifying all actions on flag manifolds.

Question 3.3 (Global rigidity). For $n \geq 3$, let Γ be a lattice subgroup in $\mathrm{SL}(n, \mathbb{R})$ and let $\mathcal{F}$ be a flag manifold (of flags in $\mathbb{R}^n$). Let $\alpha: \Gamma \to \mathrm{Diff}^\infty(\mathcal{F})$ be an action with infinite image $\alpha(\Gamma)$. Is α smoothly conjugate to the standard projective action on $\mathcal{F}$?

4. CLASSIFICATION UNDER DYNAMICAL AND TOPOLOGICAL HYPOTHESES

4.1. Classification in dimension n

Given the classification in Theorem 3.1, it is natural to ask if it is possible to classify all (possibly volume-preserving) actions $\alpha: \Gamma \to \mathrm{Diff}^\infty(M)$ when Γ is a lattice in $\mathrm{SL}(n, \mathbb{R})$ and M is a compact connected manifold of dimension n. This seems much harder since there are known examples of "exotic" actions in dimension n. In the non-volume-preserving case, there exist many nonequivalent real-analytic actions of $\mathrm{SL}(n, \mathbb{R})$ on the n-sphere constructed in [63] and the restriction to $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ yields exotic actions of Γ. Roughly, one builds a skew-product $\mathrm{SL}(n, \mathbb{R})$-action on $S^{n-1} \times (-1, 1)$ factoring onto the standard action on S^{n-1} and takes the two-point compactification. This motivates the following alternative version of Question 2.16.

Question 4.1. Let Γ be a lattice in $G = \mathrm{SL}(n, \mathbb{R})$ for $n \geq 3$. Let $\dim M = n$ and let $\alpha: \Gamma \to \mathrm{Diff}^\infty(M)$ be an action with infinite image that does not preserve any volume form (or absolutely continuous measure). Does M contain an embedded projective action or an invariant open subset that factors onto the projective action on $\mathbb{R}P^{n-1}$?

In the setting of volume-preserving actions, given the affine action of $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{T}^n$, it is possible to blowup a fixed point (or a finite Γ orbit) to obtain a smooth action on a n-manifold preserving a smooth density; in [40], A. Katok and J. Lewis showed these examples can be perturbed to preserve a smooth, nowhere vanishing density.

One might conjecture that actions of lattices $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ in dimension n are built by gluing together modifications of standard actions such as those described above. At this time though, it seems any conjectured picture is far from understood. Thus to classify actions of $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ in dimension n (and higher), it is natural to first impose additional dynamical or topological hypotheses. The remainder of this section discusses several results in this direction.

4.2. Toral homeomorphisms and Anosov diffeomorphisms

Given a homeomorphism $f \in \mathrm{Homeo}(\mathbb{T}^d)$, there is a unique matrix $A_f \in \mathrm{GL}(d, \mathbb{Z})$ such that any lift $\tilde{f}: \mathbb{R}^d \to \mathbb{R}^d$ of f is of the form $\tilde{f}(x) = A_f x + \phi(x)$ for some $\mathbb{Z}^d$-periodic $\phi: \mathbb{R}^d \to \mathbb{R}^d$. We call A_f the *linear data* of f and note that A_f induces an automorphism L_{A_f} on the torus $\mathbb{T}^d$. If $\alpha: \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ is an action we similarly obtain $\rho: \Gamma \to \mathrm{GL}(d, \mathbb{Z})$ called the *linear data* of α. A matrix $A \in \mathrm{GL}(d, \mathbb{Z})$ is *hyperbolic* if no eigenvalue of A is on the unit circle. The following theorem characterizes (up to continuous semiconjugacy) maps $f: \mathbb{T}^d \to \mathbb{T}^d$ whose linear data A_f is hyperbolic.

Theorem 4.2 (Franks, [28]). *Let $f: \mathbb{T}^d \to \mathbb{T}^d$ be a homeomorphism with hyperbolic linear data A_f. There exists a continuous, surjective $h: \mathbb{T}^d \to \mathbb{T}^d$ such that $h \circ f = L_{A_f} \circ h$.*

We recall Anosov diffeomorphisms, which provide the main example of homeomorphisms satisfying the hypotheses of Theorem 4.2.

