STABILITY AND RECURSIVE SOLUTIONS IN HAMILTONIAN PDES

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ABSTRACT

In this survey we shall consider Hamiltonian dispersive partial differential equations on compact manifolds and discuss the existence, close to an elliptic fixed point, of special recursive solutions, which are superpositions of oscillating motions, together with their stability/instability properties. One can envision such equations as chains of harmonic oscillators coupled with a small nonlinearity, thus one expects a complicated interplay between chaotic and recursive phenomena due to resonances and small divisors, which are studied with methods from KAM theory.

We shall concentrate mainly on the stability properties of the fixed point, as well as the existence and stability of quasiperiodic and almost periodic solutions. After giving an overview on the literature, we shall present some promising recent results and discuss possible extensions and open problems.

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1. INTRODUCTION

A huge variety of physical systems is modeled by Hamiltonian dispersive partial differential equations (PDEs), such as the nonlinear Schrödinger (NLS) and wave (NLW) equations, Euler and water wave equations, KdV, etc. A good point of view, which has produced many advancements in the last 30 years, is to take a dynamical systems perspective and understand qualitative behavior by studying special invariant objects, such as finite- and infinite-dimensional tori, chaotic and diffusion orbits, etc. This perspective is particularly uselful for PDEs on compact domains, where one expects a complicated interplay between chaotic and recursive phenomena. For concreteness, we shall concentrate on NLS equations on a compact Riemannian manifold $(\mathcal{M}, g)$ without boundary, namely

$$\mathrm{i}u_t - \Delta_g u + V(x)u + f\left(|u|^2\right)u = 0 \tag{NLS}$$

where $f(y)$ is analytic in a neighborhood of zero, with $f(0) = 0$, and V is an appropriately regular, real potential $V : \mathcal{M} \to \mathbb{R}$, so that $u = 0$ is an elliptic fixed point.

Of course, (NLS) is still a simplified model, since more physical examples have derivatives in the nonlinearity; this is true for most PDEs modeling hydrodynamics and, indeed, there are results in this more general setting, mostly confined to spheres or flat tori $\mathbb{T}^n := \mathbb{R}^n \setminus \mathbb{Z}^n$. In fact, in all the results we discuss, we shall impose some simplifying condition, such as choosing simple manifolds (for instance, tori or spheres, or more generally, simple compact Lie groups), and/or simplify the model, for instance, by using a convolution (instead of multiplicative) potential.

In studying the dynamics of (NLS) close to zero, one expects a complicated interplay between chaotic and recursive phenomena, with the qualitative behavior of solutions depending in a subtle way on the geometry of $\mathcal{M}$ and on V. For instance, for stability results one typically needs to use V as a "source of parameters," say by modulating V so that the eigenvalues of the elliptic operator $-\Delta_g + V$ satisfy some nonresonance conditions (such as lower bounds on the integer combinations of eigenvalues). At the same time, to deal with the nonlinearity, one needs rather precise information on products of eigenfunctions, particularly on the coefficients that give the representation in the eigenfunction basis of the product of two eigenfunctions. As a drawback, it is usually very difficult to get results for a fixed value of V, for instance, $V = 0$.

Recalling that $-\Delta_g + V(x)$ is self-adjoint and with pure point spectrum, let $(\psi_j)_{j\in I}$ be its eigenfunctions (I is some countable index set) and ω_j the corresponding eigenvalues. Then we pass to the "Fourier side," $u = \sum_{j\in I} u_j \psi_j$. Writing NLS in terms of this basis, we have an infinite chain of harmonic oscillators coupled by a nonlinear term. It is easily verified that the associated equations are Hamiltonian with respect to the standard symplectic form $\Omega := \mathrm{i}\sum_{j\in I} du_j \wedge d\bar{u}_j$, with Hamiltonian

$$H_{\mathrm{NLS}} := \sum_{j\in I} \omega_j |u_j|^2 + P(u), \quad u = (u_j)_{j\in I}, \tag{1}$$

where P is a suitable nonlinearity with a zero of order at least four.

If we ignore the nonlinearity, the dynamics is very simple. All the linear actions $|u_j|^2$ are constants of motion and the dynamics is $u_j(t) = u_j(0)e^{\mathrm{i}\omega_j t}$, hence all solutions are superpositions of oscillations. For typical V's, the linear frequencies ω_j are rationally independent and the solutions live on tori, with dimension depending on the number of nonzero actions.

Now if we take into account the nonlinearity, the u_j's interact, exchanging energy; we want to study how, close to the origin, the dynamics differs from the linear one and over which time scales. To make this quantitative, we fix a phase space $\mathtt{h} \subset L^2$ of sufficiently regular functions $\mathcal{M} \to \mathbb{C}$ (typically, a spectrally defined Sobolev space prescribing sufficiently fast decay of the linear actions $|u_j|^2$ as $j \to \infty$). If $\mathtt{h}$ is sufficiently regular then NLS is at least locally well posed and one expects the dynamics to be close to the linear one at least close to zero and for finitely long time.

If we look at a finite-dimensional truncation of (1), then the classical Kolmogorov–Arnold–Moser (KAM) theory gives a rather clear picture: under some (generic) nondegeneracy assumptions,[1] close to the origin most of the phase space is foliated by Lagrangian invariant tori (with dimension half of that of the phase space). In particular, the system is not ergodic and most initial data give rise to quasiperiodic solutions that densely fill some invariant torus and are, therefore, perpetually stable.[2] Possible chaotic behavior is restricted outside a set of asymptotically full measure at the origin. Moreover, the origin and the maximal tori are stable, with nearby trajectories staying close for exponentially long times. All the finite-dimensional results strongly depend on the dimension, and one cannot naïvely perform finite-dimensional truncations in (1) and then take limits. In fact, in the infinite-dimensional setting, the general picture is so far rather obscure and the main questions still remain unanswered. All linear solutions are perpetually stable and typical ones lie on maximal infinite-dimensional invariant tori. What is their fate under perturbation? Is it still true that typical initial data produce perpetually stable solutions? What are the stability times?

Of course, even the concept of a "typical solution" depends on our choice of the measure on the phase space. Moreover, even at the linear level, simple topological issues such as whether the maximal tori give a foliation, or whether the dynamics on the tori is dense, depend strongly on the choice of the phase space and its topology.

In this survey we discuss some partial answers to these fundamental questions. We concentrate on three issues:

1. *Stability of zero.* A good way to capture the transfer of energy between Fourier modes is to study the time evolution of the norm $|\cdot|_{\mathtt{h}}$; indeed, if $\mathtt{h}$ is sufficiently regular, a growth in the norm represents transfers between low and high modes. With this in mind, we take any initial datum $u_0 \in B_\delta(\mathtt{h})$ and give estimates on the time $T(\delta)$ such that the flow $u(t,\cdot)$ of the NLS equation is well defined and belongs to $u_0 \in B_{2\delta}(\mathtt{h})$. A rough estimate[3] gives $T(\delta) \geq \delta^{-2}$; to get better lower bounds, a good strategy is to find a change of variables on $B_\delta(\mathtt{h})$ which conjugates the Hamiltonian to $N + R$, where N is *the normal form* preserving the norm $|\cdot|_{\mathtt{h}}$ while R is *the remainder* which is small and affects the dynamics

1 On ω and/or on the nonlinearity.

2 Namely the linear actions have a small variation for all times.

3 Coming from well posedness and the fact that the nonlinearity is at least cubic.

over very long times. In constructing such a change of variables, one encounters small divisors, i.e., in their analytic expression one has integer combinations of linear frequencies in the denominator, so a major point will be to impose sufficiently strong irrationality conditions and ensure some lower bounds. This is done by appropriately modulating the external parameters.

