

GEOMETRIC METHODS IN HOLOMORPHIC DYNAMICS

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ABSTRACT

In this text we review a selection of contemporary research themes in holomorphic dynamics. The main topics that will be discussed are geometric (laminar and woven) currents and their applications, bifurcation theory in one and several variables, and the problem of wandering Fatou components.

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Holomorphic dynamics was once part of classical complex analysis, but since its rebirth in the 1980s it keeps enlarging its scope, integrating new ideas, and developing new interactions. Some main tendencies of contemporary holomorphic dynamics are the convergence between its one- and higher-dimensional aspects and its ever deeper interconnection with algebraic and arithmetic dynamics. As a consequence, there is an endless diversification of the available mathematical techniques. Besides the classical methods from dynamics and complex analysis, its modern toolbox now comprises sophisticated tools and ideas imported from complex geometry, pluripotential theory (and its latest advances for currents of higher bidegree), algebraic geometry and commutative algebra, non-Archimedean analysis and geometry, arithmetic geometry (in particular, arithmetic equidistribution theory), Teichmüller theory, geometric group theory, etc. Conversely, each of these domains benefits from its interaction with holomorphic dynamics, by gaining new problems and examples. Many (though not all!) of these connections were reported in recent ICMs [21,25,34,40,71,77]. Our purpose here is to present a few contemporary research themes whose common thread—if one were to find one—is an emphasis on “soft” geometric techniques, such as the basic geometry of analytic subsets in \mathbb{C}^n . These represent only a tiny piece of the domain, reflecting, of course, the author’s own taste and research interests. The main topics that will be discussed are geometric currents, bifurcation theory, and the problem of wandering Fatou components. The reader will soon notice that these three subjects are largely interrelated. Many open problems have also been included, as a motivation for future investigations.

Let us describe in more detail the contents of this paper. Section 1 is a short survey on positive closed currents with “geometric structure”. The use of geometric currents in holomorphic dynamics was pioneered by Bedford, Lyubich, and Smillie in their seminal work [9] on complex Hénon maps. Since then they have turned into a very versatile tool, with many applications. Here we intend to give the flavor of a few specific results and how they are used in dynamical problems, so this part of the paper will be a bit more technical than the remaining sections.

Holomorphic dynamics is equally about the dynamics of a holomorphic map f and about the evolution of this dynamical behavior when f depends on certain parameters. The basic stability/bifurcation theory of rational maps in one variable was designed by Mañé, Sad, Sullivan, and Lyubich [69,70,73] in the 1980s, who showed that one-dimensional rational maps are generically structurally stable, using surprisingly elementary arguments. For the quadratic family $z^2 + c$, $c \in \mathbb{C}$, the bifurcation locus is the celebrated Mandelbrot set, whose intricate structure was thoroughly studied since then, using a variety of combinatorial and geometric methods. This research area was profoundly renewed in the 2000s by the systematic investigation of higher-dimensional phenomena, and in particular with the introduction of bifurcation currents by DeMarco [32]. The bifurcation theory of holomorphic dynamical systems is nowadays a very active research domain, and a meeting point between the communities of one and several variable dynamicists. We relate this continuing story in Section 2.

Finally, one recent breakthrough is the construction of wandering Fatou components in higher-dimensional polynomial dynamics, which at the same time solves an old problem and raises many questions. We review these recent developments in Section 3.

Let us conclude this introduction with a little notice. Some important theorems will be mentioned only in passing, while others are isolated within numbered environments: this is meant to keep the reading flow, not to reflect a hierarchy of importance. Likewise, the list of references is already quite long, but not exhaustive, and we apologize in advance for any serious omission.

1. GEOMETRIC CURRENTS

1.1. Definitions

This part assumes some familiarity with positive currents and pluripotential theory (see, e.g., Demailly [31] for basics). All the definitions here are local, so we work in some bounded open set $\Omega \subset \mathbb{C}^k$. Let T be a positive closed current of bidimension (p, p) in Ω . Following Bedford, Lyubich, and Smillie [9], we say that T is *locally uniformly laminar* if there exists a lamination by complex submanifolds of dimension p embedded in Ω such that the restriction of T to any flow box B of the lamination is of the form

$$T|_B = \int_{\tau} [\Delta_t] d\nu(t). \tag{1.1}$$

Here τ is a global transversal in the flow box B , the Δ_t are the plaques of the lamination in the flow box, and ν is a positive measure on τ . The word “uniform” here refers to the local uniformity of the geometry of the plaques Δ_t . We say that T is *laminar* if there exists a sequence of open subsets Ω_k , together with a sequence of currents T_k , locally uniformly laminar in Ω_k , such that T_k increases to T . The Ω_k should be thought of as a union of many small polydisks, whose complement has a small mass. The key word in the definition is “increases.” Intuitively, this definition should be understood as follows: T_k represents all the disks contained in T of some given size (say 2^{-k}); then, to T_k we add $T_{k+1} - T_k$ which is made of disks of size $2^{-(k+1)}$ (which may have nonempty boundary in Ω_k , but form a lamination in $\Omega_{k+1} \subset \Omega_k$), and so on. The sequence T_k is not canonical, and has to be understood as the choice of a “representation” of T as a laminar current. From this we can deduce another representation of T as an integral over an abstract family of compatible holomorphic disks, namely

$$T = \int_{\mathcal{A}} [D_{\alpha}] d\mu(\alpha). \tag{1.2}$$

Here *compatible* means that two disks can only intersect along some relatively open subset, but there is no further restriction on the geometry of the D_{α} . Even if this definition is rather restrictive, it can lead to pathological examples, and for dynamical applications we will have to constrain it further (see the notion of “strongly approximable” current below).

It was observed by Dinh [39] that in many situations it is more natural to let the disks admit nontrivial intersections. One then defines *uniformly woven* currents by replacing “lamination” by “web” in (1.1), where a web is locally given by a family of disks of

dimension p with uniformly bounded volume or, more generally, a family of holomorphic chains of dimension p with uniformly bounded volume (any such family is precompact for the Hausdorff topology, so it makes sense to define a measure on a set of such disks). Then, *woven* currents are defined from uniformly woven ones as in the laminar case. A difference between laminar and woven currents is that in the woven case the measures in (1.1) and (1.2) are not determined by T (e.g., the standard Kähler form in \mathbb{C}^2 admits several representations as a uniformly woven current), so a woven current has to be thought of as “marked” by such a measure μ . It is not completely obvious to show that not every positive closed current is woven; we leave this as an exercise to the reader!

There is no unified reference for the basic properties of laminar and woven currents. Besides [9] and [39], the information in this paragraph was extracted from various papers, notably by De Thélin and the author [28, 30, 37, 43, 44, 46]. In the following we use the word *geometric* as a synonym of “laminar or woven.”