Definition 4.3. A C^1 diffeomorphism $f: M \to M$ of a compact Riemannian manifold M is *Anosov* if there is a continuous, Df-invariant splitting of the tangent bundle $TM = E^s \oplus E^u$ and constants $0 < \kappa < 1$ and $C \geq 1$ such that for every $x \in M$ and every $n \in \mathbb{N}$,

$$\|D_x f^n(v)\| \leq C\kappa^n \|v\|, \quad \text{for } v \in E^s(x), \qquad \|D_x f^{-n}(w)\| \leq C\kappa^n \|w\|, \quad \text{for } w \in E^u(x).$$

All known examples of Anosov diffeomorphisms occur on finite factors of tori and nilmanifolds. From [28,49] we have a complete classification of Anosov diffeomorphisms on tori (and nilmanifolds) up to homeomorphism.

Theorem 4.4 (Franks–Manning, [28,49]). *If $f: \mathbb{T}^n \to \mathbb{T}^n$ is Anosov, then f is homotopic to L_A for some hyperbolic $A \in \mathrm{GL}(n, \mathbb{Z})$; moreover, there is a homeomorphism $h: \mathbb{T}^n \to \mathbb{T}^n$ such that $h \circ f = L_A \circ h$.*

4.3. Global topological and smooth rigidity of Anosov actions

For simplicity, consider $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$. An action $\alpha: \Gamma \to \mathrm{Diff}(M)$ is *Anosov* if $\alpha(\gamma_0)$ is Anosov for some $\gamma_0 \in \Gamma$; see Definition 4.3. We state the following conjecture which is motivated in part by the works of Feres–Labourie [23] and Goetze–Spatzier [33].

Conjecture 4.5 ([24, CONJECTURE 1.3]). *If Γ is a lattice in $\mathrm{SL}(n, \mathbb{R})$ where $n \geq 3$, then any C^∞, volume-preserving, Anosov action by Γ on a compact manifold is smoothly conjugate to an action by affine automorphisms of an infranilmanifold.*

See also [35, CONJECTURE 1.1] and [40, CONJECTURE 1.1] for related conjectures. The assumption that the action preserves a volume is standard though results discussed below suggest that such a hypothesis may be unnecessary. Most progress on this conjecture requires additional strong dynamical hypotheses on the action, low dimensionality of the manifold, or assumptions on the topology of the underlying manifold.

We note that affine Anosov actions of higher-rank lattices are known to be local rigid by the work of A. Katok and R. Spatzier [44], extending many earlier results including [34,39,58]. Several partial results towards Conjecture 4.5 appear in [23,33,34,40,41,51,57]. In [9], a new topological and smooth classification of higher-rank lattice actions on tori and nilmanifolds was established. A novelty of the approach in [9] is that no invariant measure is assumed unlike many prior global rigidity results including those in [27,41,51]. For simplicity, we state the following result for actions on tori though versions on nilmanifolds also hold.

Theorem 4.6 ([9, THEOREM 1.3]). *Let* Γ *be a lattice in* $\mathrm{SL}(n,\mathbb{R})$ *for* $n \geq 3$. *Let* $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ *be an action by homeomorphisms with linear data* $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$. *Suppose*

(1) *the matrix* $\rho(\gamma_0)$ *is hyperbolic for some* $\gamma_0 \in \Gamma$, *and*

(2) *for some finite-index subgroup* $\Gamma' \subset \Gamma$, *the action* $\alpha\colon \Gamma' \to \mathrm{Homeo}(\mathbb{T}^d)$ *lifts to an action* $\tilde{\alpha}\colon \Gamma' \to \mathrm{Homeo}(\mathbb{R}^d)$.

Then there is a continuous, surjective $h\colon \mathbb{T}^d \to \mathbb{T}^d$ *such that*

$$h \circ \alpha(\gamma) = \rho(\gamma) \circ h \tag{4.1}$$

for all γ *in a finite-index subgroup* $\Gamma'' \subset \Gamma$. *In particular, the action* $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ *is semiconjugate to an action by affine maps of* $\mathbb{T}^d$.

Sufficient conditions for the lifting hypothesis (2) are known; see [9, REMARK 1.5] and references therein. In particular, this automatically holds if $\Gamma = \mathrm{SL}(d,\mathbb{Z})$ acts on $\mathbb{T}^d$ for $d \geq 5$, Γ is cocompact, or α preserves a probability measure μ.

Assuming that $\alpha(\gamma_0)$ is Anosov for some $\gamma_0 \in \Gamma$, Theorem 4.4 implies the map h in Theorem 4.6 is a homeomorphism. For actions by higher-rank lattices, the map h is, in fact, smooth, thus classifying all Anosov actions on tori up to smooth coordinate change.