2. *Small quasiperiodic solutions.* We look for special global solutions living on finite-dimensional invariant tori. More precisely, we look for a sufficiently regular map $U : \mathbb{T}^n \to \mathtt{h}$ and a frequency $\omega \in \mathbb{R}^n$ such that $U(\mathbb{T}^n)$ is invariant under the NLS dynamics, which, when restricted to the torus, is the linear translation by ωt. We work close to zero in order to take advantage of the fact that all the solutions of the linearized equation which are supported on finitely many Fourier modes are indeed quasiperiodic. Then starting from such approximately invariant tori, one wishes to prove that nearby there exist truly invariant ones. This is done by iterative approximations using a quadratic scheme. Again one needs to control small divisors by modulating the external parameters (typically, one only needs as many parameters as the dimension of the torus but there are a number of results where one only needs one parameter, or even none). Note that these solutions are very special, even in the case of an integrable PDE they are not typical.

3. *Small almost-periodic solutions.* Starting from quasiperiodic solutions with an arbitrary number n of frequencies, it is very natural to wonder whether one can pass to the limit as $n \to \infty$ thus obtaining an almost-periodic solution. Since almost-periodic solutions are "typical" for integrable systems, a main question is how rare such solutions are in a nonintegrable setting. Unfortunately, up to now all results are for PDEs on the circle and show the existence of few and very regular solutions.

Having proved the existence of these special global solutions, an interesting point is to study their stability properties, thus giving an insight on the nearby dynamics for finite but long times. A strategy is to perform changes of variables to put the system in normal form in a neighborhood of the solution. A dual point of view is to look for unstable/chaotic orbits driven by the presence of resonant terms in the nonlinearity.

We shall concentrate on (NLS); however, all the questions described above are tied mainly to the Hamiltonian formulation (1), thus they can be reformulated for PDEs on unbounded domains, when $-\Delta_g + V(x)$ has a pure point spectrum $\omega_j \to \infty$. An interesting (and widely studied) example is the harmonic oscillator, namely $\mathcal{M} = \mathbb{R}^n$ and $V(x) = |x|^2$.

In the next sections we shall give a brief (and necessarily incomplete) survey on the three questions described above, together with some open problems.

2. LONG TIME STABILITY

The problem of *long-time* stability for infinite-dimensional dynamical systems has been studied by many authors, starting from [13] for infinite chains with a finite range coupling. In the PDE context, after the first results in [5,6,28], a breakthrough was in the papers [7,9] where the authors proved polynomial bounds on the stability times for a rather wide class of *tame-modulus* PDEs depending on parameters. Their result applies to the NLS equations on tori, where they show that for any $\mathtt{N} \gg 1$ there exist many values of the parameters for which any δ-small initial datum in H^p (with $p = p(\mathtt{N})$ tending to infinity as $\mathtt{N} \to \infty$) stays 2δ small for times $T \geq C(\mathtt{N}, p)\delta^{-\mathtt{N}}$.

An interesting question is how such results perform in applications to PDEs with derivatives in the nonlinearity; a series of results in this direction were proved for the Klein–Gordon equation on Zoll manifolds in [8,41–43]. The method developed in these papers, based on a good control of the small divisors, together with ideas from paradifferential calculus, does not apply to the case where the PDE has a superlinear dispersion relation.

Recently there has been a lot of progress regarding[4] quasilinear and fully nonlinear PDEs on $\mathcal{M} = \mathbb{S}^1$, we mention [20,21] on the water waves and [52] for quasilinear NLS. Regarding higher-dimensional quasilinear PDEs, we mention [50,53] for Klein–Gordon and Schrödinger equations on higher-dimensional tori. While most results deal with parameter families of PDEs (and hold for most values of the parameters), we mention [14] on perturbations of the integrable 1D NLS.

If one wants to go beyond polynomial bounds, up to now the literature is restricted to PDEs on tori, with initial data which are at least C^∞. In [47] the authors considered the case of analytic initial data and proved subexponential bounds of the form $T \geq e^{c\ln(\frac{1}{\delta})^{1+\beta}}$ for classes of NLS equations in $\mathbb{T}^d$. Such bounds have been discussed also in [25,36] in Gevrey class for the 1D NLS.

In order to describe the results more in detail, let us restrict to the simplest possible case of a translation invariant NLS with a convolution potential when $\mathcal{M} = \mathbb{S}^1$ so the Fourier decomposition is $u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}$. We consider

$$\mathrm{i}u_t - u_{xx} + V \star u + f\big(|u|^2\big)u = 0, \quad V \star u := \sum_{j\in\mathbb{Z}} u_j V_j e^{\mathrm{i}jx}, \quad (V_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \tag{2}$$

so the NLS Hamiltonian (1) has frequencies $\omega_j = \omega_j(V) = j^2 + V_j$ and $P := \int_{\mathbb{T}} F(|u(x)|^2)dx$ with $F(y) := \int_0^y f(s)ds$. An important feature is that the equation now has two constants of motion

$$L = \sum_{j\in\mathbb{Z}} |u_j|^2, \quad M = \sum_{j\in\mathbb{Z}} j|u_j|^2,$$

corresponding respectively to gauge invariance $u(x) \to e^{\mathrm{i}\tau}u(x)$ and translation invariance $u(x) \to u(x+\tau)$.

As we have explained before, the stability results depend on imposing sufficiently good nonresonance conditions, otherwise one can produce counterexamples where the actions have a *fast drift*; see, for instance, [57]. For this purpose, we shall assume a very strong condition, proposed by Bourgain in [32], which is tailored to 1D PDEs and gives good estimates for many choices of the phase space. More precisely, recalling that $\omega_j(V) = j^2 + V_j$,

4 We say that a PDE is semilinear if the highest order derivatives occur in the linear part, quasilinear if the same order derivatives appear in the linear and nonlinear parts but with degree one, otherwise fully nonlinear if the highest derivative has degree higher than one.

for $\gamma > 0$ we define the set of Diophantine frequencies

$$\mathtt{D}_\gamma := \left\{ V \in \left[-\frac{1}{2}, \frac{1}{2}\right]^{\mathbb{Z}} : \left|\omega(V)\cdot \ell\right| > \gamma \prod_{j\in\mathbb{Z}} \frac{1}{1+|\ell_j|^2\langle j\rangle^2},\ \forall \ell \in \mathbb{Z}^{\mathbb{Z}} : 0 < |\ell| < \infty \right\}. \quad (3)$$

It results (see [25,31]) that $\mathtt{D}_\gamma$ is large with respect to the natural probability product measure on $[-\frac{1}{2}, \frac{1}{2}]^{\mathbb{Z}}$. From now on we shall assume that $V \in \mathtt{D}_\gamma$.

Theorem 1 (Sobolev stability, [9]). *For any p large enough and any initial datum $u(0) = u_0$ satisfying*

$$|u_0|_{H^p} := |u_0|_{L^2} + \left|\partial_x^p u_0\right|_{L^2} \le \delta \le \delta_0 \sim p^{-3p}, \quad (4)$$

the solution $u(t)$ of $(\mathrm{NLS})_V$ *with initial datum $u(0) = u_0$ exists and satisfies*

$$\left|u(t)\right|_{H_p} \le 4\delta \quad \textit{for all times } |t| \le T \sim p^{-5p}\delta^{-\frac{2(p-1)}{\tau_S}}. \quad (5)$$

An interesting feature of this result is that the stability time is related to the regularity. The estimates given here are those for (2) with the Diophantine conditions (3); however, a similar phenomenon appears in all the known literature.

Let us now increase the regularity and consider Gevrey initial data, let us fix $0 < \theta < 1$, and define the function space

$$\mathtt{H}_{s,a} := \left\{ u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx} \in L^2 : |u|^2_{s,a} := \sum_{j\in\mathbb{Z}} |u_j|^2 \langle j\rangle^2 e^{2a|j|+2s\langle j\rangle^\theta} < \infty \right\}, \quad (6)$$

with the assumption $a \ge 0$, $s > 0$. We remark that if $a > 0$, this is a space of analytic functions, while if $a = 0$ the functions have Gevrey regularity.

