TOPOLOGICAL ENTROPY AND PRESSURE FOR FINITE-HORIZON SINAI BILLIARDS

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ABSTRACT

This brief survey describes recent progress in our understanding of a variety of equilibrium states for finite-horizon dispersing billiard maps in two dimensions. In particular, we review formulations of topological entropy and pressure for the family of geometric potentials $-t \log J^u T$, where $J^u T$ denotes the unstable Jacobian of the map and $t \in \mathbb{R}$. We summarize recent results, proving the existence and uniqueness of related equilibrium states for some range of $t \geq 0$, including $[0, 1]$. In this family, $t = 0$ corresponds to the measure of maximal entropy, while $t = 1$ corresponds to the smooth invariant measure for the billiard map. In addition, variational principles are presented which express topological notions of pressure and entropy as the supremum of their measure-theoretic counterparts.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 37C83; Secondary 37D35, 37C30, 37B40

KEYWORDS

Dispersing billiard, topological pressure, topological entropy, variational principle, transfer operator, anisotropic Banach spaces

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 5, pp. 3432–3459
DOI 10.4171/ICM2022/52

Published by EMS Press

1. INTRODUCTION

The study of mathematical billiards as prototypical examples of mechanical systems with frictionless collisions was introduced by Sinai [57] and subsequently developed by many authors. In such models, a finite number of convex obstacles are placed on a two-dimensional torus, forming the billiard table, and a point particle is set in motion, moving with constant velocity between collisions, and undergoing elastic reflections at the boundary. The billiard map is the discrete-time map which takes the particle from one collision to the next. Despite the presence of singularities (the map is discontinuous and its derivative is unbounded near tangential collisions) the map preserves a smooth invariant measure, and the ergodic properties of the map with respect to this measure have been studied extensively through a variety of techniques, including Markov partitions [14] and sieves [16], Young towers [21,60], and the spectral analysis of the associated transfer operator [32–34].

It is also possible to introduce variations on the dynamics, by varying the shape of the boundary or by including external forces such as electric fields and potentials which act on the particle between collisions [24,25,29], or twists and kicks at the moment of collision [45,61]. For small forces, the dynamics resemble those of the classical billiard [8,22,23], while for large forces the dynamics may change significantly [36,45,52]. The subject quickly becomes vast and technical, so in this note we will focus on the dynamics of the classical Sinai billiard without external forces. The book [27] by Chernov and Markarian provides an excellent introduction to the subject.

The purpose of this expository note is to introduce the reader, without delving into too many technicalities, to recent developments in the study of a family of equilibrium states for this class of billiards. Traditionally, and in all the references listed above, the focus has been on the ergodic and statistical properties of the map with respect to the Sinai–Ruelle–Bowen (SRB) measure, which in the unperturbed case has a smooth density with respect to Lebesgue measure, such as ergodicity, mixing and the Bernoulli property [37,57], rate of decay of correlations [21,60], dynamical Central Limit Theorem [14], and related limit theorems [32,46,51]. Here, instead, we outline the progress made in [3,4] regarding the family of geometric potentials, $-t \log J^u T$, $t \in \mathbb{R}$, determining the existence and uniqueness of the associated equilibrium states. The importance of this family lies in the fact that $t = 1$ corresponds to the SRB measure while $t = 0$ corresponds to the measure of maximal entropy. More generally, the parameter t has been linked to the Hausdorff dimension of certain invariant sets [12,42].

Despite this, geometric potentials have received relatively little attention in the context of billiards. For $t = 0$, the topological entropy of a finite-horizon Sinai billiard map T was studied in [20] by identifying a full Lebesgue measure set of points M_1 that can be coded via a countable Markov partition. Chernov showed that the topological entropy of T restricted to M_1 is equal to the topological entropy of the induced topological Markov chain and used this to obtain a lower bound on the growth of periodic orbits. Yet, no invariant measure achieving this topological entropy was constructed and whether the set M_1 saw the full topological entropy of the system was left open. For t near 1, the preprint [19] obtains results regarding equilibrium states for this class of geometric potentials. Yet, uniqueness is not proved, and left open is the possible connection to a topological notion of pressure.

With these questions in mind, the papers [3, 4] represent a significant advance in our understanding of topological entropy and pressure and related equilibrium states. It is these ideas and the techniques involved that we hope to illuminate in this note. These, naturally, lead to further questions, several of which we formulate at the end of this review.

The paper is organized as follows. In Section 2, we present the minimal background on dispersing billiards necessary for the subsequent discussion, and state the main results from [3, 4]. In Section 3 we outline the main approach and principle estimates needed to prove the variational principle for $t > 0$, and in Section 4 we do the same for the case $t = 0$. In Section 5, we formulate some open problems related to the equilibrium states we will construct.

2. PRELIMINARIES AND STATEMENTS OF MAIN THEOREMS

A Sinai billiard table is a subset of the torus $\mathbb{T}^2$ obtained by removing finitely many pairwise disjoint, closed convex sets B_i, i.e., $Q = \mathbb{T}^2 \setminus (\bigcup_{i=1}^d B_i)$. The B_i are called scatterers and are assumed to have C^3 boundaries with strictly positive curvature $\mathcal{K}$. The billiard flow is the motion of a point particle in Q traveling at unit speed and undergoing specular reflections (angle of incidence equals angle of reflection) at collisions with the scatterers.

We introduce coordinates on ∂Q by parametrizing ∂B_i according to arclength and recording at each collision the position r and the angle φ made by the postcollision velocity vector with the outward pointing normal to the boundary. Thus the phase space for the map, $M = (\bigcup_{i=0}^d \partial B_i) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, is a union of cylinders, and for each $x = (r, \varphi) \in M$, the billiard map $T(r, \varphi) = (r_1, \varphi_1)$ maps one collision to the next. The map preserves a smooth probability measure, $d\mu_{\mathrm{SRB}} = (2|\partial Q|)^{-1} dr d\varphi$, which is ergodic, indeed Bernoulli, and enjoys exponential decay of correlations on smooth observables, as described in the Introduction.

Let $\tau(x)$ denote the (Euclidean) distance from x to $T(x)$ in Q. We say the billiard has *finite horizon* if there is no trajectory making only tangential collisions. This implies, in particular, that $\tau_{\max} := \sup \tau < \infty$. In addition, the fact that the scatterers are disjoint guarantees $\tau_{\min} := \inf \tau > 0$. Setting $\mathcal{K}_{\min} = \inf \mathcal{K} > 0$ and $\mathcal{K}_{\max} = \sup \mathcal{K} < \infty$, it follows [27, SECT. 4.4] that the stable and unstable cones in the tangent space $\mathbb{R}^2$,

$$\mathcal{C}^s = \left\{ (dr, d\varphi) \;\middle|\; -\mathcal{K}_{\max} - \frac{1}{\tau_{\min}} \le \frac{d\varphi}{dr} \le -\mathcal{K}_{\min} \right\},$$
$$\mathcal{C}^u = \left\{ (dr, d\varphi) \;\middle|\; \mathcal{K}_{\min} \le \frac{d\varphi}{dr} \le \mathcal{K}_{\max} + \frac{1}{\tau_{\min}} \right\}$$

are strictly invariant under DT^{-1} and DT, respectively, whenever the derivatives exist. Away from tangential collisions, T is uniformly hyperbolic, i.e., for $\Lambda := 1 + 2\tau_{\min}\mathcal{K}_{\min}$, there exists $C_0 > 0$ such that for all $n \ge 0$,

$$\begin{aligned} \|DT^n(x)v\| &\ge C_0 \Lambda^n \|v\|, \quad \forall v \in \mathcal{C}^u, \quad \text{and} \\ \|DT^{-n}(x)v\| &\ge C_0 \Lambda^n \|v\|, \quad \forall v \in \mathcal{C}^s. \end{aligned} \tag{2.1}$$

2.1. Singularities and distortion

Denote the set of tangential collisions by $\mathcal{S}_0 = \{x = (r, \varphi) \in M \mid \varphi = \pm\frac{\pi}{2}\}$. The singularity set for T^n, $n \in \mathbb{Z}$, is defined by

$$\mathcal{S}_n = \bigcup_{i=0}^{n} T^{-i}\mathcal{S}_0.$$

Assuming finite horizon, $\mathcal{S}_n$ comprises a finite collection of C^2 curves for each n. Indeed, the hyperbolicity of T implies an alignment property: for $n > 0$, $\mathcal{S}_n \setminus \mathcal{S}_0$ comprises decreasing curves in $\mathcal{C}^s$, while for $n < 0$, $\mathcal{S}_n \setminus \mathcal{S}_0$ comprises increasing curves in $\mathcal{C}^u$ **[27, PROP. 4.45]**.

While the set $\bigcup_{n\in\mathbb{Z}} \mathcal{S}_n$ is dense in M **[27, SECT. 4.11]**, its μ_{SRB}-measure is 0. Setting $M' = M \setminus \bigcup_{n\in\mathbb{Z}} \mathcal{S}_n$, it follows that the stable/unstable subspaces $E^s(x)$ and $E^u(x)$ are defined for all $x \in M'$. Thus we may define the stable/unstable Jacobians of T by

$$J^sT(x) = \left\|DT(x)|_{E^s(x)}\right\| \quad \text{and} \quad J^uT(x) = \left\|DT(x)|_{E^u(x)}\right\| \quad \forall x \in M'.$$

If x has a stable/unstable manifold of positive length (which also occurs on a full-measure set $M'' \subset M'$ **[27, THM. 4.66]**), then J^sT and J^uT serve as the Jacobians for a change of variables when integrating along these manifolds with respect to arclength. We let $\mathcal{W}^s$ denote the set of local stable manifolds of T with length at most $\delta_0 > 0$, which is chosen to guarantee a local complexity condition. See Lemma 3.1 for $t > 0$ and Section 4.1.1 for $t = 0$.

In fact, $DT(x)$ becomes unbounded as $T(x)$ approaches $\mathcal{S}_0$. To compensate for this, the standard technique is to introduce *homogeneity strips* that partition the space into a countable set of horizontal strips accumulating on $\mathcal{S}_0$ and which are effectively treated as singularity curves in exchange for providing some control of distortion. Specifically, choosing[1] $q > 1$ and an index $k_0 \in \mathbb{N}$, one defines for $k \geq k_0$,

$$\mathbb{H}_k = \left\{(r, \varphi) \mid (k+1)^{-q} \leq \frac{\pi}{2} - \varphi \leq k^{-q}\right\},$$

with a similar definition for $\mathbb{H}_{-k}$ approaching $\varphi = -\pi/2$. Let $\mathcal{W}^s_{\mathbb{H}} \subset \mathcal{W}^s$ denote the set of curves $W \in \mathcal{W}^s$ which lie in a single homogeneity strip. Such curves are called *weakly homogeneous* stable manifolds for T.

It follows that there exists $C_d > 0$ such that for all $n \geq 0$, if $T^i(x), T^i(y) \in T^iW \in \mathcal{W}^s_{\mathbb{H}}$ for each $0 \leq i \leq n-1$, then

$$\left|\frac{J^sT^n(x)}{J^sT^n(y)} - 1\right| \leq C_d\, d(x, y)^{1/(q+1)}, \tag{2.2}$$

which is the desired distortion control (see **[27, LEMMA 5.27]** or **[4, LEMMA 2.1]**).