1.2. Construction and approximation

Positive closed currents often appear as limits of sequences of normalized currents of integration. Furthermore, by a classical theorem of Lelong, any positive closed current of bidegree $(1, 1)$ is locally of this form. In this section we explain how, under appropriate hypotheses, a geometric structure can be extracted from such an approximation.

Still working locally in some open set $\Omega \subset \mathbb{C}^k$, endowed with its standard Hermitian structure, we say that a submanifold V of dimension p in Ω has size r at $x \in V$ if it contains a graph over a ball of radius r of its tangent space $T_x V$, relative to the orthogonal projection to $T_x V$, with slope (i.e., the norm of the derivative of the graphing map) bounded by 1. In particular, V has no boundary in $B(x, cr)$ for some constant c depending only on p and k . This notion of size makes sense in any compact complex manifold, up to uniform constants, by choosing a finite covering by coordinate charts and a Hermitian metric. Note that we may relax this definition by allowing V to be an analytic set: then V can have several irreducible components at x , some of which being of size r .

If V is any submanifold (or subvariety) of Ω , possibly with boundary, and $r > 0$, we denote by V^r the set of $x \in V$ such that V has size r at x . In this way we get a tautological decomposition, $V = V^r \cup (V \setminus V^r)$, which is reminiscent of the thin–thick decomposition of hyperbolic manifolds.

Assume now that V_n is a sequence of p -dimensional subvarieties of volume v_n , such that $v_n^{-1}[V_n]$ converges to a positive closed current T . If $\text{Vol}(V_n^r) \geq v_n(1 - \varepsilon(r))$ where ε is a function independent of n and such that $\varepsilon(r) \rightarrow 0$ as $r \rightarrow 0$, then one may extract a subsequence so that $v_n^{-1}[V_n^r]$ converges to a geometric current $T^r \leq T$ with the mass estimate $\mathbf{M}(T - T^r) \leq \varepsilon(r)$. This endows T with a geometric structure: if $p \leq k - 2$, we obtain a woven current and, if $p = k - 1$, this current is laminar. Indeed, if $p = k - 1$, by the persistence of proper intersections, the limiting graphs cannot intersect nontrivially. (Note that when $p \leq k - 2$, intersections can appear at the limit even if the V_n are submanifolds. Conversely, if in codimension 1 we allow the V_n to admit self-intersections, we obtain woven currents also in this case.)

A technically convenient option is to further assume that the disks constituting V_n^r are submanifolds (without boundary) in a subdivision of Ω by cubes of size cr (for some constant $c > 0$). This is consistent with the manner in which the V_n^r are constructed in practice, and the resulting definition is equivalent (see [46]). In this way the limiting currents T^r are uniformly geometric in the cubes of this subdivision.

There are several easily checkable geometric and/or topological criteria ensuring this condition, which sometimes give an explicit bound on $\varepsilon(r)$:

- If $\psi : \mathbb{C} \rightarrow X$ is an entire curve in a projective manifold, then by Ahlfors' theory of covering surfaces, for well-chosen sequences $R_n \rightarrow \infty$, $V_n := \psi(D(0, R_n))$ satisfies $v_n^{-1}[\partial V_n] \rightarrow 0$ and $\text{Vol}(V_n^r) \geq v_n(1 - \varepsilon(r))$ for $\varepsilon(r) = O(r^2)$. Thus the cluster values of $v_n^{-1}[V_n]$ are closed woven currents; if, in addition, ψ is injective and $\dim(X) = 2$, then they are laminar (Bedford–Lyubich–Smillie [9], Cantat [24]).
- If V_n is a sequence of algebraic curves in a projective surface whose geometric genus is $O(v_n)$, then $\text{Vol}(V_n^r) \geq v_n(1 - \varepsilon(r))$ for $\varepsilon(r) = O(r^2)$, therefore the limiting currents of $v_n^{-1}[V_n]$ are woven; under a mild additional condition on the singularities of V_n , they are laminar (Dujardin [43]).
- If $\iota_n : \mathbb{P}^p \rightarrow X$ is a sequence of holomorphic mappings of generic degree 1 to a projective manifold X of dimension $k > p$ and $V_n = \iota_n(\mathbb{P}^p)$, then the limiting currents of $v_n^{-1}[V_n]$ are woven (Dinh [39]). In addition, $\varepsilon(r) = O(r^2)$ [46].
- If V_n is a sequence of smooth curves in the unit ball in \mathbb{C}^2 , whose genus is $O(v_n)$, then the limiting currents of $v_n^{-1}[V_n]$ are laminar (De Thélin [28]). A version of this result in arbitrary dimension is given by De Thélin in [30].

In all these papers, the geometric structure is obtained by projecting V_n in several directions and keeping only from V_n the graphs over these directions with bounded diameter or volume. The bound $\varepsilon(r) = O(r^2)$ plays an important role in applications as we shall see below.

1.3. Geometric intersection

The main interest of geometric currents is the possibility of a geometric interpretation of their wedge products. This technique was introduced in [9], and it was systematized and generalized in several subsequent works. Such results are so far essentially available in dimension 2; again since the problem is local, we work in some open set $\Omega \subset \mathbb{C}^2$, say a ball. If T_1 and T_2 are closed positive $(1, 1)$ currents in Ω , we say that the wedge product $T_1 \wedge T_2$ is well defined if $u_1 \in L_{\text{loc}}^1(T_2)$, where u_i is a local potential of T_i , in which case we set $T_1 \wedge T_2 = dd^c(u_1 T_2)$. This condition and the resulting wedge product are actually symmetric in T_1 and T_2 . We also say that such a current is *diffuse* if it gives no mass to curves.

For uniformly laminar and woven currents, geometric intersection is easy and basically follows from Fubini's theorem. Indeed, assume that T_1 and T_2 are uniformly geometric $(1, 1)$ -currents in Ω , which locally in Ω admit the representation $T_i = \int [\Delta_i^j] dv_j(t)$. Then,

if the wedge product $T_1 \wedge T_2$ is well defined, locally we have that

$$T_1 \wedge T_2 = \int [\Delta_t^1 \cap \Delta_s^2] dv_1(t) dv_2(s), \tag{1.3}$$

where $[\Delta_t^1 \cap \Delta_s^2]$ is the sum of point masses at isolated intersection points, counting multiplicities (see [38, 44]). In addition, if T_1 and T_2 are laminar and diffuse, nontransverse intersections do not contribute to the integral, so we can restrict to transverse intersections. Note the intermediate “semigeometric intersection” result

$$T_1 \wedge T_2 = \int ([\Delta_t^1] \wedge T_2) dv_1(t), \tag{1.4}$$

which makes sense for an arbitrary positive closed current T_2 .