Theorem 2 (Gevrey stability, [25,36]). *Fix any $a \ge 0$, $s > 0$. For any u_0 such that*

$$|u_0|_{s,a} \le \delta \le \delta_0 \ll 1,$$

the solution $u(t)$ of (2) with initial datum $u(0) = u_0$ exists and satisfies

$$\left|u(t)\right|_{s,a} \le 2\delta \quad \textit{for all times } |t| \le \frac{\mathtt{T}_0}{\delta^2} e^{(\ln \frac{\delta_0}{\delta})^{1+\theta/4}}.$$

As can be expected, as $s \to 0$ one has $\delta_0, T_0^{-1} \to 0$, on the other hand, modulating the parameter a does not give significantly improved bounds. This leads to two very natural questions: Can one get better bounds for analytic initial data? Conversely, can we lower the regularity still obtaining superpolynomial stability times?

A reasonable strategy for tackling the second question is proposed in [25] where we discussed a BNF approach for (2) on abstract weighted functions spaces. Given a positive sequence $\mathtt{w} = (\mathtt{w}_j)_{j\in\mathbb{Z}}$, with $1 \le \mathtt{w}_j \nearrow \infty$, let us consider the Hilbert space

$$\ell^2_{\mathtt{w}} := \left\{ u := (u_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{C}) : |u|^2_{\mathtt{w}} := \sum_{j\in\mathbb{Z}} \mathtt{w}_j^2 |u_j|^2 < \infty \right\}. \quad (7)$$

By the Fourier transform, such spaces identify corresponding function spaces of periodic functions. For instance, if $\mathtt{w}_j = \langle j\rangle^p$, then[5] $\mathcal{F}(\ell^2_{\mathtt{w}})$ identifies with the Sobolev space H_p. Similarly, if $\mathtt{w}_j := \langle j\rangle e^{a|j|+s\langle j\rangle^\theta}$, we are in the Gevrey/analytic case of $H_{s,a}$.

5 The Fourier transform $\mathcal{F}$ identifies sequences with functions $\mathcal{F}((u_j)_{j\in\mathbb{Z}}) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}$.

In this context we gave some computationally heavy, but very explicit conditions on $\mathtt{w}$ which ensure that a BNF theorem can be applied and allows computing the stability times. We concentrated on the two cases above, but if one runs the same computations with $\mathtt{w}_j := \langle j \rangle e^{p \log(1+\langle j \rangle)^2}$ then one gets times of order $\delta^{-\ln\ln(\delta^{-1})}$.

It would be interesting to understand whether such bounds are optimal. A natural strategy would be to construct solutions to (NLS) whose Sobolev norm increases in time. There has been a lot of interest in this question and in particular on whether one can construct solutions whose norm becomes arbitrarily large, or even diverges as $t \to \infty$. A mechanism for ensuring finite, but arbitrarily large growth was constructed in [33] for the cubic parameterless NLS on $\mathbb{T}^2$ (see also [61] for the noncubic case and [59] for a case with convolution potential). The idea is to look for solutions which are approximately supported on Fourier modes $\mathcal{S} \subset \mathbb{Z}^2$ which are resonant (i.e., some linear combinations of ω_j's with $j \in \mathcal{S}$ are zero or very small). Then the interactions between Fourier modes due to the nonlinearity become dominant and the Sobolev norm varies. The very beautiful approach of [33] seems strongly tied to the NLS equation, if one only wants to find solutions which only, say, double their Sobolev norm then there are more robust mechanisms. One idea, see [63], is to prove the existence of secondary tori which transfer energy between two sets of Fourier modes periodically in time. Another very interesting approach, see [58], is to construct chaotic orbits generalizing "Arnold diffusion" to infinite dimension.

2.1. Questions and open problems

Q1. Can one obtain stability times on a fixed Sobolev space H_p, with $T(\delta)$ growing faster than polynomially as $\delta \to 0$?

Q2. Can one prove stability for most V for the NLS with a multiplicative potential?

This was considered in [9]. However, in order to obtain a stability time of order $\delta^{-\mathtt{N}}$, the authors had to restrict the potential to a small ball (in an appropriate norm) with radius going to zero as $\mathtt{N} \to \infty$. The delicate point here is what kind of irrationality conditions can be imposed on the linear frequencies ω_j, which in this case are the periodic spectrum of the Sturm–Liouville operator $-\partial_{xx} + V(x)$ where V is an analytic function.

Q3. Can one extend the stability results to general manifolds in higher dimension?

Q4. Can one extend the subexponential Gevrey bounds to quasilinear PDEs?

Q5. What kind of bounds can be given on instability times?

2.2. An idea of the strategies

To conclude this section, let us briefly illustrate the Birkhoff normal form procedure in its simplest form applied to (2). For this purpose, we consider an analytic translation-invariant Hamiltonian written as an absolutely convergent power series

$$H(u) = \sum_{(\alpha,\beta)\in\mathcal{M}} H_{\alpha,\beta} u^\alpha \bar{u}^\beta, \quad u^\alpha := \prod_{j\in\mathbb{Z}} u_j^{\alpha_j}, \tag{8}$$

where $\mathcal{M} := \{(\alpha,\beta) \in \mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}} \mid |\alpha| = |\beta| < +\infty,\ \sum_{j\in\mathbb{Z}} j(\alpha_j - \beta_j) = 0\}$, satisfying the reality condition $H_{\alpha,\beta} = \overline{H}_{\beta,\alpha}$, $\forall\,(\alpha,\beta) \in \mathcal{M}$.

Given a Hamiltonian as in (8), we denote by X_H its Hamiltonian vector field with respect to the symplectic form $\Omega := \mathrm{i}\sum_{j\in I} du_j \wedge d\bar{u}_j$. We say that $H \in \mathcal{H}_r(\ell^2_{\mathrm{w}})$ for $r > 0$ if the Hamiltonian vector field $X_{\underline{H}}$ of the Cauchy majorant of the Hamiltonian is a bounded analytic map $B_r(\ell^2_{\mathrm{w}}) \to \ell^2_{\mathrm{w}}$:

$$|H|_{r,\ell^2_{\mathrm{w}}} := \frac{1}{r}\Big(\sup_{|u|_{\ell^2_{\mathrm{w}}}\le r} |X_{\underline{H}}|_{\ell^2_{\mathrm{w}}}\Big) < \infty, \quad \underline{H}(u) := \sum_{(\alpha,\beta)\in\mathcal{M}} |H_{\alpha,\beta}| u^\alpha \bar{u}^\beta. \tag{9}$$

The space $\mathcal{H}_r(\ell^2_{\mathrm{w}})$ is closed with respect to Poisson brackets and, moreover, if $S \in \mathcal{H}_r(\ell^2_{\mathrm{w}})$ has a sufficiently small norm then it generates a well-defined time-one flow $B_r(\ell^2_{\mathrm{w}}) \to \ell^2_{\mathrm{w}}$. Finally, we say that a Hamiltonian H has scaling (degree) $\mathtt{d}(H) \ge \mathtt{d}$ if[6]

$$H = \sum_{(\alpha,\beta)\in\mathcal{M}:|\alpha|+|\beta|\ge \mathtt{d}+2} H_{\alpha,\beta} u^\alpha \bar{u}^\beta;$$

note that the scaling degree is additive with respect to Poisson brackets.

Now we recall that the NLS Hamiltonian (1) has the form $H = D_\omega + P$ with P of scaling ≥ 2 and $D_\omega := \sum_{j\in\mathbb{Z}} \omega_j |u_j|^2$ having scaling zero.