2.2. Measure-theoretic pressure for geometric potentials

Let $t \in \mathbb{R}$ and μ be an invariant probability measure for T. Define the *pressure of* μ with respect to the geometric potential $-t \log J^uT$ to be

$$P_\mu(-t \log J^uT) = h_\mu(T) - t \int_M \log J^uT \, d\mu,$$

1 The standard choice for dispersing billiards is $q = 2$, yet here we will choose q depending on the parameter t in our potential.

where $h_\mu(T)$ denotes the Kolmogorov–Sinai entropy[2] of μ. If μ satisfies

$$P_\mu(-t \log J^u T) = P(t) := \sup\{P_\nu(-t \log J^u T) \mid \nu \in \mathcal{I}\},$$

where $\mathcal{I}$ is the set of invariant probability measures for T, then μ is called an *equilibrium state* for the potential, and $P(t)$ is the *pressure of* $-t \log J^u T$.

The theory of equilibrium states has been well established for Hölder-continuous potentials, first for Anosov and Axiom A systems [11, 53, 58], and then for nonuniformly hyperbolic systems using a variety of techniques [17, 18, 44, 50, 56]. Much less is known in the case of billiards. For $t = 1$, it is known that $P_{\mu_{\text{SRB}}}(-\log J^u T) = 0$; this is the so-called Pesin entropy formula [27, THM. 3.42]. Yet, the uniqueness of this equilibrium state was not proved until [4]. For t near 1, Chen–Wang–Zhang [19] prove existence, but not uniqueness, of equilibrium states using Young towers.

One of the complications of studying equilibrium states associated with this potential is that the unstable Jacobian is not Hölder continuous; indeed, $J^u T$ is not continuous on any open set and it is not bounded (for x near $\mathcal{S}_1$, $J^u T(x) \sim 1/\cos\varphi(Tx)$). Yet, it is regular along homogeneous unstable manifolds (as the time reversal of (2.2) demonstrates) and can be approximated by smooth functions in the distributional norms we will define in Sections 3.2 and 4.2, permitting the analysis we will describe here.

In addition to proving the existence and uniqueness of equilibrium states for the family of geometric potentials, we are interested in expressing the pressure $P(t)$ in terms of topological notions of entropy for $t = 0$ and pressure for $t > 0$. We define such notions precisely in the next two subsections.

Remark 2.1. For $t < 0$, $P(t) = \infty$ if there is a periodic orbit making a grazing collision. In this case, if ν is the atomic measure supported on such a periodic orbit, then $P_\nu(t) = \infty$ as well. Thus the (possibly many) measures maximizing the pressure are simple to describe, so we will not discuss the case $t < 0$ here.

2.3. Topological entropy and variational principle for $t = 0$

Following [3], define for $n, k \geq 0$,

$$\mathcal{M}_{-k}^n = \{\text{maximal connected components of } M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_n)\}. \tag{2.3}$$

Thus elements of $\mathcal{M}_0^n$ are the (open) domains of continuity for T^n and $\mathcal{M}_{-n}^0$ plays the analogous role for T^{-n}. We define the topological entropy of T to be the exponential rate of growth of $\#\mathcal{M}_0^n$, where $\#A$ denotes the cardinality of the set A.

Definition 2.2 (Topological entropy). Define $h_* := \lim_{n\to\infty} \frac{1}{n} \log(\#\mathcal{M}_0^n)$.

The limit above exists due to the submultiplicativity of $\#\mathcal{M}_0^n$ [3, LEMMA 3.3]. Note also that if $A \in \mathcal{M}_0^n$, then $T^n A \in \mathcal{M}_{-n}^0$, so that $\#\mathcal{M}_0^n = \#\mathcal{M}_{-n}^0$ and hence $h_*(T) = h_*(T^{-1})$.

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2 Since T admits a finite generating partition, $h_\mu(T)$ is necessarily finite for any T-invariant probability measure.

One can connect h_* to the usual Bowen definitions of topological entropy via both ε-separated and ε-spanning sets, whose definitions we do not recall here. Although such definitions are usually made for continuous maps, it is a consequence of **[3, THM. 2.3]** that both of the Bowen definitions coincide with h_*.

The main result in **[3]** is the following.

Theorem 2.3 (Measure of maximal entropy and variational principle). *Let T be a finite-horizon Sinai billiard map as defined above. Under a sparse recurrence condition on the singularity set, defined in (4.1), there exists a unique measure μ_0 such that*

$$h_* = h_{\mu_0}(T) = \sup_{\mu \in \mathcal{I}} h_\mu(T).$$

Moreover, μ_0 is hyperbolic and Bernoulli,[3] *has no atoms and is positive on open sets.*

A more complete set of properties for μ_0 can be found in **[3, THM. 2.6]**. That $h_* \geq P_\mu(0)$ for any T-invariant probability measure μ is due to a soft, classical argument (see, for example, **[59, PROP. 9.10]**) since the sequence $\mathcal{M}_0^n$ is related to a finite generating partition for T **[3, LEMMA 3.3]**. The main work of **[3]** is to construct an invariant measure μ_0 whose entropy equals h_*. This requires a precise understanding of the geometry of the sets $\mathcal{M}_0^n$ combined with some functional analytic techniques, whose main ideas are described in Section 4.

2.4. Topological pressure and variational principle for $t > 0$

Before defining our notion of topological pressure for the potential $-t \log J^u T$, it is convenient to consider the corresponding potential in the associated transfer operator. Indeed, arguing by analogy to smooth hyperbolic systems, the transfer operator $\tilde{\mathcal{L}}_t$ with spectral radius $e^{P(t)}$ is defined, for example, on bounded, measurable functions, by

$$\tilde{\mathcal{L}}_t f = \frac{f \circ T^{-1}}{((J^u T)^t J^s T) \circ T^{-1}}.$$

For a Sinai billiard, setting $E(x) = \sin(\angle(E^s(x), E^u(x)))$ to denote the sine of the angle between the stable and unstable subspaces at x, and denoting by $J_{\mathrm{Leb}} T$ the Jacobian of T with respect to Lebesgue measure on M, we have

$$\frac{\cos\varphi(x)}{\cos\varphi(Tx)} = J_{\mathrm{Leb}} T(x) = J^s T(x) J^u T(x) \frac{E(Tx)}{E(x)}$$
$$\implies (J^u T)^t J^s T = \left(\frac{E \cos\varphi}{(E\cos\varphi)\circ T}\right)^t (J^s T)^{1-t}.$$

Since the two potentials are related by a coboundary, the associated transfer operators will have the same spectral radius, so we will study instead the operator

$$\mathcal{L}_t f = \frac{f \circ T^{-1}}{(J^s T)^{1-t} \circ T^{-1}}. \tag{2.4}$$

Remark that $J^s T \approx \cos\varphi$ so that the potential is unbounded whenever $t \neq 1$.

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3 By hyperbolic, we mean that μ_0-a.e. point has stable and unstable manifolds of positive length. By Bernoulli, we mean that it is isomorphic to a Bernoulli shift, which implies also that μ_0 is ergodic and K-mixing.

2.4.1. Weight function for the topological pressure

In order to control the evolution of $\mathcal{L}_t f$ in Section 3.2, it will be necessary to control integrals of the form

$$\int_W \mathcal{L}_t^n f \psi \, dm_W = \int_{T^{-n}W} f \psi \circ T^n \left|J^s T^n\right|^t \, dm_{T^{-n}W},$$

where $W \in \mathcal{W}^s$, m_W is (unnormalized) arclength on W, ψ is a Hölder-continuous test function, and f is an element of the Banach space we will construct.

In order for $(J^s T^n)^t$ to play the role of a test function, (2.2) suggests that we decompose $T^{-1}W$ into a countable collection of maximal curves $W_i^1 \in \mathcal{W}^s_{\mathbb{H}}$ and then iterate these, subdividing into homogeneous components at each step until time n. We denote this collection of curves comprising $T^{-n}W$ by $\mathcal{G}_n^{\mathbb{H}}(W)$. Then the spectral properties of $\mathcal{L}_t$ depend on the growth of

$$\sum_{W_i \in \mathcal{G}_n^{\mathbb{H}}(W)} \left|J^s T^n\right|^t_{C^0(W_i)} \qquad \text{as a function of } n \text{ and } W. \tag{2.5}$$

This toy calculation suggests the weight we use to define the topological pressure below.

2.4.2. Definition of topological pressure

Define $\mathcal{S}_0^{\mathbb{H}} = \mathcal{S}_0 \cup (\bigcup_{|k| \geq k_0} \partial H_k)$ and for $n \in \mathbb{Z}$, $\mathcal{S}_n^{\mathbb{H}} = \bigcup_{i=0}^n T^{-i} \mathcal{S}_0^{\mathbb{H}}$. This will act as an extended singularity set for T^n where we introduce artificial cuts in order to preserve bounded distortion. Let

$$\mathcal{M}_0^{n,\mathbb{H}} := \left\{\text{maximal connected components of } M \setminus (\mathcal{S}_{n-1}^{\mathbb{H}} \cup T^{-n}\mathcal{S}_0)\right\}.$$

We define the weighted sum

$$Q_n(t) = \sum_{A \in \mathcal{M}_0^{n,\mathbb{H}}} \sup_{x \in A \cap M'} \left|J^s T^n(x)\right|^t,$$

and the topological pressure for $t > 0$ is the exponential rate of growth of $Q_n(t)$.

Definition 2.4 (Topological pressure). We let $P_*(t) := \lim_{n\to\infty} \frac{1}{n} \log Q_n(t)$.

As with Definition 2.2, the limit above exists and equals the lim inf due to the submultiplicativity of $Q_n(t)$. It follows that $Q_n(t) \geq e^{nP_*(t)}$ for each $n \geq 0$.

The first theorem from [4] says that $P_*(t)$ dominates the metric pressures.

Theorem 2.5 (Variational inequality). *Let T be a finite-horizon Sinai billiard map. Then $P_*(t)$ is a convex, continuous, decreasing function for $t > 0$, and $P(t)$ satisfies*

$$P_*(t) \geq P(t) = \sup\left\{P_\nu(-t \log J^u T) \mid \nu \text{ is an invariant probability measure for } T\right\}.$$

Remark that for any T-invariant measure μ, $\int \log J^u T \, d\mu = -\int \log J^s T \, d\mu$, which is useful for relating $P(t)$ with $P_*(t)$. As with Theorem 2.3, the inequality $P_*(t) \geq P(t)$ is straightforward, while the main work lies in constructing a measure μ_t such that $P_{\mu_t}(-t \log J^u T) = P_*(t)$. This again requires a detailed analysis of the growth rate of $Q_n(t)$ and the pressure of $\mathcal{G}_n^{\mathbb{H}}(W)$ from (2.5) as a function of n and $W \in \mathcal{W}^s_{\mathbb{H}}$.

2.4.3. Equilibrium state and variational principle

To prove that, in fact, $P(t) = P_*(t)$ and produce an equilibrium state, we restrict our range of t. Recalling $\Lambda > 1$ from (2.1), define

$$t_* := \sup\{t > 0 \mid P(t) > -t \log \Lambda\}.$$

Remark 2.6. The definition of t_* is motivated by the fact that Λ^{-t} controls the local growth in complexity due to singularities (and homogeneity strips), while $e^{P_*(t)}$ controls the global growth in complexity via $Q_n(t)$. Then $t < t_*$ implies the "pressure gap" condition $\Lambda^{-t} < e^{P(t)} \leq e^{P_*(t)}$.