Now assume that T is a geometric positive closed current in $\Omega \subset \mathbb{C}^2$ and S is an arbitrary positive closed current in Ω such that the wedge product $S \wedge T$ is well defined. We say that $T \wedge S$ is *semigeometric* if there is a representation $T = \lim_{r \rightarrow 0} T^r$ as an increasing limit of uniformly geometric currents, such that $T^r \wedge S$ increases to $T \wedge S$ as $r \rightarrow 0$. Thanks to (1.4), $T^r \wedge S$ admits a geometric interpretation. If now S itself is a geometric current, we say that the wedge product $T \wedge S$ is *geometric* if there are representations $T^r \nearrow T$ and $S^r \nearrow S$ such that $T^r \wedge S^r$ (which has a geometric interpretation by (1.3)) increases to $T \wedge S$.

We say that a geometric current is *strongly approximable* if there is a representation $T^r \nearrow T$ where T^r is uniformly geometric in a subdivision Ω^r of Ω into cubes of size r , and $\varepsilon(r) = \mathbf{M}(T - T^r) = O(r^2)$. As we have seen in Section 1.2, this estimate is commonly satisfied in practice. (Technically, some freedom on the choice of Ω^r is also necessary, but we do not dwell on this point.) The sharpest version of the geometric intersection theorem for geometric currents in dimension 2 is the following:

Theorem 1.1 (Dujardin [38, 44, 45]). *Let S and T be closed positive (1, 1) currents in $\Omega \subset \mathbb{C}^2$, such that the wedge product $T \wedge S$ is well defined. Assume that T is a strongly approximable geometric current. Then, if S has locally bounded potentials, or if $T \wedge S$ gives no mass to pluripolar sets, then $T \wedge S$ is semigeometric.*

A consequence of this theorem, which is often as useful as the result itself, is that if T was obtained as the limit of $v_n^{-1}[V_n]$ as in Section 1.2, then $v_n^{-1}[V_n] \wedge S$ is close to $T \wedge S$ for small r and large n .

Applying Theorem 1.1 to $T \wedge S$ and $S \wedge T$, we get:

Corollary 1.2. *If in Theorem 1.1 both S and T are strongly approximable geometric currents and $T \wedge S$ gives no mass to pluripolar sets, then $T \wedge S$ is geometric.*

The main open problem at this stage is the extension of these results to higher dimensions.

Question 1.3. Is there a version of Theorem 1.1 for geometric currents of arbitrary codimension?

While the case of uniformly geometric currents and the case where T is of bidimension $(1, 1)$ follow without serious difficulties (see [46] and [47] for details), the general case remains a challenge so far. The crucial mass estimate $\mathbf{M}(T - T^r) = O(r^2)$ is known to hold in some significant cases (see [46]), but it does not appear to be sufficient to conclude for currents of arbitrary bidimension.

1.4. Dynamical applications

The first application of laminar currents by Bedford, Lyubich, and Smillie [9] was to prove that certain intersections are nonempty. A typical example is the following: assume that we are given an entire curve $\psi : \mathbb{C} \rightarrow X$ in some projective manifold, and let T be a closed current obtained from ψ by Ahlfors' construction. Let S be a current of bidegree $(1, 1)$ with bounded potentials. If we know that $\int T \wedge S > 0$ (for instance, for cohomological reasons), then by Theorem 1.1, this intersection is semigeometric, therefore $S|_{\psi(D(0, R_n))}$ is nonzero for large n . (A version of this result which does not apply to laminarity was proved by Dinh and Sibony [42].) This fact (as well as some variants) plays an important role in the dynamics of automorphisms and birational maps on complex surfaces, where it is used as a tool to create intersections between stable and unstable manifolds. This is used in [9] to establish that any saddle point belongs to the support of the maximal entropy measure; this technique also appears in the work of Cantat, Favre, Lyubich, and the author [24, 26, 52, 53], among others. Note also that the failure of Theorem 1.1 for unbounded potentials can be viewed as the main reason why the uniqueness of the measure of maximal entropy for general birational maps of surfaces remains an unsolved problem.

Another use of geometric intersection, which was initiated in [45], concerns the dynamical analysis of wedge products of dynamically defined currents. Indeed, suppose that f is a self-map of some complex manifold X , and $f^n(L)$ is a sequence of iterated curves such that $d^{-n} f^n(L)$ converges to a geometric current T , with a control of the asymptotic geometry of $f^n(L)$ as in Section 1.2. Assume also that S is some invariant current of bidegree $(1, 1)$: $f^* S = dS$ and that $T \wedge S$ is a semigeometric intersection. Then for large n , the action of f^k on the bounded geometry part of $d^{-n}[f^n(L)] \wedge S$ is a good approximation of the action of f^k on $T \wedge S$, and its expansion properties “in the direction of T ” can be analyzed geometrically by “soft” methods, such as counting disjoint disks of size r and length–area estimates (see below Theorem 2.4 for a worked out example). This idea was used in various contexts by De Thélin and others [29, 36, 37, 45, 46].

1.5. Foliations

Foliated Ahlfors currents play an important role in the work of Brunella and McQuil-lan on singular holomorphic foliations (see, e.g., [20]). Geometric intersection has been applied in foliation theory to prove the vanishing of certain self-intersections. For a positive current directed by a holomorphic foliation on a compact Kähler surface, this vanishing can in turn be used to infer dynamical properties of the foliation such as the nonexistence of invariant transverse measures (for closed currents) or the uniqueness of harmonic measures (for dd^c -closed currents), according to a Hodge-theoretic formalism for dd^c -closed

currents devised by Fornæss and Sibony [58]. Proving that the self-intersection of harmonic currents directed by holomorphic foliations vanishes is a very difficult problem in the presence of singularities. On \mathbb{P}^2 this can be treated by regularizing with global automorphisms, the general case makes use of the theory of densities of Dinh and Sibony (see [41]). Here we want to mention a more elementary-looking problem:

Question 1.4. Does there exist a diffuse (closed) uniformly laminar current on \mathbb{P}^2 ?

The expected answer to the question is “no,” since it is generally expected that there does not exist a Riemann surface lamination embedded in \mathbb{P}^2 . The above question is supposed to be the “easy case” of this deep conjecture (since it deals with laminations with transverse measures), and it admits a straightforward approach: if T is such a current, then $T \wedge T = 0$ because of the laminar structure, which is impossible on \mathbb{P}^2 . This approach works well as soon as $T \wedge T$ is well defined in the sense of pluripotential theory (but it does not for a curve!), or when the holonomy of the induced lamination is Lipschitz [58]. But in general the holonomy of a Riemann surface lamination in \mathbb{C}^2 (that is, a holomorphic motion) is less regular and, surprisingly enough, the problem is still open so far. (See Kaufmann [66] for a discussion of the higher-dimensional case.)