Let us conjugate H by the time-one flow Φ^1_S with generating Hamiltonian S. Denoting[7] $\mathrm{ad}_S : H \mapsto \{S, H\}$, the Lie exponentiation formula reads

$$H \circ \Phi^1_S = e^{\{S,\cdot\}} H = D_\omega + P + \{S, D_\omega\} + \sum_{h=2}^{\infty} \frac{\mathrm{ad}_S^{h-1}}{h!}\{S, D_\omega\} + \sum_{k=1}^{\infty} \frac{\mathrm{ad}_S^k}{k!} P.$$

Now at least at the level of formal power series, the last two summands have scaling ≥ 4 (just by the additivity of the scaling degree), so our goal is to cancel the term $P + \{S, D_\omega\}$ (which has scaling ≥ 2) up to a remainder which is either action preserving or of scaling ≥ 4. Let

$$\mathcal{R}_r\big(\ell^2_{\mathrm{w}}\big) := \Big\{H \in \mathcal{H}_r\big(\ell^2_{\mathrm{w}}\big) \;\Big|\; H = \sum_{\alpha\ne\beta} H_{\alpha,\beta} u^\alpha \bar{u}^\beta\Big\}, \tag{10}$$

introduce the decomposition $\mathcal{H}_r(\ell^2_{\mathrm{w}}) = \mathcal{R}_r(\ell^2_{\mathrm{w}}) \oplus \mathcal{K}_r(\ell^2_{\mathrm{w}})$ and the continuous projections $\Pi_{\mathcal{K}} H := \sum_{\alpha=\beta} H_{\alpha,\beta} u^\alpha \bar{u}^\beta$, $\Pi_{\mathcal{R}} H := \sum_{\alpha\ne\beta} H_{\alpha,\beta} u^\alpha \bar{u}^\beta$. Now all Hamiltonians in $\mathcal{K}_r(\ell^2_{\mathrm{w}})$ are action preserving while for any $R \in \mathcal{R}_r(\ell^2_{\mathrm{w}})$, at least formally, one has

$$R + \{S, D_\omega\} = 0 \quad\Leftrightarrow\quad S = -\mathrm{i} \sum_{(\alpha,\beta)\in\mathcal{M}} \frac{R_{\alpha,\beta}}{\omega\cdot(\alpha-\beta)} u^\alpha \bar{u}^\beta,$$

this is called the "homological equation." Thus we choose S_0 so that $\{S_0, D_\omega\} + \Pi_{\mathcal{R}} P = 0$ and, provided that we can show that it is well defined and has a sufficiently small norm, we have found a change of variables $e^{\mathrm{ad}_{S_0}} : D_\omega + P \rightsquigarrow D_\omega + Z_1 + P_1$ where Z is action

6 Note that saying that H has scaling $\ge \mathtt{d}$ means that its Taylor series has minimal degree of homogeneity $\ge \mathtt{d} + 2$.

7 The Poisson brackets are defined as $\{S, H\} := dS(X_H)$.

preserving and now P_1 has scaling ≥ 4. Following the same scheme, if we choose S_1 so that $\{S_1, D_\omega\} + \Pi_{\mathcal{R}} P_1 = 0$ and again S_1 has sufficiently small norm then, composing the two changes of variables, we conjugate $D_\omega + P \rightsquigarrow D_\omega + Z_2 + P_2$ where now P_2 has scaling ≥ 6.

Assuming that $P \in \mathcal{H}_{r_0}(\ell^2_{\mathtt{w}_0})$, for some r_0, $\mathtt{w}_0 = (\mathtt{w}_{0,j})_{j\in\mathbb{Z}}$, does not imply that S_1, S_2 are such. We have reduced the problem to finding a correct weighted space such that $S_1, S_2 \in \mathcal{H}_r(\ell^2_{\mathtt{w}})$ for r small. Note that since they have scaling ≥ 2 and ≥ 4, respectively, once S_1, S_2 are well defined their norm can be made arbitrarily small by just taking r small.

Let us consider the simple example of $\mathtt{w} = \mathtt{w}_s = (\langle j\rangle^2 e^{s\langle j\rangle^\theta})_{j\in\mathbb{Z}}$. Direct computations show that $P \in \mathcal{H}_{r_0}(\ell^2_{\mathtt{w}_0})$, for some $r_0 > 0$. Now there are two key points:

Immersions. If $H \in \mathcal{H}_{r_0}(\ell^2_{\mathtt{w}_0})$ then $H \in \mathcal{H}_r(\ell^2_{\mathtt{w}_s})$ for all $r \leq r_0$ and $s \geq 0$ and the norm is decreasing in s and increasing in r.

Homological equation. If $R \in \mathcal{H}_r(\ell^2_{\mathtt{w}_s})$ then the solution S of the homological equation belongs to $\mathcal{H}_r(\ell^2_{\mathtt{w}_{s+\sigma}})$ for all $\sigma > 0$ and

$$|S|_{\ell^2_{\mathtt{w}_{s+\sigma}}} \leq e^{C\sigma^{-\frac{3}{\theta}}} |R|_{\ell^2_{\mathtt{w}_s}}. \tag{11}$$

Thus for any given $\sigma > 0$, there exists r_2 such that, for $|r| \leq r_2$, both S_1, S_2 are well defined and small and the composition of their time-one flows maps $B_r(\ell^2_{\mathtt{w}_{2\sigma}}) \to \ell^2_{\mathtt{w}_{2\sigma}}$. This gives all the necessary estimates and one can repeat this procedure $\mathtt{N}$ times. At the end, for $|r| \leq r_{\mathrm{fin}}$, we get a change of variables $B_r(\ell^2_{\mathtt{w}_{\mathtt{N}\sigma}}) \to \ell^2_{\mathtt{w}_{\mathtt{N}\sigma}}$ which conjugates the Hamiltonian to $D_\omega + Z_{\mathtt{N}} + R_{\mathtt{N}}$ where $Z_{\mathtt{N}}$ depends only on the actions and $R_{\mathtt{N}}$ has scaling $2\mathtt{N} + 2$. Of course, we have also estimates on the norms of $Z_{\mathtt{N}}$, $R_{\mathtt{N}}$, and $R_{\mathtt{N}} \sim r^{2\mathtt{N}+2}$. Now if we want a stability estimate in $\ell^2_{\mathtt{w}_s}$, we first leave $\mathtt{N}$ as a free parameter and fix $\sigma = s\mathtt{N}^{-1}$. This gives a stability estimate $\sim r^{-(2\mathtt{N}+2)}$. Finally, by optimizing $\mathtt{N}$, one gets the subexponential bounds. Now if we take any weight $\mathtt{w}$ and follow the same strategy, we only need to verify the immersions and control the homological equation, this is what we do in **[25]**.

The main difference in the Sobolev case is that in solving the homological equation, if $R \in \mathcal{H}(\ell^2_{\mathtt{w}})$ with $\mathtt{w}_j = \langle j\rangle^p$, then $S \in \mathcal{H}(\ell^2_{\mathtt{w}'})$ with $\mathtt{w}'_j = \langle j\rangle^{p+\tau}$ with τ fixed. This is a typical feature in the setting with finite regularity, in this context it produces the relation between stability time appearing in **[9]**.

3. QUASIPERIODIC SOLUTIONS

There is by now a vast literature on quasiperiodic solutions for NLS (mainly confined to the case when $\mathcal{M}$ is a torus or a sphere), covering also PDEs without external parameters and quasilinear PDEs. The first results in this direction (in the early 1990s, we mention, for example, Kuksin, Wayne, Craig, Bourgain, Pöschel) were for semilinear 1D PDEs with with periodic or Dirichlet boundary conditions. There were essentially two approaches, both quadratic iteration schemes generalizing Newton's steepest descent method:

(1) Extend *KAM theory of elliptic tori* to the infinite-dimensional setting (see **[67–70, 73, 80]**) thus proving not only existence but also linear stability.

This amounts to looking for an analytic symplectic change of variables which conjugates the Hamiltonian to a normal form where the invariant torus is flat, namely there exist a set of indexes $\mathcal{S} \subset \mathbb{Z}$ of cardinality n and a set of symplectic variables such that the torus in these variables is $u_j = 0$ for all $j \notin \mathcal{S}$ and $|u_j| = \text{const.}$ for $j \in \mathcal{S}$. Finally, the dynamics on the torus is a linear translation (with Diophantine frequency) and linearized dynamics in the normal directions to the torus is diagonal and elliptic.