Note that since $P(1) = 0$, $\Lambda > 1$ and $P(t)$ is decreasing, it must be that $t_* > 1$.

The main result in this setting from [4] is the following.

Theorem 2.7 (Unique equilibrium state and variational principle). *Let T be a finite-horizon Sinai billiard and let $t \in (0, t_*)$. Then $P_*(t) = P(t)$ and there exists a unique equilibrium state μ_t for the potential $-t \log J^u T$, i.e.,*

$$P_{\mu_t}(-t \log J^u T) = P_*(t) = P(t).$$

Moreover, μ_t is hyperbolic, has no atoms, is positive on every open set, and enjoys exponential decay of correlations against Hölder observables.

The main technique used in the proof of the theorem is the construction of anisotropic Banach spaces of distributions, adapted to t, on which the transfer operator $\mathcal{L}_t$ has a spectral gap. Then the measure μ_t is constructed as a product of left and right eigenvectors of $\mathcal{L}_t$, following the standard Parry construction (see, for example, [43, SECT. 4.4] for an introduction, or [41] for an application in the case of Anosov diffeomorphisms). Indeed, our control of the spectrum of $\mathcal{L}_t$ also implies the following theorem.

Theorem 2.8 (Analyticity of $P(t)$). *The pressure function $P(t)$ is analytic for $t \in (0, t_*)$, with*

$$P'(t) = \int \log J^s T \, d\mu_t = -\int \log J^u T \, dm_t < 0 \quad \textit{and}$$
$$P''(t) = \sum_{k \geq 0} \left[\int (\log J^s T \circ T^k) \log J^s T \, d\mu_t - \big(P'(t)\big)^2 \right] \geq 0.$$

Moreover, $P''(t) = 0$ if and only if $\log J^s T = f - f \circ T + P'(t)$ for some $f \in L^2(\mu_t)$.

If there exists $s \neq t \in (0, t_*)$ such that $\mu_s = \mu_t$, then $P(t)$ is affine on $(0, t_*)$ and $\log J^s T$ is μ_t-a.e. cohomologous to a constant for all $t \in (0, t_*)$.

Finally, under the sparse recurrence condition (4.1), $\lim_{t \downarrow 0} P(t) = P(0) = h_*$.

We conjecture that, in fact, $\mu_s \neq \mu_t$ for any $s \neq t$ in $(0, t_*)$, i.e., $J^s T$ cannot be cohomologous to a constant for a Sinai billiard. If it were, then the theorem would imply that $P(t)$ is affine and, by uniqueness, $\mu_t = \mu_{\text{SRB}}$ for each $t \in [0, t_*)$. See Section 5.

3. IDEAS FROM THE PROOF OF THEOREM 2.7

In this section, we present some of the key ideas in the proof of Theorem 2.7. In light of Theorem 2.5, they divide into two principal parts: (1) geometric estimates that control local complexity and establish a uniform exponential rate of growth for $Q_n(t)$; (2) the functional-analytic framework needed to construct a measure μ_t with pressure $P_*(t)$.

The development of a functional-analytic framework in which to study transfer operators for hyperbolic systems has a well-established history. After some early success in the hyperbolic analytic case [54,55], intense interest was generated by the paper of Blank, Keller, and Liverani [10], which launched a series of subsequent papers developing a variety of Banach spaces for Anosov and Axiom A maps [1,7,40,41]. This was later extended to piecewise hyperbolic maps [5,6,31] and ultimately to a variety of hyperbolic billiards [3,32–34]. See [2] for a comprehensive survey or [30] for a gentle introduction to the use of such spaces. The norms we shall define in Section 3.2 are a natural adaptation of these ideas to the family of geometric potentials.

3.1. Growth lemmas and exact exponential growth of $Q_n(t)$

Our first goal is to establish precise bounds on the exponential growth of $Q_n(t)$ as well as the growth of the pressure over $\mathcal{G}_n^{\mathbb{H}}(W)$ from (2.5). In order to accomplish this, we fix $t_0 > 0$ and $t_1 < t_*$ and obtain uniform bounds for t in the closed interval $[t_0, t_1]$.

Fix $q \geq 2/t_0$ and let $\theta \in (\Lambda, 1)$ be such that $\theta^{t_1} < e^{P_*(t_1)}$. The latter choice is possible by definition of t_*, and implies by the convexity of $P_*(t)$ that $\theta^t < e^{P_*(t)}$ for all $t \in [t_0, t_1]$. Next we adapt the usual one-step expansion (see [27, LEMMA 5.56]) to our potential.

Lemma 3.1. *There exist k_0, $\delta_0 > 0$ such that*

$$\sup_{\substack{V \in \mathcal{W}^s \\ |V| \leq \delta_0}} \sum_{V_i} \left| J^s T \right|^t_{*, C^0(V_i)} < \theta^t \quad \text{for all } t \geq t_0,$$

where V_i are the maximal, connected homogeneous components of $T^{-1}V$ and $|\cdot|_*$ denotes the sup *norm with respect to an adapted metric.*[4]

Sketch of proof. Due to the finite-horizon condition, a short stable manifold V can be cut by at most $\tau_{\max}/\tau_{\min}$ tangential collisions under T^{-1} and all but one of these collisions are nearly grazing. Near grazing collisions, $V_k \subset \mathbb{H}_k$ and, since $J^s T \sim \cos\varphi$,

$$\sum_{k \geq k_0} \left| J^s T \right|^t_{*, C^0(V_k)} \leq C \sum_{k \geq k_0} k^{-qt} \leq C' k_0^{-1} \quad \text{since } qt \geq 2.$$

Thus setting $\varepsilon = \theta - \Lambda^{-1} > 0$, we choose k_0 in the definition of homogeneity strips so large that $C' k_0^{-1} \frac{\tau_{\max}}{\tau_{\min}} < \varepsilon$. Finally, choose δ_0 so small that if $|V| \leq \delta_0$, then $T^{-1}V$ can intersect only homogeneity strips of index at least k_0 at the nearly tangential collisions. This is possible since $|T^{-1}V| \leq C|V|^{1/2}$ [27, EXERCISE 4.60]. ■

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4 The adapted metric is defined as in [27, SECT. 5.10] so that in (2.1) the constant $C_0 = 1$, i.e., the expansion is seen in one step. The lemma applies equally well to more general cone-stable curves and its time reversal to cone-unstable curves.

The one-step expansion expressed by Lemma 3.1 guarantees that the expansion provided by the weight $1/(J^sT)^t$ along stable manifolds mapped by T^{-1} is strong enough to overcome the effect of cutting by both the primary and secondary discontinuities of T^{-1}. This can be iterated inductively to obtain statements regarding the prevalence of long pieces in both $T^{-n}W$ and $\mathcal{M}_0^{n,\mathbb{H}}$, as summarized in Lemma 3.2 below.

Recalling the definition of $\mathcal{G}_n^{\mathbb{H}}(W)$ from (2.5), for $\delta_1 < \delta_0$ and $W \in \mathcal{W}^s$, define $\mathcal{G}_n^{\delta_1,\mathbb{H}}(W)$ as a decomposition of $T^{-n}W$ in an analogous manner with $\mathcal{G}_n^{\mathbb{H}}(W)$, but with pieces longer than length δ_1 subdivided into length between $\delta_1/2$ and δ_1 at each step (rather than length δ_0).

For a set $E \subset M$, we use $\mathrm{diam}^u(E)$ to denote the length of the longest cone-unstable curve in E, and $\mathrm{diam}^s(E)$ to denote the length of the longest cone-stable curve in E.

Lemma 3.2. (a) *$\forall \varepsilon > 0\ \exists \delta_1, n_1 > 0$ such that $\forall W \in \mathcal{W}^s$ with $|W| \geq \delta_1/3$ and all $n \geq n_1$,*

$$\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1,\mathbb{H}}(W_i) \\ |W_i| < \delta_1/3}} \left|J^sT^n\right|_{C^0(W_i)}^t \leq \varepsilon \sum_{W_i \in \mathcal{G}_n^{\delta_1,\mathbb{H}}(W)} \left|J^sT^n\right|_{C^0(W_i)}^t.$$

(b) *For $A \in \mathcal{M}_0^{n,\mathbb{H}}$, let $B_{n-1}(A)$ denote the connected component of $M \setminus (\bigcup_{i=0}^{n-1} T^i \mathcal{S}_0^{\mathbb{H}})$ containing $T^{n-1}A$. Define $\mathcal{A}_n(\delta) = \{A \in \mathcal{M}_0^{n,\mathbb{H}} : \mathrm{diam}^u(B_{n-1}(A)) \geq \delta/3\}$. There exist $\delta_2 \leq \delta_1$ and $c_0 > 0$ such that*

$$\sum_{A \in \mathcal{A}_n(\delta_2)} \sup_{x \in A \cap M'} \left|J^sT^n(x)\right|^t \geq c_0 Q_n(t), \quad \forall n \in \mathbb{N}, \forall t \in [t_0, 1].$$

Comments on proof. The proof of part (a) relies on iterating Lemma 3.1 combined with the following lower bound on growth valid for $t \leq 1$ and $V \in \mathcal{W}^s$,

$$\begin{aligned}\sum_{W_i \in \mathcal{G}_k^{\delta_1,\mathbb{H}}(V)} \left|J^sT^k\right|_{C^0(W_i)}^t &= \sum_{W_i \in \mathcal{G}_k^{\delta_1,\mathbb{H}}(V)} \left|J^sT^k\right|_{C^0(W_i)} \left|J^sT^k\right|_{C^0(W_i)}^{t-1} \\ &\geq C_1 \Lambda^{k(1-t)} \sum_{W_i \in \mathcal{G}_k^{\delta_1,\mathbb{H}}(V)} \frac{|T^kW_i|}{|W_i|} \geq C_1 \Lambda^{k(1-t)} |V| \delta_1^{-1},\end{aligned}$$

which guarantees that long pieces most continue to produce a sufficient number of long pieces. Once (a) is proved for $t \leq 1$, we extend it to $t \in (1, t_1]$ via interpolation (see [4, SECT. 3.4]).

(b) The proof of (b) follows the same lines as (a), using a version of Lemma 3.1 for elements of $\mathcal{M}_0^{n,\mathbb{H}}$ and a generalization of bounded distortion which says that $J^sT^n(x)$, $J^sT^n(y)$ are comparable when x, y belong to the same element of $\mathcal{M}_0^{n,\mathbb{H}}$. ■

Using Lemma 3.2, we can prove the following key results regarding the uniform growth of $W \in \mathcal{W}^s$ and a type of supermultiplicativity for $Q_n(t)$.

