SELF-SIMILAR SETS AND MEASURES ON THE LINE

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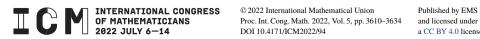**ABSTRACT**

We discuss the problem of determining the dimension of self-similar sets and measures on $\mathbf{R}$. We focus on the developments of the last four years. At the end of the paper, we survey recent results about other aspects of self-similar measures including their Fourier decay and absolute continuity.

A (self-similar) iterated function system, IFS for short, is a finite collection

$$\Phi = \{\varphi_i : i \in \Lambda\}$$

of contractive similarities of $\mathbf{R}^d$. A contractive similarity is a map $x \mapsto \lambda \cdot Ux + t$, where $\lambda \in (0, 1)$, $U \in \mathrm{O}(d)$ is a rotation, and $t \in \mathbf{R}$. We call λ the contraction factor of the similarity. Given such an IFS, there is a unique self-similar set, that is, a compact set $K \subset \mathbf{R}^d$ such that

$$K = \bigcup_{i \in \Lambda} \varphi_i(K).$$

This set K is also known as the attractor of the IFS. Furthermore, given an IFS and a probability vector $\{p_i : i \in \Lambda\}$, there is a unique self-similar measure, that is, a probability measure μ on $\mathbf{R}^d$ such that

$$\mu = \sum_{i \in \Lambda} p_i \varphi_i(\mu).$$

Here $\varphi_i(\mu)$ denotes the push-forward of μ under φ_i. In other words, μ is the unique stationary measure for the Markov chain on $\mathbf{R}^d$ with transitions φ_i executed with probability p_i. The support of μ equals the self-similar set K provided $p_i > 0$ for all i.

Self-similar sets and measures are central objects of interest in fractal geometry and they include many classical examples of fractals. For example, the attractor of the IFS

$$\{x \mapsto \lambda x - 1, x \mapsto \lambda x + 1\}$$

is (a scaled copy of) the middle $1 - 2\lambda$ Cantor set for $\lambda \in (0, 1/2)$, while for $\lambda \geq 1/2$, the attractor is an interval. The self-similar measure associated to the same IFS with equal probability weights $p_i = 1/2$ is called the Bernoulli convolution and is denoted by ν_λ. They can also be defined as the distribution of the random variables

$$\sum_{n=0}^{\infty} \pm \lambda^n,$$

where the $\pm$ are independent fair coin tosses. The study of these measures go back at least to Wintner and his collaborators in the 1930s. See [38] for more on Bernoulli convolutions. Other classical self-similar sets include the Sierpiński triangle and (a side of) the Koch snowflake curve.

The systematic study of self-similar sets and measures was initiated by Hutchinson [25]. We refer to his paper and Falconer's book [15, CHAPTER 9] for thorough treatments of the fundamental properties of these objects.

Determining the dimension of self-similar sets and measures is a central problem in fractal geometry. While there are several competing notions of dimension for sets and measures, most of them coincide in the self-similar case. In this paper, for self-similar sets, by dimension we mean the common value of the Minkowski and Hausdorff dimensions.

The local dimension of a measure μ in $\mathbf{R}^d$ at a point x is

$$\lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

provided the limit exists, where $B(x, r)$ is the ball of radius r around x. We say that the measure is exact dimensional if its local dimension exists and is constant μ-almost everywhere. By the dimension of an exact dimensional measure μ we mean this μ-almost constant value of its local dimension. It is known that self-similar measures are exact dimensional (see [18]).

