DYNAMICS AND "ARITHMETICS" OF HIGHER GENUS SURFACE FLOWS

CORINNA ULCIGRAI

ABSTRACT

We survey some recent advances in the study of (area-preserving) flows on surfaces, in particular on the typical dynamical, ergodic, and spectral properties of smooth area-preserving (or *locally Hamiltonian*) flows, as well as recent breakthroughs on *linearization* and *rigidity* questions in higher genus. We focus in particular on the *Diophantine-like conditions* which are required to prove such results, which can be thought of as a generalization of *arithmetic conditions* for flows on tori and circle diffeomorphisms. We will explain how these conditions on higher genus flows and their Poincaré sections (namely generalized interval exchange maps) can be imposed by controlling a renormalization dynamics, but are of more subtle nature than in genus one since they often exploit features which originate from the nonuniform hyperbolicity of the renormalization.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 37-02; Secondary 37E35, 37E05, 11J70, 37A25, 37A30

KEYWORDS

Flows on surfaces, ergodic theory, area-preserving, locally Hamiltonian, multi-valued Hamiltonian, parabolic dynamics, interval exchange transformations, generalized interval exchange transformations, linearization, Diophantine conditions, renormalization, Rauzy-Veech induction, Rauzy-Veech cocycle

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 5, pp. 3576–3609
DOI 10.4171/ICM2022/143

1. INTRODUCTION

Flows on surfaces are among the most basic and fundamental examples of dynamical systems. First of all, they are among the lowest possible dimensional smooth systems; furthermore, many models of systems of physical origin are described by flows on surfaces, starting from celestial mechanics, up to solid state physics or statistical mechanics models. The beginning of the study of surface flows can be dated back to Poincaré [63] at the end of the 19th century, and coincides with the birth of dynamical systems as a research field. Poincaré was in particular interested in the study of flows on *tori*, or surfaces of *genus one*. Several famous systems in physics lead naturally to the study of flows on surfaces of *higher genus*, which, in this survey, will mean genus $g \geq 2$. Examples include the Ehrenfest model in statistical mechanics (related to a linear flow on a translation surface of genus five), or the Novikov model in solid state physics, which is described by locally Hamiltonian flows, a class which will be one of the central themes of this survey (see Section 3.1).

There is a rich history of results on the topological and qualitative behavior of trajectories (see, for example, [60] and the references therein), as well as on the ergodic theory of certain well-studied classes of flows (for example, in genus one, in relation with KAM theory, see Section 2, and linear flows on translation surfaces, whose study is intertwined with Teichmüller dynamics, see Section 3). Many fundamental problems, though, in particular on the mathematical characterization of chaos (such as dynamical, spectral, and rigidity questions) in various natural classes of surface flows, in particular smooth flows preserving a smooth measure, were only recently understood and many others are still open (see Section 3.1).

One of the reasons for this late development is perhaps that, in order to investigate fine chaotic or rigidity properties of flows in higher genus, one needs to impose quite delicate assumptions on the behavior of orbits on different scales. To capture these multiscale features, the concept of *renormalization* plays a crucial role (see Section 4). In the case of genus one, the assumptions on the flow often take the form of *Diophantine conditions* or, more generally, of *arithmetic conditions* on the rotation number (see Section 5) and control how well the flow orbits are approximated by *periodic orbits*. The renormalization point of view on these conditions is that they can be described in terms of continued fraction theory and therefore studying the dynamics of the Gauss map, or, equivalently, geometrically, studying the geodesic flow on the modular surface, both of which are classically well understood.

In higher genus, on the other hand, one had to wait for the development of the rich and fruitful theory of renormalization in Teichmüller dynamics (see Section 4). This theory provides a renormalization framework (initially developed to study ergodic properties of rational billiards, interval exchange transformations, and translation flows), which can be exploited to understand when a surface flow is *renormalizable* (see Sections 3.2 and 4) and when it preserves a smooth invariant measure; in the latter case, then, it allows imposing conditions on a (smooth) surface flows to guarantee the presence of particular chaotic properties (see Section 3.1). The type and nature of what we refer to as *Diophantine-like conditions* in higher genus, which is much more delicate than in genus one and often involves assumptions on *hyperbolicity* of the renormalization, will be the leading theme of this survey. These conditions are sometimes also called *arithmetic conditions*, by analogy with the genus one case, even though the relation with classical arithmetic and Diophantine equations is lost when the genus is greater than one.

In what follows, we first start in the next Section 2 with the classical case of flows on *genus one surfaces*, recalling some of the classical results on the *linearization problem* and ergodic properties and discussing the related arithmetic conditions. Then, in Section 3, we will briefly overview some of the rapid developments in our understanding of ergodic, spectral, and disjointness properties of (smooth) area-preserving flows on higher genus surfaces (see Section 3.1), as well as linearization and rigidity problems in higher genus (in Section 3.2). After having introduced the notion of *renormalization* in this setting (see Section 4), we then focus in Section 5 on the Diophantine-like conditions behind these results.

2. FLOWS ON SURFACES OF GENUS ONE AND CLASSICAL ARITHMETIC CONDITIONS

A central idea introduced by Poincaré was that the study of a surface flow can be often *reduced* to the study of a one-dimensional discrete dynamical system, by taking what we nowadays call a *Poincaré* section and considering the *Poincaré* first return map of the flow to the section (when and where it is defined). If we start from a flow $\varphi_{\mathbb{R}} := (\varphi_t)_{t\in\mathbb{R}}$ on a torus, i.e., on a compact, orientable surface S of genus one, and assume that it does not have fixed points, or closed orbits (or, more generally, Reeb components, see [60]), there is a (global) section given by a closed transverse curve and the Poincaré first return map to it is a diffeomorphism $f : S^1 \to S^1$ of the circle $S^1 \cong \mathbb{R}/\mathbb{Z}$. The simplest example of *circle diffeomorphism* (or *circle diffeo* for short) is a (rigid) *rotation*, i.e., the map $R_\alpha(x) = x + \alpha \mod 1$ on $\mathbb{R}/\mathbb{Z} = [0,1]/\sim$. A key concept associated to circle diffeomorphisms is that of *rotation number*: if μ is an *invariant* probability *measure* for the circle diffeo f (which always exists by Krylov–Bogolyubov theorem), the rotation number $\rho(f)$ of f can be seen as an *average displacement* of points, namely $\rho(f) = \int_0^1 (F(x) - x)\, \mathrm{d}\mu(x) \mod 1$ where $F : \mathbb{R} \to \mathbb{R}$ is a lift of $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. The rotation R_α can be seen as the *linear model* of a circle diffeo with rotation number α.

The *topological behavior* of trajectories of $(\varphi_t)_{t\in\mathbb{R}}$ can be completely understood and classified exploiting the *rotation number* (this is essentially the content of *Poincaré classification theorem*, see [36] for an expository account): when $\rho(f) \in \mathbb{Q}$, there exist periodic orbits (which either *foliate* the surface S, or are attracting or repelling). On the other hand, when $\rho(f) \notin \mathbb{Q}$, the dynamics of $(\varphi_t)_{t\in\mathbb{R}}$ is either *minimal* on the whole surface (i.e., all orbits are *dense*), or minimal when restricted to a *Cantor-like* invariant limit set (locally a product of a Cantor set with $\mathbb{R}$). In the latter case, we speak of *Denjoy-counterexamples*; their existence is ruled out when the diffeo (and the flow) is sufficiently smooth, for example, $\mathscr{C}^2$ in view of Denjoy's work [15] (less regularity, in particular $\mathscr{C}^1$ with bounded variation derivative, suffices, see, e.g., [36] for more details).

Arithmetic conditions for linearization of circle diffeomorphisms. To gain a finer understanding of the dynamics and describe the ergodic behavior of almost-every trajectory with respect to a smooth measure, one has to address the *linearization problem*, a classical question which is at the heart of the theory of circle diffeomorphisms. Namely, one wants to understand when a circle diffeomorphism T is *linearizable*, i.e., conjugate to a rigid rotation R_α (i.e., when there exists a homeomorphism $h : S^1 \to S^1$, called the *conjugacy*, such that $R_\alpha \circ h = h \circ T$) and what is the *regularity* of the conjugacy h. To address this question, one needs to put further assumptions both on the *regularity* of the diffeo and, in relation to it, the irrationality of the *rotation number*.

We recall that *arithmetic conditions* are conditions that prescribe how well (or how *badly*) the irrational rotation number $\alpha \in \mathbb{R}$ is approximated by *rational* numbers and morally control how well the flow orbits are approximated by *periodic orbits*. The best known such condition is perhaps the (classical) *Diophantine condition* (or DC, for short): $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is said to be *Diophantine* (of exponent $\tau \geq 0$) iff there exists $C > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^{2+\tau}}, \quad \text{for all } p, q \in \mathbb{Z},\ q \neq 0.$$

If the above condition holds for $\tau = 0$, we say that α is *badly approximable* or *bounded-type*. Equivalently, the DC can be rephrased in terms of the continued fraction expansion $[a_0, a_1, \dots, a_n, \dots]$ of α: if q_n denotes the *convergents* of α, namely the denominators of the partial approximations $p_n/q_n := [a_0, a_1, \dots, a_n]$, the DC is equivalent to the growth control $a_{n+1} = O(q_n^\tau)$. In particular, α is of bounded type iff a_n are uniformly bounded.

The *local theory* of linearization of circle diffeos, which treats the case of diffeos $f : S^1 \to S^1$ which are $\mathscr{C}^\infty$*-close* (or analytically, or $\mathscr{C}^r$-close) to a circle rotation R_α, where $\alpha = \rho(f)$, is a rather classical application of KAM theory. The prototype result is the *local rigidity* theorem of Arnold [1], who showed that if α is Diophantine, circle diffeos which are a sufficiently small analytic deformations of R_α and have rotation number equal to α, must be *analytically* conjugate to R_α. Among the few *global results* (which do not assume that f is close to a rotation), we recall the celebrated theorem by Michael Herman [30] and Jean-Christophe Yoccoz [77], answering a question by Arnold, showing that if f is $\mathscr{C}^\infty$ (or analytic) and its rotation number $\rho(f)$ satisfies the DC, the conjugacy is $\mathscr{C}^\infty$ (resp. analytic). Furthermore, the DC turns out to be the optimal arithmetic condition for global smooth linearization. Another, more subtle arithmetic condition, called "*condition* H" in honor of Herman, was introduced by Yoccoz as the optimal condition for global *analytic* linearization of analytic diffeos, see [79].

Another famous arithmetic condition is the *Roth-type* condition, which is satisfied by irrationals $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ such that $a_n = O_\varepsilon(q_n^\varepsilon)$ for all $\varepsilon > 0$. A crucial step in the KAM approach developed by Arnold for circle diffeomorphisms is to solve a *linearized* version of the conjugacy equation $R_\alpha \circ h = h \circ T$, namely the *cohomological equation*: given a smooth $\phi : I \to \mathbb{R}$, one looks for a smooth solution $\varphi : I \to \mathbb{R}$ to the equation $\varphi \circ R_\alpha - \varphi = \phi$. The Roth-type condition turns out to be the optimal one needed to solve this cohomological equation with optimal loss of differentiability: for any $r > s + 1 \geq 1$, one can find a solution $\varphi \in \mathscr{C}^s$ for any $\phi \in \mathscr{C}^r$ as long as $\int \phi = 0$ (which is a trivial necessary condition) *if and only* *if* α is Roth-type: this equivalent characterization provides a remarkable connection between dynamical and arithmetical properties.

We remark that the Diophantine condition, the H condition, and the Roth-type condition can all be proved to have *full measure*, namely they hold for a set of $\alpha \in [0,1]$ of Lebesgue measure one (the set of badly approximable $\alpha \in [0,1]$, on the other hand, has Lebesgue measure *zero*, although full Hausdorff dimension). While full measure of the Diophantine and Roth conditions can be proved in an elementary way, it is an instructive exercise to derive it from the properties of the Gauss map $G : [0,1] \to [0,1]$ and of the Gauss invariant measure $\mathrm{d}x/\log 2(1+x)$, since this point of view can be applied to show full measure of other arithmetic conditions as well and it can be furthermore generalized to higher genus (see Sections 4 and 5).

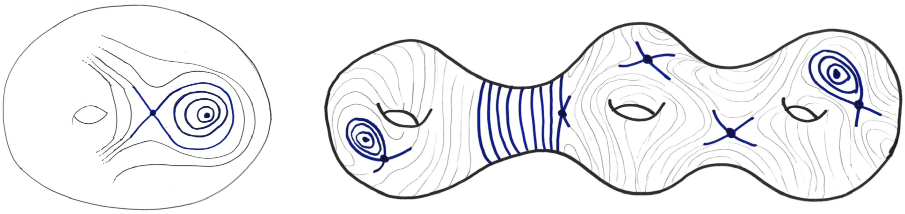
In view of this remark, we conclude this section with a reinterpretation of Herman's linearization theorem in the language of *foliations* into flow trajectories. In this setting, the linear model of a flow on a torus is a *linear flow* on $\mathbb{R}^2/\mathbb{Z}^2$ (i.e., the flow which arises as solution of $(\dot{x_1}, \dot{x_2}) = (\theta_1, \theta_2)$, which moves points with unit speed along lines of slope θ_2/θ_1).

Theorem 2.1 (Reformulation of Herman's global theorem **[30]**). *For a full measure set of real numbers α, a foliation on a genus one surface which is topologically conjugate to the foliation given by a linear flow with rotation number α is also $\mathscr{C}^\infty$-conjugate to it.*

Since the regularity of a conjugacy between foliations, which sends leaves into leaves, is defined in terms of the transverse structure, this result is just a restatement of the result for the Poincaré maps of the two flows (which are circle diffeomorphisms and rotations respectively).