Proposition 3.3. (a) *$\exists c_1 > 0$ such that $\forall W \in \mathcal{W}^s$ with $|W| \geq \delta_1/3$,*

$$\sum_{W_i \in \mathcal{G}_n^{\mathbb{H}}(W)} \left|J^sT^n\right|_{C^0(W_i)}^t \geq c_1 Q_n(t), \quad \forall n \geq 1, \forall t \in [t_0, t_1].$$

(b) *$\exists c_2 > 0$ such that for all $k, n \geq 1$ and all $t \in [t_0, t_1]$, $Q_{n+k}(t) \geq c_2 Q_n(t) Q_k(t)$.*

Sketch of proof. Let $L_n^{\delta_1}(W)$ denote the elements of $\mathcal{G}_n^{\delta_1,\mathbb{H}}(W)$ longer than $\delta_1/3$. Then (b) follows from (a) and Lemma 3.2 (choosing $\varepsilon = 1/2$ there) since,

$$\sum_{W_i \in \mathcal{G}_{n+k}^{\delta_1,\mathbb{H}}(W)} |J^s T^{n+k}|^t_{C^0(W_i)} \geq C \sum_{V_j \in L_n^{\delta_1}(W)} |J^s T^n|^t_{C^0(V_j)} \sum_{W_i \in \mathcal{G}_k^{\delta_1,\mathbb{H}}(V_j)} |J^s T^k|^t_{C^0(W_i)}$$
$$\geq \frac{C}{2} \sum_{V_j \in \mathcal{G}_n^{\delta_1,\mathbb{H}}} |J^s T^n|^t_{C^0(V_j)} c_1 Q_k(t) \geq \frac{C}{2} c_1^2 Q_n(t) Q_k(t).$$

The proof of (a) relies on covering a full μ_{SRB}-measure set of M with a finite collection of Cantor rectangles, formed by maximal intersections of local stable and unstable manifolds so that each rectangle has a hyperbolic product structure. By [27, LEMMA 7.87], we may choose a finite collection of such rectangles, $\mathcal{R}(\delta_2) = \{R_i\}_{i=1}^{N_{\delta_2}}$, such that any cone-stable or cone-unstable curve of length at least $\delta_2/3$ properly crosses at least one R_i. Let

$$\mathcal{A}_n^i := \{A \in \mathcal{A}_n(\delta_2) \subset \mathcal{M}_0^{n,\mathbb{H}} \mid B_{n-1}(A) \text{ properly crosses } R_i\}.$$

By Lemma 3.2(b), there must exist i_* such that $\sum_{A \in \mathcal{A}_n^{i_*}} \sup_A |J^s T^n|^t \geq \frac{c_0}{N_{\delta_2}} Q_n(t)$.

Let $W \in \mathcal{W}^s$ with $|W| \geq \delta_1/3 \geq \delta_2/3$. W must properly cross one R_j. Since μ_{SRB} is mixing, we may ensure that $V = T^{-N} W$ properly crosses R_{i_*}, where N depends only on δ_2. This proper crossing ensures that $\sum_{W_i \in \mathcal{G}_n^{\mathbb{H}}(V)} |J^s T^n|^t_{C^0(W_i)}$ will be comparable to $\sum_{A \in \mathcal{A}_n^{i_*}} \sup_A |J^s T^n|^t$, and then, adjusting for N, we conclude that $\sum_{W_i \in \mathcal{G}_n^{\mathbb{H}}(W)} |J^s T^n|^t_{C^0(W_i)}$ grows at the rate $Q_n(t)$. ■

Remark 3.4. Proposition 3.3(a) says that the pressure of all long (in the scale δ_1) local stable manifolds grows at a uniform exponential rate (not just asymptotically the same rate).

A corollary of Proposition 3.3(b) is the exact exponential growth of $Q_n(t)$,

$$e^{nP_*(t)} \leq Q_n(t) \leq 2c_2^{-1} e^{nP_*(t)} \quad \forall n \geq 1, \forall t \in [t_0, t_1],$$

where the lower bound follows from the submultiplicativity of $Q_n(t)$ and the upper bound follows from its (approximate) supermultiplicativity. This bound is essential in proving the requisite spectral properties of $\mathcal{L}_t$ in Section 3.2.2.

3.2. Banach spaces adapted to $t \in [t_0, t_1]$

The Banach spaces adapted to the operator $\mathcal{L}_t$ for $t \in (0, t_*)$ are similar to those used in [33] for the case $t = 1$. For convenience, we identify $f \in C^1(M)$ with the measure $d\mu = f d\mu_{\mathrm{SRB}}$. With this identification, the transfer operator defined on distributions μ by

$$\mathcal{L}_t \mu(\psi) = \mu\big(\psi \circ T \cdot (J^s T)^{t-1}\big) \quad \text{for suitable test functions } \psi, \tag{3.1}$$

coincides with the pointwise definition of $\mathcal{L}_t f$ acting on measurable functions from (2.4). As in Section 3.1, we fix $[t_0, t_1] \subset (0, t_*)$ and obtain uniform estimates for $t \in [t_0, t_1]$.

3.2.1. Definition of norms

Since $\mathcal{L}_t f$ has a deregularizing effect in the stable direction, but improves regularity in the unstable direction, the norms defined below have two important properties: they integrate along local stable manifolds to average out the action of $\mathcal{L}_t f$ in the stable direction, while requiring $\mathcal{L}_t f$ to have a form of average regularity in the unstable direction (see the definition of $\|\cdot\|_u$ below). Integrating along local stable manifolds (as opposed to the cone-stable curves used in [32–34]) also allows us to take advantage of the fact that J^sT is Hölder continuous along such manifolds.

Fix $0 < \alpha \le 1/(q+1)$. For $f \in C^1(M)$, define the *weak norm* of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s_{\mathbb{H}}} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha} \le 1}} \int_W f \psi \, dm_W. \tag{3.2}$$

Define $\mathcal{B}_w$ to be the completion of $C^1(M)$ in the $|\cdot|_w$ norm.

For the strong norm, we need additional parameters. Choose

$$p > q+1 \quad \text{such that } \theta^{t_1 - 1/p} < e^{P_*(t_1)}, \beta \in (1/p, \alpha) \text{ and } \gamma < \min\{1/p, \alpha - \beta\}.$$

Define the *strong stable norm* of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s_{\mathbb{H}}} \sup_{\substack{\psi \in C^\beta(W) \\ |\psi|_{C^\beta} \le |W|^{-1/p}}} \int_W f \psi \, dm_W.$$

The strong unstable norm measures the integral of f on two curves that are close together. To define this, we need notions of distance between curves and test functions. Since the stable cone $\mathcal{C}^s$ is bounded away from the vertical, we view $W \in \mathcal{W}^s$ as the graph of a function of the r-coordinate over an interval I_W,

$$W := \{G_W(r) \mid r \in I_W\} := \{(r, \varphi_W(r)) \mid r \in I_W\}.$$

Now given $W_1, W_2 \in \mathcal{W}^s$ defined by $\varphi_{W_1}, \varphi_{W_2}$, define

$$d(W_1, W_2) = |I_{W_1} \triangle I_{W_2}| + |\varphi_{W_1} - \varphi_{W_2}|_{C^1(I_{W_1} \cap I_{W_2})},$$

if W_1, W_2 lie in the same homogeneity strip, and $d(W_1, W_2) = \infty$ otherwise. If $d(W_1, W_2) < \infty$, we define a distance between test functions $\psi_k \in C^0(W_k)$ by

$$d_0(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{C^0(I_{W_1} \cap I_{W_2})}.$$

With these definitions, we are able to define the *strong unstable norm* of f as

$$\|f\|_u = \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s_{\mathbb{H}} \\ d(W_1, W_2) \le \varepsilon}} \sup_{\substack{|\psi_i|_{\mathcal{C}^\alpha(W_i)} \le 1 \\ d_0(\psi_1, \psi_2) = 0}} \varepsilon^{-\gamma} \left| \int_{W_1} f \psi_1 \, dm_{W_1} - \int_{W_2} f \psi_2 \, dm_{W_2} \right|,$$

where $\varepsilon_0 > 0$ is a small constant depending on the table. Finally, define $\mathcal{B}$ to be the closure of $C^1(M)$ in the *strong norm* $\|\cdot\|_{\mathcal{B}}$, defined by $\|f\|_{\mathcal{B}} = \|f\|_s + c_u \|f\|_u$, where c_u is chosen so that the inequalities in Theorem 3.7 provide contraction in the strong norm (see [4, SECT. 4.3]).

Remark 3.5. The choices of parameters are motivated as follows: $\alpha \le 1/(q+1)$ due to the Hölder exponent in (2.2). Then $\beta < \alpha$ is required for relative compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$. The weight $|W|^{-1/p}$ weakens the contraction of the one-step expansion to

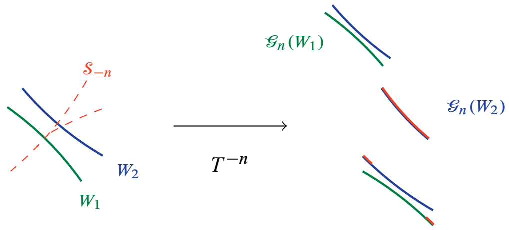

FIGURE 1
Two stable manifolds W_1 (green) and W_2 (blue) and their images under T^{-n}. Green and blue pieces are matched, while red curves are not matched due to cuts introduced by $\mathcal{S}_{-n}$.

$\theta^{t-1/p}$, so p is chosen large enough that this is still small compared to the pressure $e^{P_*(t)}$. Finally, the regularity exponent γ is chosen sufficiently small that the unmatched pieces created by discontinuities in the Lasota–Yorke inequalities are rendered negligible by the weight $|W|^{1/p}$.

Proposition 3.6. *With the correct choices of parameters above, we have a sequence of continuous inclusions, $C^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^\alpha(M))^*$.*

Moreover, the embedding of the unit ball of $\mathcal{B}$ into $\mathcal{B}_w$ is compact.

3.2.2. A spectral gap for $\mathcal{L}_t$

Since $\mathcal{B}$ is defined as the completion of $C^1(M)$ in $\|\cdot\|_{\mathcal{B}}$, a priori it is not clear that $\mathcal{L}_t$ acts continuously on $\mathcal{B}$ since J^sT is not even piecewise Hölder continuous; however, [27, THEOREM 5.66] and [4, LEMMA 4.10] show that J^sT varies sufficiently regularly on hyperbolic Cantor rectangles so that if $f \in C^1(M)$, then $\mathcal{L}_t f$ can be approximated by C^1 functions in the $\|\cdot\|_{\mathcal{B}}$-norm, i.e., $\mathcal{L}_t f \in \mathcal{B}$. Thus we are able to prove:

Theorem 3.7 ([4]). *Operator $\mathcal{L}_t$ acts continuously on $\mathcal{B}$ and satisfies the following Lasota–Yorke (or Doeblin–Fortet) inequalities: there exist $C, C_n > 0$ such that for all $f \in \mathcal{B}$, $n \geq 0$,*

$$\begin{aligned}
&\left|\mathcal{L}_t^n f\right|_w \leq C Q_n(t) |f|_w, \\
&\left\|\mathcal{L}_t^n f\right\|_s \leq C\left(\Lambda^{-(\beta-1/p)n} Q_n(t) + \theta^{(t-1/p)n}\right)\|f\|_s + C_n |f|_w, \\
&\left\|\mathcal{L}_t^n f\right\|_u \leq C Q_n(t)\left(n^\gamma \Lambda^{-\gamma n}\|f\|_u + C_n \|f\|_s\right).
\end{aligned}$$

Furthermore, $\mathcal{L}_t$ has a spectral gap: $e^{P_*(t)}$ is the eigenvalue of maximum modulus, it is simple, and the rest of the spectrum of $\mathcal{L}_t$ is contained in a disk of radius $\sigma e^{P_*(t)}$, where $\sigma < 1$ is uniform for $t \in [t_0, t_1]$.