Theorem 4.7 ([9, THEOREM 1.7]). *Let* Γ *be a lattice in* $\mathrm{SL}(n,\mathbb{R})$ *for* $n \geq 3$. *Let* $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ *be an action with linear data* $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$. *Suppose that*

(1) *the diffeomorphism* $\alpha(\gamma_0)$ *is Anosov for some* $\gamma_0 \in \Gamma$, *and*

(2) *for some finite-index subgroup* $\Gamma' \subset \Gamma$, *the action* $\alpha\colon \Gamma' \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ *lifts to an action* $\tilde{\alpha}\colon \Gamma' \to \mathrm{Diff}^\infty(\mathbb{R}^d)$.

Then, there is a C^∞ *diffeomorphism* $h\colon \mathbb{T}^d \to \mathbb{T}^d$ *such that*

$$h \circ \alpha(\gamma) = \rho(\gamma) \circ h$$

for all γ *in a finite-index subgroup* $\Gamma'' \subset \Gamma$. *In particular, the action* $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ *is smoothly conjugate to an action by affine maps of* $\mathbb{T}^d$.

Again, similar results hold for lattices in other higher-rank simple Lie groups and for Anosov actions on nilmanifolds. To establish Theorem 4.7, we need only show the homeomorphism h in (4.1) given by Theorem 4.6 and Theorem 4.4 is C^∞. Roughly, this follows by studying the restriction of the action α to a higher-rank abelian subgroup $\Sigma \subset \Gamma$. For Anosov actions of higher-rank abelian groups, the map intertwining the action with the linear data is often smooth as shown in [25,26] with the most general result obtained in [60]. The main work to establish Theorem 4.7 is to find $\gamma \in \Gamma$ (which may be different from γ_0) with sufficiently large centralizer in Γ and for which $\alpha(\gamma)$ is Anosov.

Returning to the setting of Theorem 4.6, we might ask if it is possible to classify all (non-Anosov) C^∞ actions on tori with hyperbolic linear data; it seems plausible that all such actions are obtained by a blow-up or slow-down procedure of affine Anosov actions. This suggests the following.

Problem 4.8. Classify all C^∞ actions satisfying the hypotheses of Theorem 4.6.

Specifically, the following may give a possible approach to Problem 4.8.

Question 4.9. Let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ be an action satisfying the hypotheses of Theorem 4.6. Is there an α-invariant open set $U \subset \mathbb{T}^d$ such that for the map h satisfying (4.1), $h(U)$ is dense, $h\restriction_U$ is injective, and $h\restriction_U$ is smooth?

In the proof of Theorem 4.7, one shows that every Anosov action of a higher-rank lattice Γ on $\mathbb{T}^d$ preserves a volume form. It is natural to ask if the same holds for the actions as in Theorem 4.6 and ask if a weaker version of Question 4.9 holds. We note that this holds for $\mathrm{SL}(n,\mathbb{Z})$ acting on $\mathbb{T}^n$ by discussion and references in [9, THEOREM 1.6] and [42].

Question 4.10. Let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ be an action satisfying the hypotheses of Theorem 4.6. Does α preserve an absolutely continuous probability measure μ on $\mathbb{T}^d$? If so, is there a set $U \subset \mathbb{T}^d$ of full μ-measure such that for h satisfying (4.1), $h(U)$ has full Lebesgue measure, $h\restriction_U$ is injective, and h is smooth along Pesin unstable manifolds?

4.4. Anosov actions in dimension n

We return to the motiving problem of classifying actions of $\mathrm{SL}(n,\mathbb{Z})$ on n-manifolds. In [41], volume-preserving Anosov actions of $\mathrm{SL}(n,\mathbb{Z})$ on n-tori were shown to be smoothly conjugate to affine actions for $n \geq 3$. Recently, H. Lee considered the same problem but without any assumption on the topology of the underlying manifold.

Theorem 4.11 ([47, THEOREM 1.5]). *For $n \geq 3$, let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$. Suppose $\dim(M) = n$ and let $\alpha\colon \Gamma \to \mathrm{Diff}^1_{\mathrm{vol}}(M)$ be an action such that $\alpha(\gamma_0)$ is Anosov for some $\gamma_0 \in \Gamma$. Then there is a homeomorphism $h\colon M \to \mathbb{T}^n$ such that $h \circ \alpha(\gamma) \circ h^{-1}$ is affine for every $\gamma \in \Gamma$. Moreover, if $\alpha\colon \Gamma \to \mathrm{Diff}^\infty_{\mathrm{vol}}(M)$ then h is C^∞.*

It is natural to ask if the assumption that the action preserves a volume form in Theorem 4.11 can be removed.