Ergodic properties in genus one and exceptional behavior. From the existence (and abundance) of smooth (or at least continuously *differentiable*, i.e., $\mathscr{C}^1$) linearizations, one can infer many of the smooth measure-theoretical ergodic properties of flows of genus one. In particular, one sees that, for a full measure set of rotation numbers, flows in genus one are *ergodic* (since irrational rotations are) with respect to a *smooth* invariant measure of full support (the $\mathscr{C}^1$-regularity of the conjugacy allows us indeed to *transport* the Lebesgue invariant measure to obtain the invariant measure for the diffeo, which in turns gives a *transverse measure* for the flow). Furthermore, they are *uniquely ergodic* (in view of Kronecker–Weyl theorem for rotations, e.g., **[14]**), i.e., this natural invariant measure is the *unique* invariant measure (up to scaling).

We remark that *exceptional* ergodic behaviors in genus one (smooth) surface flows, can be constructed for flows whose rotation numbers are irrational but not Diophantine, i.e., the so-called *Liouvillean* (rotation) numbers. When α is Liouville, exploiting the abundance of good rational approximations $(p_n/q_n)_n$ to α, for example, using the method of *periodic approximations* pioneered by Anosov and Katok and later revived by Fayad, Katok et al. (see **[36]** or the survey **[18]**), one can construct many examples with *pathological* behavior, for example, flows with a *singular* invariant measures and *time-reparametrizations* (also



(a) (b)

FIGURE 1
Pictorial representation of locally Hamiltonian flows on a surfaces: in (a) an Arnold flow ($g = 1$) and in (b) a flow in $g = 3$ with two minimal components and 3 periodic components.

called *time-changes*) which are weakly mixing or which have mixed spectrum (see [18] and the references therein).

Finally, before moving to higher genus, we remark that another possible way to introduce interesting dynamical features for *typical* rotation numbers is to consider flows on tori *with singularities*. The simplest type of singularity is a *stopping point*. Already such a simple perturbation, which is only a time-reparametrization of the flow, can lead to flows which are typically *mixing* (see [43]) and even to flows with *Lebesgue spectrum* (see [17]). Smooth measure preserving flows on a torus with one center and one simple saddle (see Figure 1a) were first studied by Arnold in [2] and constitute one of the most studied examples in the class of flows known as *locally Hamiltonian*: we return to them and to their typical ergodic properties in Section 3.1.

3. DYNAMICS OF FLOWS ON SURFACES OF HIGHER GENUS

Let us now consider the *higher genus* case, namely consider now a (smooth) flow $\varphi_\mathbb{R} := (\varphi_t)_{t\in\mathbb{R}}$ on a compact, connected orientable (closed) surface S of genus $g \geq 2$. Notice that in this case, by Euler characteristic restrictions, the flow *always* has *fixed points* (see Figure 2 for some examples). We require that singularities be *isolated* (so that in particular, by compactness, the set $\mathrm{Fix}(\varphi_\mathbb{R})$ of fixed points is *finite*).

Topological dynamics and quasiminimal sets. The *topological classification* of the possible behavior of trajectories of a flow on a surface (and, more generally, of surface *foliations* which are not necessarily orientable) has been a topic of research in the 20th century (starting from the 1930–1940s, up to the 1970s). In particular, through the works of Maier, Levitt, Gutierrez, Gardiner et al. (see [60] for references), one could obtain results on what possible *orbit closures* are, as well as a classification of *quasiminimal sets*, which can be defined as possible *ω-limit sets* of *nontrivial* recurrent trajectories, i.e., set of accumulation points of trajectories different from a fixed point or a closed, periodic orbit. Quasiminimal sets can be the whole surface, subsurfaces with boundary, or a Cantor-like invariant sets. Moreover, one

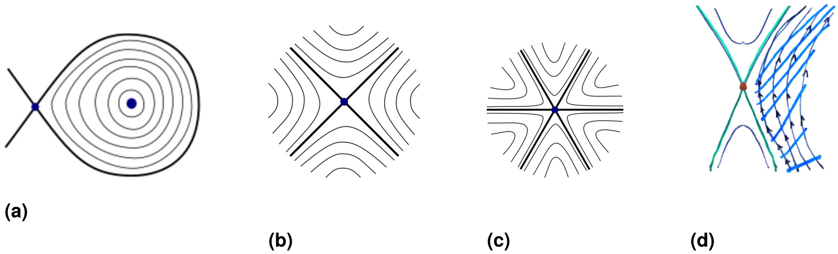


FIGURE 2
Type of singularities of a locally Hamiltonian flow: a center in (a), a simple saddle in (b) and a multisaddle in (c). Decelerations and shearing near a Hamiltonian saddle in (d).

can prove *decomposition theorems* showing that one can *cut* the surface S into subsurfaces each of which contains at most one quasiminimal set (see in particular the work by Levitt [49]). We do not enter here into the details of these topological results, but refer the interested reader, for example, to the monograph [60] and the references therein.

Interval exchanges and generalized IETs as Poincaré sections. As in the case of genus one, an essential tool to study a higher genus flow is to consider a (local) *transversal* $I \subset S$ to the flow and the *Poincaré first return map* T of the flow on I (when it is defined, for example, almost everywhere when the flow preserves a finite measure with full support; for more general situations, see [60]). Such first return maps $T : I \to I$ are one-to-one *piecewise diffeomorphisms* known as *generalized interval exchange transformations*: a map $T : I \to I$ is a generalized interval exchange transformations or, for short, a GIET, if one can partition I into intervals $I_1, \dots, I_d$ (finitely many since we are assuming that $\varphi_{\mathbb{R}}$ has finitely many fixed points) so that the restriction T_i of T to I_i, for each $1 \le i \le d$, is a diffeomorphism onto its image which extends to a diffeo of the closure $\overline{I}_i$ (see, e.g., [55]). We say in this case that T is a d-GIET. We say, furthermore, that T is *of class* $\mathcal{C}^r$ if the restriction of T to each I_i extends to a $\mathcal{C}^r$-diffeomorphism onto the closed interval $\overline{I}_i$. The adjective *generalized* is used to distinguish them from the more commonly studied (standard) interval exchange transformations (or simply IETs), which are one-to-one piecewise *isometries*, namely GIETs such that the derivative T_i' of each branch is constant and equal to one.

Standard IETs are a generalization of circle rotations (since an IET is a rotation when $d = 2$) and play an analogous role in higher genus, providing the natural *linear model* of a GIET (see Section 3.2). Furthermore, as rotations are Poincaré maps of *linear flows* on the torus $\mathbb{R}^2/\mathbb{Z}^2$, IETs arise naturally as Poincaré maps of *linear flows* on *translation surfaces* (see the ICM proceeding [12] for an introduction to the latter).

3.1. Locally Hamiltonian flows

We will be mostly concerned with flows which preserve a (probability) measure μ of *full support*, for example, an area-form, since this is a natural setup for *ergodic theory*. Given a surface S with a fixed smooth area form ω, a *smooth area-preserving flow* $\varphi_{\mathbb{R}} = (\varphi_t)_{t\in\mathbb{R}}$ on S is a smooth flow on S which preserves the measure μ given integrating a smooth density with respect to ω. The interest in the study of these flows and, in particular, in their ergodic and mixing properties, was revived by Novikov [61] in the 1990s, in connection with problems arising in solid-state physics, as well as in pseudoperiodic topology (see, e.g., the survey [84] by A. Zorich). Smooth area-preserving flows are also called *locally Hamiltonian flows* or *multivalued Hamiltonian flows* in the literature, in view of their interpretation as flows locally given by Hamiltonian equations: one can find local coordinates (x_1, x_2) on each open set $U \subsetneq S$ in which $\varphi_{\mathbb{R}}$ is given by the solution to the equations:

$$\begin{cases} \dot{x_1} = \partial H/\partial x_2, \\ \dot{x_2} = -\partial H/\partial x_1, \end{cases}$$

where $H : U \to \mathbb{R}$ is a real-valued (*local*) Hamiltonian. For simplicity, we will assume here that H is infinitely differentiable, even though for several results $\mathcal{C}^3$ (or also $\mathcal{C}^{2+\varepsilon}$ for every $\varepsilon > 0$) suffices. It turns out that such smooth area-preserving flows on S are in one-to-one correspondence with smooth *closed* real-valued differential 1-forms: given such a 1-form η, we can associate to it the integral flow $\varphi^\eta_{\mathbb{R}}$ of the vector field X such that $\eta = i_X\omega$, where i_X denotes the contraction operator. Since η is closed, $\varphi^\eta_{\mathbb{R}}$ is area-preserving; conversely, every smooth area-preserving flow can be obtained in this way.

Topology and measure class. Let $\mathcal{F}$ denote the set of smooth closed 1-forms on S with isolated zeros. On $\mathcal{F}$ (which we can think of as the space of locally Hamiltonian flows) one can define a *topology* as well as a measure class. The *topology* is obtained by considering perturbations of closed smooth 1-forms by (small) closed smooth 1-forms. We will often restrict our attention to the subset $\mathcal{M} \subset \mathcal{F}$ of *Morse* closed 1-forms (i.e., forms which are locally the differential of a *Morse function*), which is *open and dense* in $\mathcal{F}$ with respect to this topology (see, e.g., [64]). Locally Hamiltonian flows corresponding to forms in $\mathcal{M}$ have only *nondegenerate fixed points*, i.e., *centers* and *simple saddles* (as in Figures 2a and 2b), as opposed to degenerate *multisaddles* (as in Figure 2c). Furthermore, if $\mathcal{F}_{s,l}$ denote the flows which correspond to flows in $\mathcal{M}$ with s saddle points and l centers, each $\mathcal{F}_{s,l}$ is open and their union is dense in $\mathcal{F}$ (see [64]).

A *measure-theoretical notion of typical* can be defined on each $\mathcal{F}_{s,l}$ using the *Katok fundamental class* (introduced by Katok in [35], see also [60]), i.e., the cohomology class of the 1-form η which defines the flow. Let $\mathrm{Fix}(\varphi_{\mathbb{R}})$ denote the set of *fixed points* (also called *singularities*) of the flow $\varphi_{\mathbb{R}}$ and let $k = s + l$ be its cardinality (recall that it is finite since the flow is in $\mathcal{F}$ and $k \geq 1$ when $g \geq 2$). If we fix a base $\gamma_1, \ldots, \gamma_n$ of the relative homology $H_1(S, \mathrm{Fix}(\varphi_{\mathbb{R}}), \mathbb{R})$ (where $n = 2g + k - 1 = 2g + s + l - 1$) and consider the period map Per given by $\mathrm{Per}(\eta) = (\int_{\gamma_1} \eta, \ldots, \int_{\gamma_n} \eta) \in \mathbb{R}^n$, we say that a property holds for a *typical* locally Hamiltonian flow in $\mathcal{F}_{s,l}$ if it holds for all η such that $\mathrm{Per}(\eta)$ belongs to a full measure set with respect to the Lebesgue measure on $\mathbb{R}^n$.

Minimal components and ergodicity. To describe (typical) chaotic behavior in locally Hamiltonian flows, it is crucial to distinguish between two open sets (complementary, up to measure zero, see [75] or [64] for more details): in the first open set, which we will denote by $\mathcal{U}_{\min}$, the typical flow is *minimal* (the term *quasiminimal* is also used in the literature), in the sense that the orbits of all points which are not fixed points are *dense* in S; flows in $\mathcal{U}_{\min}$ have only saddles, since the presence of centers prevents minimality. On the other open set, that we call $\mathcal{U}_{\neg\min}$, the flow is not minimal (there are saddle loops homologous to zero which disconnect the surface), but one can decompose the surface into a finite number of subsurfaces with boundary S_i, $i = 1, \dots, N$ such that for each i either S_i is a *periodic component*, i.e., the interior of S_i if foliated into closed orbits of $\varphi_{\mathbb{R}}$ (in Figure 1b one can see three periodic components, namely two disks and one cylinder), or S_i is such that the restriction of $\varphi_{\mathbb{R}}$ to S_i is minimal in the sense above, as pictured in the remaining two subsurfaces in Figure 1b. These are called *minimal components* and there are at most g of them (where g is the genus of S), see Section 3.1.

Notice that minimality and ergodicity of a (minimal component of a) locally Hamiltonian flow are equivalent to minimality or respectively ergodicity of an (and hence any) interval exchange transformation which appears as the Poincaré map. Classical results proved in the 1980s guarantee that almost every IET (with respect to the Lebesgue measure on the interval lengths, assuming that the permutation is irreducible) is minimal (as showed by Keane [37], see also [35]) and (uniquely) ergodic (as proved in the works by Masur [50] and Veech [76], considered early milestones of the successful application of Teichmüller dynamics to the study of IETs and translation surfaces, see the ICM proceeding [12] or the survey [85]). It then follows from definition of Katok measure class that a typical local Hamiltonian flow in $\mathcal{U}_{\min}$ is minimal and ergodic and, given a typical local Hamiltonian flow in $\mathcal{U}_{\neg\min}$, its restriction on each minimal component is ergodic.

Classification of mixing properties. Finer chaotic features of locally Hamiltonian flows, in particular mixing and spectral properties, change according to the type of singularities and depend crucially on the locally Hamiltonian parametrization of saddle points. For a (nongeneric) locally Hamiltonian flow with at least one *degenerate saddle* (an example of such a saddle is shown in Figure 2c), *mixing* (for the definition, see (3.1) with $n = 2$) was proved in the 1970s by Kochergin [43]. When, on the other hand, $\eta \in \mathcal{M}$ is a Morse 1-form, so that all saddles are *simple*, one has a dichotomy: inside the open set $\mathcal{U}_{\min}$ in which the typical flow is minimal, almost every locally Hamiltonian flow is *weakly mixing*, but it is *not mixing*; both results follow from work by the author [72,73]. On the other hand, for a full measure set of flows in $\mathcal{U}_{\neg\min}$, the restriction to each of minimal components is mixing (as proved by Ravotti [64] extending the previous work [71] by the author).