(2) Look for the torus embedding $U : \mathbb{T}^n \mapsto \mathtt{h}$ (the phase space) as the solution of a nonlinear functional equation $F(U) = 0$. Apply a Newton method to construct successive approximations, provided that one has some control on the left inverse for the linearized operator $\mathrm{d}F(U)$ at an approximate quasiperiodic solution. The main difficulty is that $\mathrm{d}F(U)$ is a small perturbation of a diagonal operator whose spectrum accumulates to zero, thus there is a small divisor problem which is dealt with by a multiscale analysis. This is the so-called *Craig–Wayne–Bourgain* (CWB) approach, see [28, 40] and the papers [16, 18] for a more modern point of view. Of course, these two approaches have many similarities and can be combined in an effective way; see, for instance, [17].

To make the statements more concrete, let us restrict to the NLS equation (2). We fix a set $\mathcal{S} \subset \mathbb{Z}$ of cardinality n and assume, for simplicity, that $V_j = 0$ for all $j \notin \mathcal{S}$; finally, we fix an appropriate phase space (say, $\ell^2_{\mathtt{w}}$ for some weight, e.g., $\mathtt{w}_j = \langle j \rangle$). We look for solutions close to the n-dimensional approximately invariant torus $\mathcal{T}_n$ such that $|u_j| = 0$ for all $j \notin \mathcal{S}$ and $|u_j|^2 = I_j > 0$ otherwise. In a neighborhood of such torus, we can pass to "elliptic-action angle variables" $\chi : (\theta, y, z) \to u$, with $u_j = \sqrt{I_j + y_j} e^{\mathrm{i}\theta_j}$, for $j \in \mathcal{S}$ while $z_j = u_j$ for $j \notin \mathcal{S}$. In these variables the NLS Hamiltonian reads

$$H_{\mathrm{NLS}} = \sum_{j \in \mathcal{S}} (j^2 + V_j) y_j + \sum_{j \notin \mathcal{S}} j^2 |u_j|^2 + \mathcal{P}. \tag{12}$$

Now the KAM scheme ensures, for many values of V, the existence of a bounded symplectic change of variables, defined in a neighborhood of $\mathcal{T}_n$, which conjugates the Hamiltonian to

$$\widetilde{\omega}(V) \cdot y + \sum_{j \in \mathbb{Z} \setminus \mathcal{S}} \Omega_j(V) |z_j|^2 + \mathcal{P}_{\mathrm{fin}}, \quad \mathcal{P}_{\mathrm{fin}} = O(y^2 + yz + z^3), \tag{13}$$

where $\widetilde{\omega}, \Omega$ are appropriate real functions of V. This means not only that $\mathcal{T}_n$ is now invariant, but also that the dynamics in the normal directions z_j is (at least at the linear level) the rotation by $e^{\mathrm{i}\Omega_j t}$.

Conversely, with the CWB method one can conjugate the NLS Hamiltonian to a normal form like (12), but where the quadratic terms in z are neither diagonal nor independent of θ.

While the first approach is technically simpler and gives a stronger result, it requires stronger hypotheses which give some control on the difference of distinct eigenvalues of the linearized equation at an approximately invariant torus, that are not verified for many physically interesting PDEs. Indeed, in the case of manifolds of dimension greater than one, the first results were by the CWB method, we mention, for instance, [29, 31] for the NLS on tori and [19] for a forced NLS on simple compact Lie groups.

Regarding linear stability issues, note that if one proves existence of solutions (via CWB) then one can prove the linear stability a posteriori, for instance, by proving that the PDE linearized at the quasiperiodic solution is "reducible," i.e., can be conjugated to constant coefficients (or even diagonalized) via a time quasiperiodic change of variables on the phase space. Then the stability can be inferred by solving the linear dynamics, which becomes trivial. In this setting if one wants to conjugate via a close to identity change of variables (since the solution is small, the linearized operator is close to diagonal, and one hopes to apply some perturbative argument), one has to deal with small divisors related to the differences of eigenvalues, just as in the KAM case. This is just like diagonalization algorithms for finite-dimensional matrices close to a diagonal one, where one needs distinct eigenvalues in order to apply perturbative arguments. Of course, in infinite dimension, differences of eigenvalues may also accumulate to zero (and typically do in our setting), so the best hope is to impose some nonuniform lower bound. Thus, proving linear stability for the solutions of PDEs in dimension higher than one is typically a rather difficult question, due to the multiplicity of the eigenvalues, the idea is to introduce a partition of the eigenvalues into clusters so that one has control on the difference of eigenvalues in different clusters while the dynamics inside a cluster is stable.

A breakthrough was in [44, 45] where the authors proved reducibility for the NLS equation with a convolution potential on $\mathbb{T}^d$. This requires a subtle analysis and the introduction of the class of Töplitz–Lipschitz functions. Their approach is based on a good control of the asymptotics of eigenvalues of operators of the form $-\Delta + V(x, \omega t)$ where V is periodic in all its variables and $x \in \mathbb{T}^d$, see also [12] for a discussion on general flat tori. As far as I am aware, the only other manifolds on which there are reducibility results are spheres (see [51] for Zoll manifolds). Instead of the reducibility, one can concentrate on the control of Sobolev norms for the corresponding linear operator. This has been discussed by many authors, see [10, 11, 24, 30, 41].

Regarding the question of parameterless PDEs, most results are in the 1D case starting from [70, 75]. Let us briefly discuss the completely resonant (NLS) with $V = 0$ and $\mathcal{M} = \mathbb{T}^d$. The idea is to first perform one step of Birkhoff normal form in order to extract parameters from the initial data. Unfortunately, if $d > 1$, the normal form is *not integrable* and actually has a rather complicated structure. This is well known and used, for instance, in [33] in order to prove explosion of Sobolev norms. Building on the paper [56] for the case $d = 2$, in [76–78] we discussed this problem and showed that in the neighborhood of appropriately chosen initial data the NLS Hamiltonian after one step of BNF is indeed integrable, satisfies the twist condition, and has appropriately controlled distinct eigenvalues.

Interestingly, the good initial data are found by first choosing the Fourier support $\mathcal{S} \subset \mathbb{Z}$ in a generic way (i.e., outside the zero set of some nontrivial polynomial) and then by choosing the actions on such support in some Cantor set. This allows proving the following theorem for any equation of the type (NLS) with $\mathcal{M} = \mathbb{T}^d$ and $V = 0$ (see also [79]):

Theorem ([78]). *Fix any n and any choice of generic frequencies $S = \{\mathtt{j}_1, \dots, \mathtt{j}_d\} \subset \mathbb{Z}^n$. For ε sufficiently small, there exists a compact set $\mathcal{C}_\varepsilon \in [\varepsilon/2, \varepsilon]^n$ of positive measure,* *parametrizing bijectively a set of analytic quasiperiodic solutions of NLS of the type*

$$\mathcal{C}_\varepsilon \ni \xi \mapsto u(\xi, x, t) = \sum_{j \in \mathcal{S}} \sqrt{\xi_j} e^{\mathrm{i}t(|j|^2 + \omega_j(\xi))} e^{\mathrm{i}j \cdot x} + O(\xi^2).$$

Moreover, the linearized NLS operator at a quasiperiodic solution is conjugated to a constant coefficient block-diagonal form with uniform bounds on the dimension of the blocks.

In (NLS), the nonlinearity is analytic and so the quasiperiodic solutions we have discussed are at least C^∞. If one considers nonlinearities with finite (but rather high) regularity then one can obtain analogous results (both KAM and Nash–Moser) for finite regularity solutions.

All the results described above are for semilinear PDEs. In order to deal with the quasilinear case, where the derivatives in the nonlinearity have the same order of the linear part, one needs to introduce new perspectives. A real breakthrough appeared in [2], where the authors introduced ideas from pseudodifferential calculus (see also [64]) to produce a general method applicable to PDEs on the circle, we mention [2,3,49,54], as well as [1,22,48] for the water wave equation. There have been some extensions of these results to higher dimensions; we mention [4,38,71].