Conjecture 4.12. *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice. Let $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ be an action such that $\alpha(\gamma_0)$ is Anosov for some $\gamma_0 \in \Gamma$. Then Γ preserves a smooth (nowhere vanishing) volume form on M.*

In a recent collaboration, we were able to verify this conjecture in certain situations.

Theorem 4.13 (A. Brown and H. Lee; in preparation). *Conjecture 4.12 holds for C^∞ Anosov actions on n-manifolds by cocompact lattices in $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 4$.*

We also expect that Theorem 4.13 holds for nonuniform lattices. In Theorem 4.11, the only possible lattice subgroups $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ admitting Anosov actions on $\mathbb{T}^n$ are (up to conjugacy) commensurable with $\mathrm{SL}(n,\mathbb{Z})$. Combined with Theorem 4.11, this would imply that the only lattices $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ that admit a C^∞ Anosov action in dimension n are commensurable with $\mathrm{SL}(n,\mathbb{Z})$.

5. TOOLS USED IN PROOFS

5.1. Suspension space and induced G-action

Let G be a Lie group and let Γ be a lattice subgroup of G. Let M be a compact manifold and let $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ be an action. A well-known construction translates between the Γ-action on M and an equivariant G-action on a fiber-bundle X over G/Γ with fibers diffeomorphic to M: On $G \times M$ consider the right Γ-action and the left G-action:

$$(g,x)\cdot\gamma = \big(g\gamma, \alpha(\gamma^{-1})(x)\big), \quad a\cdot(g,x) = (ag,x).$$

Define the quotient manifold $X := (G \times M)/\Gamma$. The G-action on $G \times M$ descends to a G-action on X. For $g \in G$ and $x \in X$, denote the action by $g \cdot x$ and denote the derivative of the diffeomorphism $x \mapsto g \cdot x$ at $x \in X$ by $D_x g\colon T_x X \to T_{g\cdot x} X$.

The space X is a fiber bundle over G/Γ. Let $\pi\colon X \to G/\Gamma$ be the projection and let $\mathcal{E} := \ker D\pi$ denote the *fiberwise tangent bundle*. That is, $\mathcal{E} = (G \times TM)/\Gamma$.

We write $\mathcal{A}\colon G \times \mathcal{E} \to \mathcal{E}$ for the *fiberwise derivative* cocycle over the G-action X: given $x \in X$, if $\mathcal{E}(x)$ is the fiber of $\mathcal{E}$ over x then $\mathcal{A}(g,x)\colon \mathcal{E}(x) \to \mathcal{E}(g\cdot x)$ is the restriction to $\mathcal{E}(x)$ of the derivative of translation by g:

$$\mathcal{A}(g,x) = D_x g \restriction_{\mathcal{E}(x)}.$$

When Γ is cocompact, we equip TX and $\mathcal{E}$ with any choice of Riemannian metric. When Γ is nonuniform, we use arithmeticity of Γ and Siegel domains in G to equip TX and $\mathcal{E}$ with Riemannian metrics adapted to the geometry of Γ in G.

5.2. Common themes

Let $A = \exp\mathfrak{a}$ be a maximal $\mathbb{R}$-split Cartan subgroup of G; when $G = \mathrm{SL}(n,\mathbb{R})$, we take $A = \{\mathrm{diag}(e^{t_1},\dots,e^{t_n}) : t_1 + \cdots + t_n = 0\}$, the subgroup of positive diagonal matrices whence $A \simeq \mathbb{R}^{n-1}$. Most results discussed above follow from precise formulations of the following 2 heuristics. In the remainder of this section, we discuss concrete examples of these. Throughout, we always assume $\dim(A) \geq 2$.

Theme 1. *Dynamical properties of the Γ-action on M induce A-invariant probability measures μ on X factoring onto the Haar measure on G/Γ with corresponding dynamical properties.*

Theme 2. *A-invariant probability measures μ on X factoring onto the Haar measure on G/Γ are expected to be very "rigid."*

Theme 2 often leads to extra invariance or homogeneity of the measure μ. Combined with dynamical structures associated with μ in Theme 1, this often constrains possible dynamical properties of Γ on M or reveals some homogeneous structures associated with the Γ-action on M.

Below we describe one instance of Theme 1 and several instances of Theme 2. We also outline cohomological versions of Theme 1 and Theme 2 that are used in the proof of Theorem 4.6.