The question of mixing in higher genus was raised by V. Arnold in the 1990s, when he conjectured (see [2]) that the restriction of a typical smooth flow on a torus with one center and one simple saddle to its minimal component (namely for what we nowadays call an *Arnold flow*) was indeed mixing. His conjecture was proved shortly after by Khanin and Sinai in [67], who showed mixing under the assumption that the rotation number α is such that the entries a_n of the continued fraction expansion of α do not grow too fast, namely there exist a power $1 < \tau < 2$ and $C > 0$ such that $|a_n| \leq C n^{\tau}$. One can show (for example, exploiting the

Gauss map G and the finiteness of $\int_0^1 a_0(x)\,\mathrm{d}\,\mu_G(x)$, where μ_G is the Gauss measure, via a standard Borel–Cantelli argument) that this arithmetic condition holds for a full measure set of α. The condition was later improved by Kocerghin, see **[44]**. Also in the case of absence of mixing, a prototype result for flows over a full measure set of rotation numbers was proved by Kochergin **[42]** already in the 1970s (and much more recently extended in **[45]** to all irrational rotation numbers), much earlier than results in higher genus **[65,70,73]**.

In higher genus, the above mentioned results on mixing/absence of mixing require the introduction of Diophantine-like conditions, which describe the full measure set of locally Hamiltonian flows for which the results hold. In **[71]**, for example, we introduced a condition on a IET (see Section 5 for more details) called *Mixing Diophantine Condition* (or MDC, for short). Let us say that the restriction of a locally Hamiltonian flow $\varphi_{\mathbb{R}}$ to one of its minimal components S_i satisfies the MDC if one can find a section $I \subset S_i$ (in *good position* in the sense of **[55]**) such that the IET which arises as Poincaré map of $\varphi_{\mathbb{R}}$ to I satisfies the MDC. One can then prove:

Theorem 3.1 (Ulcigrai **[71]**, Ravotti **[64]**). *Let $\varphi_{\mathbb{R}}$ be a flow in $\mathcal{U}_{\neg\min}$ and let S_i be a minimal component. If the restriction of $\varphi_{\mathbb{R}}$ to S_i satisfies the Mixing Diophantine Condition, then $\varphi_{\mathbb{R}}$ restricted to S_i is mixing.*

We then show in **[71]** (exploiting results from **[3]**, see Section 5) that the MDC is satisfied by a full measure set of IETs. Similarly, to prove that a typical flow in $\mathcal{U}_{\min}$ is *not* mixing, a Diophantine-like condition is introduced and proved to be of full measure in **[73]**. Special cases of the absence of mixing result for surfaces with $g = 2$ and two isometric saddles were proved in **[70]** and by Scheglov in **[65]**. We remark that in $\mathcal{U}_{\min}$ there exist, nevertheless, exceptional mixing flows, as shown by Chaika and Wright in **[13]**, who produced sporadic examples in $g = 5$.

Parabolic dynamics and slow chaos. Smooth area-preserving flows on surfaces also provide one of the fundamental classes of *parabolic*, or *slowly chaotic*, dynamical systems (see, e.g., the survey **[75]**). In systems which display *sensitive dependence* on initial conditions (the so-called *butterfly effect*), one can find many nearby initial conditions whose trajectories *diverge* with time. Contrary to hyperbolic systems, where this divergence happens (infinitesimally) at *exponential speed*, in parabolic systems the divergence speed is *slow*, namely subexponential, and in all known examples *polynomial* or subpolynomial. Slow divergence in locally Hamiltonian flows is created by Hamiltonian saddles, which create different deceleration rates of nearby trajectories and produce a form of (local) *shearing*, by *tilting* in the flow direction the image under the flow of arcs initially transverse to the dynamics, as illustrated in Figure 2d. Shearing happens not only locally, near a saddle, but globally for typical flows in $\mathcal{U}_{\neg\min}$, which (in view of the presence of saddle loops) display a global *asymmetry* in the prevalent direction of shearing. It is this geometric mechanism which is behind the proof of mixing (in this setting, but also for many other classes of parabolic flows, see the survey **[74]** and the references therein). Under the assumption that the restriction of $\varphi_{\mathbb{R}}$ to a minimal component S_i satisfies the Mixing Diophantine Condition, one can produce quantitative estimates on shearing of transverse arcs and, as shown by Ravotti in [64], prove quantitative mixing estimates, which show that mixing happens (at least) at *subpolynomial speed*, i.e., for any two smooth observables $f, g : S_i \to \mathbb{R}$ supported outside the saddles in $\mathrm{Fix}(\varphi_{\mathbb{R}}) \cap S_i$,

$$\left| \int_{S_i} f(\varphi_t(x)) g(x) \mathrm{d}\mu - \int_{S_i} f \mathrm{d}\mu \int_{S_i} f \mathrm{d}\mu \right| \le \frac{C_{f,g}}{(\log t)^{\gamma}}, \quad t \ge 0.$$

This is expected to be also the optimal nature of the estimates, namely the decay is *not* expected to be polynomial or faster in this setting, but no lower bounds on the decay of correlations are currently available.

Ratner's forms of shearing. Striking consequences of shearing (such as measure and joining rigidity) were proved for another famous class of parabolic flows, namely horocycle flows on hyperbolic surfaces and their time-changes, by exploiting a *quantitative shearing* property introduced by Marina Ratner and nowadays known as *Ratner property* (or RP). In view of its importance, in the study of horocycle flows and, more generally, unipotent flows in homogeneous dynamics, it is natural to ask whether this property can be proved and exploited in other parabolic (non homogeneous) settings. For locally Hamiltonian flows, which are natural candidates, the original Ratner property is believed to fail due to the presence of singularities (see [16]). Nevertheless, a variant of the RP which has the same dynamical consequences, called *Switchable Ratner Property* (or SRP, for short), was introduced by B. Fayad and A. Kanigowski [16] and showed to hold for typical Arnold flows (as well as some flows in genus one with one degenerate singularity). As an abstract consequence of the SRP property, one can conclude that typical Arnold flows are not only mixing, but *mixing of all orders*, namely for any $n \ge 2$ and any n-tuple $A_0, \dots, A_{n-1}$ of measurable sets,

$$\mu(A_0 \cap \varphi_{t_1}(A_1) \cap \cdots \cap \varphi_{t_1+\cdots+t_{n-1}}(A_{n-1})) \xrightarrow{t_1,t_2,\dots,t_{n-1}\to\infty} \mu(A_0)\cdots\mu(A_{n-1}). \quad (3.1)$$

Notice that this definition reduces to the classical definition of mixing in the special case $n = 2$; whether mixing implies mixing of all orders in general is still an open problem, known as *Rohlin conjecture*.

To prove the SRP property, one needs to assume that the rotation number $\alpha = [a_0, a_1, \dots, a_n, \dots]$ satisfies an ad hoc arithmetic condition, namely, if q_n are the denominators of α, one requires that, for some $0 < \xi, \eta < 1$ (taken to be $\xi = \eta = 7/8$ in [16]) the following series is finite:

$$\sum_{k \notin K(\alpha)} \frac{1}{(\log q_n)^{\eta}} < +\infty, \quad \text{where } K(\alpha) := \{k \in \mathbb{N}, \ a_{k+1} \le C(\log q_k)^{\xi}\}. \quad (3.2)$$

In a joint work with A. Kanigowski and J. Kułaga-Przymus [33], we were able to generalize this result to higher genus. To do so, it is once again crucial to introduce a suitable Diophantine-like condition, which we called in [33] the *Ratner Diophantine Condition* (or RDC) and we describe in Section 5. The main result we prove is the following.

Theorem 3.2 (Kanigowski, Kułaga-Przymus, Ulcigrai [33]). *If the restriction of $\varphi_{\mathbb{R}} \in \mathcal{U}_{\neg\min}$ to a minimal component S_i satisfies the Ratner Diophantine Condition, $\varphi_{\mathbb{R}} : S_i \to S_i$ satisfies the Switchable Ratner Property and is mixing of all orders.*

We then show that the RDC is satisfied by almost every IET and therefore can conclude that, for a full measure set of locally Hamiltonian flows in $\mathcal{U}_{\neg\min}$, each restriction to a minimal component is mixing of all orders.

Quantitative estimates on slow, Ratner-type shearing were recently used (in the joint work [34] with A. Kanigowski and M. Lemańczyk) to study *disjointness of rescalings*, a property that has recently received a revival of attention in view of its role as possible tool to prove Sarnak Möbius orthogonality conjecture (see the ICM proceedings survey [48] and the references therein). In [34] we introduce a disjointness criterium based on Ratner shearing and use it (as one of the applications) to show that, in genus one, typical Arnold flows have *disjoint rescalings* and satisfy Moebius orthogonality. Disjointness of rescalings seems to be an important feature of parabolic dynamics: while specific parabolic flows may fail to be disjoint from their rescalings (primarily the horocyle flow on a hyperbolic surface), several recent results seem to indicate that this property is indeed widespread among parabolic flows (see, e.g., the results in [34] on time-changes of horocycle flows). In the context of surface flows, disjointness of rescalings has been verified in [4] for *von Neumann flows* (which can be realized as translation flows on surfaces with boundary). Whether one can extend the disjointess result proved in [34] for Arnold flows to higher genus smooth flows, remains an open problem and is likely to require a delicate control of Diophantine-like properties.

Polynomial deviations of ergodic averages. Slow chaotic behavior manifests itself not only through *slow* mixing, but also through *slow* convergence of ergodic integrals: given an ergodic area-preserving flow $\varphi_{\mathbb{R}}$ (or its restriction to an ergodic minimal component $S' \subset S$) and a real valued observable f with zero-mean, the ergodic integrals $I_T(f,x) := \int_0^T f(\varphi_t(x))\,\mathrm{d}t$ decay to zero *polynomially* with some exponent $0 < \nu < 1$ for almost every initial point, i.e., $|I_T(f,x)| \sim O(T^\nu)$ in the sense that

$$\limsup_{T\to\infty} \frac{\log |I_T(f,x)|}{\log T} = \nu.$$

This phenomenon, known as *polynomial deviations of ergodic averages*, was discovered experimentally in the 1990s by A. Zorich and explained (for linear flows on translation surfaces and observables corresponding to cohomology classes) in seminal work by Kontsevitch and Zorich [46,83] relating power deviations to Lyapunov exponents of renormalization (see Section 4). Forni in [23] could extend this result to integrals of sufficiently regular functions over translation flows and show that ergodic integrals can display a *power spectrum* of behaviors, i.e., there are exactly g positive exponents $0 < \nu_g \leq \cdots \leq \nu_2 < \nu_1 := 1$ (which correspond to the positive Lyapunov exponents of renormalization) and for each a subspace of finite codimension of smooth observables that present polynomial deviations as above with exponent $\nu = \nu_i$. A finer analysis of the behavior of Birkhoff sums or integrals, beyond the *size* of oscillations, appears in the works [7,54]: Bufetov in [7] shows in particular that (for *typical* translation flows and sufficiently regular observables) the *asymptotic behavior* of ergodic integrals can be described in terms of g (where g is the genus of the surface) *cocycles* $\Phi_i(t,x)$, $1 \leq i \leq g$ (also called *Bufetov functionals*): each $\Phi_i : \mathbb{R} \times S' \to \mathbb{R}$ is a cocycle over the flow $\varphi_{\mathbb{R}}$ (in the sense that $\Phi_i(t+s,x) = \Phi_i(t,x) + \Phi_i(s,\varphi_t(x))$ for any $x \in S'$

and $t \in \mathbb{R}$), $\Phi_1(T,x) \equiv T$ and each Φ_i has power deviations $|\Phi_i(T,x)| \sim O(T^{\nu_i})$ with exponent ν_i. Together, the cocycles encode the *asymptotic behavior* of the ergodic integrals up to subpolynomial behavior, in the sense that, for some constants $c_i = c_i(f)$,

$$\int_0^T f(\varphi_t(x))\mathrm{d}t = c_1 T + c_2\Phi_2(T,x) + \cdots + c_g\Phi_g(T,x) + \mathrm{Err}(f,T,x), \tag{3.3}$$

where for almost every $x \in S'$ the *error term* $\mathrm{Err}(f,T,p)$ is subpolynomial, i.e., for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|\mathrm{Err}(f,T,p)| \le C_\varepsilon T^\varepsilon$. In a joint work with Frączek, we recently gave a new proof of this result in [26], which extends the result to the setting of smooth observables over locally Hamiltonian flows with Morse singularities (in $\mathcal{U}_{\min}$ as well as in $\mathcal{U}_{\neg\min}$) and also shows that the set of locally Hamiltonian flows for which the result holds can be described in terms of a Diophantine-like condition. More precisely, we define in [26] the *Uniform Diophantine Condition* (or UDC, for short; see Section 5) and show that it has full measure. We then prove the following.

Theorem 3.3 (Frączek–Ulcigrai [26]). *If the restriction of the locally Hamiltonian flow $\varphi_\mathbb{R} \in \mathcal{M}$ on a minimal component S' satisfies the* Uniform Diophantine Condition, *for each $\mathcal{C}^3$ observable $f : S' \to \mathbb{R}$, there exist g exponents ν_i and corresponding cocycles Φ_i such that the expansion* (3.3) *holds.*

Spectral theory. The study of the spectrum of the unitary operators acting on $L^2(S,\mu)$ given by $f \mapsto f \circ \varphi_t$ can shed further light on the chaotic features of the dynamics of the flow $\varphi_\mathbb{R} := (\varphi_t)_{t\in\mathbb{R}}$ and is at the heart of the study of spectral theory of dynamical systems (see [48] or [75] and the references therein). While the classification of mixing properties of locally Hamiltonian flows is essentially complete, very little is known about their spectral properties beyond the case of genus one (and some sporadic examples, such as *Blokhin examples*, essentially built gluing genus one flows, see the work [25]). The recent result [17] by Fayad, Forni, and Kanigowski for genus one suggests that it may be possible to prove that the spectrum is countable Lebesgue also in higher genus when in presence of degenerate, sufficiently strong (multisaddle) singularities. In the nondegenerate case, though, we recently proved in joint work with Chaika, Frączek, and Kanigowski [10] that a *typical* locally Hamiltonian flow on a *genus two* surface with two isomorphic *simple saddles* has *purely singular* spectrum. This result does not use explicit Diophantine-like conditions, but rather geometry and, in particular, a special symmetry (the hyperelliptic involution) that surfaces in genus two are endowed with; Liouville-type Diophantine conditions are here imposed by requesting the presence on the surface of large flat cylinders close to the direction of the flow, whose existence for typical flows is then proved by a Borel–Cantelli-type of argument (see [10] for details). Extending this result beyond genus two, though, will probably require the use of Rauzy–Veech induction (see Section 4) and the introduction of new Diophantine-like conditions, which impose some controlled form of degeneration. The nature of the spectrum of minimal components of locally Hamiltonian flows in $\mathcal{U}_{\neg\min}$ (even in genus one, i.e., for Arnold flows) is a completely open problem.