2. BIFURCATION THEORY IN ONE AND SEVERAL DIMENSIONS

Let $(f_\lambda)_{\lambda \in \Lambda}$ be a family of rational maps on \mathbb{P}^1 of degree d , holomorphically parameterized by some complex manifold Λ . Then the well-known Fatou–Julia decomposition of the phase space is mirrored by a stability–bifurcation dichotomy of the parameter space. The proper definition of stability in this context was found simultaneously by Mañé–Sad–Sullivan and Lyubich [69, 70, 73]: the family $(f_\lambda)_{\lambda \in \Lambda}$ is *J-stable* over some domain $\Omega \subset \Lambda$ if one of the following equivalent conditions holds over Ω :

- (i) the periodic points of (f_λ) do not collide or, equivalently, the nature (attracting, repelling, indifferent) of each periodic point remains the same in the family;
- (ii) the Julia set $\lambda \mapsto J_\lambda$ moves continuously for the Hausdorff topology;
- (iii) for any two parameters λ, λ' in Ω , $f_\lambda|_{J_\lambda}$ is topologically conjugate to $f_{\lambda'}|_{J_{\lambda'}}$;
- (iv) the orbits of the critical points f_λ do not bifurcate.

The equivalence between these properties relies on the notion of *holomorphic motion* (also known as *holomorphic families of injections*) of a subset of the Riemann sphere, and the simple, yet powerful idea of automatic extension of a holomorphic motion to its closure (the “ λ -lemma”). Condition (iv), together with the finiteness of the critical set, easily implies that in any such parameterized family (f_λ) , *the stability locus is open and dense in Λ* . In other words, *one-dimensional polynomial and rational maps are generically stable*.

For the emblematic family $f_\lambda(z) = z^2 + \lambda$ of quadratic polynomials, the bifurcation locus is the boundary of the Mandelbrot set M (connectivity locus). Even if its interior is

empty, ∂M is still quite large, as shown by the following famous result of Shishikura [88]: ∂M has Hausdorff dimension 2. This property was extended to arbitrary families of rational maps by Tan Lei and McMullen [67, 75]. The basic technical tool underlying Shishikura’s theorem is the phenomenon of *parabolic implosion*, which will also play an important role below. Note that it is still unknown whether ∂M has zero or positive Lebesgue measure.

This research area was renewed in the last 20 years as the result of several tendencies: (1) the use of positive closed currents, and (2) the move towards higher dimensions (both in dynamical and parameter spaces). In the next few pages, we review some of these developments; in particular, we will see how these influential one-dimensional results translate to new settings. Lack of space prevents us from giving a complete treatment, and some important results will barely be mentioned. Also, we do not discuss the profound connection with arithmetic dynamics, for which the reader is referred, e.g., to [34], or bifurcations of Kleinian groups (see [35, 48]).

2.1. Bifurcation currents in one-dimensional dynamics

Let as above $(f_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of rational maps of degree d . The following addition to the list of equivalent conditions to stability was found by DeMarco [33]:

- (v) the Lyapunov exponent of the unique measure of maximal entropy $\chi(\mu_{f_\lambda})$ is a pluriharmonic function of λ .

The *bifurcation current* is then defined by $T_{\text{bif}} := dd^c_\lambda \chi(\mu_{f_\lambda})$. For the family of quadratic polynomials, $T_{\text{bif}} (= \mu_{\text{bif}}$, see below) is the harmonic measure of the Mandelbrot set.

The original definition of the bifurcation current in [32] can be interpreted geometrically as follows (see [51]). Consider the fibered dynamical system in $\Lambda \times \mathbb{P}^1$ defined by $\hat{f} : (\lambda, z) \mapsto (\lambda, f_\lambda(z))$. It admits a natural invariant current \hat{T} of bidegree $(1, 1)$, satisfying $\hat{f}^* \hat{T} = d \hat{T}$, whose restriction to a generic vertical line $\{\lambda\} \times \mathbb{P}^1$ is the maximal entropy measure μ_{f_λ} . Now, take a holomorphically moving (or “marked”) point $\lambda \mapsto a(\lambda)$ in \mathbb{P}^1 , and denote by Γ_a its graph in $\Lambda \times \mathbb{P}^1$. If $\pi : \Lambda \times \mathbb{P}^1 \rightarrow \Lambda$ is the natural projection, we obtain a current in Λ associated to a by slicing \hat{T} by Γ_a and projecting down to Λ : $T_a := \pi_*(\hat{T} \wedge [\Gamma_a])$. If in a holomorphic family (f_λ) , the critical points are marked by holomorphic functions $\lambda \mapsto c_i(\lambda)$ (this is always possible up to replacing Λ by some branched cover), we thus obtain the corresponding bifurcation currents T_{c_i} . It turns out that $T_{\text{bif}} = \sum T_{c_i}$: this follows from a variant of the Manning–Przytycki formula for the Lyapunov exponent $\chi(\mu_{f_\lambda})$, which in the case of polynomials is written as

$$\chi(\mu_f) = \log d + \sum_i G_f(c_i),$$

where G_f is the dynamical Green function (which satisfies $dd^c G_f = \mu_f$).

Bifurcation currents have turned into a fundamental tool for exploring higher dimensional issues in parameter spaces. Here is a sample problem: consider a critically marked family $(f_\lambda, c_i(\lambda))$ and suppose that for some parameter $\lambda_0 \in \Lambda$, the critical point $c_1(\lambda)$ bifurcates at $\lambda = \lambda_0$. Then a simple application of Montel’s theorem shows that there is

a sequence of parameters $\lambda_n \rightarrow \lambda_0$ such that, for $\lambda = \lambda_n$, c_1 is preperiodic. Now assume that several (say all) critical points bifurcate at λ_0 : is it then possible to approximate λ_0 by parameters such that the corresponding critical points are preperiodic? Of course, in this question one has to discard a few “trivial” obstructions, e.g., when $\dim(\Lambda)$ is too small, so that there are not enough degrees of freedom to hope for an independent behavior of the critical points. Still after excluding these counterexamples, the answer to this problem is “no” (see [51, EXAMPLE 6.13]), the fundamental reason for this being the failure of Montel’s theorem in higher dimension. Using currents is a known way of circumventing this problem in higher-dimensional dynamics, and, as a matter of fact, the following theorem holds:

Theorem 2.1 (Bassanelli–Berteloot [8], Dujardin–Favre [51]). *Let $(f_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of rational maps of degree $d \geq 2$. Then for every $k \leq \dim(\Lambda)$,*

$$\text{Supp}(T_{\text{bif}}^k) \subset \overline{\{\lambda, f_\lambda \text{ admits } k \text{ periodic critical points}\}}. \quad (2.1)$$

(This result was actually not stated explicitly in [8, 51], see [48] for this formulation. The converse inclusion is studied below.)