5.3. Theme 1 and subexponential growth

Let $X = (G \times M)/\Gamma$ denote the induced G-space and let $\mathcal{A}$ denote the corresponding fiberwise derivative cocycle. Given $a \in G$ and an a-invariant probability measure μ on X, we define the *average top Lyapunov exponent of* $\mathcal{A}$ by

$$\lambda_{\mathrm{top},a,\mu,\mathcal{A}} := \liminf_{n\to\infty} \frac{1}{n} \int \log \left\| \mathcal{A}(a^n, x) \right\| \, d\mu(x).$$

This is finite whenever the function $x \mapsto \log \|\mathcal{A}(a, x)\|$ is $L^1(\mu)$ which—by the choice of norm on $\mathcal{E}$—holds for any probability measure μ on X that factors onto the normalized Haar measure on G/Γ. We recall Definition 2.5. The main technical theorem established in the papers [5–7] is the following precise version of Theme 1 which, under the assumption that the conclusion of Theorem 2.6 fails, builds a measure on the suspension space X with certain dynamical properties.

Theorem 5.1 ([7, THEOREM D]). *Let G be a connected semisimple Lie group with finite center,*[2] *without compact factors, and with* $\operatorname{rank}_{\mathbb{R}} G \geq 2$. *Let Γ be an irreducible lattice subgroup in G, let M be a compact manifold, and let $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ be an action. If the action α fails to have uniform subexponential growth of derivatives then there exists a maximal $\mathbb{R}$-split Cartan subgroup A of G and a probability measure μ on X such that*

(1) *μ is A-invariant,*

(2) *μ projects to the Haar measure on G/Γ, and*

(3) *for some $a \in A$, the average top Lyapunov exponent $\lambda_{\mathrm{top},a,\mu,\mathcal{A}}$ is positive.*

We remark that there are no constraints on the dimension of M in the statement of Theorem 5.1. In particular, Theorem 5.1 serves as the starting point for the proofs of Theorems 2.4 and 2.6 as well as Theorems 2.11, 2.13, and 3.1 and may serve as a starting point for future results.

5.4. Theme 2 and invariance of measures

In Theme 2, we consider an A-invariant probability measure μ on X factoring onto the Haar measure on G/Γ. One precise version of Theme 2 produces extra invariance of μ by certain subgroups of G normalized by A. See especially [8, PROPOSITION 5.1]. This has the following corollary used to prove Theorem 2.6.

Theorem 5.2. *Let G be a higher-rank simple Lie group, let Γ be a lattice in G, let M be a compact manifold, and let $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ be an action for $r > 1$. Then*

2 For simplicity of statement, we assume the center of G is finite though that is not necessary for applications.

(1) *if* $\dim(M) \leq r(G) - 1$*, every* A*-invariant probability measure on* X *that projects to the Haar measure on* G/Γ *is* G*-invariant;*

(2) *if* $\dim(M) \leq r(G)$ *and* α *is volume-preserving, every* A*-invariant probability measure on* X *that projects to the Haar measure on* G/Γ *is* G*-invariant.*

To prove Theorem 2.6, if $\dim M < n(G)$ and if μ is a G-invariant probability measure on X, then Zimmer's cocycle superrigity implies $\lambda_{\text{top},a,\mu,\mathcal{A}} = 0$ for every $a \in G$. Combined with Theorems 5.1 and 5.2, we obtain a contradiction unless the conclusion of Theorem 2.6 holds.

In the setting of C^1 actions, we have the following weaker version of Theorem 5.2 which follows from mild modifications of the invariance principle in [1] (extending results of [46]).

Theorem 5.3 ([4, PROPOSITION 3]). *Let* G *be a higher-rank simple Lie group, let* Γ *be a lattice in* G*, let* M *be a compact manifold, and let* $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ *be an action. Then*

(1) *if* $\dim(M) \leq \operatorname{rank}(G) - 1$*, every* A*-invariant probability measure on* X *that projects to the Haar measure on* G/Γ *is* G*-invariant;*

(2) *if* $\dim(M) \leq \operatorname{rank}(G)$ *and* α *is volume-preserving, every* A*-invariant probability measure on* X *that projects to the Haar measure on* G/Γ *is* G*-invariant.*

The appearance of $r(G)$ (rather than $v(G)$ or $n(G)$) in Theorem 5.2 is the main reason why $r(G)$ appears in Theorem 2.4. However, one might expect an analogue of Theorem 5.2 with dimension bounds corresponding to those in Conjecture 2.2 holds.