Comments on the proof. (1) *The estimate on unmatched pieces.* For a proof of the Lasota–Yorke inequalities, the reader is referred to [4, SECT. 4.3]. Here we comment only on the control of "unmatched pieces" in the estimate of the strong unstable norm since this leads to essential changes in the case $t = 0$. We must estimate $|\int_{W_1} \mathcal{L}_t^n f \psi_1 - \int_{W_2} \mathcal{L}_t^n f \psi_2|$.

Changing variables, we see that $\mathcal{G}_n^{\mathbb{H}}(W_1)$ comprises matched pieces (that are close to a corresponding curve in $\mathcal{G}_n^{\mathbb{H}}(W_2)$), and unmatched pieces (which are not, due to cuts by the singularity set $\mathcal{S}_{-n}$). See Figure 1. The distance between matched pieces contracts due to the hyperbolicity of T, but the unmatched pieces do not contract. Yet, unmatched pieces have length at most $\Lambda^{-j}\varepsilon$ if they are cut by a singularity curve at time $-j$, so we may use the strong stable norm to estimate

$$\int_{W_i} \mathcal{L}_t^n f \psi = \int_{V_j} \mathcal{L}_t^{n-j} f \psi \circ T^j \left|J_{V_j} T^j\right|^t \le \Lambda^{-j/p} \varepsilon^{1/p} \left\|\mathcal{L}_t^{n-j} f\right\|_s \left|J_{V_j} T^j\right|_{C^0}^t$$

In this sense, $\|\cdot\|_s$ acts as a “weak norm” for $\|\cdot\|_u$ to control unmatched pieces. This is the reason why the weight $|W|^{-1/p}$ must be included in the definition of $\|\cdot\|_s$, and is an essential difference with the case $t=0$ in Section 4.2.

(2) *Quasicompactness of* $\mathcal{L}_t$. The Lasota–Yorke inequalities imply that the spectral radius of $\mathcal{L}_t$ on $\mathcal{B}$ is at most $e^{P_*(t)}$ and its essential spectral radius $< e^{P_*(t)}$ if $\theta^t < e^{P_*(t)}$. This is the pressure gap condition guaranteed by choice of θ for all $t \in [t_0, t_1]$. In order to conclude quasicompactness, however, we need a *lower bound* on the spectral radius. This follows from Proposition 3.3(a). Indeed, let $W \in \mathcal{W}_{\mathbb{H}}^s$ with $|W| \ge \delta_1/3$, and choose $\psi \equiv 1$. For any $n \ge 1$,

$$\begin{aligned}\int_W \mathcal{L}_t^n 1 &= \sum_{W_i \in \mathcal{G}_n^{\mathbb{H}}(W)} \int_{W_i} \left|J^s T^n\right|^t \ge e^{-C_d} \frac{\delta_1}{3} \sum_{W_i \in L_n^{\delta_1}(W)} \left|J^s T^n\right|_{C^0(W_i)}^t \ge C c_1 Q_n(t) \\ &\ge C' e^{n P_*(t)}.\end{aligned}$$

Thus $\|\mathcal{L}^n 1\|_s \ge C' e^{n P_*(t)}$, and so the spectral radius of $\mathcal{L}$ is $e^{P_*(t)}$.

(3) *A spectral gap for* $\mathcal{L}_t$. Exact exponential growth of $Q_n(t)$ (see Remark 3.4) implies $\|\mathcal{L}_t^n\|_{\mathcal{B}} \le C Q_n(t) \le C' e^{n P_*(t)}$, so that the peripheral spectrum of $\mathcal{L}_t$ has no Jordan blocks. Then the expression

$$\nu_t(\psi) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-k P_*(t)} \mathcal{L}_t 1(\psi) \tag{3.3}$$

defines a finite Borel measure in $\mathcal{B}$ satisfying $\mathcal{L}_t \nu_t = e^{P_*(t)} \nu_t$, where, according to our identification of functions with densities with respect to μ_{SRB}, we set

$$\mathcal{L}_t 1(\psi) := \int_M \psi \mathcal{L}_t 1 \, d\mu_{\mathrm{SRB}}. \tag{3.4}$$

Using (3.3) and the uniform control provided by Proposition 3.3, one shows that all eigenvectors corresponding to the peripheral spectrum are measures absolutely continuous with respect to ν_t and all eigenvalues are roots of unity. Finally, the topological mixing of T implies that there can be no other eigenvalues of modulus $e^{P_*(t)}$. ■

3.2.3. An equilibrium state and a variational principle

Let ν_t be defined as in (3.3) and let $\tilde{\nu}_t \in \mathcal{B}^*$ denote the analogous construction with the dual operator $\mathcal{L}_t^*$. Define

$$\mu_t(\psi) = \frac{\langle \nu_t, \psi \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle}, \quad \psi \in C^\alpha(M).$$

The normalization $\langle \nu_t, \tilde{\nu}_t \rangle \neq 0$ by Proposition 3.3(a). Then it is a standard calculation that μ_t is an invariant probability measure for T, and due to the spectral gap of $\mathcal{L}_t$, μ_t enjoys exponential decay of correlations against Hölder observables.

The facts that μ_t has no atoms, gives 0 weight to any C^1 curve, is positive on open sets, and has stable and unstable manifolds of positive length all follow from the regularity of $\nu_t \in \mathcal{B}$.

Finally, we comment on the entropy of μ_t and conclude the variational principle stated in Theorem 2.7. To this end, define the Bowen balls for T^{-n} by

$$B(x,n,\varepsilon) = \{y \in M : d(T^{-i}x, T^{-i}y) \leq \varepsilon, \forall i \in [0,n]\}. \tag{3.5}$$

Proposition 3.8 (Measure of Bowen balls). *There exists $C > 0$ such that for all $x \in M$, $n \geq 1$, and $y \in B(x,n,\varepsilon)$,*

$$\mu_t(B(x,n,\varepsilon)) \leq Ce^{-nP_*(t)+t \log J^s T^n(T^{-n}y)}.$$

Then **[13, MAIN THEOREM]** implies that for μ_t-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_t(B(x,n,\varepsilon)) = h_{\mu_t}(T).$$

This, together with Proposition 3.8, implies

$$h_{\mu_t}(T) \geq P_*(t) - t \int \log J^s T \, d\mu_t = P_*(t) + t \int \log J^u T \, d\mu_t.$$

But $P_*(t) \geq h_{\mu_t}(T) - t \int \log J^u T \, d\mu_t$ since $P_*(t) \geq P(t)$ by Theorem 2.5. We conclude that $P_*(t) = h_{\mu_t}(T) - t \int \log J^u T \, d\mu_t = P(t)$, which implies both that μ_t is the measure maximizing the pressure and that the topological pressure satisfies a variational principle, despite the effect of singularities.

The last item of Theorem 2.7, the uniqueness of μ_t, uses the concept of a tangent measure. The argument exploits in particular the differentiability of the pressure for $t > 0$. We refer the interested reader to **[4, SECT. 5.5]**.

4. IDEAS FROM THE PROOF OF THEOREM 2.3

In this section, we provide a parallel presentation to Section 3 for the case $t = 0$, i.e., the construction of the measure of maximal entropy. As before, we divide the ideas into two parts: (1) geometric estimates to control local complexity and a uniform rate of growth for $\#\mathcal{M}_0^n$; (2) a functional-analytic framework needed to construct the equilibrium state μ_0.

In contrast to Section 3, we cannot use homogeneity strips and must drastically alter the weights in the strong norm. These changes are sufficiently severe to prevent us from obtaining a spectral gap for $\mathcal{L}_0$ and exponential mixing for μ_0.

4.1. Complexity and exact exponential growth of $\#\mathcal{M}_0^n$

Recall the toy calculation in Section 2.4.1 for deriving the correct weight for the topological pressure. If we consider (2.5) with $t = 0$, we have $\#\mathcal{G}_n^{\mathbb{H}}(W) = \infty$ whenever

$T^{-n}W$ crosses infinitely many homogeneity strips. Thus we cannot use homogeneity strips when studying the case $t = 0$. Moreover, the one-step expansion Lemma 3.1 does not hold.

Instead, we use the linear complexity bound due to Bunimovich. For $x \in M$, let $N(\mathcal{S}_n, x)$ denote the number of singularity curves in $\mathcal{S}_n$ that meet at x. Define $N(\mathcal{S}_n) = \sup_{x\in M} N(\mathcal{S}_n, x)$.

Lemma 4.1 ([15]). *Assume finite horizon. There exists $K > 0$, depending only on the configuration of scatterers, such that $N(\mathcal{S}_n) \le Kn$ for all $n \ge 1$.*

Sketch of proof from [28]. Suppose $x, x' \in M$ lie on a straight billiard trajectory with one or more tangential collisions between them. Let A, A' be neighborhoods of x, x' in M, partitioned into sectors $A_1, \dots A_k \subset A$ and $A'_1, \dots A'_k \subset A$ such that $T^{n_j} A_j = A'_j$. Define $\hat{T}|_{A_j} := T^{n_j}$. See Figure 2.

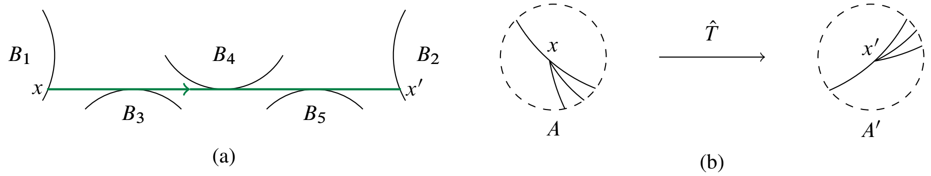

FIGURE 2
(a) A trajectory with multiple tangencies. (b) Neighborhood A of x with elements of $\mathcal{S}_n$ and neighborhood A' of x' with their images in $\mathcal{S}_{-n}$.

We prove the statement by induction. For $n = 1$ it is trivial. Now assume $N(\mathcal{S}_{n-1}) \le K(n-1)$ for some $K > 0$. Let $N(\mathcal{S}_i | A'_j, x')$ denote the number of curves in $\mathcal{S}_i$ passing through x' and lying in A'_j. Since curves in $\mathcal{S}_i \setminus \mathcal{S}_0$ (stable) and $\mathcal{S}_{-n} \setminus \mathcal{S}_0$ (unstable) are uniformly transverse, each sector created by $\mathcal{S}_i$ can only intersect one sector created by $\mathcal{S}_{-n}$. Then pulling back the picture from x' to x and recalling that k is the number of tangencies meeting at x, we have (here we are using continuity of the flow)

$$N(\mathcal{S}_n, x) \le k + \sum_j N(\mathcal{S}_{n-n_j} | A'_j, x') \le k + \sum_j N(\mathcal{S}_{n-1} | A'_j, x'),$$

and, using the inductive assumption on $n-1$, this yields $N(\mathcal{S}_n, x) \le k + K(n-1)$, which is less than Kn if $k \le K$. Due to the finite horizon condition, the number of tangencies intersecting at a point $x \in M$ has a finite upper bound depending only on the table. Thus choosing K to be this upper bound completes the proof of the lemma. ■

4.1.1. Fragmentation lemmas

Choose $n_0 \in \mathbb{N}$ such that $n_0^{-1} \log(Kn_0 + 1) < h_*$. Due to Lemma 4.1, we may choose $\delta_0 > 0$ such that any stable curve of length $\le \delta_0$ is cut into at most $Kn_0 + 1$ pieces by $\mathcal{S}_{-n_0}$. We use this choice of δ_0 in our definition of $\mathcal{W}^s$, the set of local stable manifolds with which we work. (In Section 4.2.2 we will shrink δ_0 further depending on the parameters in our norms.) This choice of δ_0 will ensure that the growth in $\mathcal{G}_n(W)$ due to local complexity will be slower than e^{nh_*}. We make this precise below.