When Λ is the moduli space \mathcal{P}_d of polynomials of degree d with marked critical points (which is a finite quotient of \mathbb{C}^{d-1}) or the moduli space \mathcal{M}_d of rational maps of degree d with marked critical points (which is of dimension $2d - 2$), we define the *bifurcation measure* μ_{bif} to be the maximal exterior power of T_{bif} , that is, $\mu_{\text{bif}} = T_{\text{bif}}^{d-1}$ or $\mu_{\text{bif}} = T_{\text{bif}}^{2d-2}$, respectively. The following neat dynamical characterization of $\text{Supp}(\mu_{\text{bif}})$ can be obtained:

Theorem 2.2 (Dujardin–Favre [51], Buff–Epstein [22]). *For $\Lambda = \mathcal{P}_d$ or \mathcal{M}_d , the support of μ_{bif} is the closure of (non-Lattès) strictly postcritically finite parameters, that is, parameters for which all critical points are preperiodic to a repelling cycle.*

A version of this result for intermediate powers of T_{bif} was obtained in [47], which explains to what extent the converse inclusion in (2.1) holds.

Sketch of proof. The most delicate point is to show that any non-Lattès postcritically finite parameter λ_0 belongs to $\text{Supp}(\mu_{\text{bif}})$. To fix the ideas, assume that $\Lambda = \mathcal{M}_d$. Observe that λ_0 is an intersection point of a family of $(2d - 2)$ hypersurfaces of the form

$$\{\lambda \in \mathcal{M}_d, f_\lambda^n(c_i(\lambda)) = f_\lambda^{n+k}(c_i(\lambda))\}$$

(one for each critical point). The proof in [22] is based on two important ideas. The first one consists in proving that these hypersurfaces are smooth and transverse at λ_0 : this is based on Teichmüller-theoretic ideas. Then, using this transversality, a version of Tan Lei’s transfer principle between dynamical and parameter space allows comparing the mass of μ_{bif} in a carefully scaled small polydisk about λ_0 with the mass of $\mu_{f_{\lambda_0}}$ near the $f^n(c_i)$, and conclude that this mass is positive. ■

In the space of polynomials of degree d , Theorem 2.2, together with other characterizations of $\text{Supp}(\mu_{\text{bif}})$, e.g., in terms of landing of parameter rays, makes $\text{Supp}(\mu_{\text{bif}})$ the natural analogue of the boundary of the Mandelbrot set for polynomials of higher degree.

This motivates an investigation of its topological and geometric properties. First, it is a compact set, which, for $d \geq 3$, is strictly contained in the boundary of the locus \mathcal{C}_d of polynomials with connected Julia set. A topological consequence of Theorem 2.1 is that $\text{Supp}(\mu_{\text{bif}})$ is contained in the closure of $\text{Int}(\mathcal{C}_d)$; on the other hand, it is unknown whether \mathcal{C}_d is the closure of its interior. Gauthier [59] extended Shishikura’s theorem to show that $\text{Supp}(\mu_{\text{bif}})$ has maximal Hausdorff dimension at each of its points. Let us also note that by using advanced nonuniform hyperbolicity techniques, it was shown by Astorg, Gauthier, Mihalache, and Vigny [6] that in the space \mathcal{M}_d of rational maps of degree d , $\text{Supp}(\mu_{\text{bif}})$ has positive volume.

The technical core of Theorems 2.1 and 2.2 is the fact that T_{bif} and its exterior powers describe the asymptotic distribution of families of dynamically defined hypersurfaces in the parameter space, like parameters with a preperiodic critical point, or parameters with a periodic point of a given multiplier. Initiated in [7, 8, 51], this research theme has gradually evolved in scope and sophistication, notably through its connections with arithmetic equidistribution (see [54]).

A striking and unexpected consequence of this technology is an asymptotic estimate for the number of hyperbolic components in \mathcal{M}_d , which is so far not accessible by other means. Recall that a hyperbolic component is a connected component of the stability locus in which the dynamics is uniformly expanding on the Julia set. We say that a hyperbolic component Ω is of disjoint type (n_1, \dots, n_{2d-2}) if the critical points are attracted by distinct attracting cycles of respective exact period n_i .

Theorem 2.3 (Gauthier, Okuyama, and Vigny [60]). *The number $N(n)$ of hyperbolic components of disjoint type (n, \dots, n) in \mathcal{M}_d satisfies*

$$N(n) \underset{n \rightarrow \infty}{\sim} \frac{d^{(2d-2)n}}{(2d-2)!} \int_{\mathcal{M}_d} \mu_{\text{bif}}.$$

(An analogous formula holds for arbitrary disjoint type (n_1, \dots, n_{2d-2}) .) Note that the corresponding result in \mathcal{P}_d is much easier and follows essentially from Bézout’s theorem (together with a transversality argument). The value of $\int_{\mathcal{M}_d} \mu_{\text{bif}}$ is known only for $d = 2$ [60].

Once the bifurcation measure is constructed on \mathcal{P}_d or \mathcal{M}_d , it is natural to inquire about the dynamics of a μ_{bif} -typical parameter. In \mathcal{M}_d this question is completely open so far. For the family of quadratic (and more generally unicritical) polynomials, it was shown by Graczyk–Świątek [62] and Smirnov [81] in the late 1990s that a μ_{bif} -typical parameter satisfies the Collet–Eckmann condition; in particular, the local geometry of its Julia set is well understood. These results are based on combinatorial techniques and the landing of external and parameter rays, and the method carries over for degree d polynomials (see [51, THM. 10]). Interestingly, a completely new approach to the results of [62, 81] was recently found, which applies to arbitrary families of rational maps.

Theorem 2.4 (De Thélin, Gauthier, and Vigny [36]). *Let $(f_\lambda)_{\lambda \in \Lambda}$ be an algebraic family of rational maps of degree d with a marked critical point $c(\lambda)$. Let T_c be the bifurcation current associated to c and $\|T_c\|$ be the associated total variation measure. Then for $\|T_c\|$ -a.e. λ ,*

$$\liminf_{n \rightarrow \infty} |Df_\lambda^n(c(\lambda))| \geq \frac{1}{2} \log d > 0. \tag{2.2}$$

For the unicritical family $z^d + \lambda$, this statement is precisely the typicality of the Collet–Eckmann expansion property.