Conjecture 5.4. *Let* G *be a higher-rank simple Lie group, let* Γ *be a lattice in* G*, let* M *be a compact manifold, and let* $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ *be an action.*

(1) *If* $\dim(M) \leq v(G) - 1$*, every* A*-invariant probability measure on* X *that projects to the Haar measure on* G/Γ *is* G*-invariant.*

(2) *If* $\dim(M) \leq v(G)$ *and if* α *is volume-preserving, every* A*-invariant probability measure on* X *that projects to the Haar measure on* G/Γ *is* G*-invariant.*

One might further conjecture the following.

(3) *If* $\dim(M) \leq n(G) - 1$ *and if* α *is volume-preserving, every* A*-invariant probability measure on* X *that projects to the Haar measure on* G/Γ *is* G*-invariant.*

For many non-$\mathbb{R}$-split groups G, (1) and (2) of Conjecture 5.4 can be established using tools of measure rigidity and cocycle superrigidity. We discuss this in the next section.

5.5. Theme 2: measure and cocycle rigidity; homogeneous structures

Let A be a maximal (connected) $\mathbb{R}$-split Cartan subgroup of G; since $A \simeq \mathbb{R}^{\operatorname{rank}(G)}$, A is a higher-rank abelian group if G is higher-rank. Measures invariant under higher-rank abelian groups (with positive entropy) are expected to exhibit some degree of homogeneity unless they factor onto an action of a rank-1 quotient of A. Such results have been established in the setting of homogenous dynamics, see especially [19–21, 43, 48, 61], and in the setting of smooth non-linear dynamics, see especially [36, 37].

To prove Theorem 2.11, it remains to establish relevant cases of Conjecture 5.4; this implies an analogous version of Theorem 2.6 and allows one to complete the outline in Section 2.2. An argument discovered by J. An shows that one may assume the A-invariant measure μ in the conclusion of Theorem 5.1 is invariant under a parabolic subgroup $Q \subset G$ containing A. When the Levi component of Q is sufficiently large, Zimmer's cocycle superrigidity constrains the combinatorics of the Lyapunov spectrum of the cocycle $\mathcal{A}$ (over the action of A and the measure μ); this, combined with [8, PROPOSITION 5.1], yields many cases of Conjecture 5.4 including (among others) the ranges in Theorem 2.11. Adaptations of the nonlinear measure rigidity arguments in [36, 37] yield further constraints on the combinatorics of the Lyapunov spectrum of $\mathcal{A}$ solving additional cases of Conjecture 5.4. We summarize with following.

Theorem 5.5 (J. An, A. Brown, and Z. Zhang; in preparation).

(1) *Conjecture 5.4(1) holds for lattices in complex simple Lie groups.*

(2) *Conjecture 5.4(2) holds for lattices in* $\mathrm{SL}(n, \mathbb{C})$ *for* $n \geq 4$.

(3) *Conjecture 5.4(1) holds for lattices in the groups appearing in Theorem 2.11.*

While this establishes Conjecture 5.4 in many cases, there are many higher-rank simple groups for which Conjecture 5.4 is unresolved.

Problem 5.6. Find a new mechanism to obtain extra invariance of A-invariant measures that allows us to establish additional cases of Conjecture 5.4.

The result announced in Theorem 4.13 follows by similarly adapting the measure rigidity arguments of [36] as well as a version of Theme 1 involving topological entropy.

The proof of Theorem 3.1 follows from further adapting the techniques of measure rigidity. Very roughly, starting from the A-invariant measure μ in the conclusion of Theorem 5.1, we have that μ is invariant under a parabolic subgroup $Q \subset G$ containing A. Locally, every point $x \in X$ has a neighborhood parameterized as $U \times M$ where $U \subset G$ is an open neighborhood of the identity. If $V \subset G$ denotes the unipotent subgroup transverse to Q, one shows the restriction of the measure μ to such parameterized neighborhoods coincides with the graph of an injective, C^r function $V \to M$. These graphs then assemble coherently to give local homogeneous coordinates relative to which M admits the structure of a Γ-equivariant covering space of G/Q.

5.6. Cohomological versions of Theme 1 and Theme 2

We end this note with a reformulation of Theme 1 and Theme 2 used in the proof of Theorem 4.6. Let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a lattice, let $\alpha \colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ be as in Theorem 4.6, and let $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{Z})$ be the linear data of α. Passing to compact extensions and subgroups of finite index, assume α lifts to an action $\tilde{\alpha}\colon \Gamma \to \mathrm{Homeo}(\mathbb{R}^d)$ and that ρ coincides with the restriction to Γ of a continuous representation $\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$.