3.2. Linearization and rigidity in higher genus

A different line of problems in which Diophantine-like conditions in higher genus play a crucial role are conjectures concerning *linearization* and *rigidity* properties of higher genus flows and their Poincaré sections, GIETs (defined in Section 3). In analogy with the case of circle diffeos, we say that a GIET T is *linearizable* if it is topologically conjugate to a linear model, namely to a (standard) IET T_0.

Topological conjugacy and wandering intervals. To generalize Poincaré and Denjoy work, one needs first of all a combinatorial invariant which extends the notion of rotation number. Such an invariant can be constructed by recording the combinatorial data of a renormalization process, as we explain in Section 4. One of the crucial differences between GIETs and circle diffeomorphisms, though, is the failure of a generalization of Denjoy theorem: there are smooth GIETs that are semiconjugate to a minimal IET for which the semiconjugacy is *not* a conjugacy; in other words, they have wandering intervals (see the examples found in [6,8] in the class of periodic-type (affine) IETs and, more generally, [54]). It is important to stress that this is *not* a low-regularity phenomenon, nor is it related to special arithmetic assumptions: as shown by the key work [54] by Marmi, Moussa, and Yoccoz, wandering intervals exist even for piecewise *affine* (hence analytic) GIETs (called AIETs), for almost every topological conjugacy class. The presence of wandering intervals is on the contrary expected to be *typical* (see, e.g., the conjectures in [27,55]) and it is closely interknit with the absence of a *Denjoy Koksma inequality* and, more generally, *a priori bounds* for renormalization, see [29].

Local obstructions to linearization. As an important first step towards local linearization, we already mentioned the *cohomological equation* $\varphi \circ T - \varphi = \phi$ in Section 2, where $T = R_\alpha$ was a rotation. Whether the cohomological equation could be solved when T is an IET, under suitable assumptions, was unknown until the pioneering work of Forni [21], who brought to light the existence of a *finite* number of obstructions to the existence of a (piecewise finite differentiable) solution. We remark that obstructions to solve the cohomological equation have been since then discovered to be a characteristic phenomenon in *parabolic dynamics* (e.g., their existence have been proved by Flaminio and Forni for horocycle flows [19] and nilflows on nilmanifolds [20], see also the ICM talk [22]). Forni's work is a breakthrough that paved the way for the development of a linearization theory in higher genus.

Another breakthrough, which put the stress on the *arithmetic* aspect of linearization in higher genus, was achieved by Marmi–Moussa–Yoccoz in their work [55] (and related works [53, 57]). In [53], in particular, they reproved and extended Forni's result using the IETs renormalization described in Section 5 and introduced the *Roth-type* condition (see also Section 5), as an explicit Diophantine-like condition on the IET needed to solve the cohomological equation $\varphi \circ T - \varphi = \phi - \xi$, where ξ is a piecewise constant function which embodies the finite-dimensional *obstructions*. This result, combined with a generalization of Herman's *Schwarzian derivative trick*, then led to the proof in [55] by the same authors that, for any $r \geq 2$, the $\mathscr{C}^r$ *local conjugacy class* of almost every IET T (more precisely, of any T of *restricted* Roth-type, see Section 5) is a *submanifold of finite codimension*. Marmi, Moussa, and Yoccoz also conjectured that for $r = 1$ it is a submanifold of codimension $(d-1)+(g-1)$, where d is the number of exchanged intervals and g the genus of the surface of which T is a Poincaré section. For the measure zero class of IETs of *hyperbolic periodic type* (see Section 5), this conjecture has recently been proved by Ghazouani in **[28]**. The proof of this result for almost every IET will require the introduction of a new suitable Diophantine-like condition on IETs.

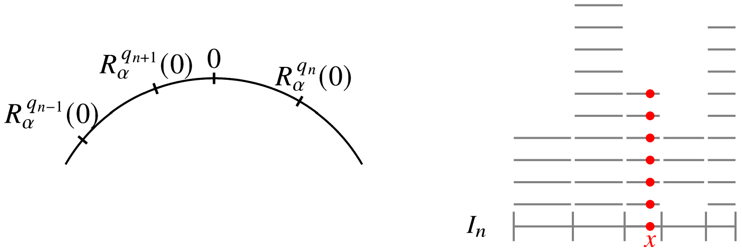
Rigidity of GIETs. We say that a class of (dynamical) systems is *geometrically rigid* (or also $\mathscr{C}^1$-rigid), if the existence of a topological conjugacy between two objects in the class automatically imply that the conjugacy is actually $\mathscr{C}^1$. The global linearization results by Herman and Yoccoz recalled in Section 2 shows that the class of (smooth, or at least $\mathscr{C}^3$) circle diffeomorphisms with Diophantine rotation number is geometrically rigid (and actually $\mathscr{C}^\infty$-rigid, i.e., if a smooth circle diffeo is conjugated via a homeomorphism h to R_α with α satisfying the DC, then h is $\mathscr{C}^\infty$). We already saw that this can be reinterpreted as a rigidity result for flows on surfaces of genus one (see Theorem 2.1). In joint work with S. Ghazouani, we recently proved a generalization of this result to genus two.

Theorem 3.4 (Ghazouani, Ulcigrai **[29]**). *Under a full measure Diophantine-like condition, a foliation on a* genus two *surface which is topologically conjugate to the foliation given by a linear flow with Morse saddles is also $\mathscr{C}^1$ conjugate to it.*

Here full measure refers to the Katok measure class on the linear flow models (see the definition given earlier in this section). For simplicity, we stated the result for flows with simple, Morse-type saddles; degenerate saddles can also be considered, but then one has to further assume that the foliations are *locally* $\mathscr{C}^1$ conjugated in a neighborhood of the multisaddle. Both these results can be reformulated at level of Poincaré sections: we introduce more precisely a rather subtle Diophantine-like conditions on (irreducible) IETs of any number of intervals $d \geq 2$, that we call the *Regular Diophantine Condition*, or RDC (we comment on it in Section 3) and show that it is satisfied by almost every (irreducible) IET on d. We then prove:

Theorem 3.5 (Ghazouani, Ulcigrai **[29]**). *If an irreducible d-IET T_0 with $d = 4$ or $d = 5$ satisfies the RDC, then any $\mathscr{C}^3$-generalized interval exchange map T which is topologically conjugate to T_0, and whose* boundary $B(T)$ vanishes*, is actually conjugated to T_0 via a $\mathscr{C}^1$ diffeomorphism.*

The *boundary operator* $B(T)$ which appears in this statement is a $\mathscr{C}^1$-conjugacy invariant introduced in **[55]**; it encodes the holonomy at singular points of the surface of which T is a Poincaré section. Requesting that $B(T)$ vanishes is therefore a necessary condition for the existence of a conjugacy of class $\mathscr{C}^1$. Theorem 3.5 solves for $d = 4, 5$ one of the open problems suggested by Marmi, Moussa, and Yoccoz in **[55]**, where they conjecture the result to hold also for any other larger d. The result which is missing to prove the conjecture in



(a) **(b)**

FIGURE 3
Renormalization algorithms for rotations and IETs.

its generality is a generalization of an estimate used in **[54]** to show existence of wandering intervals in affine IETs. The main result in **[29]**, on the other hand (namely a dynamical dichotomy for the orbit of T under renormalization) is already proved for IETs which satisfy the RDC for any $d \geq 2$.

4. RENORMALIZATION AND COCYCLES

In this section we introduce the renormalization dynamics which is used as main tool to impose Diophantine-like conditions in higher genus. Renormalization in dynamics is a powerful tool to study dynamical systems which present forms of self-similarity (exact or approximate) at different scales. A map $T : I \to I$ of the unit interval which is (infinitely) *renormalizable* is such that one can find a (infinite) sequence of nested subintervals $I_{n+1} \subset I_n \subset \cdots \subset I$ such that the *induced dynamics* $T_n : I_n \to I_n$ (obtained by considering the first return map of T on I_n) is well defined and, up to *rescaling*, belongs to the same class of dynamical systems of the original T. Here, the rescaling, which is done so that the *rescaled* (or *renormalized*) map acts again on an interval of unit length, is given by the map $x \mapsto T_n(|I_n|x)/|I_n|$. We will now describe renormalization in the context of rotations first and then IETs. In both cases, at the level of (minimal) flows (or equivalently orientable foliations) on surfaces, the inducing process corresponds to taking shorter and shorter Poincaré sections of a given surface flow (on the torus or on a higher genus surface).

Renormalization algorithms. If $T = R_\alpha$ is a rotation by an irrational α and $q_n, n \in \mathbb{N}$, are the denominators of the convergents p_n/q_n of α, then one can consider as sequence $(I_n)_{n\in\mathbb{N}}$ the shrinking arcs on S^1 which have as endpoints $R_\alpha^{q_n}(0)$ and $R_\alpha^{q_{n+1}}(0)$. These endpoints correspond dynamically to consecutive closest returns of the orbit of 0 (see Figure 3a). The induced map T_n is then again a rotation R_{α_n}, with rotation number $\alpha_n = \mathcal{G}^n(\alpha)$, where $\mathcal{G}$ is the Gauss map $\mathcal{G}(x) = \{1/x\}$ and $\{\cdot\}$ denotes the fractional part.

Similarly, for a d-IET T, one wants to choose the nested sequence $(I_n)_{n\in\mathbb{N}}$ of inducing intervals so that the induced maps T_n are all IETs of the same number d of subintervals. Given any minimal T (or more generally any IET satisfying the *Keane condition* [37], i.e., such that the orbits of its discontinuity points are infinite and distinct), classical algorithms which produce such an infinite sequence $(I_n)_{n\in\mathbb{N}}$ are the *Rauzy–Veech induction algorithm* (see Veech [76] or [81] and the references therein) and *Zorich induction*, an acceleration of the same algorithm introduced by Zorich in [82]. For the definitions of these algorithms, which we will not use in the following, we refer the interested reader to the lecture notes [81]. One can show that, for $d = 2$, Zorich induction corresponds to the renormalization of rotations given by the Gauss map.

On the parameter space $\mathcal{I}_d$ of all d-IETs, these algorithms induce renormalization operators $\mathcal{R} : \mathcal{I}_d \to \mathcal{I}_d$, which associate to T the d-IET $\mathcal{R}(T)$ obtained by applying one step of the corresponding induction and then renormalizing the induced map to act on $[0, 1]$. Veech showed that Rauzy–Veech renormalization admits a conservative absolutely continuous invariant measure, that induces a *finite* invariant measure for the *Zorich acceleration*, as proved in [82]. The ergodic properties of the renormalization dynamics in parameter space have been intensively studied and are by now well understood, see, e.g., [80] and the references therein for a brief survey.

Rohlin towers and matrices. After n steps of induction, one can recover the original dynamics through the notion of *Rohlin towers* as follows: if I_n^i is one of the subintervals of T_n and $r := r_n^i$ is its first return time to I_n under the action of T, the intervals $I_n^i, T(I_n^i), \dots, T^{r-1}(I_n^i)$ are disjoint. Their union is called a *Rohlin tower* of step n and each of them is called a *floor* (see Figure 3b for a graphical depiction of floors and towers). Given an infinitely renormalizable T, for any n one can see $[0, 1]$ as a union of d Rohlin towers of step n, as shown in Figure 3b. Rohlin towers thus produce a sequence of *partitions* of $[0, 1]$ (into floors of towers of step n).

Renormalization produces also a sequence of $d \times d$ matrices A_n, $n \in \mathbb{N}$, with integer entries, which should be thought of as *multidimensional continued fraction* digits and describe *intersection numbers* of Rohlin towers. The matrices $(A_n)_{n\in\mathbb{N}}$ are defined so that the entries of the product $A^n := A_n \cdots A_1$ have the following dynamical meaning: the (i, j) entry $(A^n)_{ij}$ is the number of visits of the orbit of any point $x \in I_n^j$ to the initial subinterval I_0^i until its first return time r_n^j; in other words, $(A^n)_{ij}$ is the number of floors of the jth tower of level n which are contained in I_0^i. These entries generalize the classical continued fraction digits: for $d = 2$, indeed, the matrices $(A_n)_{n\in\mathbb{N}}$ associated to R_α, for n of alternate parity, have respectively the form

$$\begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix},$$

where a_n are the entries of the continued fraction expansion $\alpha = [a_0, a_1, \dots, a_n, \dots]$. Diophantine-like conditions for IETs are defined by imposing conditions on these matrices, on their growth as well as on their hyperbolicity, see in Section 5. The matrices $(A_n)_{n\in\mathbb{N}}$ are produced by the renormalization dynamics: for rotations, the entries $(a_n)_{n\in\mathbb{N}}$ of the continued fraction expansion of α satisfy $a_n = a(\mathcal{G}^n(\alpha))$, where $a(\cdot)$ is an integer-valued function on $[0,1]$. Similarly, one has now that $A_n = A(\mathcal{R}^n(T))$, where $A : \mathcal{I}_d \to \mathrm{SL}(d,\mathbb{Z})$ is a matrix-valued function on the space $\mathcal{I}_d$ of d-IETs, i.e., a *cocycle* (known as the *Rauzy–Veech cocycle*, or *Zorich cocycle* if considering the Zorich acceleration).