Sketch of proof. This is an application of the techniques of Section 1.4. We may assume that Λ is of dimension 1, so that T_c is just a positive measure on Λ . Consider the sequence of iterated graphs $\Gamma_{f^n(c)}$, parameterized by $\gamma_n : \lambda \mapsto (\lambda, f_\lambda^n(c(\lambda)))$. Then, as explained above, $T_c = \pi_*(\hat{T} \wedge [\Gamma_c])$, where $\pi : \Lambda \times \mathbb{P}^1 \rightarrow \Lambda$ is the first projection and \hat{T} is the natural \hat{f} -invariant current in $\Lambda \times \mathbb{P}^1$. Using the \hat{f} -invariance of \hat{T} , we infer that

$$T_c = \pi_*(d^{-n}[\Gamma_{f^n(c)}] \wedge \hat{T}), \text{ and conversely } (\gamma_n)_*(T_c) = d^{-n}[\Gamma_{f^n(c)}] \wedge \hat{T}.$$

Since the $\Gamma_{f^n(c)}$ are algebraic curves of uniformly bounded genus, by the results of Section 1.2, the part $(d^{-n}[\Gamma_{f^n(c)}])^r$ of these curves made of disks of size r has mass $1 - O(r^2)$, and since \hat{T} has continuous potential, by Theorem 1.1 the intersection $d^{-n}[\Gamma_{f^n(c)}] \wedge \hat{T}$ is carried by $(d^{-n}[\Gamma_{f^n(c)}])^r$, up to a small error $\eta(r)$. But to fill up a set of measure $1 - \eta(r)$ of $d^{-n}[\Gamma_{f^n(c)}] \wedge \hat{T}$, at least $c(r)d^n$ disjoint such disks are required, and, pulling them back by γ_n , we get a set of $c(r)d^n$ disjoint disks in Λ , covering a set of measure $1 - \eta(r)$ for T_c , each of which mapped under γ_n to a disk of size r . Being disjoint, most of the pulled-back disks in Λ have area at most Cd^{-n} , so the derivative of γ_n there must typically be larger than $Cd^{n/2}$. Analyzing how the derivative of γ_n is expressed in terms of the $Df_\lambda^k(c(\lambda))$, for $0 \leq k \leq n$, finally leads to (2.2). ■

As already mentioned, the theory of bifurcation currents has deep connections with arithmetic dynamics, and related rigidity problems in moduli spaces. A typical problem in this context is the classification of families with a marked point $(f_\lambda, a(\lambda))$ for which the bifurcation current T_a is “abnormally regular.” The reader is referred to the recent monograph [55] by Favre and Gauthier for more on this topic.

2.2. Stability/bifurcation theory in higher dimension

Moving to higher dimension, it is tempting to imitate the definition of J -stability by coining a definition of stability from the noncollision of periodic points. An obvious difficulty is that in this context the automatic extension of holomorphic motions fails and the relevance of this definition needs to be justified, for instance, by proving its equivalence with other natural ones. Due to the variety of possible situations, in higher dimension the details depend on the category of maps under study. So far, this program has been fulfilled in two cases: polynomial automorphisms of \mathbb{C}^2 (by Lyubich and the author), and holomorphic maps on \mathbb{P}^k (by Berteloot, Bianchi, and Dupont).

2.2.1. Polynomial automorphisms of \mathbb{C}^2

For a polynomial automorphism f of \mathbb{C}^2 , we can define Julia sets J^+ and J^- respectively associated to forward and backward iteration, as well as the “small Julia set” $J = J^+ \cap J^-$, and $J^* \subset J$ the closure of the set of saddle periodic points, which is also the support of the maximal entropy measure [9]. Following [53], we say that a holomorphic

family $(f_\lambda)_{\lambda \in \Lambda}$ of polynomial automorphisms of fixed dynamical degree d is *weakly J^* -stable* if (i) its saddle points do not bifurcate, hence (under mild assumptions) so do all periodic points. (Here the numbering of properties corresponds to that of the 1-dimensional case at the beginning Section 2.) Then the holomorphic motion of saddle points extends to a *branched holomorphic motion* of J^* and the condition is equivalent to (ii) $\lambda \mapsto J^*(f_\lambda)$ is continuous. Furthermore, the branched holomorphic motion extends to the “big Julia set” $J^+ \cup J^-$. It remains an open question whether weak J^* -stability yields a conjugacy on J^* or J (that is, whether an analogue of (iii) holds). It is proved in [13] that weak J^* -stability implies a probabilistic form of structural stability, that is, a conjugacy can be defined on a full measure subset for any hyperbolic measure. Also, weak J^* -stability preserves uniform hyperbolicity [13, 50], so the familiar concept of hyperbolic component makes sense in this setting.

Even if strictly speaking polynomial automorphisms have no critical points, the main issue in [53] is about condition (iv) (stability of critical points). Indeed, it a popular analogue of a prerepelling critical point for a 2-dimensional diffeomorphism is a heteroclinic tangency, so we are looking for a characterization of stability in terms of (absence of) tangencies. It is well known that in dissipative dynamics, homoclinic tangencies yield bifurcations from saddles to sources, and the main point of [53] is to find a mechanism for the converse implication. The key is the phenomenon of *semiparabolic implosion*.

Before moving on to this topic, let us point out that so far there is no theory of bifurcation currents for automorphisms of \mathbb{C}^2 .

Question 2.5. For polynomial automorphisms of \mathbb{C}^2 , is stability characterized by the harmonicity of the Lyapunov exponents of the maximal entropy measure? In other words, does an analogue of condition (v) above hold?

2.2.2. Semiparabolic implosion and tangencies

Parabolic implosion refers to a set of phenomena, discovered by Douady and Lavaurs, occurring when unfolding a periodic point with a rational indifferent multiplier. To be specific, consider a family of the form

$$f_\lambda(z) = (1 + \lambda)z + z^2 + \text{h.o.t.}$$

in a neighborhood of the origin, for small λ . For $\lambda = 0$, the fixed point 0 admits a basin of attraction \mathcal{B} . Now If λ approaches the origin tangentially to the imaginary axis, we can track precisely how the parabolic basin \mathcal{B} “implodes” by “passing through the eggbeater” created between two slightly repelling fixed points $p_\lambda = 0$ and $q_\lambda \approx -\lambda$. More precisely, for well-chosen λ_n , $f_{\lambda_n}^n$ converges locally uniformly in \mathcal{B} to a nonconstant *Lavaurs map* $\psi : \mathcal{B} \rightarrow \mathbb{C}$, depending on (λ_n) . Of course, for $\lambda_n \equiv 0$, $\psi = 0$: in this sense the limiting dynamics of f_λ as $\lambda \rightarrow 0$ is richer than that of f_0 . This gives rise to a wealth of dynamical phenomena at a such a parabolic bifurcation, like the discontinuity of the Julia set or the birth of hyperbolic set of large Hausdorff dimension, which are instrumental in Shishikura’s theorem that the boundary of the Mandelbrot set has dimension 2.

Bedford, Smillie and Ueda [11] extended this analysis to the unfolding of a semiparabolic fixed point of multiplicity 2 in \mathbb{C}^2 , that is, of the form

$$f_\lambda(z, w) = ((1 + \lambda)z + z^2 + \text{h.o.t.}, b_\lambda w + \text{h.o.t.}), \quad \text{with } |b_0| < 1. \quad (2.3)$$

In this dissipative situation, as before the Lavaurs map is a limit of iterates of the form $f_{\lambda_n}^n$, its domain is the attracting basin \mathcal{B} of the origin, but its values are contained a curve: the *repelling petal* of the semiparabolic point. For polynomial automorphisms, this leads to a precise description of the discontinuity of the Julia sets J and J^+ at $\lambda = 0$. (See also Bianchi [15] for some results about the implosion of general parabolic germs.)