For $\delta \le \delta_0$, let $\mathcal{G}_n^\delta(W)$ denote the collection of curves in $T^{-n}W$ analogous to $\mathcal{G}_n^{\mathbb{H}}(W)$, but without using homogeneity strips, and with pieces longer than δ subdivided into curves between length $\delta/2$ and δ at each step. Define $L_n^\delta(W) = \{W_i \in \mathcal{G}_n^\delta(W) : |W| \ge \delta/3\}$ and $Sh_n^\delta(W) = \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W)$.

Lemma 4.2 ([3]). *For all $\varepsilon > 0$, there exist $n_1, \delta > 0$ such that, for all $n \ge n_1$,*

$$\#Sh_n^\delta(W) \le \varepsilon \#\mathcal{G}_n^\delta(W) \quad \textit{for all } W \in \hat{\mathcal{W}}^s \textit{ with } |W| \ge \delta/3.$$

Idea of proof. Recalling (2.1), choose $\varepsilon > 0$ and n_1 such that $3C_0^{-1}(Kn_1 + 1)\Lambda^{-n_1} < \varepsilon$. Choose $\delta > 0$ such that if $|W| < \delta$ then $T^{-n_1}W$ comprises at most $Kn_1 + 1$ connected components of length at most δ_0. Then $Sh_{n_1}^\delta(W)$ contains at most $Kn_1 + 1$ elements. On the other hand, $|T^{-n_1}W| \ge C_0\Lambda^{n_1}\delta/3$, where $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$ is from (2.1). Thus $\#\mathcal{G}_{n_1}^\delta(W) \ge C_0\Lambda^{n_1}/3$, and so $\#Sh_{n_1}^\delta(W) \le \varepsilon \#\mathcal{G}_{n_1}^\delta(W)$ by the choice of n_1.

The argument can be iterated, grouping each collection of pieces at time kn_1 by the most recent time jn_1, $j \le k$, that each piece was contained in an element of $L_{jn_1}^\delta(W)$. ■

As in Lemma 3.2(b), some control of short pieces can also be extended to elements of $\mathcal{M}_0^n$ and $\mathcal{M}_{-n}^0$. Let $\delta_1, n_1 \ge n_0$ correspond to $\varepsilon = 1/4$ in Lemma 4.2. Define

$$L_s(\mathcal{M}_0^n) = \left\{A \in \mathcal{M}_0^n : \mathrm{diam}^s(A) \ge \delta_1/3\right\} \quad \text{and}$$
$$L_u(\mathcal{M}_{-n}^0) = \left\{B \in \mathcal{M}_{-n}^0 : \mathrm{diam}^u(B) \ge \delta_1/3\right\}.$$

Lemma 4.3 ([3]). *There exists $c_0 > 0$ such that, for all $n \ge 1$,*

$$\#L_s(\mathcal{M}_0^n) \ge c_0\delta_1 \#\mathcal{M}_0^n \quad \textit{and} \quad \#L_u(\mathcal{M}_{-n}^0) \ge c_0\delta_1 \#\mathcal{M}_{-n}^0.$$

4.1.2. Uniform bounds on growth

As in Section 3.1, the fragmentation lemmas above imply uniform bounds on the growth of $\#\mathcal{G}_n(W)$ and $\#\mathcal{M}_0^n$.

Proposition 4.4. (a) *There exists $c_1 > 0$ such that, for any $W \in \mathcal{W}^s$ with $|W| \ge \delta_1/3$,*

$$\#\mathcal{G}_n(W) \ge c_1 \#\mathcal{M}_0^n \quad \forall n \ge 1.$$

(b) *There exists $c_2 > 0$ such that for all $k, n \ge 1$,*

$$\#\mathcal{M}_0^{n+k} \ge c_2 \#\mathcal{M}_0^n \cdot \#\mathcal{M}_0^k.$$

Idea of proof. Claim (b) follows from (a) and Lemma 4.2 since $\#\mathcal{M}_0^{n+k} \ge 2\delta_0^{-1}\#\mathcal{G}_{n+k}(W)$ and

$$\#\mathcal{G}_{n+k}(W) \ge \sum_{V_j \in L_n^{\delta_1}(W)} \#\mathcal{G}_k(V_j) \ge \#L_n^{\delta_1}(W) c_1 \#\mathcal{M}_0^k \ge \frac{3c_1}{4}\#\mathcal{G}_n^{\delta_1}(W)\#\mathcal{M}_0^k \ge \frac{3c_1^2}{4}\#\mathcal{M}_0^n\#\mathcal{M}_0^k.$$

The proof of (a) follows the same lines as the proof of Proposition 3.3(a), covering M with a finite number N_{δ_1} of Cantor rectangles depending on the length scale δ_1. Then

Lemma 4.3 implies at least one of these rectangles, R_{i_*}, is fully crossed (in the unstable direction) by at least $\frac{c_0\delta_1}{N_{\delta_1}}\#\mathcal{M}_{-n}^0$ "long" elements of $\mathcal{M}_{-n}^0$. Any $W \in \mathcal{W}^s$ of length at least $\delta_1/3$ crosses one rectangle R_j. Then there exists N, depending only on δ_1, such that $T^{-N}W$ properly crosses R_{i_*} in the stable direction. Thus $T^{-N-n}W$ intersects at least $\frac{c_0\delta_1}{N_{\delta_1}}\#\mathcal{M}_0^n$ elements of $\mathcal{M}_0^n$ since $\#\mathcal{M}_0^n = \#\mathcal{M}_{-n}^0$. Adjusting for N (which only affects c_1) proves (a). ■

Remark 4.5. Proposition 4.4(b) implies the exact exponential growth of $\#\mathcal{M}_0^n$,

$$e^{nh_*} \le \#\mathcal{M}_0^n \le 2c_2^{-1}e^{nh_*} \quad \text{for all } n \ge 1.$$

As in Section 3.2.2, this will be essential to controlling the peripheral spectrum of $\mathcal{L}_0$.

A second corollary of our uniform bounds is the uniform growth rate of $|T^{-n}W|$ in terms of the topological entropy h_*, i.e., there exists $C > 0$ such that, for all $W \in \mathcal{W}^s$ with $|W| \ge \delta_1/3$,

$$Ce^{nh_*} \le \left|T^{-n}W\right| \le C^{-1}e^{nh_*} \quad \text{for all } n \ge n_1.$$

This is precisely the rate of growth one sees in smooth hyperbolic systems, despite the fact that in this context h_* also counts cuts due to discontinuities.

To prove this bound, the previous remark, together with Proposition 4.4(a), gives $Ce^{nh_*} \le \#\mathcal{G}_n(W) \le C^{-1}e^{nh_*}$. But $|T^{-n}W| \le \delta_0\#\mathcal{G}_n(W)$ since curves in $\mathcal{G}_n(W)$ have length at most δ_0, proving the upper bound. Finally, the lower bound follows from Lemma 4.2 with $\varepsilon = 1/4$,

$$\left|T^{-n}W\right| = \sum_{W_i \in \mathcal{G}_n^{\delta_1}} |W_i| \ge \frac{\delta_1}{3}\#L_n^{\delta_1}(W) \ge \frac{\delta_1}{4}\#\mathcal{G}_n^{\delta_1}(W).$$

4.2. Banach spaces adapted to $t = 0$

We define $\mathcal{L}_0$ acting on functions as in (2.4) and on distributions as in (3.1).

Unfortunately, the Hölder weight $|W|^{1/p}$ in the strong stable norm from Section 3.2.1 is disastrous when $t = 0$. This is because if $W \in \mathcal{W}^s$ and $T^{-1}W$ has a single component near a tangential collision so that $|T^{-1}W| \sim |W|^{1/2}$, then, if $\psi = |W|^{-1/p}$,

$$\int_W \mathcal{L}_0 f\psi = \int_{T^{-1}W} f\psi \circ T \le \|f\|_s \frac{|T^{-1}W|^{1/p}}{|W|^{1/p}} \sim \|f\|_s|W|^{-1/2p}.$$

Taking the supremum over $W \in \mathcal{W}^s$ yields ∞ and hence $\mathcal{L}_0$ is not a bounded operator.

Yet, we cannot abandon the weight entirely due to the need to control the unmatched pieces in the Lasota–Yorke estimates, see Figure 1 and the proof of Theorem 3.7. These considerations force us to adopt a weak logarithmic weight $|\log|W||$ in the definition of $\|\cdot\|_s$, which in turn forces a logarithmic modulus of continuity in $\|\cdot\|_u$. This last change prevents a genuine contraction in the Lasota–Yorke inequality, which prevents us from proving that $\mathcal{L}_0$ is quasicompact with a spectral gap.

Nevertheless, under a sparse recurrence condition to the singular set (4.1), we show that the spectral radius of $\mathcal{L}_0$ on $\mathcal{B}$ is e^{h_*} and we obtain left and right eigenvectors of $\mathcal{L}_0$ as limit points using compactness, from which we construct the measure μ_0 with entropy h_*.

4.2.1. Sparse recurrence to singularities

In order to control the evolution of $\mathcal{L}_0^n$ in the strong norm, we shall need the following condition on the rate of recurrence to the singular set $\mathcal{S}_0$, which corresponds to tangential, or grazing, collisions. Note that our results up until now have not needed this condition.

Choose $n_0 \in \mathbb{N}$ and an angle φ_0 close to $\pi/2$. Let $s_0 \in (0,1)$ be the smallest number such that any orbit of length n_0 has at most $s_0 n_0$ collisions with $|\varphi| \geq \varphi_0$. The finite horizon condition guarantees that we can always choose n_0 and φ_0 so that $s_0 < 1$. Indeed, if there are no trajectories with three consecutive tangencies on the table (a generic condition), then one may choose n_0 and φ_0 so that $s_0 \leq \frac{2}{3}$. Our assumption is the following:

$$h_* > s_0 \log 2. \tag{4.1}$$

The $\log 2$ comes from the fact that if W is a local stable manifold that makes a nearly tangential collision under T^{-1} then $|T^{-1}W| \sim |W|^{1/2}$. Thus our assumption ensures that the growth due to tangential collisions along sufficiently long orbit segments does not exceed the exponential rate of growth given by h_*.

We remark that there is no known table for which the condition $h_* > s_0 \log 2$ fails. Indeed, since $h_* \geq h_{\mu_{\mathrm{SRB}}}$ and $h_{\mu_{\mathrm{SRB}}} = \int \log J^u T \, d\mu_{\mathrm{SRB}}$ by the Pesin entropy formula, it suffices to check that $\chi^+_{\mu_{\mathrm{SRB}}} > s_0 \log 2$, where $\chi^+_{\mu_{\mathrm{SRB}}}$ is the positive Lyapunov exponent of T with respect to μ_{SRB}, in order to conclude that (4.1) holds. Using this criterion, all examples computed numerically in [9] for a triangular lattice and [38] for a rectangular lattice satisfy (4.1). Furthermore, it is possible to prove analytically that (4.1) holds for large open sets of such billiard configurations. See [3, SECT. 2.4] for a more detailed discussion.