Positive and balanced accelerations. It turns out though that Zorich acceleration is often not sufficient (see, for example, [41] and [40] where it is shown that the classical Diophantine notions of bounded- [41] and Diophantine-type [40] do not generalize naturally when using Zorich acceleration). Two accelerations which play a key role in Diophantine-like conditions are the *positive* and the *balanced acceleration*. By *accelerations* we mean here an induction which is obtained by considering only a subsequence $(n_k)_{k\in\mathbb{N}}$ of Rauzy–Veech times. The associated (accelerated) cocycle is then obtained considering products

$$A(n_k, n_{k+1}) := A_{n_{k+1}-1} \cdots A_{n_k+1} A_{n_k}.$$

The *positive acceleration* appears in the works by Marmi, Moussa, and Yoccoz [53, 55, 57]. They showed that if T satisfies the Keane condition, for any n there exists $m > n$ such that $A(n,m)$ is a strictly positive matrix. The accelerated algorithm then corresponds to choosing the sequence $(n_k)_{k\in\mathbb{N}}$ setting $n_0 := 0$ and then, for $k \geq 1$, choosing n_k to be the smallest integer $n > n_{k-1}$ such that $A(n_{k-1}, n)$ is strictly positive. On the other hand, to define the *balanced acceleration*, one considers a subsequence $(n_k)_{k\in\mathbb{N}}$ of Rauzy–Veech times n for which the corresponding Rohlin towers are *balanced*, in the sense that ratios of widths $|I_n^i|/|I_n^j|$ and heights r_n^i/r_n^j are uniformly bounded above and below. We will return to these accelerations and some instances in which they are helpful in Section 5.

Combinatorial rotation numbers. We remark that the definition of Rauzy–Veech induction can be extended also to a GIET T (under the Keane condition, which guarantees that $\mathcal{R}^n(T)$ can be defined for every $n \in \mathbb{N}$) and then exploited to give a combinatorial notion of *rotation number* as well as a definition of *irrationality* in higher genus (following [55, 57], see also [81]). As one computes the induced maps $(T_n)_{n\in\mathbb{N}}$, one can indeed record the sequence $(\pi_n)_{n\in\mathbb{N}}$ of permutations of the GIETs $(T_n)_{n\in\mathbb{N}}$: this sequence provides the desired *combinatorial rotation number* for $d > 2$. We say that a GIET is *irrational* if the sequence of matrices $(A_n)_{n\in\mathbb{N}}$ have a positive acceleration (or equivalently, in the terminology introduced by Marmi, Moussa, and Yoccoz, the path described by $(\pi_n)_{n\in\mathbb{N}}$ is *infinitely complete*). One can then show that two *irrational* GIETs with the same rotation number are *semiconjugated* (see, e.g., [81]), a result that generalizes a property of rotations numbers and circle diffeos and hence explains the choice of calling this higher genus combinatorial object the "*rotation number*" of a GIET.

Renormalization of Birkhoff sums. Given $T : I \to I$ and a function $f : I \to \mathbb{R}$, we denote by $S_n f := \sum_{k=0}^{n-1} f \circ T^k$ the nth Birkhoff sum (of the function f under the action of T). When $T = R_\alpha$ is a rotation (or a circle diffeo), it is standard to study first Birkhoff sums of the form $S_{q_n} f$ for q_n convergent of α, corresponding to closest returns, and then use them to *decompose* more general Birkhoff sums. Similarly, renormalization for (G)IETs can be exploited to produce *special Birkhoff sums*, namely Birkhoff sums of a special form that can be understood first, exploiting renormalization, and then used to decompose and study general Birkhoff sums. For each $n \in \mathbb{N}$, if $T_n : I_n \to I_n$ is the induced map after n steps of renormalization, the nth *special Birkhoff sum* is the induced function $S(n)f : I_n \to I_n$, defined by $S(n)f(x) := S_{r_n^i} f(x)$ if $x \in I_n^i$. Thus, since r_n^i is the height of the Rohlin tower over I_n^i, the value $S(n)f(x)$ is obtained summing the orbit *along the tower* which has x in the base, see Figure 3b. Notice that for $d = 2$, when considering Zorich acceleration, these reduce to sums of the form $S_{q_n} f(x)$. The associated *special Birkhoff sums operators* $S(n)$, $n \in \mathbb{N}$, map f : $I \to \mathbb{R}$ to $S(n)f : I_n \to \mathbb{R}$. When f is piecewise constant and takes a constant value f^i on each I^i, $S(n)$ can be identified with a linear operator given by the (studied acceleration of the) Rauzy–Veech cocycle $A^n = A_n \cdots A_1$ as follows: one can show that $S(n)f$ takes constant values f_n^i on each I_n^i and the column vectors $\mathbb{f} := (f^i)_{i=1}^d$ and $\mathbb{f}_n := (f_n^i)_{i=1}^d$ are related by $\mathbb{f}_n = A^n\,\mathbb{f}$. Thus, special Birkhoff sums operators can be seen as infinite-dimensional extensions of the Rauzy–Veech cocycle (and its accelerations).

When considering a rotation R_α, to decompose $S_n f(x)$ into Birkhoff sums of the form $S_{q_k} f(y)$ where $y \in I_k$, one can write $n = \sum_{k=0}^{k_n} b_k q_k$, where k_n is the smallest integer k such that $n < q_k$ and b_k are integers such that $0 \le b_k \le a_k$ (a presentation sometimes known as *Ostrowsky decomposition*). Correspondingly, recalling that $S_{q_k} f(y) = S(k)f(y)$ when $y \in I_k$, we can write

$$S_n f(x) = \sum_{k=0}^{k_n} \sum_{j=0}^{b_k-1} S(k)f\left(x_j^k\right), \quad \text{where } x_j^k \in I_k, \text{ for all } 0 \le j < b_k. \tag{4.1}$$

For IETs one can also get an analogous decomposition of any Birkhoff sums $S_n f(x)$ into special Birkhoff sums, which has the same form (4.1), but where $0 \le b_k \le \|A^n\| := \sum_{i,j}(A^n)_{ij}$ and the decomposition is obtained *dynamically*, by decomposing the orbit of x until time n into blocks, each of which is contained in a tower and hence corresponds to a special Birkhoff sums.

Renormalization in moduli spaces. We conclude this section mentioning that these renormalization algorithms (for rotations and IETs) describe a discretization of a renormalization dynamics on the moduli space of surfaces. In genus one, the Gauss map is well known to be related to the geodesic flow on the modular surface (which can be seen as the moduli space of flat tori), see, e.g., [66]. Similarly, (an extension of) Rauzy–Veech induction can be obtained as Poincaré map of the *Teichmüller geodesic flow* on the moduli space of translation surfaces (see, e.g., [85]).

The full measure Diophantine-like conditions that we discuss in this survey are satisfied by (Poincaré maps of) linear flows in almost every direction on almost every translation surface in these moduli space (with respect to the Lebesgue, or Masur–Veech measure, see [12]). A different question is whether these properties hold for a *given* surface in almost every direction, in particular if the surface has special properties, for example, is a torus cover (i.e., it is a square-tiled surface), or has special symmetries (e.g., it is a *Veech surface* or it belongs to an $\mathrm{SL}(2,\mathbb{R})$-invariant locus, see [12]). In these settings, while some results can be obtained by general measure-rigidity techniques (in particular, from the work [9] by

Chaika and Eskin, see also the ICM proceedings [12] and the references therein), to describe explicit Diophantine-like conditions, it is often helpful to exploit or develop *ad hoc* renormalization algorithms (for example, one can use finite extensions of the Gauss map to study square-tiled surfaces, see, e.g., [59], or construct Gauss-like maps for some Veech surfaces, see, e.g., [69]).

5. DIOPHANTINE-LIKE CONDITIONS IN HIGHER GENUS

We finally describe in this section some of the Diophantine-like conditions which were introduced to prove some of the results on typical ergodic and spectral properties of smooth area-preserving flows on surfaces (see Section 3.1) and on *linearization* (such as solvability of the cohomological equation and rigidity questions in higher genus, see Section 3.2).

5.1. Bounded-type IETs and Lagrange spectra

We start with two important classes of IETs, namely *periodic-type* and *bounded-type* IETs, both of which have measure zero in the space $\mathcal{I}_d$ of IETs (although full Hausdorff dimension in the case of bounded-type IETs), but often constitute an important class of IETs in which dynamical and ergodic properties can be tested.

One of the simplest requests on a (G)IET is that its orbit under renormalization is *periodic*, so that the sequence of Rauzy–Veech cocycle matrices $(A_n)_{n\in\mathbb{N}}$ introduced in the previous Section 4 is *periodic*, i.e., there exists $p > 0$ such that $A_{n+p} = A_n$ for every $n \in \mathbb{N}$. We will furthermore request that the *period matrix* $A := A_p \cdots A_2 A_1$ is strictly positive. These IETs are called in the literature *periodic-type* IETs (see, e.g., [68]), in analogy with *periodic-type* rotation numbers (quadratic irrationals like the golden mean $(\sqrt{5}-1)/2 = [1, 1, \dots, 1, \dots]$ which have a periodic continued fraction expansion). By construction they are *self-similar*, and one can also show that they arise as Poincaré section of foliations which are fixed by a *pseudo-Anosov* surface diffeomorphism. Notice that d-IETs of periodic type form a *measure zero set* in $\mathcal{I}_d$ (they are actually countable). One can show (in view of a Perron–Frobenius argument, e.g., following [76]) that periodic-type IETs are always *uniquely ergodic* with respect to the Lebesgue measure.

Periodic-type IETs are often the very first type of IETs used to construct *explicit examples*; see, e.g., the explicit examples of weakly mixing periodic-type IETs in [68] or the explicit examples of Roth-type IETs build in the Appendix of [53]. On the other hand, among periodic-type IETs one can also find examples with exceptional behavior. A further request, that is used to guarantee that a periodic-type T displays features similar to those of typical (in the measure theoretical sense) IETs is that T is of *hyperbolic periodic-type*: this means that the periodic matrix A has g eigenvalues of modulus greater than 1, where g is the genus of the surface of which T is a Poincaré section. Notice that g is the largest possible number of such eigenvalues, as it can be shown by either geometric or combinatorial arguments (in particular, exploiting the symplectic features of the cocycle matrices, which come from their interpretation as action of renormalization on the *relative* homology $H_1(S, \mathrm{Fix}(\varphi_{\mathbb{R}}), \mathbb{R})$, one can show that A has also g eigenvalues of modulus less than 1, while the transpose A^T acts as a permutation on a subspace of dimension $k := d - 2g$ which gives rise to a k-dimensional central space).

Bounded-type IETs equivalent characterizations. Periodic-type IETs are a special case of so called bounded-type IETs: we say that a (Keane) IET T is of *bounded-type* if the matrices of the *positive acceleration* $P_k := A(n_k, n_{k+1})$ are uniformly bounded, i.e., there exists a constant $M > 0$ such that $\|P_k\| \leq M$ for every $k \in \mathbb{N}$. From this point of view, bounded-type IETs can be seen as a generalization of *bounded-type* rotation numbers (which, recalling Section 2, are $\alpha = [a_0, a_1, \dots, a_n, \dots]$ such that for some $M > 0$ we have $|a_n| \leq M$). It turns out that this renormalization-based definition characterizes a natural class of IETs (and corresponding surfaces) from the combinatorial and geometric point of view: bounded-type IETs are *linearly recurrent* (i.e., satisfy an important notion of low complexity in word-combinatorics) and surfaces which have a bounded-type IET as a section give rise to *bounded* Teichmüller geodesics in the moduli space of translation surfaces (see, e.g., [31] for the proof of the equivalences). These natural characterizations show once more how the *positive* acceleration (and not simply Zorich acceleration) is the good one to use in this setting (see also [41] where it is shown that asking that Zorich matrices are bounded leads to a different, strictly larger class).

Furthermore, from the point of view of renormalization, the uniform bounds on the norm of the matrices P_k imply that the partitions into Rohlin towers produced by Rauzy–Veech renormalization are all *balanced* (see Section 4). From a purely dynamics perspective, the orbits of a bounded-type IET are *well-spaced*: there are uniform constants $c, C > 0$ such that, for any point x and any n, the *gaps* (i.e., the distances between closest point) of the orbit $\{T^i x, 0 \leq i < n\}$ are all comparable to n, i.e., are bounded below by c/n and above by C/n. Yet another characterization is in terms of orbits of discontinuities: if $\delta_n(T)$ denotes the smallest length of a continuity interval for T_n, $\liminf_{n\in\mathbb{N}} n\delta_n(T) > 0$, see [31] and the reference therein.

Several results in the literature were proved first assuming bounded-type (for example, the absence of mixing for flows in $\mathcal{U}_{\min}$, see [70], preceding [73]) and some properties are currently known only under the assumption of being bounded-type, for example, *absence of partial rigidity* and *mild-mixing* (see [47] and [32], respectively) for flows in $\mathcal{U}_{\min}$ (it is possible, but an open question, that these two properties fail without assuming that a Poincaré section is of bounded-type), or ergodicity of typical skew-product extensions of IETs by piecewise constant cocycles (see [11]).

Bounded-type uniform contraction and deviations estimates. One of the way in which the bounded-type assumption can be exploited is the following. It is well known that iterates of a *positive* $d \times d$ matrix $A > 0$ act on the positive cone $\mathbb{R}^d_+$ as a *strict* contraction (e.g., with respect to the Hilbert projective metric): this is the phenomenon behind the proof of Perron–Frobenius theorem, showing that A has a unique (positive) eigenvector with maximal eigenvalue. More generally, the projective action of any matrix A_i with $\|A_i\| \leq M$ has a contraction rate which depends on M only; this, in view of the connection between the entries of the cocycle products $A^n := A_n \cdots A_1$ and (special) Birkhoff sums (see Section 4), can be used, given a bounded-type IET, to prove unique ergodicity and to give uniform estimates on the rate of convergence of ergodic averages: one can, for example, show that there is a uniform constant (which can be taken to be 1) and a uniform exponent γ_M such that, for any bounded-type IET with $\|P_k\| \leq M$ and any mean zero (piecewise) smooth $f : I \to \mathbb{R}$, $|S_n f(x)| \leq n^{\gamma_M}$ for all $x \in I$ (see the Appendix of [11]).