Before we state the main conjectures in the dimension theory of self-similar sets and measures on $\mathbf{R}$, which will be the main focus of this paper, we make some simple observations to motivate them. Let K be the attractor of a self-similar IFS $\{\varphi_i : i \in \Lambda\}$. We write H^s for the s-dimensional Hausdorff measure. Now suppose that the sets $\varphi_i(K)$ are pairwise disjoint for $i \in \Lambda$ and that $0 < H^s(K) < \infty$ for some s. Then we can write

$$H^s(K) = \sum_{i \in \Lambda} H^s\big(\varphi_i(K)\big) = H^s(K) \sum_{i \in \Lambda} \lambda_i^s,$$

where λ_i is the contraction factor of φ_i. It follows that s must be the unique solution of

$$1 = \sum_{i \in \Lambda} \lambda_i^s. \tag{1}$$

While an s with $0 < H^s(K) < \infty$ may not exist in general, if it does, then it must equal the Hausdorff dimension of K. Therefore, the above considerations suggest that a reasonable guess for $\dim(K)$ is the unique solution of the equation (1). It is a classical result going back to Moran [35] in some form, that this guess is correct when the IFS satisfies the so-called open set condition, which is a mild relaxation of requiring that the sets $\varphi_i(K)$ are pairwise disjoint. See [15, CHAPTER 9] for a precise definition.

It turns out that the unique solution of (1) is always an upper bound for $\dim(K)$ and it is natural to ask to what extent it is possible to drop the open set condition without turning this upper bound into a strict inequality. There are two immediate obstructions to this. First, the solution of (1) may be larger than d, but the dimension of K will never exceed d which is the dimension of the ambient space $\mathbf{R}^d$. Second, (1) depends on the IFS and not only on the set K. It may be possible to realize K as the attractor of another IFS such that the corresponding (1) has a smaller solution. This happens, for example, if the IFS contains exact overlaps, which we define now.

Definition 1. An IFS $\{\varphi_i : i \in \Lambda\}$ contains exact overlaps if there is some $n \in \mathbf{Z}_{\geq 1}$ and $(i_1, \ldots, i_n) \neq (\tilde{i}_1, \ldots, \tilde{i}_n) \in \Lambda^n$ such that

$$\varphi_{i_1} \circ \cdots \circ \varphi_{i_n} = \varphi_{\tilde{i}_1} \circ \cdots \circ \varphi_{\tilde{i}_n}. \tag{2}$$

In other words, the IFS contains no exact overlaps if and only if the semigroup generated by the maps in the IFS with respect to the composition operation is free. We note that it does not make a difference in the definition whether or not we require that we have the same number of composition factors on the two sides of (2).

The next conjecture due to Simon (see [47]) predicts that apart from the above two obstructions, $\dim(K)$ equals the unique solution of (1) in the $d = 1$ case.

Conjecture 2. *Let K be the attractor of a self-similar IFS $\{\varphi_i : i \in \Lambda\}$ on $\mathbf{R}$ that contains no exact overlaps. Let λ_i be the contraction factor of φ_i. Then*

$$\dim K = \min(1, s),$$

where s is the unique solution of the equation

$$\sum_i \lambda_i^s = 1.$$

The conjecture also has a counterpart for measures.

Conjecture 3. *Let μ be the self-similar measure on $\mathbf{R}$ associated to an IFS $\{\varphi_i : i \in \Lambda\}$ without exact overlaps and a probability vector $\{p_i\}$. Let λ_i be the contraction factor of φ_i. Then*

$$\dim \mu = \min\left(1, \frac{\sum_i p_i \log p_i^{-1}}{\sum_i p_i \log \lambda_i^{-1}}\right).$$

Self-similar measures are of interest in their own right, but a major motivation for Conjecture 3 is that it implies Conjecture 2. To see this, recall that if a set K supports an exact dimensional measure μ of dimension s, then the Hausdorff dimension of K is at least s, see [**15, PRINCIPLE 4.2**]. This is a common way of giving lower bounds on the Hausdorff dimension. Now let s be the solution of (1), and consider the probability weights $p_i = \lambda_i^s$. Observe that this choice yields

$$s = \frac{\sum_i p_i \log p_i^{-1}}{\sum_i p_i \log |\lambda_i|^{-1}},$$

showing that Conjecture 3 indeed implies Conjecture 2.