If (f_λ) is an arbitrary family of dissipative polynomial automorphisms, semiparabolic bifurcations (of possibly arbitrary multiplicity) occur densely in the bifurcation locus by definition. A mechanism producing homoclinic tangencies from semiparabolic implosion was designed in [53]. Besides the analysis of Lavaurs maps (which is not as precise as in the multiplicity 2 case (2.3)), this involves a construction of “critical points” in semiparabolic basins, which by definition are tangencies between unstable manifolds (associated to some given saddle point) and the foliation of the basin by strong stable manifolds. Surprisingly, this construction is based on Wiman’s classical theorem on entire functions of slow growth, and requires a stronger dissipativity condition: $|\text{Jac}(f_\lambda)| < d^{-2}$ (*substantially dissipative regime*). Altogether we obtain the following theorem, which confirms a classical conjecture of Palis in this setting:

Theorem 2.6 (Dujardin and Lyubich [53]). *In a substantially dissipative family of polynomial automorphisms of \mathbb{C}^2 , parameters with homoclinic tangencies are dense in the bifurcation locus.*

It is expected that this result holds without the substantial dissipativity assumption. Also, it is an open question whether quadratic tangencies are always created in this process. A positive answer would yield an interesting link with the quadratic family, and add further evidence to the universality of the Mandelbrot set.

2.2.3. Holomorphic maps on \mathbb{P}^k

The case of families of holomorphic maps on \mathbb{P}^k was studied by Berteloot, Bianchi, and Dupont in [14]. Here, as in the one-dimensional case, one starts with the stability of repelling periodic points. More precisely, one has to restrict to repelling points contained in the “small Julia set” J^* (which by definition is the support of the maximal entropy measure μ), since there can be a number of “spurious” repelling points outside J^* . Then Berteloot, Bianchi, and Dupont obtain an almost complete generalization of the results of Mañé–Sad–Sullivan, Lyubich, and DeMarco (that is, of the above equivalent conditions (i) to (v)). As before, a remaining issue is whether this notion of weak J^* -stability implies structural stability on J^* . A main difference with the 1-dimensional case is that the characterization of bifurcation in terms of currents is now essential to establish the equivalence between the remaining conditions. More precisely, the link between the instability of critical

orbits and that of periodic points is provided by a formula à la Manning–Przytycki for the Lyapunov exponent of the maximal entropy measure.

We saw in Theorems 2.1 and 2.2 that the higher bifurcation currents T_{bif}^k describe accurately certain higher-codimensional phenomena in the parameter space. It seems that the distinction between T_{bif} and its powers is not as clear in higher-dimensional dynamics: in a recent work, Astorg and Bianchi [3] showed that in a large portion of the family of polynomial skew products of \mathbb{C}^2 , the supports of all currents T_{bif}^k coincide with the bifurcation locus. So the significance of these higher bifurcation currents in this context is yet to be explored.

2.3. Robust bifurcations

As said before, due to the finiteness of the critical locus, one-dimensional polynomial and rational maps are generically stable. Intuition from real dynamics suggests that this is not anymore the case in higher dimension. As in the previous paragraph, we discuss separately the cases of polynomial automorphisms and of holomorphic maps on \mathbb{P}^k .

2.3.1. Polynomial automorphisms

Given the characterization of weak J^* -stability in [53], a straightforward adaptation of the one-dimensional argument for the density of stability shows that in any holomorphic family (f_λ) of polynomial automorphisms of \mathbb{C}^2 , the union of (weakly J^* -)stable parameters together with parameters with infinitely many sinks is dense. Prior to [53], it was actually already known that stability is not a dense phenomenon in this context, due to the following remarkable result:

Theorem 2.7 (Buzzard [23]). *There exist $d > 1$ and an open subset $\Omega \subset \text{Aut}_d(\mathbb{C}^2)$ contained in the bifurcation locus. In particular, maps with infinitely many sinks are dense in Ω .*

Here $\text{Aut}_d(\mathbb{C}^2)$ is the space of polynomial automorphisms of \mathbb{C}^2 of degree d . This deep theorem is nothing but the adaptation to the complex setting of Newhouse’s theorem (see [76]) on the existence of surface diffeomorphisms with persistent homoclinic tangencies. It is obtained by first constructing transcendental examples and then approximating them by polynomial ones, hence the degree d is unknown and presumably very large. The existence of this complex Newhouse phenomenon in arbitrary degree is a major open problem.

Question 2.8. Is the bifurcation locus of nonempty interior in $\text{Aut}_d(\mathbb{C}^2)$ for any $d \geq 2$?

As in the real case (cf. [76]), one may even expect that robust bifurcations (that is, interior points of the bifurcation locus) are dense in the bifurcation locus, at least in the dissipative regime. For this, it is tempting to imitate the approach of Shishikura’s theorem on the Hausdorff dimension of ∂M and use semiparabolic implosion to construct large bifurcation sets from a single parabolic bifurcation: in this sense the density of robust bifurcations would be the optimal generalization of Shishikura’s theorem to automorphisms of \mathbb{C}^2 . An interesting first step would be to show that the bifurcation locus has maximal Hausdorff dimension at every point. More advanced techniques will certainly be needed to get open subsets: an ambi-

tious research program on the intersection of complex Cantor sets was initiated by Araujo, Moreira, and Zamudio towards this perspective (see [1, 2]).

Biebler observed in [18] that the existence of robust bifurcations is actually more tractable in higher dimensions and showed that: *for every $d \geq 2$, the bifurcation locus has nonempty interior in $\text{Aut}_d(\mathbb{C}^3)$* . This is based on a distinct mechanism for robust bifurcation, namely the *blenders* of Bonatti and Diaz [19]. These are dynamically defined Cantor sets which are so fat in a certain “direction” that they intersect an open set of curves. The point of [18] is to use this feature as a building block for persistent tangencies.

Finally, let us point out a recent beautiful result by Yampolsky and Yang [85]: *the one-dimensional family of degree 2 Hénon maps with a golden mean Siegel disk*

$$f_a(x, y) = (x^2 + c_a - ay, x),$$

$$\text{with } c_a = (1 + a) \left(\frac{\mu}{2} + \frac{a}{2\mu} \right) - \left(\frac{\mu}{2} + \frac{a}{2\mu} \right)^2 \text{ and } \mu = e^{\pi(1+\sqrt{5})i},$$

is structurally unstable at every parameter with small enough Jacobian $|a|$. This relies on a completely different approach to persistent tangencies, based on Siegel renormalization.