4.2.2. Definition of norms

Choose $\alpha, \beta, \gamma > 0$, and $p > 1$ such that

$$\beta < \alpha \leq 1/3, \quad 2^{s_0 p} < e^{h_*}, \quad \gamma < p.$$

Enlarge n_0 so that

$$\frac{1}{n_0} \log(K n_0 + 1) < h_* - p s_0 \log 2,$$

where K is from Lemma 4.1. Choose $\delta_0 > 0$ as in Section 4.1.1 so that any stable manifold of length $\leq \delta_0$ is cut into at most $K n_0 + 1$ pieces by $\mathcal{S}_{-n_0}$.

The weak norm $|\cdot|_w$ and $\mathcal{B}_w$ are defined precisely as in (3.2), so we focus on the strong norm. For $f \in C^1(M)$, define the *strong stable norm* of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\beta(W) \\ |\psi|_{\mathcal{C}^\beta(W)} \leq |\log |W||^p}} \int_W f \psi \, dm_W$$

Recalling the distance between curves $d(W_1, W_2)$ and between test functions $d_0(\psi_1, \psi_2)$ from Section 3.2.1, we define the *strong unstable norm* of f by

$$\|f\|_u = \sup_{\varepsilon \leq \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d(W_1, W_2) \leq \varepsilon}} \sup_{\substack{|\psi_i|_{\mathcal{C}^\alpha(W_i)} \leq 1 \\ d_0(\psi_1, \psi_2) = 0}} |\log \varepsilon|^\gamma \left| \int_{W_1} f \psi_1 - \int_{W_2} f \psi_2 \right|.$$

The *strong norm* of f is defined to be $\|f\|_{\mathcal{B}} = \|f\|_s + \|f\|_u$, and $\mathcal{B}$ is the completion of $C^1(M)$ in the $\|\cdot\|_{\mathcal{B}}$ norm.

4.2.3. Spectrum of $\mathcal{L}_0$ and construction of an invariant measure

Proposition 3.6 still holds true with these new norms and most importantly, the unit ball of $\mathcal{B}$ is compact in $\mathcal{B}_w$. However, due to the logarithmic modulus of continuity in the definition of $\|\cdot\|_u$, the strong unstable norm does not contract. Indeed, recalling Figure 1, when we compare $\int_{W^1} \mathcal{L}_0 f \psi_1 - \int_{W^2} \mathcal{L}_0 f \psi_2$, the matched pieces $W_i^k \in \mathcal{G}_n(W^k)$ have contracted to a distance $d(W_i^1, W_i^2) \le C\Lambda^{-n}\varepsilon$. Yet, the contraction in the norm is given by $\frac{|\log C\Lambda^{-n}\varepsilon|^\gamma}{|\log \varepsilon|^\gamma}$, and taking the supremum over $\varepsilon > 0$ yields 1. The inequalities we can prove are the following.

Proposition 4.6. *Assume $h_* > s_0 \log 2$. There exists $C > 0$ such that, for all $f \in \mathcal{B}$, $n \ge 0$,*

$$\left|\mathcal{L}^n f\right|_w \le C|f|_w \#\mathcal{M}_0^n,$$
$$\left\|\mathcal{L}^n f\right\|_s \le C\left(\sigma^n \|f\|_s + |f|_w\right)\#\mathcal{M}_0^n, \quad \textit{for some } \sigma < 1,$$
$$\left\|\mathcal{L}^n f\right\|_u \le C\left(\|f\|_u + \|f\|_s\right)\#\mathcal{M}_0^n.$$

Although the bounds of Proposition 4.6 are not sufficient to prove the quasicompactness of $\mathcal{L}_0$ on $\mathcal{B}$, they, together with Proposition 4.4, do provide good control of $\|\mathcal{L}_0^n\|_{\mathcal{B}}$.

Using Remark 4.5, we have $\|\mathcal{L}_0^n\|_{\mathcal{B}} \le Ce^{nh_*}$, for all $n \ge 1$. Moreover, our lower bounds on $\#L_n^{\delta_1}(W)$ from Lemma 4.2 and $\#\mathcal{G}_n(W)$ from Proposition 4.4 imply that

$$\left\|\mathcal{L}_0^n 1\right\|_s \ge \left|\mathcal{L}_0^n 1\right|_w \ge \int_W \mathcal{L}_0^n 1 \ge \sum_{W_i \in L_n^{\delta_1}(W)} |W_i| \ge \frac{\delta_1}{3}\frac{3}{4}\#\mathcal{G}_n^{\delta_1}(W) \ge Ce^{nh_*}. \tag{4.2}$$

These estimates imply not only that the spectral radius of $\mathcal{L}_0$ on $\mathcal{B}$ is e^{h_*}, but also that the sequence $e^{-nh_*}\mathcal{L}_0^n 1$ is uniformly bounded away from 0 and ∞ in the strong norm. We now use this fact to construct an eigenmeasure for $\mathcal{L}_0$ with eigenvalue e^{h_*}.

By the observation above, for $n \ge 1$ the sequence

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}\mathcal{L}_0^k 1 \quad \text{is uniformly bounded in } \mathcal{B}.$$

Since any ball of finite size in $\mathcal{B}$ is compact in $\mathcal{B}_w$, a subsequence converges in $\mathcal{B}_w$. Let $\nu_0 \in \mathcal{B}_w$ be a limit point of ν_n. A priori, ν_0 is only a distribution; yet, recalling (3.4), the calculation

$$\left|\nu_0(\psi)\right| \le \lim_{j\to\infty} \frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kh_*}\left|\mathcal{L}_0^k 1(\psi)\right| \le |\psi|_\infty \nu_0(1)$$

shows that, indeed, ν_0 can be extended as a bounded operator on continuous functions, i.e., ν_0 is a measure and, indeed, a nonnegative measure since the ν_n are nonnegative. A similar calculation shows that $\mathcal{L}_0\nu_0 = e^{h_*}\nu_0$.

Similarly, let $\tilde{\nu}_0 \in (\mathcal{B}_w)^*$ be a limit point of the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}(\mathcal{L}_0^*)^k(d\mu_{\text{SRB}}),$$

which is again a measure. Define the pairing

$$\mu_0(\psi) = \frac{\langle \nu_0, \psi \tilde{\nu}_0 \rangle}{\langle \nu_0, \tilde{\nu}_0 \rangle}, \quad \text{for } \psi \in C^1(M).$$

Since $\mathcal{L}_0 \nu_0 = e^{h_*} \nu_0$ and $\mathcal{L}_0^* \tilde{\nu}_0 = e^{h_*} \tilde{\nu}_0$, it is a standard calculation that $\mu_0(\psi \circ T) = \mu_0(\psi)$, i.e., μ_0 is an invariant probability measure for T. We remark that, by definition of ν_0 and $\tilde{\nu}_0$, the normalization $\langle \nu_0, \tilde{\nu}_0 \rangle$ can be computed as the average of the terms $e^{-kh_*} \int_M \mathcal{L}_0^k 1 \, d\mu_{\mathrm{SRB}}$. Thus the fact that $\langle \nu_0, \tilde{\nu}_0 \rangle \neq 0$ follows from the lower bound (4.2) (see **[3, PROOF OF PROP. 7.1]**).

4.2.4. Properties of μ_0

The key observation for proving all subsequent properties of μ_0 is that, although $\nu_0 \in \mathcal{B}_w$, it inherits stronger regularity as a limit point of the sequence $(\nu_n)_{n \in \mathbb{N}}$, which is uniformly bounded in the $\| \cdot \|_{\mathcal{B}}$-norm. In particular, the convergence of (ν_{n_j}) to ν in $\mathcal{B}_w$ implies

$$\lim_{j \to \infty} \sup_{W \in \mathcal{W}^s} \sup_{|\psi|_{C^\alpha(W)} \le 1} \left(\int_W \nu \psi \, dm_W - \int_W \nu_{n_j} \psi \, dm_W \right) = 0,$$

and, since $\|\nu_{n_j}\|_u \le C$ for some $C > 0$, we conclude that $\|\nu\|_u \le C$ as well. Similarly, $\int_W \nu \le C |\log |W||^{-p}$ from the uniform bound on $\|\nu_{n_j}\|_s$. This regularity then opens the door to a host of properties for μ_0.

(1) Hyperbolicity. For $k \in \mathbb{Z}$, $\varepsilon > 0$, letting $\mathcal{N}_\varepsilon(\mathcal{S}_k)$ denote the ε-neighborhood of $\mathcal{S}_k$ in M, the strong norm bound implies that there exists $C_k > 0$ such that

$$\nu_0\big(\mathcal{N}_\varepsilon(\mathcal{S}_k)\big) \le C_k |\log \varepsilon|^{-p} \quad \text{and} \quad \mu_0\big(\mathcal{N}_\varepsilon(\mathcal{S}_k)\big) \le C_k |\log \varepsilon|^{-p}. \tag{4.3}$$

This implies in turn that μ_0 is T-adapted, i.e., $\int_M -\log d(x, \mathcal{S}_{\pm 1}) \, d\mu_0(x) < \infty$, and that μ_0-a.e. $x \in M$ has a stable and unstable manifold of positive length. The same is true for ν_0.

(2) Ergodicity. Since μ_0 is hyperbolic, we may cover a full measure set of M with Cantor rectangles comprising intersections of stable and unstable manifolds, and study the properties of μ_0 on each rectangle. In particular, the fact that $\|\nu_0\|_u < \infty$ allows us to prove the following (but note that μ_0 itself is singular with respect to Lebesgue measure).

Lemma 4.7 (Absolute continuity of holonomy). *On each Cantor rectangle R, the holonomy map sliding along unstable manifolds in R is absolutely continuous with respect to the conditional measures of μ_0 on stable manifolds.*

Using a Hopf argument and the above lemma, we show that each Cantor rectangle R belongs to one ergodic component. Then since T is topologically mixing, we can force images of rectangles to overlap and thus conclude that (T^n, μ_0) is ergodic for all n.

(3) Mixing and Bernoulli property. The local product structure of the Cantor rectangles, together with a global argument showing that a full measure set of points on each component

of M can be connected by a network of stable and unstable manifolds, enables us to prove that (T, μ_*) is K-mixing,[5] following techniques of Pesin [48, 49].

Then adapting the approach of [26] (carried out there for μ_{SRB}), which uses all the properties we have established thus far: K-mixing, hyperbolicity, the absolute continuity of Lemma 4.7, and our bounds on $\mu_0(\mathcal{N}_\varepsilon(\mathcal{S}_{\pm 1}))$, we prove that the partition $\mathcal{M}_{-1}^1$ is *very weakly Bernoulli*. Since $\bigvee_{n=-\infty}^{\infty} T^{-n}(\mathcal{M}_{-1}^1)$ generates the full σ-algebra for T, this implies by [47] that (T, μ_0) is Bernoulli.