Almost all of this paper is concerned only with self-similar measures on $\mathbf{R}$. Some difficulties arise when one tries to formulate versions of Conjectures 2 and 3 for self-similar sets and measures in higher-dimensional ambient spaces due to the presence of affine subspaces of intermediate dimension. For a discussion of these issues and results in higher dimension, we refer to [**22**].

The purpose of this paper is to survey results towards Conjectures 2 and 3. Since this subject has already been exposed by Hochman in his ICM lecture in 2018 [**24**], we focus on the developments of the last four years and discuss earlier results only to the extent necessary to keep our presentation self-contained.

We will outline some ideas from the proofs of these results; however, we will not give full details, and some of our discussion will be imprecise. Our aim is to overview the theory and give insight into the role played by its components. For details and a rigorous discussion of the proofs we refer to the original papers.

In the final section, we briefly survey some further recent developments on Fourier decay and absolute continuity of self-similar measures.

1. EXPONENTIAL SEPARATION PROPERTY

The exponential separation property was introduced by Hochman [**23**] who showed that Conjectures 2 and 3 hold when the IFS satisfies this property. This property can be verified in many cases of interest. While these results have been already discussed in [**24**], we recall them now because they are of crucial importance to later developments both logically and for the motivation of ideas.

We begin with the definitions. We introduce a distance function on the group of similarities on $\mathbf{R}$. Let $\varphi_i : x \mapsto \lambda_i x + t_i$ be similarities for $i = 1, 2$. We define

$$\mathrm{dist}(\varphi_1, \varphi_2) = \begin{cases} |t_1 - t_2| & \text{if } \lambda_1 = \lambda_2, \\ \infty & \text{if } \lambda_1 \neq \lambda_2. \end{cases}$$

Given an IFS $\Phi := \{\varphi_i : i \in \Lambda\}$, we define its level n separation by

$$\Delta_n(\Phi) := \min_{(i_1,\ldots,i_n) \neq (\tilde{i}_1,\ldots,\tilde{i}_n) \in \Lambda^n} \mathrm{dist}(\varphi_{i_1} \circ \cdots \circ \varphi_{i_n}, \varphi_{\tilde{i}_1} \circ \cdots \circ \varphi_{\tilde{i}_n}).$$

We say that the IFS satisfies the exponential separation property if there is a constant $c > 0$ such that $\Delta_n(\Phi) > c^n$ for infinitely many n's.

We observe that the IFS contains exact overlaps if and only if $\Delta_n = 0$ for some and hence for all sufficiently large n. The exponential separation property is a quantitative strengthening of the condition that the IFS contains no exact overlaps. Hochman proved that Conjectures 2 and 3 hold under this strengthening of the hypothesis.

Theorem 4 (Hochman [23]). *Let $\{\varphi_i : i \in \Lambda\}$ be an IFS that satisfies the exponential separation property and let K be its attractor. Write λ_i for the contraction factor of φ_i. Then*

$$\dim K = \min(1, s),$$

where s is the unique solution of the equation

$$\sum_i \lambda_i^s = 1.$$

Let μ be the self-similar measure associated to the above IFS and a probability vector $\{p_i\}$. Then

$$\dim \mu = \min\left(1, \frac{\sum_i p_i \log p_i^{-1}}{\sum_i p_i \log \lambda_i^{-1}}\right).$$

It can be shown that the exponential separation property holds in parametric families of IFSs for all but possibly a (packing or Hausdorff) codimension 1 subset of exceptions. This shows that Conjectures 2 and 3 hold generically in a very strong sense. We refer to [23] for details and more precise results.

We also note that a stronger version of Conjecture 3 involving the L^q dimension instead of local dimension of measures was established subsequently by Shmerkin [46] under the exponential separation property. This result has very important and far reaching applications, see also [45] and Shmerkin's paper in this volume.

Our main focus here are explicit cases and families of IFSs for which the exponential separation property can be verified. We first observe that the exponential separation property holds always whenever all contraction and translation parameters in the IFS are rational and the IFS contains no exact overlaps. Indeed, writing Q for the least common denominator of all parameters, a simple calculation shows that the translation parameters of n-fold compositions of maps in the IFS have denominators that divide Q^n. This means that for all n, we have $\Delta_n \geq Q^{-n}$ or $\Delta_n = 0$. The second possibility is excluded by the absence of exact overlaps.