3.1. Questions and open problems

Q6. Can one develop a "general" pseudodifferential approach to deal with quasilinear dispersive PDEs in high dimension?

Even on tori, the results up to now rely on special features of the equations. An interesting strategy was developed in [11] for a linear NLS (see also [72]).

Q7. Can one study the NLS with external parameters or even a multiplicative potential as in [16] when $\mathcal{M}$ is a compact Lie group? And having done this, can one prove a reducibility result? Can one deal with the parameterless case?

These questions are largely open and interesting even in the case when $\mathcal{M}$ is an irrational torus.

Q8. It is expected that the solutions described in this section are linearly stable or have at most a finite number of linearly unstable directions. What kind of normal form can be achieved close to the tori? This was discussed, for instance, in [15]. What can be said about nonlinear stability/instability?

In [46] the authors discuss polynomial stability times close to a periodic plane wave solution. For the NLS on $\mathbb{T}^2$, there are a number of instability results, stemming from the paper [33]; we mention [62] close to the plane wave solutions and [60] close to one-dimensional quasiperiodic solutions.

4. ALMOST PERIODIC SOLUTIONS

By definition, *almost-periodic solutions* are solutions which are limits (in the uniform topology in time) of quasiperiodic solutions. A very naïve approach would be to find them by just constructing quasiperiodic solutions supported on invariant tori of dimension n and then take the limit $n \to \infty$. Unfortunately, the KAM procedure (of, say, [70,74]) is not uniform in the dimension n, and, by taking the limit, one just falls on the elliptic fixed point.

A refined version of this very natural idea is to construct a sequence of invariant tori of growing dimension using at each step the invariant torus of the previous one as an unperturbed solution: in this way, the $(n+1)$th and nth tori are extremely close, leading to very regular solutions. This was done by Pöschel [75] by using the KAM method and by Bourgain [28] via the Nash–Moser approach, getting solutions which decay at least superexponentially (see also [55] for solutions with exponential decay).

A different approach was proposed by Bourgain in [32] to study the translation-invariant NLS (2). The idea is to construct a converging sequence of infinite-dimensional approximately-invariant manifolds and prove that the limit is the support of the desired almost-periodic solution. The fact that one does not restrict to neighborhoods of finite-dimensional tori allows for a better control of the small-divisors and hence the construction of more general, i.e., *less regular* solutions in spaces of Gevrey regularity. A drawback of Bourgain's approach is that it works only for "maximal tori," namely one has to impose some lower bounds on the actions of the approximately invariant ∞-dimensional tori. His statement concerns the quintic NLS on $\mathbb{S}^1$, i.e., (2) with $f(y) = 3y^2$. Here we give a slightly more general version taken form [35], where the authors prove also stability of the tori.

Theorem (Gevrey almost periodic solutions, [32]). *Fix $s > 0$, $0 < \theta < 1$. For all $0 < r \ll 1$ small enough, and any "approximate initial datum"*

$$u_0(x) = \sum_{j \in \mathbb{Z}} \sqrt{I_j} e^{\mathrm{i} j x} \quad \text{such that } r/2 < \sqrt{I_j} e^{s\langle j \rangle^\theta} \langle j \rangle^2 < r, \tag{14}$$

there exists a set of positive measure in $[-1/2, 1/2]^{\mathbb{Z}}$ (depending on u_0) such that for all V in such a set there exists one almost-periodic solution with $|\sqrt{I_j} - |u_j(t)|| \ll r e^{-s\langle j \rangle^\theta} \langle j \rangle^{-2}$.

Similar results were proved in [37] for the wave equation.

In [26] we gave a more precise description of these solutions, proving that for all frequencies $\omega \in \mathtt{D}_\gamma$ and *for any* approximate initial datum u_0 in a small ball in $\ell^\infty_{\mathtt{w}_s} := \{u : (e^{s\langle j \rangle^\theta} \langle j \rangle^2 u_j)_{j \in \mathbb{Z}} \in \ell^\infty\}$, there exists a potential $V = V(\omega, u_0) \in \ell^\infty$ such that the corresponding NLS equation has an almost periodic solution of frequency ω close to $\sqrt{I}$. Furthermore, in [26] we developed a strategy which allows constructing in a unified context tori of Gevrey regularity and of any dimension essentially supported on the Fourier modes belonging to any subset $\mathcal{S} \subseteq \mathbb{Z}$. Essentially, this amounts to Bourgain's result, but in (14) we only need $\sqrt{I_j} e^{s\langle j \rangle^\theta} \langle j \rangle^2 < r$. This is an interesting novelty because, with Bourgain's condition, the acceptable u_0 are of zero measure.

In [26] we discussed Gevrey solutions, but one can find even less regular solutions, see [34]. However, the question of finding maximal tori which are not C^∞ is still open. If one looks for "nonmaximal" tori, approximately supported on an infinite set $\mathcal{S}$, then one can reach very low regularity. Again, just as in the quasiperiodic case, the choice of the support can be used as a precious additional source of parameters. Given a function $u : \mathbb{R}^2 \to \mathbb{C}$

which is 2π-periodic in x and such that the map $t \mapsto u(t,\cdot) \in \mathcal{F}(\ell^1)$ is continuous,[8] we say that u is a weak solution of (2) if, for any smooth compactly supported function $\chi : \mathbb{R}^2 \to \mathbb{R}$, one has

$$\int_{\mathbb{R}^2} (-\mathrm{i}\chi_t + \chi_{xx})u - \big(V * u + f\big(|u|^2\big)u\big)\chi \, dx \, dt = 0. \tag{15}$$

Theorem ([27]). *For almost every Fourier multiplier V, there exist infinitely many small-amplitude weak almost-periodic solutions of* (2). *Infinitely many of such solutions are not classical and infinitely many are classical.*

Unfortunately, such solutions are not in any way typical and, in fact, correspond to very special infinite-dimensional elliptic tori.

The question whether full-dimensional tori exist in the Sobolev class is still open. Apart from the interest *per se*, these low regularity solutions could be used in order to find solutions for parameterless PDEs. Essentially one wants to solve the "countertem equation" $V(\omega, u_0) = 0$ by finding $u_0 = u_0(\omega)$.

4.1. Questions and open problems

Q9. Are almost-periodic solutions generic in some Banach space? For example, is it true that for many convolution potentials the tori cover a positive measure set (with respect to the probability product measure in the Gevrey space $B_r(\ell^\infty_{\mathrm{w}_s})$)?

Q10. Can one construct maximal tori with Sobolev regularity?

Q11. Can one construct almost-periodic solutions for the NLS on higher-dimensional manifolds?

At least in the case of tori, most of the strategy proposed by Bourgain can be generalized, the main point here seems to be the choice of a smart Diophantine condition.

Q12. Can one construct almost-periodic solutions for parameterless NLS equations? Here even the case of 1D NLS with generic multiplicative potentials would be interesting.

Q13. Can one deal with unbounded nonlinearities?

This has been discussed in the case of a forced quasilinear Airy equation in [39], generalizing the approach for the quasiperiodic case.

Q14. Can one construct almost-periodic solutions for small perturbations of integrable PDEs?

In the case of quasiperiodic solutions, there are a number of results, we mention [23,65,66,68]. In order to cover the almost-periodic case, the main point is to control convexity properties for the Hamiltonian in action angle variables.

4.2. An idea of the strategies

Let us first discuss the linear case. Recall that we are restricting to 1D NLS with convolution potential so that the linear actions $|u_j|^2$ are constants of motions and the dynamics

8 Here $\mathcal{F}$ is the usual Fourier transform.

is

$$u_j(t) = u_j(0)e^{\mathrm{i}\omega_j t}, \quad j \in \mathbb{Z}, \quad \omega = (\omega_j)_{j\in\mathbb{Z}},\ \omega_j := j^2 + V_j.$$

Let us call $\mathcal{S}_0 := \{j \in \mathbb{Z} \mid u_j(0) \neq 0\}$.