Cohomological reformulation. Consider the identify map $h_0\colon \mathbb{T}^d \to \mathbb{T}^d$; the defect of h_0 satisfying (4.1) determines a continuous, ρ-twisted 1-cocycle $c\colon \Gamma \times \mathbb{T}^d \to \mathbb{R}^d$, given by $c(\gamma, x) = \tilde{\alpha}(\gamma)(\tilde{x}) - \rho(\gamma)\tilde{x}$ for any lift $\tilde{x} \in \mathbb{R}^d$ of x. In particular, for $\gamma_1, \gamma_2 \in \Gamma$,

$$c(\gamma_1\gamma_2, x) = \rho(\gamma_1)c(\gamma_2, x) + c\big(\gamma_1, \alpha(\gamma_2)(x)\big). \tag{5.1}$$

Suppose that c is a coboundary; that is, suppose there exists a continuous function $\eta\colon \mathbb{T}^d \to \mathbb{R}^d$ such that for every $\gamma \in \Gamma$ and $x \in \mathbb{T}^d$, $c(\gamma, x) = \rho(\gamma)\eta(x) - \eta(\alpha(\gamma)(x))$. The function $h(x) = h_0(x) + \eta(x) = x + \eta(x)$ then satisfies (4.1).

Cohomological version of Theme 1. Rather than study the Γ-action on $\mathbb{T}^d$, we pass to the G-action on the suspension space X and define a related cocycle $\tilde{c}\colon G \times X \to \mathbb{R}^d$. While $\tilde{c}$ is continuous in every fiber of X, it is only Borel measurable over G/Γ. Nonetheless, to establish Theorem 4.6, it suffices to show $\tilde{c}$ is a coboundary: for every $g \in G$,

$$\tilde{c}(g, x) = \rho(g)\tilde{\eta}(x) - \tilde{\eta}(g \cdot x) \tag{5.2}$$

where $\tilde{\eta}\colon X \to \mathbb{R}^d$ is a measurable function that is continuous in Haar-almost every fiber. Using that $\rho(\gamma_0)$ is hyperbolic for some $\gamma_0 \in \Gamma$, a modification of the proof of Theorem 4.2 produces such a function $\tilde{\eta}\colon X \to \mathbb{R}^d$ such that (5.2) holds for all $g \in A$ and almost every fiber.

Cohomological version of Theme 2. It remains to show the function $\tilde{\eta}$ solves equation (5.2) for all $g \in G$. We identify finitely many unipotent subgroups $\mathcal{U} = \{U_1, \dots, U_\ell\}$ (specifically, 1-parameter root subgroups) of $\mathrm{SL}(n,\mathbb{R})$, each of which is normalized by A, and show

(1) the cocycle equation (5.2) holds for all $g \in U_j$, and

(2) the group G is generated by the subgroups $\mathcal{U} = \{U_1, \dots, U_\ell\}$ and A.

For the case of $G = \mathrm{SL}(n,\mathbb{R})$ we may take $\mathcal{U}$ to contain all root subgroups normalized by A. However, for certain higher-rank simple groups G (such as $\mathrm{Sp}(4,\mathbb{R})$ of real rank 2), it may be that (5.2) only holds for g in a subset of the root groups; nonetheless, these groups still generate all of G.

The above outline should apply to any ρ-twisted cocycle $c\colon \Gamma \times \mathbb{T}^d \to \mathbb{R}^k$, assuming $\rho(\gamma_0)$ is hyperbolic for some $\gamma_0 \in \Gamma$, and show c is a coboundary. However, this is a large restriction on the class of representations ρ considered; for instance, it does not include the case that ρ is the adjoint representation. Still, using that c is a cocycle for an action of a large group, it may be possible to solve the following.

Question 5.7. Let $c\colon \Gamma \times \mathbb{T}^d \to \mathbb{R}^k$ be a ρ-twisted cocycle where $\rho\colon G \to \mathrm{GL}(k,\mathbb{R})$ is a nontrivial irreducible representation (such as the adjoint). Is c a coboundary?

A. NUMEROLOGY ASSOCIATED WITH ZIMMER'S CONJECTURE

We compute the numbers $n(G)$, $d(G)$, $v(G)$, and $r(G)$ for various classical real Lie groups. These numbers depend only on the Lie algebra of G.