In fact, the above reasoning can be extended to the case when the parameters are algebraic numbers and not necessarily rational. To do this, one need to work with heights instead of denominators, or see [**23, THEOREM 1.5**] for a more elementary argument. This leads to the following result

Corollary 5 (Hochman). *Conjectures* 2 *and* 3 *hold for IFSs in which all contraction and translation parameters are algebraic numbers.*

The exponential separation property can be verified also for certain IFSs that involve transcendental parameters. One such example is the family IFSs

$$\left\{ x \mapsto \frac{x}{3}, x \mapsto \frac{x}{3} + 1, x \mapsto \frac{x}{3} + t \right\}, \tag{3}$$

where $t \in \mathbf{R}$ is a parameter. It can be seen that the attractors of these IFSs are the linear projections of the Sierpiński triangle.

Another corollary of Theorem 4 is the following.

Corollary 6 (Hochman). *Conjectures* 2 *and* 3 *hold for the IFS* (3) *for all values of the parameter* $t \in \mathbf{R}$.

We sketch the proof of the exponential separation property for the family (3), as these ideas will recur later. For details, see [**23, THEOREM 1.6**], where this argument is attributed to Solomyak and Shmerkin. The translation component of an n-fold composition of maps from the above IFS is of the form

$$\sum_{j=0}^{n-1} \alpha_j 3^{-j},$$

where each α_j is equal to 0, 1, or t. Based on this observation, it can be seen that for each t and for each n, there are some integers $a_1, a_2 \in \mathbf{Z}$ not both 0 with $|a_1|, |a_2| \leq 3^{n-1}$ such that

$$\Delta_n = \frac{a_1}{3^{n-1}} - \frac{a_2}{3^{n-1}} t.$$

Assuming $a_2 \neq 0$, which holds whenever $\Delta_n \leq 3^{-n+1}$, we get

$$\left| t - \frac{a_1}{a_2} \right| \leq 3^n \Delta_n.$$

Now fix the value of the parameter t such that the IFS (3) contains no exact overlaps. Suppose $\Delta_n < 27^{-n-1}$ for some n. Then there is a rational number a_1/a_2 as above such that $|t - a_1/a_2| < 9^{-n-1}$. Let $\tilde{n}$ be such that $9^{-\tilde{n}-1} < |t - a_1/a_2| \leq 9^{-\tilde{n}}$. (Note that $t \neq a_1/a_2$, for otherwise we would have $\Delta_n = 0$ and the IFS would contain exact overlaps.) We observe that there is no rational $\tilde{a}_1/\tilde{a}_2$ with $|t - \tilde{a}_1/\tilde{a}_2| < 9^{-\tilde{n}-1}$ and $|\tilde{a}_1|, |\tilde{a}_2| \leq 3^{\tilde{n}-1}$. Indeed, if such a rational existed, we would have

$$\left| \frac{a_1 \tilde{a}_2 - a_2 \tilde{a}_1}{a_2 \tilde{a}_2} \right| = \left| \frac{a_1}{a_2} - \frac{\tilde{a}_1}{\tilde{a}_2} \right| \leq \left| \frac{a_1}{a_2} - t \right| + \left| t - \frac{\tilde{a}_1}{\tilde{a}_2} \right| \leq 2 \cdot 9^{-\tilde{n}}.$$

Since $|a_2 \tilde{a}_2| \leq 9^{\tilde{n}-1}$, this would yield $a_1 \tilde{a}_2 - a_2 \tilde{a}_1 = 0$, leading to $a_1/a_2 = \tilde{a}_1/\tilde{a}_2$ and contradicting

$$\left| t - \frac{\tilde{a}_1}{\tilde{a}_2} \right| < 9^{-\tilde{n}-1} < \left| t - \frac{a_1}{a_2} \right|.$$

This shows that $\Delta_{\tilde{n}} \geq 27^{-\tilde{n}-1}$, and the exponential separation property follows.