If $\mathcal{S}_0$ is a finite set, the corresponding solution $u(t,x) := \sum_{j\in\mathbb{Z}} u_j(t)e^{\mathrm{i}jx}$ is quasi-periodic and analytic both in time and space.

If $\mathcal{S}_0$ is infinite, the regularity of $u(t,x)$ obviously depends on that of the initial datum. If $u(0) := (u_j(0))_{j\in\mathbb{Z}} \in \ell^1$ then $u(t,x)$ is a weak solution of (2). Moreover, such a solution is a time almost-periodic function, being the limit of the quasiperiodic truncations $\sum_{|j|\le n} u_j(0)e^{\mathrm{i}\omega_j t + \mathrm{i}jx}$ as $n \to \infty$. Note, finally, that the regularity (in any reasonable weighted space) is that of the initial datum. The support of each solution is an invariant torus: given an initial datum $u(0)$, set $I := (I_j)_{j\in\mathbb{Z}}$ with $I_j = |u_j(0)|^2$, then the motion is supported on $\mathcal{T}_I := \{u : |u_j|^2 = I_j\ \forall j \in \mathbb{Z}\}$.

Now the nature of these invariant tori strongly depends on the choice of the phase space. When discussing the stability of zero, a natural context was to work with the Hilbert spaces ℓ^2_{w}, which induce on the tori the product topology. In this context, however, this produces a number of problems, related to the density of finite-dimensional tori. In KAM algorithms, one typically wants to "Taylor expand" close to the approximately-invariant tori, but this requires a Banach manifold structure, so even though the product topology is the natural one with respect to the group structure, the KAM algorithm seems to require a finer choice, e.g., weighted spaces based on ℓ^∞,

$$\ell^\infty_{\mathrm{w}} := \Big\{u := (u_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{C}) : |u|_{\mathrm{w}} := \sup_{j\in\mathbb{Z}} \mathrm{w}_j |u_j| < \infty\Big\}. \tag{16}$$

Given a sequence $I = (I_j)_{j\in\mathbb{Z}}$ with $I_j \ge 0$ and $\sqrt{I} := (\sqrt{I_j})_{j\in\mathbb{Z}} \in \ell^\infty_{\mathrm{w}}$, we consider the torus

$$\mathcal{T}_I := \big\{u \in \ell^\infty_{\mathrm{w}} : |u_j|^2 = I_j\ \forall j \in \mathbb{Z}\big\}. \tag{17}$$

Now the map

$$\mathrm{i} : \mathbb{T}^{\mathcal{S}_0} \to \mathcal{T}_I \subset \mathrm{w}_p, \quad \varphi = (\varphi_j)_{j\in\mathcal{S}_0} \mapsto \mathrm{i}(\varphi), \quad \mathrm{i}_j(\varphi) := \begin{cases} \sqrt{I_j}e^{\mathrm{i}\varphi_j} & \text{for } j \in \mathcal{S}_0, \\ \mathrm{i}_j(\varphi) := 0 & \text{otherwise}, \end{cases} \tag{18}$$

is an analytic immersion provided that we endow $\mathbb{T}^{\mathcal{S}_0}$ with the ℓ^∞-topology. Note that, assuming also that $\inf_j \sqrt{I_j}\mathrm{w}_j > 0$, the map i is an embedded torus, in a neighborhood of which one can construct local action angle variables. By construction, the linear dynamics on the torus $\mathcal{T}_I$ is $\varphi \to \varphi + \omega t$.

Since the map $t \mapsto \omega t \in \mathbb{T}^{\mathcal{S}_0}$ is not even continuous (endowing $\mathbb{T}^{\mathcal{S}_0}$ with the ℓ^∞-topology and recalling that $\omega_j \sim j^2$), the regularity of $t \mapsto \mathrm{i}(\omega t)$ depends on the choice of the actions I_j. If we assume $\inf_j \sqrt{I_j}\mathrm{w}_j > 0$, then it is not continuous with respect to the strong[9] topology.

9 Note that the map is continuous with respect to the product topology, which coincides with the weak-$*$ topology on bounded sets.

In contrast with the finite-dimensional case, even if ω has rationally independent entries, it is not straightforward to understand whether this invariant torus is densely filled[10] by the solution's orbit or not. In fact, this issue is related to the asymptotic behavior of ω. For example, if we require that $\mathcal{S}_0$ is a very sparse set then the density follows, see [27].

We say that $\mathcal{T}_I$ is a KAM torus of frequency $\omega \in \mathbb{R}^{\mathbb{Z}}$ for the Hamiltonian N if it has the form $\sum_{j\in\mathbb{Z}} \omega_j |u_j|^2 + P$, with $P = O(|u|^2 - I)^2$, so that the Hamiltonian vector field X_P vanishes on the torus $\mathcal{T}_I$. Indeed, under the hypotheses above, $\mathcal{T}_I$ is invariant and the restricted dynamics is linear with frequency ω, namely

$$u_j(t) = u_j(0)e^{\mathrm{i}\omega_j t}, \quad \left|u_j(0)\right|^2 = I_j, \quad j \in \mathbb{Z}. \tag{19}$$

Note that in this definition the only relevant frequencies are those corresponding to nonzero actions.

Let us now fix the support of the solution by taking a subset $\mathcal{S} \subset \mathbb{Z}$ and consider $\sqrt{I} \in \bar{B}_r(\ell^\infty_{\mathrm{w}})$ with $I_j = 0$ for $j \in \mathcal{S}^c$. We say that the torus $\mathcal{T}_I$ is an elliptic KAM torus of frequency $\nu \in \mathbb{R}^{\mathcal{S}}$ for the Hamiltonian N with normal frequency $(\Omega_j)_{j\in\mathbb{Z}\setminus\mathcal{S}}$ if, setting for notational convenience $u_j = v_j$ for $j \in \mathcal{S}$ and $u_j = z_j$ otherwise, one has (compare with (13) with $y = |v|^2 - I$)

$$N = \sum_{j\in\mathcal{S}} \nu_j |v_j|^2 + \sum_{j\in\mathbb{Z}\setminus\mathcal{S}} \Omega_j |z_j|^2 + R, \quad R = O\big(\big(|v|^2 - I\big)^2 + \big(|v|^2 - I\big)z + z^3\big).$$

We can now state our version of KAM theorem for infinite tori. We shall concentrate on the elliptic tori and in particular on the low regularity case. To this purpose, for $p > 1$, we consider the Sobolev space $\ell^\infty_{\mathrm{w}_p}$ where now $\mathrm{w}_p = \langle j\rangle^p$. In order to work in this low regularity setting, we need to impose some conditions on $\mathcal{S}$ requiring that it is sufficiently sparse (for instance, $\mathcal{S} = 2^{\mathbb{N}}$). For any such $\mathcal{S}$, for all $r > 0$ sufficiently small, for every $\sqrt{I} \in B_r(\ell^\infty_{\mathrm{w}_p})$ with $I_j = 0$ for $j \in \mathcal{S}^c$, we have

Theorem (Sobolev case [27]). *There exists a positive measure Cantor-like set in* $[-1/2, 1/2]^{\mathbb{Z}}$ *and for all V in this set there exists a close to identity change of variables* $\Phi : B_r(\ell^\infty_{\mathrm{w}_p}) \to \ell^\infty_{\mathrm{w}_p}$ *such that $\mathcal{T}_I$ is an elliptic KAM torus* $H_{\mathrm{NLS}} \circ \Phi$.