(4) Entropy of μ_0. For $x \in M$, define the ε-Bowen ball for T^{-n} as in (3.5). Using the fact that ν_0 scales by e^{nh_*} under a change of variables, we are able to prove:

Proposition 4.8 (Measure of Bowen balls). *There exists $C > 0$ such that, for all $x \in M$ and $n \geq 1$,*

$$\mu_0\big(B(x, n, \varepsilon)\big) \leq Ce^{-nh_*}.$$

As in Section 3.2.3, [13, MAIN THEOREM] implies that for μ_0-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_0\big(B(x, n, \varepsilon)\big) = h_{\mu_0}(T^{-1}) = h_{\mu_0}(T).$$

This, together with Proposition 4.8, implies $h_{\mu_0}(T) \geq h_*$. But $h_* \geq h_\mu(T)$ for all T-invariant probability measures as stated in Section 2.3. We conclude that $h_* = h_{\mu_0}(T)$, so μ_0 has maximal entropy.

4.2.5. Uniqueness of μ_0

Finally, we discuss the proof of uniqueness of the measure of maximal entropy from Theorem 2.3. This is essentially a modification of the classical Bowen argument, which uses a uniform lower bound on the measure of Bowen balls,

$$\forall \varepsilon > 0, \quad \exists C > 0 \text{ such that for } \mu_0\text{-a.e. } x \in M, \mu_*(B(x, n, \varepsilon)) \geq Ce^{-nh_*}$$

(see, for example, [43, SECT. 20.3]).

Unfortunately, this lower bound fails for billiards due to the rate of approach of typical points to the singularity set. We can prove, rather, that $\forall \eta > 0$ and μ_0-a.e. $x \in M$,

$$\exists C = C(\eta, x) > 0 \quad \text{such that } \mu_0(B(x, n, \varepsilon)) \geq Ce^{-n(h_* + \eta)}. \tag{4.4}$$

But even this arbitrarily small error in the exponent is not sufficient for the Bowen argument. Instead, we prove a version of the lower bound that "most" $x \in M$ "often" belong to an element of $\mathcal{M}_0^j$ satisfying good lower bounds.

To make this precise, let $\bar{n} \in \mathbb{N}$ be such that $(K\bar{n} + 1)^{1/\bar{n}} < e^{h_*/2}$. Then using Lemma 4.2, choose $\delta_2 > 0$ such that if $A \in \mathcal{M}_{-k}^n$ satisfies

$$\max\big\{\operatorname{diam}^u(A), \operatorname{diam}^s(A)\big\} \leq \delta_2,$$

5 If $\mathcal{A}$ denotes the Borel sigma-algebra on M, then K-mixing means that there exists a sub-sigma algebra $K \subset \mathcal{A}$ such that (1) $K \subset TK$; (2) $\bigvee_{n=0}^{\infty} T^n K = \mathcal{A}$; (3) $\bigcap_{n=0}^{\infty} T^{-n} K = \{X, \emptyset\}$.

then $A \setminus \mathcal{S}_{\pm\bar{n}}$ consist of at most $K\bar{n}+1$ connected components. Define

$$Sh_0^{2n} := \{A \in \mathcal{M}_0^{2n} : \forall j, 0 \le j \le n/2, T^j A \subset E \in \mathcal{M}_0^{2n-j} \text{ such that } \operatorname{diam}^s(E) < \delta_2\},$$

with a similar definition for Sh_{-2n}^0 with $\operatorname{diam}^u(E)$ replacing $\operatorname{diam}^s(E)$. These are the "persistently short" elements of $\mathcal{M}_0^{2n}$ and $\mathcal{M}_{-2n}^0$, respectively, which have not belonged to a "long" element within the past $n/2$ iterates.

The next lemma demonstrates that persistently short pieces make up a small proportion of $\mathcal{M}_0^{2n}$ and that long elements that also have long images satisfy strong lower bounds.

Lemma 4.9. (a) *Let $B_{2n} = \{A \in \mathcal{M}_0^{2n} :$ either $A \in Sh_0^{2n}$ or $T^{2n}A \in Sh_{-2n}^0\}$. There exists $C > 0$ such that, for all $n \ge 1$, $\#B_{2n} \le Ce^{7nh_*/4}$.*

(b) *For all $k \ge 1$, if $E \in \mathcal{M}_0^k$ with $\operatorname{diam}^s(E) \ge \delta_2$ and $\operatorname{diam}^u(T^kE) \ge \delta_2$,*

$$\text{then} \quad \mu_0(E) \ge C_{\delta_2} e^{-kh_*}, \quad \text{for some } C_{\delta_2} > 0.$$

Some comments on the proof. Claim (a) follows by iterating the complexity bound given by Lemma 4.2, using the fact that, by the choice of $\bar{n}$ and δ_2, persistently short pieces cannot grow at a rate faster than $(K\bar{n}+1)^{n/(2\bar{n})} < e^{nh_*/4}$ over the most recent $n/2$ iterates.

Claim (b) rests on the fact that if E is long in the stable direction and T^kE is long in the unstable direction, then both E and T^kE cross Cantor rectangles of a fixed size, depending on δ_2. Then the lower bound (4.2) is used to derive (b). ■

The importance of Lemma 4.9 lies in the fact that if $A \in G_{2n} := \mathcal{M}_0^{2n} \setminus B_{2n}$, then there exists $j, k \le n/2$ such that $T^jA \subset E \in \mathcal{M}_0^{2n-j-k}$ and E satisfies Lemma 4.9(b), i.e., $\mu_0(E) \ge C_{\delta_2} e^{-(j+k)h_*}$. Thus, apart from a set of "bad" elements B_{2n} whose size is relatively small, most elements of $\mathcal{M}_0^{2n}$ belong to the "good" set G_{2n} and are contained in a larger set that has good lower bounds. This, together with a time shift to group elements of $\mathcal{M}_0^{2n}$ according to their good counterparts in $\mathcal{M}_0^{2n-j-k}$, is sufficient to adapt the Bowen argument for uniqueness. The reader interested in more details is referred to **[3, SECT. 7]**.

5. OPEN QUESTIONS

We conclude by formulating several open questions relating to the family of geometric potentials we have discussed.

(1) Is $\mu_0 = \mu_{\mathrm{SRB}}$, or, more generally, is $\mu_s = \mu_t$ for $s \ne t$? If there exist $s, t > 0$, $s \ne t$, such that $\mu_s = \mu_t$, then Theorem 2.8 implies that $P(t)$ is affine on $(0, t_*)$, so that μ_{SRB} would be the equilibrium state for all $t \in (0, t_*)$, and assuming the sparse recurrence condition (4.1), $\mu_{\mathrm{SRB}} = \mu_0$ as well.

This seems highly unlikely. Indeed, suppose that z is a periodic orbit with no grazing collisions and let χ_z^+ be its positive Lyapunov exponent. Then our estimates on Bowen balls such as Proposition 4.8 and (4.4), in addition to analogous ones for μ_{SRB}, imply that $h_* = \chi_z^+$ **[3, PROP. 7.13]**. Thus if we can find two periodic orbits with different Lyapunov exponents, we can conclude that $\mu_0 \ne \mu_{\mathrm{SRB}}$, and in turn $\mu_s \ne \mu_t$ for all $s \ne t$. There are no known Sinai billiard tables in which all periodic orbits have the same Lyapunov exponent, yet it is not proved that this cannot happen. See [35, SECT. 4.4] for a related class of models (a type of open billiard) for which such anomalous behavior has been effectively ruled out.

(2) Can one establish a rate of mixing for μ_0? While exponential mixing for μ_t, $t \in (0, t_*)$, follows from the spectral gap for $\mathcal{L}_t$, no such gap is available for $\mathcal{L}_0$. The question arises whether this is a consequence of the technique or whether there is a genuine failure of exponential mixing at $t = 0$. The fact that the pressure $P(t)$ is finite for $t \geq 0$ and infinite for $t < 0$ whenever there is a periodic orbit with a grazing collision suggests that there is, indeed, a phase transition at $t = 0$, so a loss of exponential mixing would not be out of place. On the other hand, for many expanding systems, the measure of maximal entropy has a faster rate of mixing than the SRB measure, not a slower one.

(3) What are other limit theorems and properties of μ_0? Once a rate of mixing has been established, other limits theorems might follow, for example, a dynamical Central Limit Theorem, which generally requires a summable rate of decay of correlations. Other limit theorems might include invariance principles or large deviation estimates. These are all available for μ_t, $t > 0$, by spectral techniques (see [39] or [32, SECT. 6]), but not for μ_0 at this time.

(4) Can one find a finite horizon Sinai billiard table such that the sparse recurrence condition fails? In other words, can one find a table with $h_* \leq s_0 \log 2$? If so, does a measure of maximal entropy still exist and is it T-adapted, i.e., does it satisfy bounds of the form (4.3)?

(5) Does $\mu_t \to \mu_0$ as $t \to 0$? For $t \in (0, t_*)$, continuity of μ_t and differentiability of $P(t)$ follow from perturbation theory. Assuming the sparse recurrence condition (4.1), Baladi and Demers [4, PROP. 5.5] prove that $\lim_{t \downarrow 0} P(t) = P(0) = h_*$, yet the question of whether the equilibrium states converge remains open.

(6) Is $P(t)$ analytic for all $t > 0$ or is there a phase transition at some $t_\star > 1$? If so, how does $t_\star$ depend on the configuration of scatterers? It is clear from the definition of t_* that it is not an optimal condition for most billiard tables since the hyperbolicity constant $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$ is a lower bound, which in general may not be attained along most or even all orbits.

A more refined attempt would be to define $\chi_{\min} \geq \Lambda$ to be the minimal positive Lyapunov exponent over all periodic orbits. Then we could define

$$t_\star = \sup\{t > 0 : P(t) > -t\chi_{\min}\},$$

and try to show that the spectral techniques described here go through for all $t < t_\star$ (note that $t_\star \geq t_*$). This should involve in particular working with higher iterates of T and proving a version of (2.1) with Λ replaced by $\chi_{\min}$. (In some works on the thermodynamic formalism, the value of $t_\star$ is called the freezing point of the geometric family, in analogy with $1/t$ being thought of as temperature.)

Some inspiration for $\chi_{\min}$ being the correct quantity to use can be found in finite-horizon billiards in a triangular lattice. All scatterers on such tables are circles of equal radius R, thus the positive Lyapunov exponent of a period 2 orbit between two scatterers at minimal distance from one another is precisely $\Lambda = 1 + \frac{2\tau_{\min}}{R}$. In this case, $\Lambda = \chi_{\min}$ and so $t_* = t_\star$. If δ is the atomic invariant measure supported on this period 2 orbit, then $P_\delta(t) = -t \log \Lambda$, so certainly $P(t) \geq -t \log \Lambda$ for all $t > 0$. Yet, it is not known even in this special case whether in fact $P(t) = -t \log \Lambda$ at some $t = t_\star$, or whether $t_\star = \infty$.

ACKNOWLEDGMENTS

I would like to thank Viviane Baladi, with whom the papers [3, 4] surveyed in this note were written. I would also like to thank Carlangelo Liverani, who has taught me much about transfer operators and the construction of Banach spaces, as well as Hong-Kun Zhang and Péter Bálint, who have repeatedly explained to me many things about billiards.

FUNDING

This work was partially supported by U. S. National Science Foundation grants DMS 1800321 and DMS 2055070.

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