2.3.2. Holomorphic maps on \mathbb{P}^k

From the work of Berteloot, Bianchi, and Dupont, we know that the basic phenomenon responsible for bifurcations for holomorphic maps on \mathbb{P}^k is when the postcritical set intersects the small Julia set J^* . Thus, to obtain robust bifurcations, it is enough to find a mechanism ensuring a robust intersection between the postcritical set and J^* . A convenient tool for this is the Bonatti–Diaz blender, which leads to:

Theorem 2.9 (Dujardin [49]). *For every $k \geq 2$ and $d \geq 2$, the bifurcation locus has nonempty interior in $\text{Hol}_d(\mathbb{P}^k)$.*

Here, $\text{Hol}_d(\mathbb{P}^k)$ is the space of holomorphic maps on \mathbb{P}^k of degree d . A specific one-dimensional family of holomorphic maps of \mathbb{P}^2 with a full bifurcation locus was found independently by Bianchi and Taflin [16]. After this result, a natural question is that of the abundance of robust bifurcations in $\text{Hol}_d(\mathbb{P}^k)$. Taflin [83] showed that robust bifurcations are abundant near product polynomial maps of \mathbb{C}^2 , and Biebler [17] showed that Lattès maps of sufficiently large degree are accumulated by robust bifurcations. Blenders are involved directly or indirectly in both cases, and seem to appear quite naturally when a repelling periodic point bifurcates to a saddle. Still, the general picture remains elusive.

Question 2.10. Is the bifurcation locus in $\text{Hol}_d(\mathbb{P}^k)$ the closure of its interior?

Lastly, a celebrated theorem of McMullen asserts that any stable algebraic families of rational maps on \mathbb{P}^1 is either isotrivial or a family of flexible Lattès examples [74]. Extending this result to higher dimensions is a promising research problem; one main obstacle is that part of the argument relies on Thurston’s topological characterization of rational functions. Related preliminary results have been obtained by Gauthier and Vigny [61].

3. (NON-)WANDERING FATOU COMPONENTS

The classification of Fatou components is a basic chapter of holomorphic dynamics. For rational maps in dimension 1, periodic Fatou components can be classified into attracting basins, parabolic basins, and rotation domains (Siegel disks and Herman rings). The crowning achievement of this classification is the celebrated nonwandering domain theorem of Sullivan [82]: *for a one-dimensional rational map, any Fatou component is preperiodic.*

In higher dimensions, techniques from geometric function theory may be applied to classify periodic Fatou components. It is convenient to distinguish between recurrent and nonrecurrent periodic components: a fixed Fatou component Ω is *recurrent* if for some $x \in \Omega$, the ω -limit set $\omega(x)$ is not completely contained in $\partial\Omega$. Recurrent Fatou components were classified in various classes of rational maps in [10, 56, 57, 84]. The upshot is that in such a component either there is an transversely attracting submanifold (possibly a point) or the dynamics is of rotation type. The situation is far less understood in the nonrecurrent case. A notable exception is that of substantially dissipative automorphisms of \mathbb{C}^2 , for which it was shown by Lyubich and Peters [72] that any nonrecurrent Fatou component is the basin of a semiparabolic periodic point.

On the other hand, it is immediately clear that the quasiconformal techniques used in Sullivan’s proof are not generalizable to higher dimension. As it turns out, wandering components do exist in 2-dimensional polynomial dynamics:

Theorem 3.1 (Astorg, Buff, Dujardin, Peters, and Raissy [5]). *If $0 < a < 1$ is sufficiently close to 1, the polynomial mapping of \mathbb{C}^2 defined by*

$$f : (z, w) \mapsto (p(z, w), q(w)) = \left(z + z^2 + az^3 + \frac{\pi^2}{4}w, w - w^2 \right)$$

admits a wandering Fatou component.

The proof is based on an original idea of M. Lyubich, and relies on a skew product version of parabolic implosion. It was further implemented in other situations in [4, 63].

Sketch of proof. Write $p(z, w) = p_0(z) + \varepsilon(z, w)$, with $p_0(z) = z + z^2$ and $\varepsilon(z, w)$ being thought of as a perturbative term. Start with an initial point (z_0, w_0) such that z_0 belongs to the parabolic basin of attraction of 0 for p_0 and w_0 a small positive number, and let as usual $(z_n, w_n) = f^n(z_0, w_0)$. Then $w_n = q^n(w)$ converges to 0 along the positive real axis, and $p_0^n(z_0)$ converges to 0 along the negative real axis. Therefore $z_n = p_0^n(z_0) + \varepsilon_n$ is pushed a little faster towards the origin by the term ε_n . The terms in $\varepsilon(z, w)$ are crafted so that if z_0 is chosen carefully in some open set of initial conditions, the iterates z_n indeed pass the origin by going “through the eggbeater” and come back close to their initial position. So we can repeat this process and conclude that (z_0, w_0) belongs to some Fatou component. But since the returning time increases with the number of iterations, this Fatou component is not periodic, and we are done. ■

At this stage the following natural questions arise:

Question 3.2. (1) Are there other dynamical mechanisms leading to wandering Fatou components?

(2) Find substantial families of higher-dimensional rational mappings without wandering domains.

Regarding the first question, a mechanism for constructing wandering domains in 2-dimensional smooth dynamics, based on the Newhouse phenomenon, was devised by Colli and Vargas [27]. Berger and Biebler recently proved that this mechanism can be implemented in certain 5-dimensional families of Hénon maps, leading to the following stunning theorem:

Theorem 3.3 (Berger and Biebler [12]). *There exists a polynomial automorphism of \mathbb{C}^2 of degree 6 with a wandering Fatou component.*

This solves the existence problem for wandering Fatou components for plane polynomial automorphisms, which does not seem to be amenable to the techniques of [5].

For the second question, it is a classical fact that hyperbolic dynamics prevents the existence of wandering domains. Besides this observation, not much is known. In view of Theorem 3.1, it is natural to investigate the case of skew products with a fixed attracting fiber, that is, of the form

$$f(z, w) = (p(z), q(z, w)), \quad \text{with } p(0) = 0 \text{ and } |p'(0)| < 1. \quad (3.1)$$

In this case it could be expected that Sullivan's theorem, together with the attracting nature of the invariant fiber, should be enough to prevent the existence of wandering domains. Embarrassingly enough, even in such a simple situation, there is no definitive answer so far, and furthermore it was shown by Peters and Vivas [79] that the above naive intuition does not lead to a proof. Here is the current status of the problem:

Theorem 3.4 (Lilov, Peters-Smit, Ji). *If f is an attracting skew product as in (3.1), then there are no wandering components near the attracting fiber, whenever:*

- $p'(0) = 0$ [68] or, more generally, if $|p'(0)|$ is small enough (with respect to p and q) [65];
- $|p'(0)| < 1$ and $q(0, \cdot)$ satisfies some nonuniform hyperbolicity properties [64, 78].

There is currently no hope for a general understanding of the problem of wandering Fatou components in several dimensions, and even going beyond skew products seems to be a serious challenge. An interesting first case to be considered is that of Fatou components in the neighborhood of an invariant superattracting line, which would cover, for instance, the case of regular polynomial mappings of \mathbb{C}^2 near the line at infinity.

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