To give an idea of the proof, let us restrict to the maximal case. By the very definition of a KAM torus, we wish to decompose a regular Hamiltonian as a sum of regular terms with an increasing "order of zero" at $\mathcal{T}_I$. Namely, given a Hamiltonian $H \in \mathcal{H}_r(\ell^\infty_{\mathrm{w}})$, we wish to write it as sum of three terms, all in $\mathcal{H}_r(\ell^\infty_{\mathrm{w}})$,

$$H = H^{(-2)} + H^{(0)} + H^{(\geq 2)}$$

so that $X_{H^{(-2)}}$ is not tangent to $\mathcal{T}_I$, $H^{(0)}$ vanishes at $\mathcal{T}_I$ and its vector field is tangent but not necessarily null, while $H^{(\geq 2)} = O(|u|^2 - I)^2$ (this means that the corresponding vector field vanishes at $\mathcal{T}_I$). The main point is to make a power series expansion centered at I without introducing a singularity at zero. Start from a regular Hamiltonian $H(u)$ expanded in Taylor

10 In the product topology such solutions are always dense.

series at $u = 0$ and rewrite every monomial as $|u|^{2m} u^\alpha \bar{u}^\beta$ with α, β with distinct support. Then define an *auxiliary Hamiltonian* $\mathtt{H}(u, w)$ (here $w = (w_j)_{j\in\mathbb{Z}}$ are auxiliary "action" variables) by the substitution $|u|^{2m} u^\alpha \bar{u}^\beta \rightsquigarrow w^m u^\alpha \bar{u}^\beta$.

Since we are considering functions on an ℓ^∞ space, it turns out that $\mathtt{H}(u, w)$ is analytic in both u and w. In particular, we can Taylor expand with respect to w at the point $w = I$, with I being in the domain of analyticity.

Then we set $H^{(-2)}(u) := \mathtt{H}(u, I)$, $H^{(0)}(u) := D_w \mathtt{H}(u, I)[|u|^2 - I]$, and $H^{(\geq 2)}(u)$ is what is left. As an example, the Hamiltonian $H = |u_1|^2 |u_2|^4 \operatorname{Re}(u_1 \bar{u}_3)$ has auxiliary Hamiltonian $\mathtt{H}(u, w) = w_1 w_2^2 \operatorname{Re}(u_1 \bar{u}_3)$ and decomposes into

$$
\begin{aligned}
&H^{(-2)} := I_1 I_2^2 \operatorname{Re}(u_1 \bar{u}_3), \quad H^{(0)} := \big[I_2^2\big(|u_1|^2 - I_1\big) + 2 I_1 I_2 \big(|u_2|^2 - I_2\big)\big] \operatorname{Re}(u_1 \bar{u}_3), \\
&H^{(\geq 2)} := \big(|u_1|^2\big(|u_2|^2 - I_2\big) + 2 I_2 \big(|u_1|^2 - I_1\big)\big)\big(|u_2|^2 - I_2\big) \operatorname{Re}(u_1 \bar{u}_3).
\end{aligned}
$$

The above decomposition is, *at a formal level*, the same introduced by Bourgain in [32], but in [26] we show that it is, in fact, a direct sum decomposition $\mathcal{H}_r(\ell^\infty_{\mathtt{w}}) = \mathcal{H}_r^{(-2)}(\ell^\infty_{\mathtt{w}}) \oplus \mathcal{H}_r^{(0)}(\ell^\infty_{\mathtt{w}}) \oplus \mathcal{H}_r^{(\geq 2)}(\ell^\infty_{\mathtt{w}})$, with explicit control on the projections. An important point is that all our construction works independently of the "dimension" of $\mathcal{T}_I$, namely it never requires conditions of the form $I_j \neq 0$.

Let us compare this decomposition with that used for finite-dimensional tori. Note that in this case $\mathcal{S} = \mathbb{Z}$ so there are no "normal variables" z.

We consider the example above and pass to action–angle variables $\chi : (\theta, y) \to u$, with $u_j = \sqrt{I_j + y_j} e^{\mathrm{i}\theta_j}$. Then the terms canceled in a classical KAM scheme would be the first two terms in the Taylor expansion at $y = 0$ of $\mathcal{P}(\theta, y) := H \circ \chi = (I_1 + y_1)(I_2 + y_2)^2 \sqrt{(I_1 + y_1)(I_3 + y_3)} \cos(\theta_1 - \theta_3)$, namely

$$
\begin{aligned}
&\mathcal{P}(\theta, 0) = I_1^{\frac{3}{2}} I_2^2 \sqrt{I_3} \cos(\theta_1 - \theta_3), \\
&\mathcal{P}_y(\theta, 0)[y] = \left(\frac{3}{2} I_2 \sqrt{I_3} y_1 + 2 I_1 \sqrt{I_3} y_2 + \frac{I_1 I_2}{2\sqrt{I_3}} y_3\right) I_2 \sqrt{I_1} \cos(\theta_1 - \theta_3).
\end{aligned}
$$

Direct computations show that

$$
\begin{aligned}
H^{(-2)} \circ \chi + H^{(0)} \circ \chi &= I_1 I_2^2 \sqrt{(I_1 + y_1)(I_3 + y_3)} \cos(\theta_1 - \theta_3) \\
&\quad + \big(I_2^2 y_1 + 2 I_1 I_2 y_2\big) \sqrt{(I_1 + y_1)(I_3 + y_3)} \cos(\theta_1 - \theta_3) \\
&= \mathcal{P}(\theta, 0) + \mathcal{P}_y(\theta, 0)[y] + O(y^2),
\end{aligned}
$$

and, obviously, $H^{(\geq 2)} \circ \chi$ is at least quadratic in y. In conclusion, we are canceling more terms than is strictly necessary, but in doing so we avoid introducing the singularity $I = 0$.

Now our result is proved by an iterative procedure. To get a feeling of the proof, let us consider the 1D NLS case (2) (recalling that $H = D_\omega + P$, with P small and $D_\omega = \sum \omega_j |u_j|^2$) and perform the first step. Just like in the case of the stability of zero and of quasiperiodic solutions, once we have identified the terms $P^{(-2)}$, $P^{(0)}$ (which are the obstacles to $\mathcal{T}_I$ being a KAM torus), we perform a change of variables $e^{\operatorname{ad} S}$ to cancel them. It is convenient to look for $S = S^{(-2)} + S^{(0)}$ (namely such that the component of degree ≥ 2 is zero).

Our aim is to make the projections $\Pi^{(-2)}$, $\Pi^{(0)}$ of the new Hamiltonian, $e^{\mathrm{ad}_S}(D_\omega + P)$, “quadratically smaller” with respect to $P^{(-2)}, P^{(0)}$.

Now one can directly verify that this is achieved by choosing S as the solution of a homological equation, which now is (recall the projections on $\mathcal{R}$ defined in (10))

$$\Pi_{\mathcal{R}} P^{(-2)} + \{S^{(-2)}, D_\omega\} = 0, \tag{20}$$

$$\Pi_{\mathcal{R}} P^{(0)} + \{S^{(0)}, D_\omega\} + \Pi^{0,\mathcal{R}}\{S^{(-2)}, P^{\geq 2}\} = 0. \tag{21}$$

Now $P^{(-2)}, P^{(0)} \in \mathcal{H}_r(\ell^\infty_{\mathtt{w}})$ (they are analytic in a neighborhood of zero). If we chose Gevrey regularity $\mathtt{w} = \mathtt{w}_s = \langle j\rangle^2 e^{s\langle j\rangle^\theta}$, this allows us to solve the homological equation, using the estimates (11), which hold in ℓ^∞ as well. We reach a new Hamiltonian of the form $\sum_{j\in\mathbb{Z}}(\omega_j + \lambda_j)|u_j|^2 + P_1$, with $\lambda \in \ell^\infty$ and for $r_1 < r$ and $\sigma_1 > 0$, one has that $P_1^{(-2)}, P_1^{(0)}$ are quadratically smaller in the (nested) space $\mathcal{H}_{r_1}(\ell^\infty_{\mathtt{w}_{\sigma_1}})$. At the next step one repeats this procedure just with a slightly different frequency, decreasing at each step n the radius r_n and increasing σ_n in a summable way. Actually, in [26] we write all the equations in terms of the final frequency, and use a counterterm theorem á la Herman.

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