Lie algebra $\mathfrak{g}$	restricted root system	real rank	$n(G)$	$d(G)$	$v(G)$	$r(G)$
$\mathfrak{sl}(n,\mathbb{R})$ $n \geq 2$	A_{n-1}	$n-1$	n	$2n-2, n \neq 4$ $5, n = 4^{\text{(a)}}$	$n-1$	$n-1$
$\mathfrak{sp}(2n,\mathbb{R})$ $n \geq 2$	C_n	n	$2n$	$4n-4$	$2n-1$	$2n-1$
$\mathfrak{so}(n,n+1)$ $n \geq 3^{\text{(b)}}$	B_n	n	$2n+1$	$2n$	$2n-1$	$2n-1$
$\mathfrak{so}(n,n)$ $n \geq 4^{\text{(c)}}$	D_n	n	$2n$	$2n-1$	$2n-2$	$2n-2$

(a) $\mathfrak{sl}(4,\mathbb{R}) = \mathfrak{so}(3,3)$

(b) $\mathfrak{so}(1,2) = \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(2,3) = \mathfrak{sp}(4,\mathbb{R})$

(c) $\mathfrak{so}(2,2)$ is not simple and $\mathfrak{so}(3,3) = \mathfrak{sl}(4,\mathbb{R})$

TABLE 1
Numerology appearing in Zimmer's conjecture for classical $\mathbb{R}$-split Lie algebras.

Lie algebra $\mathfrak{g}$	restricted root system	real rank	$n(G)$	$d(G)$	$v(G)$	$r(G)$
$\mathfrak{sl}(n,\mathbb{C})$ $n \geq 2$	A_{n-1}	$n-1$	$2n$	$2n-2, n \neq 4$ $5, n = 4^{\text{(d)}}$	$2n-2$	$n-1$
$\mathfrak{sp}(2n,\mathbb{C})$ $n \geq 2$	C_n	n	$4n$	$4n-4$	$4n-2$	$2n-1$
$\mathfrak{so}(2n+1,\mathbb{C})$ $n \geq 3^{\text{(e)}}$	B_n	n	$4n+2$	$2n$	$4n-2$	$2n-1$
$\mathfrak{so}(2n,\mathbb{C})$ $n \geq 4^{\text{(f)}}$	D_n	n	$4n$	$2n-1$	$4n-4$	$2n-2$

(d) $\mathfrak{sl}(4,\mathbb{C}) = \mathfrak{so}(6,\mathbb{C})$

(e) $\mathfrak{so}(5,\mathbb{C}) = \mathfrak{sp}(4,\mathbb{C})$ and $\mathfrak{so}(3,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C})$.

(f) $\mathfrak{so}(6,\mathbb{C}) = \mathfrak{sl}(4,\mathbb{C})$ and $\mathfrak{so}(4,\mathbb{C})$ is not simple.

TABLE 2
Numerology appearing appearing in Zimmer's conjecture for classical complex Lie algebras.

Lie algebra $\mathfrak{g}$	restricted root system	real rank	$n(G)$	$d(G)$	$v(G)$	$r(G)$
$\mathfrak{sl}(n, \mathbb{H}), n \geq 3$	A_{n-1}	$n-1$	$4n$	$4n-2$	$4n-4$	$n-1$
$\mathfrak{so}(n, m)$ $2 \leq n \leq n+2 \leq m$	$B_n, n < m$	n	$n+m$	$n+m-1$	$n+m-2$	$2n-1$
$\mathfrak{su}(n, m)$ $2 \leq n \leq m$ $(n, m) \neq (2, 2)^{(g)}$	$(BC)_n, n < m$ $C_n, n = m$	n	$2n+2m$	$2n+2m-2$	$2n+2m-3$	$2n-1$
$\mathfrak{sp}(2n, 2m)$ $1 \leq n \leq m$	$(BC)_n, n < m$ $C_n, n = m$	n	$4n+4m$	$4n+4m-4$	$4n+4m-5$	$2n-1$
$\mathfrak{so}^*(2n)$ $n \geq 4$ even[h]	$C_{\frac{1}{2}n}$	$\frac{n}{2}$	$4n$	$2n-1$	$4n-7$	$n-1$
$\mathfrak{so}^*(2n)$ $n \geq 5$ odd	$(BC)_{\frac{1}{2}(n-1)}$	$\frac{n-1}{2}$	$4n$	$2n-1$	$4n-7$	$n-2$

(g) $\mathfrak{su}(2, 2) = \mathfrak{so}(4, 2)$

(h) $\mathfrak{so}^*(4)$ is not simple

TABLE 3
Numerology appearing in Zimmer's conjecture for classical higher-rank nonsplit real forms.

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