The role of bounded-type conditions in the study of Lagrange spectra. Periodic-type and bounded-type rotation numbers play a central role in the study of the *Lagrange spectrum* $\mathcal{L} \subset \mathbb{R} \cup \{+\infty\}$, a classical object in both number theory and dynamics (see, for example, [31] or [58] and the reference therein). It is defined as the set $\mathcal{L} := \{L(\alpha),\ \alpha \in \mathbb{R}\}$ where $L(\alpha) := \limsup_{q,p\to\infty} 1/q|q\alpha - p|$; one can show that $L(\alpha) < \infty$ exactly when α is of bounded-type, in which case $L(\alpha)^{-1}$ provides the smallest constant such that $|\alpha - p/q| < L(\alpha)^{-1}/q^2$ has infinitely many integer solutions $p, q \in \mathbb{Z}, q \neq 0$ (and it has also an interpretation in terms of depths of excursions into the cusp of hyperbolic geodesics on the modular surface). Among the many geometric and dynamical extensions of the notion of Lagrange spectrum (see some of the references in [31]), a natural generalization to higher genus leads to Lagrange spectra of IETs and translation surfaces, which we introduced in joint work with Hubert and Marchese in [31]. The finite values of these spectra are achieved exactly by bounded-type IETs and can be computed using renormalization. We show furthermore in [31] that these spectra can be obtained as the *closure* of the values achieved by periodic-type IETs.

5.2. Roth-like conditions and type

The *Roth-type* condition, to the best of our knowledge, was historically the first full measure "arithmetic" condition to be defined and exploited in higher genus.

Roth-type condition. In the seminal paper [53], Marmi, Moussa, and Yoccoz show first of all that (a predecessor of) the positive acceleration of Rauzy–Veech induction (refer to Section 4) is well defined for all Keane IETs and use this acceleration to define the Roth-type condition and prove that it has full measure; they then show that this condition is sufficient to solve the cohomological equation after removing obstructions (see Section 3.2). Since bounded-type IETs have measure zero, to describe a full measure set of IETs, one needs to allow the norms $\|P_k\|$ of the matrices $(P_k)_k$ of the positive acceleration to grow. Marmi, Moussa, and Yoccoz show in [53] that, for almost every d-IETs in $\mathcal{I}_d$, the matrices $(P_k)_k$ grow subpolynomially, i.e., for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|P_{k+1}\| \leq C_\varepsilon \|Q_k\|^\varepsilon, \quad \text{where } Q_k := P_k \cdots P_1. \tag{5.1}$$

This condition should be seen as a higher genus generalization of the classical Roth-type condition, see Section 2. A d-IETs is called *Roth-type* if it satisfies (5.1) (which is equivalent to condition (a) in [53], see [57]), and two additional conditions, which concern the contraction properties of the cocycle (condition (b) in [53] imposes that the operators $S(k)$ act as contractions on mean-zero functions and guarantees unique ergodicity and the existence of a *spectral gap*, while the last one, condition (c) or *coherence*, concerns the contraction rate of the stable space and its quotient space). The presence of additional requests that concern not only the growth of the matrices but also their hyperbolicity properties seems to be an important and new feature of several Diophantine-like conditions in higher genus, see Section 5.4. While the proof that the latter two conditions are satisfied by almost every IET is a simple consequence of Forni's work [23] and Oseledets theorem (which can be applied in view of the work by Zorich [82]), the proof that the growth condition (5.1) is typical takes a large part of [53]; a simpler proof can be now deduced (as explained in [52]) from a later result by Avila–Gouezel–Yoccoz [3].

Variations of the Roth-type condition. As we saw, the periodic-type condition can be refined to the more restrictive condition of *hyperbolic* periodic-type. In a similar way, one may further request, given a Roth-type IET T, that the stable space, i.e., the space $\Gamma_s(T)$ of vectors $v \in \mathbb{R}^d$ such that $A^n v \to 0$ exponentially as n grows (which, in the case of a periodic-type IET with period matrix A, is generated by the eigenvectors corresponding to the eigenvalues of A which have modulus greater than 1) has maximal dimension, namely g. The condition that one gets was called *restricted Roth-type* in [55]; it has full measure in view of [23] and was used to study the structure and codimension of local $\mathscr{C}^r$ conjugacy class of a (G)IET for $r > 2$. In the joint work [56] with Marmi and Yoccoz, we introduced a further weakening of the (restricted) Roth-type condition, the *absolute* (restricted) Roth type condition, expressed only in terms of the cocycle action on a $2g$-dimensional subspace which can be identified with the *absolute* homology $H_1(S, \mathbb{R})$ of the surface S of which T is section (in contrast, the original condition involves the whole cocycle, which describes the action on the *relative* homology $H_1(S, \mathrm{Fix}(\varphi_\mathbb{R}), \mathbb{R})$). Exploiting [9], one can also show that this absolute (restricted) Roth-type condition holds on every translation surface for almost every direction (see [9] and [56]). A generalization of the *restricted* Roth-type condition, the *quasi-Roth-type* condition, was introduced in [24] to extend the results of [53] and [55] to Poincaré maps of surfaces for which the stable space has dimension less than g (see [24] for details). Let us also mention that a Roth-type condition can also be imposed on the *backward rotation number* (of a translation flow), requesting a growth rate similar to (5.1) for the *dual* cocycle. The corresponding *dual Roth-type condition* was used in [56] to study the asymptotic oscillations of the error term in (3.3) (which we describe in terms of a *distributional* cocycle or *distributional limit shape*, see [56] for details).

Type and recurrence for IETs. It is not surprising that Diophantine-like conditions can also be used to study *recurrence* questions. While for rotations these reduce to Diophantine properties in the classical arithmetic sense (namely how well a number can be approximated by rationals), given an IET T, one can study either how frequently the successive iterates $(T_n(x))_{n\in\mathbb{N}}$ return close to x (see, e.g., [5]), or how close the iterates of a discontinuity come to other discontinuities, see, e.g., [51]. The (Diophantine) *type* η of a rotation R_α is defined to be $\eta := \sup\{\beta : \liminf_{n\to\infty} n^\beta \{n\alpha\} = 0\}$. Bounded- and Roth-type numbers have type $\eta = 1$ (while Liouville ones have type $\eta = \infty$). One can show (see [40] and [55]) that requesting an IET T be of Roth-type is equivalent to asking that $\sup\{\beta : \liminf n^\beta \delta_n(T) = 0\} = 1$, where here $\delta_n(T)$ is the minimum spacing between discontinuities of T_n. It also implies (but without equivalence) that the first return time $\tau_r(x)$ of x to a ball of radius $r > 0$ satisfies the logarithmic law $\lim_{r \to 0} \log \tau_r(x)/\log(1/r) = 1$ for almost every $x \in [0, 1]$ (see [40]).

5.3. Controlled growth Diophantine-like conditions

Any *balanced acceleration* of Rauzy–Veech induction (as defined in Section 4), produces, given a typical IET T, a sequence of times $(n_k)_k$ which correspond to occurrences of positive matrices A_{n_k} whose norm $\|A_{n_k}\| \leq M$ is uniformly bounded (these are, furthermore, return times to a compact subset K of the parameter space for the natural extension). As for bounded-type IETs, occurrences of these positive bounded matrices give very good control of the convergence of (special) Birkhoff sums of characteristic functions $\chi_{I_0^j}$ (see the end of Section 5.1). More generally, if $x_0 \in I_n^j$ belongs to the inducing interval I_n of a balanced return time $n := n_k$ and $q := r_n^j$ is the height of the corresponding tower, the orbit $\{x_0, T(x_0), \ldots, T^{q-1}(x_0)\}$ *along a tower* is so regularly spaced that one can get good estimates of the Birkhoff sums $S_q f(x_0)$ also for other classes of observables f. In order to estimate Birkhoff sums $S_n f(x)$ for other times $n \in \mathbb{N}$ and points $x \in [0, 1]$, one can then *interpolate* these estimates by using the decomposition (4.1) into special Birkhoff sums. It is clear now that for this interpolation to provide good estimates for any time $n \in \mathbb{N}$, one needs to impose that the balanced times $(n_k)_k$ are sufficiently frequent so that $\|A(n_k, n_{k+1})\|$ grows in a controlled way. Notice that by balance the tower heights $r_{n_k}^j$ for $1 \leq j \leq d$ are all comparable and if we set $q_n := \max_j r_n^j$, the norm $\|A(n_k, n_{k+1})\|$ is proportional to $q_{n_{k+1}}/q_{n_k}$.

Mixing Diophantine condition. The main requirement of the mixing Diophantine condition introduced in [71] is that there exist a (good) positive acceleration and $C > 0$ such that

$$\left\|A(n_k, n_{k+1})\right\| \leq C k^{\tau}, \quad \forall k \in \mathbb{N}, \text{ for some } 1 < \tau < 2. \tag{5.2}$$

This condition should be seen as a higher-genus generalization of the Khanin–Sinai condition $|a_k| \leq C k^{\tau}$ for mixing of Arnold flows, see Section 2. The proof that it is satisfied by a full measure set of IETs follows from a Borel–Cantelli argument analogous to that which can be used in genus one, but the input in higher genus are the highly nontrivial integrability estimates for balanced accelerations proved by Avila, Gouezel, and Yoccoz (which the authors proved to show in [3] that the Teichmüller geodesic flow is exponentially mixing): it is proved in [3] that for any $0 < \nu < 1$, there exists a suitable compact set K such that $\int_K \|A_K\|^{\nu} \mathrm{d}\mu$ is finite (where A_K is the accelerated cocycle and μ the Zorich measure).

In order to prove mixing of (minimal components of) locally Hamiltonian flows in $\mathcal{U}_{\neg \min}$ (i.e., Theorem 3.1), one needs good quantitative estimates on *shearing*: these are given by estimates of Birkhoff sums $S_n f$ over an IET which arise as Poincaré map, for a particular observable f (namely, f is taken to be the derivative of the roof function in the special flow representation of $\varphi_{\mathbb{R}}$), which turns out not to be in L^1 (indeed, the function f has singularities of type $1/x$, which are not integrable). When $n = n_k$ is a balanced time, one can control the corresponding special Birkhoff sums $S(n_k) f$ and show that each Birkhoff sum along a tower $S_q f(x)$, where $q = q^j_{n_k}$ and $x \in I^j_{n_k}$, can be controlled after removing the *closest point* contribution that, in this case, is simply $1/x$. One can indeed show that the *trimmed* Birkhoff sum $S_q f(x) - 1/x$ is asymptotic to $Cq \log q$. The mixing Diophantine condition allows to *interpolate* these estimates and to show that, also for any other $n \in \mathbb{N}$, $S_n f(x)$ grows asymptotically as $Cn \log n$ for all points x *with the exception* of points which belong to a set $\Sigma_n \subset [0,1]$ of measure going to zero. The set Σ_n of points which needs to be removed to get the desired control contains points whose orbits may be *resonant*, in the sense that it may contain a close-to-arithmetic progression near one of the singularities of f, with step $q_{n_k}/q_{n_{k+1}}$ (which can be a very small step if $q_{n_{k+1}}$ is much larger than q_{n_k}).

Ratner Diophantine condition. In order to prove that (minimal components of) locally Hamiltonian flows in $\mathcal{U}_{\neg\min}$ have the *switchable Ratner property* (e.g., Theorem 3.2, see Section 3), one needs more delicate quantitative shearing estimates. Such estimates are proven assuming first of all the mixing Diophantine condition, but the MDC is not sufficient. While mixing is an asymptotic condition and therefore it is sufficient, for all large n, to prove estimates for the Birkhoff sums $S_n f(x)$ (introduced in the previous subsection) on sets of measure tending to 1 (and hence one can remove a set Σ_n whose measure goes to zero), the (switchable) Ratner property requires estimates on arbitrarily large sets of initial points, for *all* large times $n \geq n_0$. If the series $\sum_{n\in\mathbb{N}} \mathrm{Leb}(\Sigma_n)$ were finite, the tail sets of the form $\bigcup_{n\geq n_0} \Sigma_n$ would have arbitrarily small measures, and thus one could throw away these unions for n_0 large. Unfortunately, one can check that the measures $(\mathrm{Leb}(\Sigma_n))_{n\in\mathbb{N}}$ are *not* summable. Instead, we consider a subset $K \subset \mathbb{N}$ such that $\sum_{n\notin K} \mathrm{Leb}(\Sigma_n) < +\infty$ and exploit the additional freedom given by the *switchable* Ratner condition to deal with points $x \in \Sigma_n$ when $n \in K$. This requires the introduction of a suitable Diophantine-like condition.

We say that an IET T satisfies the *Ratner Diophantine condition* (RDC) if T satisfies the mixing DC along the sequence $(n_k)_{k\in\mathbb{N}}$ of balanced induction times and there exist $0 < \xi, \eta < 1$ such that, if $B_k := A(n_k, n_{k+1})$ are the matrices of the accelerated cocycle and $q_k := \max_j r^j_{n_k}$ the maximum height of the corresponding towers, then we have

$$\sum_{k\notin K} 1/(\log q_k)^\eta < +\infty, \quad \text{where } K := \{k \in \mathbb{N} : \|B_k\| \leq k^\xi\}. \tag{5.3}$$

The assumption (5.3) guarantees in particular the summability of $\sum_{n\notin K} \mathrm{Leb}(\Sigma_n)$, so that tail sets of this series *can* be removed. When $k \in K$, using that n_k is a balanced time and $q_k/q_{k-1} \leq \|B_k\|$ is not too large, one can show that an arbitrarily large set of points x do not get close of order c/q_{k-1} to a singularity *twice* in time of order q_k, so by either going *forward* or *backward* in time one can avoid getting $O(q_k^{-1})$ close to singularities. This suffices to provide the control of $S_n f(x)$ (and therefore of *shearing*) required by the switchable Ratner property for all times.

Notice that if an IET T is of bounded type (so $\|B_k\|$ are bounded) then the RDC is automatically satisfied (since the complement of K in $\mathbb{N}$ is finite and therefore the series is a sum of finitely many terms). The Ratner DC imposes that the times k for which $\|B_k\|$ is large are not too frequent: in a sense if an IET satisfies the RDC, it behaves like an IET of bounded type modulo some error with small density (as a subset of $\mathbb{N}$), but this relaxation allows the property to hold for almost every IETs: we prove in [33] that, indeed, for suitable choices of ξ and η, the RDC is satisfied by a full measure set of IETs. Formally (when using the suitable acceleration), the assumption (5.3) looks like the Diophantine condition for rotations introduced by Kanigowski and Fayad in [16], see (3.2). The proof of full measure of the RDC is modeled on the proof of full measure of the arithmetic condition (3.2), with the role of the Gauss map played by the renormalization operator in the parameter space corresponding to the balanced acceleration. Key ingredients to make this proof work are once more the integrability estimates by Avila–Gouezel–Yoccoz [3], as well as a *quasi-Bernoulli* property of the balanced acceleration, see [33] for details.

Backward growth condition for absence of mixing. The Diophantine-like condition to prove absence of mixing of typical locally Hamiltonian flows in $\mathcal{U}_{\min}$ (see Section 3.1) is not explicitly stated in [73], but, from the proof, one can see that one needs the existence of a suitable acceleration of the balanced acceleration, whose matrices will be denoted by $(B_k)_{k\in\mathbb{N}}$, of a subsequence $(k_l)_l$ and of a constant $M > 0$ such that

$$\sum_{k=0}^{k_l} \frac{\|B_k\|}{\nu^{k_l-k}} = \sum_{j=0}^{k_l} \frac{\|B_{k_l-j}\|}{\nu^j} \le M < +\infty, \quad \text{for all } k \in \mathbb{N}, \tag{5.4}$$

where ν is some constant with $\nu > 1$. Such a condition has two interesting features: it requires a *backward* control of the growth of the matrices of an accelerated cocycle, which has to happen *infinitely often*. Indeed, for the series (5.4) to converge and be uniformly bounded by M, one needs to ask that the norms $\|B_k\|$ when k belongs to the sequence $(k_l)_{l\in\mathbb{N}}$ are uniformly bounded; furthermore, it is sufficient to then impose that, going backward in time, they grow slower than the denominator, namely that $\|B_{k_l-j}\| \le Ce^{\delta j}$ for $0 \le j \le k_l$ where δ is chosen so that $e^{\delta} < \nu$. These conditions can be shown to be of full measure by exploiting Oseledets integrability (for the *dual* cocycle).

Such backward conditions seem to appear naturally when one wants to provide good control of the deviations of the points in a finite segment $\{x, T(x), \dots, T_N(x)\}$ of an IET orbit from an arithmetic progression: one would like to show, for example, that, if we relabel the points in the orbit segment so that $0 < x_1 < x_2 < \dots < x_N < 1$, the points x_i display polynomial deviations from an arithmetic progression, i.e., there exist $C > 0$ and $0 < \gamma < 1$ such that $|x_i - i/N| \le C(i/N)^{\gamma}$. These estimates (which are used in [70,73] to show, through a cancelations mechanism, that there is a subsequence of times with no shearing and, as a consequence, that mixing fails) can be proved for all times for bounded type IETs (see [70]), but, for typical IETs, even for orbits along a balanced tower of some renormalization level n_{k_0}, it may not be possible to choose a constant C uniformly in i. Heuristically, the reason for this is that, to estimate the location of x_i, one can use a *spatial decomposition* of the interval $[0, x_i]$ into floors of renormalization towers which involves the entries of *backward* cocycle matrices (a decomposition similar to that in (4.1), but with the role of time now played by space; geometrically this can also be interpreted as swapping the role of the horizontal and vertical flows on a translation surface). The presence of an exceptionally large $\|A_k\|$, even if k is much smaller than k_0, can still spoil the deviations control, since it may correspond in the spatial decomposition to a *clustering* of points, close to an arithmetic progression of a very small step.

We point out that phenomena of similar nature, where the whole backward history of the continued fraction entries matters to control orbits, appears also in genus one, in the theory of circle diffeomorphisms. For example, in the paper [39] (in which the Herman's theory of linearization briefly recalled in Section 2) is revisited, following [38], through the renormalization perspective and optimal results are achieved for low regularity), the finiteness of a series of the form $\sum_{n=n_0}^{\infty} a_{n+1}\big(\sum_{i=0}^{n} \frac{l_n}{l_{n-i}}(l_{n-i-1})^{\eta}\big)$, where $(a_n)_{n\in\mathbb{N}}$ are the CF entries of $\alpha = [a_0, a_1, \dots]$ $l_n := |q_n\alpha - p_n|$ and $0 < \eta < 1$, is used to control the *spatial decomposition* of orbit segments. It would be interesting to know if the analogy, which at this level is only formal and on the *nature* of the conditions, hides a more profound similarity.

5.4. Effective Oseledets Diophantine-like conditions

To conclude, we briefly describe the uniform and regular Diophantine-like conditions (UDC and RDC, for short), introduced and used to prove Theorems 3.5 and 3.3, respectively (see Sections 3.2 and 3.1). Both these conditions present a novel aspect: not only they impose conditions which control the *growth* of cocycle matrices of a suitable acceleration (as all the conditions we have seen in Section 5.3), as well *hyperbolicity* assumptions (as, for example, the *hyperbolic* periodic-type or the *restricted* Roth-type condition, in Sections 5.1 and 5.2), but they also impose *quantitative* forms of *hyperbolicity*, by asking for *effective* bounds on the convergence rates in the conclusion of Oseledets theorem, as we now detail.

Effective Oseledets control and the UDC. Let us say that a sequence of balanced return times $(n_k)_{k\in\mathbb{N}}$ satisfies an *effective* Oseledets control if one can find a sequence of *invariant splittings* $\mathbb{R}^d = E_s^n \oplus E_c^n \oplus E_u^n$, with $\dim E_s^n = g$, such that, for some $\theta > 0$ and any $k \in \mathbb{N}$,

$$\big\|A(n_k, n)|_{E_s^{n_k}}\big\|_\infty \le Ce^{-\theta(n-n_k)} \quad \text{for every } n \ge n_k; \tag{5.5}$$

$$\big\|A(n, n_k)^{-1}|_{E_u^{n_k}}\big\|_\infty \le Ce^{-\theta(n_k-n)} \quad \text{for every } 0 \le n \le n_k. \tag{5.6}$$

Thus, the cocycle contracts the stable space $E_s^{n_k}$ in the future and the unstable space $E_u^{n_k}$ in the past with a *uniform* rate θ and a uniform constant C. These times can be produced, for example, by considering returns to a set (for the natural extension) where the conclusion of Oseledets theorem (for the cocycle and its inverse) can be made uniform. An IET satisfies the *uniform Diophantine condition* (UDC) if there exists balanced times $(n_k)_k$ with effective Oseledets control and, furthermore, for every $\varepsilon > 0$ there exist $C, c > 0$, $\lambda > 0$ and a subsequence $(k_l)_{l\in\mathbb{N}}$ which is *linearly growing* (i.e., such that $\liminf_{l\to\infty} k_l/l > 0$), for which

$$\big\|A(n_k, n_{k_l})\big\| \le C_\varepsilon e^{\varepsilon|k-k_l|} \quad \text{for all } k \ge 0 \text{ and } l \ge 0; \tag{5.7}$$

$$ce^{\lambda k} \le \big\|A(0, n_k)\big\| \le Ce^{(\lambda+\varepsilon)k} \quad \text{for all } k \ge 0. \tag{5.8}$$

One can show that assuming that T satisfies the RDC implies, in particular that T is of (restricted) Roth-type (see [26]); on the other hand, (5.5) and (5.6) are assumptions of a new nature, and furthermore (5.8) clearly excludes IETs of bounded type; thus this is a more restrictive Diophantine-like condition, although still of full measure (see [26]).

The RDC and conditions on Diophantine series. In the regular Diophantine condition (used to study rigidity of GIETs in [29] and, in particular, to prove Theorem 3.5), we assume that T is Oseledets generic and require the existence of a special sequence of balanced times $(n_k)_k$ such that the two following *forward* and *backward* series (involving the accelerated matrices $B_k := A(n_k, n_{k+1})$, their products $B(k,l) := B_l B_{l-1} \cdots B_{k+1}$, as well as the projections Π_s^k and Π_u^k to $E_s^{n_k}$ and $E_u^{n_k}$, respectively) are uniformly bounded by some constant $M > 0$ along a linearly growing subsequence $(k_l)_{l\in\mathbb{N}}$, namely, for every $l \in \mathbb{N}$,

$$\sum_{k=1}^{k_l} \big\| B(k,k_l)_{|E_s^{n_k}} \big\| \big\| \Pi_s^k \big\| \| B_{k-1} \| \leq M, \qquad \sum_{k=k_l+1}^{\infty} \big\| B(k_l,k)^{-1}_{|E_u^k} \big\| \big\| \Pi_u^k \big\| \| B_{k-1} \| \leq M. \tag{5.9}$$

We also require a uniform lower bound on the *angles* between the subspaces E_s^n, E_u^n, and E_c^n of the splitting along the subsequence $(n_{k_l})_l$ and subexponential growth of $B(k_l, k_{l+1})$. The convergence of these series can be proved assuming that the sequence $(n_k)_k$ provides effective Oseledets control; the subsequence $(k_l)_l$ is then selected so that the uniform upper bound holds. We remark that also the UDC can be used to prove the convergence and uniform boundedness (along a linearly growing subsequence) of some series of similar (although simpler) nature (that we call *Diophantine series*, see [26] for details). Notice also the similarity between the backward series in (5.9) and the series (5.4) used to prove absence of mixing, even though the latter involves only the norm of the matrices and not their hyperbolic properties.

Examples of arithmetic conditions on classical rotation numbers which do not depend only on the asymptotic behavior of the continued fraction entries (as *Diophantine* or *Roth-type* conditions) but instead depend on values or finiteness of series involving continued fraction entries include the *Brjuno*-condition (see, e.g., [78]) and the *Perez–Marco* condition [62]. Conditions which require recurrence to a set of rotation numbers with this type of control in the theory of circle diffeos seem to appear in global rigidity results, see, for example, Condition (H) defined by Yoccoz [79].

Final remarks and questions. We saw that advancements in our understanding of both chaotic properties and linearization and rigidity questions in the context of surface flows in higher genus depend crucially on sometimes delicate Diophantine-like conditions, imposed to control the renormalization dynamics. While some of these resemble the classical counterparts, others are of new nature and involve in particular hyperbolicity features which become visible only in higher genus. A downside of this new aspect is that conditions that require Oseledets genericity assumptions are not easily checkable. If there is a way of producing explicit examples with such properties which are not of periodic type, even within a locus, remains a challenge. Since many developments are still quite recent, it is possible that some conditions can be simplified or weakened and still yield the same results; furthermore, the interdependence or inclusions between the various conditions have not been fully investigated. Finally, even though, all the conditions we described, with the only exception of bounded-type conditions, are of full measure, they are likely not to be the optimal ones required for the results for which they were introduced (we know this, for example, for the absence of mixing condition, in view of [13]). Finding optimal conditions for each of these problems is certainly interesting, but probably very difficult.

ACKNOWLEDGMENTS

I would like to acknowledge Selim Ghazouani, Giovanni Forni, and Stefano Marmi for many interesting discussions, as well as for their useful feedback on an earlier version of this text. A thank goes also to Stefan Kurz for allowing me to use Figure 3 and to P. Berk, K. Frączek, D. H. Kim, F. Trujillo, and T. Kappeler for reading and commenting this text.

FUNDING

This work was partially supported by the Swiss National Science Foundation.

REFERENCES

[1] V. I. Arnold, Small denominators and problems of stability of motion in classical and celestial mechanics. *Uspekhi Mat. Nauk* **18** (1963), no. 6, 91–192 (Russian).

[2] V. I. Arnold, Topological and ergodic properties of closed 1-forms with incommensurable periods. *Funktsional. Anal. i Prilozhen.* **25** (1991), no. 2, 1–12.

[3] A. Avila, S. Gouëzel, and J.-C. Yoccoz, Exponential mixing for the Teichmüller flow. *Publ. Math. Inst. Hautes Études Sci.* **104** (2006), 143–211.

[4] P. Berk and A. Kanigowski, Spectral disjointness of rescalings of some surface flows. *J. Lond. Math. Soc. (2)* **103** (2021), no. 3, 901–942.

[5] M. Boshernitzan and J. Chaika, Diophantine properties of IETs and general systems: quantitative proximality and connectivity. *Invent. Math.* **192** (2013), no. 2, 375–412.

[6] X. Bressaud, P. Hubert, and A. Maass, Persistence of wandering intervals in self-similar affine interval exchange transformations. *Ergodic Theory Dynam. Systems* **30** (2010), no. 3, 665–686.

[7] A. I. Bufetov, Limit theorems for translation flows. *Ann. of Math. (2)* **179** (2014), no. 2, 431–499.

[8] R. Camelier and C. Gutierrez, Affine interval exchange transformations with wandering intervals. *Ergodic Theory Dynam. Systems* **17** (1997), no. 6, 1315–1338.

[9] J. Chaika and A. Eskin, Every flat surface is Birkhoff and Oseledets generic in almost every direction. *J. Mod. Dyn.* **9** (2015), 1–23.

[10] J. Chaika, K. Frączek, A. Kanigowski, and C. Ulcigrai, Singularity of the spectrum for smooth area-preserving flows in genus two and translation surfaces well approximated by cylinders. *Comm. Math. Phys.* **381** (2021), no. 3, 1369–1407.

[11] J. Chaika and D. Robertson, Ergodicity of skew products over linearly recurrent IETs. *J. Lond. Math. Soc. (2)* **100** (2019), no. 1, 223–248.

[12] J. Chaika and B. Weiss, The horocycle flow on the moduli space of translation surfaces. In *ICM 2022 Proceedings, Vol. 5*, pp. 3412–3430, EMS Press, 2022.

[13] J. Chaika and A. Wright, A smooth mixing flow on a surface with nondegenerate fixed points. *J. Amer. Math. Soc.* **32** (2019), no. 1, 81–117.

[14] I. P. Cornfeld, S. V. Fomin, and Y. G. Sinai, *Ergodic theory*. Springer, 1980.

[15] A. Denjoy, Sur les courbes definies par les équations différentielles à la surface du tore. *J. Math. Pures Appl. (9)* **11** (1932), 333–375.

[16] B. Fayad and A. Kanigowski, Multiple mixing for a class of conservative surface flows. *Invent. Math.* **203** (2016), no. 2, 555–614.

[17] B. Fayad, A. Kanigowski, and G. Forni, Lebesgue spectrum of countable multiplicity for conservative flows on the torus. *J. Amer. Math. Soc.* **34** (2021), 747–813.

[18] B. Fayad and A. Katok, Constructions in elliptic dynamics. *Ergodic Theory Dynam. Systems* **24** (2004), no. 5, 1477–1520.

[19] L. Flaminio and G. Forni, Invariant distributions and time averages for horocycle flows. *Duke Math. J.* **119** (2003), no. 3, 465–526.

[20] L. Flaminio and G. Forni, On the cohomological equation for nilflows. *J. Mod. Dyn.* **1** (2007), no. 1, 37–60.

[21] G. Forni, Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus. *Ann. of Math. (2)* **146** (1997), no. 2, 295–344.

[22] G. Forni, Asymptotic behaviour of ergodic integrals of 'renormalizable' parabolic flows. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pp. 317–326, Higher Ed. Press, Beijing, 2002.

[23] G. Forni, Deviation of ergodic averages for area-preserving flows on surfaces of higher genus. *Ann. of Math. (2)* **155** (2002), no. 1, 1–103.

[24] G. Forni, S. Marmi, and C. Matheus, Cohomological equation and local conjugacy class of diophantine interval exchange maps. 2018, arXiv:1712.02509. To appear in *Proc. Amer. Math. Soc.*, DOI 10.1090/proc/14538.

[25] K. Frączek and M. Lemańczyk, A class of special flows over irrational rotations which is disjoint from mixing flows. *Ergodic Theory Dynam. Systems* **24** (2004), no. 4, 1083–1095.

[26] K. Frączek and C. Ulcigrai, On the asymptotic growth of Birkhoff integrals for locally Hamiltonian flows and ergodicity in their extensions. 2021, arXiv:2112.05939.

[27] S. Ghazouani, Une invitation aux surfaces de dilatation. 2019, arXiv:1901.08856.

[28] S. Ghazouani, Local rigidity for periodic generalised interval exchange transformations. *Invent. Math.* **226** (2021), no. 2, 467–520.

[29] S. Ghazouani and C. Ulcigrai, A priori bounds for giets, affine shadows and rigidity of foliations in genus 2. 2021, arXiv:2106.03529.

[30] M.-R. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Publ. Math. Inst. Hautes Études Sci.* **49** (1979), 5–233.

[31] P. Hubert, L. Marchese, and C. Ulcigrai, Lagrange spectra in Teichmüller dynamics via renormalization. *Geom. Funct. Anal.* **25** (2015), no. 1, 180–255.

[32] A. Kanigowski and J. Kułaga-Przymus, Ratner's property and mild mixing for smooth flows on surfaces. *Ergodic Theory Dynam. Systems* **36** (2016), no. 8, 2512–2537.

[33] A. Kanigowski, J. Kułaga-Przymus, and C. Ulcigrai, Multiple mixing and parabolic divergence in smooth area-preserving flows on higher genus surfaces. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 12, 3797–3855.

[34] A. Kanigowski, M. Lemańczyk, and C. Ulcigrai, On disjointness properties of some parabolic flows. *Invent. Math.* **221** (2020), no. 1, 1–111.

[35] A. B. Katok, Invariant measures of flows on oriented surfaces. *Sov. Math., Dokl.* **14** (1973), 1104–1108.

[36] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*. Encyclopedia Math. Appl. 54, Cambridge University Press, Cambridge, 1995.

[37] M. Keane, Interval exchange transformations. *Math. Z.* **141** (1975), 25–31.

[38] K. M. Khanin and Y. G. Sinaĭ, A new proof of M. Herman's theorem. *Comm. Math. Phys.* **112** (1987), no. 1, 89–101.

[39] K. Khanin and A. Teplinsky, Herman's theory revisited. *Invent. Math.* **178** (2009), no. 2, 333–344.

[40] D. H. Kim, Diophantine type of interval exchange maps. *Ergodic Theory Dynam. Systems* **34** (2014), no. 6, 1990–2017.

[41] D. H. Kim and S. Marmi, Bounded type interval exchange maps. *Nonlinearity* **27** (2014), no. 4, 637–645.

[42] A. V. Kočergin, The absence of mixing in special flows over a rotation of the circle and in flows on a two-dimensional torus. *Dokl. Akad. Nauk SSSR* **205** (1972), 512–518.

[43] A. V. Kočergin, Mixing in special flows over a shifting of segments and in smooth flows on surfaces. *Mat. Sb.* **96** (1975), no. 138, 471–502.

[44] A. V. Kochergin, Some generalizations of theorems on mixing flows with nondegenerate saddles on a two-dimensional torus. *Mat. Sb.* **195** (2004), no. 9, 19–36.

[45] A. V. Kochergin, Nondegenerate saddles, and absence of mixing. II. *Mat. Zametki* **81** (2007), no. 1, 145–148.

[46] M. Kontsevich, Lyapunov exponents and Hodge theory. In *The mathematical beauty of physics (Saclay, 1996)*, pp. 318–332, Adv. Ser. Math. Phys. 24, World Sci. Publ., River Edge, NJ, 1997.

[47] J. Kułaga, On the self-similarity problem for smooth flows on orientable surfaces. *Ergodic Theory Dynam. Systems* **32** (2012), no. 5, 1615–1660.

[48] M. Lemańczyk, Furstenberg disjointness, Ratner properties and Sarnak's conjecture. In *ICM 2022 Proceedings, Vol. 5*, pp. 3508–3528, EMS Press, 2022.

[49] G. Levitt, La décomposition dynamique et la différentiabilité des feuilletages des surfaces. *Ann. Inst. Fourier (Grenoble)* **37** (1987), no. 3, 85–116.

[50] R. Mañé, Chap. I.8. In *Ergodic theory and differentiable dynamics*, pp. 52–57, Springer, 1987.

[51] L. Marchese, The Khinchin theorem for interval-exchange transformations. *J. Mod. Dyn.* **5** (2011), no. 1, 123–183.

[52] S. Marmi, C. Matheus, and P. Moussa, Fonctions de Brjuno, échanges d'intervalles, origamis et autres objets insolites. In *'Jean-Christophe Yoccoz', Mathématiciens*, SMF, 2018.

[53] S. Marmi, P. Moussa, and J.-C. Yoccoz, The cohomological equation for Roth-type interval exchange maps. *J. Amer. Math. Soc.* **18** (2005), no. 4, 823–872 (electronic).

[54] S. Marmi, P. Moussa, and J.-C. Yoccoz, Affine interval exchange maps with a wandering interval. *Proc. Lond. Math. Soc. (3)* **100** (2010), no. 3, 639–669.

[55] S. Marmi, P. Moussa, and J.-C. Yoccoz, Linearization of generalized interval exchange maps. *Ann. of Math. (2)* **176** (2012), no. 3, 1583–1646.

[56] S. Marmi, C. Ulcigrai, and J.-C. Yoccoz, On Roth type conditions, duality and central Birkhoff sums for I.E.M. In *Quelques aspects de la théorie des systèmes dynamiques: un hommage à Jean-Christophe Yoccoz. II*, pp. 65–132, Astérisque 416, SMF, 2020.

[57] S. Marmi and J.-C. Yoccoz, Hölder regularity of the solutions of the cohomological equation for Roth type interval exchange maps. *Comm. Math. Phys.* **344** (2016), no. 1, 117–139.

[58] C. Matheus, The Lagrange and Markov spectra from the dynamical point of view. In *Ergodic theory and dynamical systems in their interactions with arithmetics and combinatorics*, pp. 259–291, Lecture Notes in Math. 2213, Springer, Cham, 2018.

[59] C. Matheus, M. Möller, and J.-C. Yoccoz, A criterion for the simplicity of the Lyapunov spectrum of square-tiled surfaces. *Invent. Math.* **202** (2015), no. 1, 333–425.

[60] I. Nikolaev and E. Zhuzhoma, *Flows on 2-dimesional manifolds*. Lecture Notes in Math. 1705, Springer, 1999.

[61] S. P. Novikov, The Hamiltonian formalism and a multivalued analogue of Morse theory. *Uspekhi Mat. Nauk* **37** (1982), no. 5, 3–49 (Russian).

[62] R. Pérez Marco, Sur les dynamiques holomorphes non linéarisables et une conjecture de V. I. Arnol'd. *Ann. Sci. Éc. Norm. Supér. (4)* **26** (1993), no. 5, 565–644.

[63] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*. Les Grands Classiques Gauthier-Villars, Librairie Scientifique et Technique Albert Blanchard, Paris, 1987.

[64] D. Ravotti, Quantitative mixing for locally Hamiltonian flows with saddle loops on compact surfaces. *Ann. Henri Poincaré* **18** (2017), no. 12, 3815–3861.

[65] D. Scheglov, Absence of mixing for smooth flows on genus two surfaces. *J. Mod. Dyn.* **3** (2009), no. 1, 13–34.

[66] C. Series, The modular surface and continued fractions. *J. Lond. Math. Soc. (2)* **31** (1985), no. 1, 69–80.

[67] Y. G. Sinai and K. M. Khanin, Mixing for some classes of special flows over rotations of the circle. *Funktsional. Anal. i Prilozhen.* **26** (1992), no. 3, 1–21.

[68] Y. G. Sinai and C. Ulcigrai, Weak mixing in interval exchange transformations of periodic type. *Lett. Math. Phys.* **74** (2005), no. 2, 111–133.

[69] J. Smillie and C. Ulcigrai, Geodesic flow on the Teichmüller disk of the regular octagon, cutting sequences and octagon continued fractions maps. In *Dynamical numbers—interplay between dynamical systems and number theory*, pp. 29–65, Contemp. Math. 532, Amer. Math. Soc., Providence, RI, 2010.

[70] C. Ulcigrai, *Ergodic properties of some area-preserving flows*. Princeton University, 2007.

[71] C. Ulcigrai, Mixing of asymmetric logarithmic suspension flows over interval exchange transformations. *Ergodic Theory Dynam. Systems* **27** (2007), no. 3, 991–1035.

[72] C. Ulcigrai, Weak mixing for logarithmic flows over interval exchange transformations. *J. Mod. Dyn.* **3** (2009), no. 1, 35–49.

[73] C. Ulcigrai, Absence of mixing in area-preserving flows on surfaces. *Ann. of Math. (2)* **173** (2011), no. 3, 1743–1778.

[74] C. Ulcigrai, Shearing and mixing in parabolic flows. In *European congress of mathematics*, pp. 691–705, Eur. Math. Soc., Zürich, 2013.

[75] C. Ulcigrai, Slow chaos in surface flows. *Boll. Unione Mat. Ital.* **14** (2021), no. 1, 231–255.

[76] W. A. Veech, Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math.* **115** (1982), 201–242.

[77] J.-C. Yoccoz, Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition Diophantienne. *Ann. Sci. Éc. Norm. Supér. (4)* **17** (1984), no. 3, 333–359.

[78] J.-C. Yoccoz, Théorème de Siegel, nombres de Bruno et polynômes quadratiques. In *Petits diviseurs en dimension* 1, pp. 3–88, Astérisque 231, SMF, 1995.

[79] J.-C. Yoccoz, Analytic linearization of circle diffeomorphisms. In *Dynamical systems and small divisors (Cetraro, 1998)*, pp. 125–173, Lecture Notes in Math. 1784, Springer, Berlin, 2002.

[80] J.-C. Yoccoz, Échanges d'intervalles et surfaces de translation. In *Séminaire Bourbaki 2007/2008*, Astérisque, Société Mathématique De France, 2009.

[81] J.-C. Yoccoz, Interval exchange maps and translation surfaces. In *Homogeneous flows, moduli spaces and arithmetic*, pp. 1–69, Clay Math. Proc. 10, Amer. Math. Soc., Providence, RI, 2010.

[82] A. Zorich, Finite Gauss measure on the space of interval exchange transformation. Lyapunov exponents. *Ann. Inst. Fourier (Grenoble)* **46** (1996), 325–370.

[83] A. Zorich, Deviation for interval exchange transformations. *Ergodic Theory Dynam. Systems* **17** (1997), no. 6, 1477–1499.

[84] A. Zorich, How do the leaves of a closed 1-form wind around a surface? In *Pseudoperiodic topology*, pp. 135–178, Amer. Math. Soc. Transl. Ser. 2 197, Amer. Math. Soc., Providence, RI, 1999.

[85] A. Zorich, Flat surfaces. In *Frontiers in number theory, physics, and geometry. I*, pp. 437–583, Springer, Berlin, 2006.

CORINNA ULCIGRAI

Universität Zürich Institut für Mathematik, Y27-K36, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland, corinna.ulcigrai@math.uzh.ch