A key property of the IFS (3) exploited in the above argument is that exact overlaps occur for certain special values of the parameter t, in this case certain rational numbers, and these special values are very well separated from each other. This will be a recurrent concern for us in what follows.

A similar argument can be made when the contraction factor $1/3$ in (3) is replaced by another algebraic number. We omit the details.

2. BERNOULLI CONVOLUTIONS

In this section, we consider the one parameter family of IFSs

$$\Phi_\lambda := \{x \mapsto \lambda x, x \mapsto \lambda x + 1\},$$

where $\lambda \in (0, 1)$. Instead of 0 and 1 we could take any other pair of distinct real numbers as the translation parameters; we would get the same IFS up to a change of coordinates. In fact, it is more customary to take ± 1 instead of 0 and 1, but the above choice will make notation more consistent with the rest of this note.

In this case, the resulting self-similar sets have a simple structure. For $\lambda < 1/2$, it is the middle $(1 - 2\lambda)$th Cantor set, while for $\lambda \geq 1/2$ it is an interval. In both cases, Conjecture 2 is easily verified. However, the associated self-similar measures called Bernoulli convolutions are more difficult to understand. The purpose of this section is to summarize the developments that lead to the following result.

Theorem 7. *Conjecture 3 holds for the IFS Φ_λ for any value of the parameter $\lambda \in (0, 1)$.*

For algebraic parameters, this result is due to Hochman as it falls under the scope of Corollary 5. For transcendental parameters, the result has been established in [54]. Strictly speaking, only the case of uniform $(1/2, 1/2)$ probability weights is treated there, but the arguments can be extended to the general case. Moreover, one can even allow more general IFSs with an arbitrary number of maps as long as the contraction factors are the same and the translation parameters are rational. This has been demonstrated in the Appendix of [41].

To simplify the exposition, we assume in our discussion that the probability weights are uniform. We write ν_λ for the self-similar measure associated to the IFS Φ_λ. We note that ν_λ is the law of the random variable $\sum_{n=0}^\infty \xi_n \lambda^n$, where (ξ_n) is a sequence of independent random variables taking the values 0 and 1 with equal probability.

In the algebraic case, Hochman's results yield more information, which allows computing the dimension even in the presence of exact overlaps. This is in terms of the entropy rate of the IFS Φ_λ, which we define now, and which will also play an important role later

on. The entropy rate is defined as

$$h(\Phi_\lambda) := \lim_{n \to \infty} \frac{H(\sum_{j=0}^{n-1} \xi_j \lambda^j)}{n},$$

where $H(\cdot)$ stands for Shannon entropy of a discrete random variable. The numerator on the right can be shown to be a subadditive sequence, hence the limit exists and, moreover,

$$h(\Phi_\lambda) \le \frac{H(\sum_{j=0}^{n-1} \xi_j \lambda^j)}{n}$$

for each n.

See [9, SECTION 3.4] for the details of how the following follows from the main result of Hochman [23].

Theorem 8 (Hochman). *Let $\lambda \in (0, 1)$ be an algebraic number. Then*

$$\dim \nu_\lambda = \min\left(1, \frac{h(\Phi_\lambda)}{\log \lambda^{-1}}\right). \tag{4}$$

This result, together with Theorem 7, gives an almost complete solution to the problem of determining the dimension of Bernoulli convolutions. In addition, there are numerical algorithms to compute $\dim \nu_\lambda$ with arbitrary precision for any given algebraic λ, see [1,17,21,29]. However, it is still not known precisely what is the set of algebraic parameters $\lambda \in (1/2, 1)$ for which $\dim \nu_\lambda < 1$.

We turn to the case of transcendental parameters in Theorem 7. If the IFS Φ_λ satisfied the exponential separation property whenever it does not contain exact overlaps, then Theorem 7 would follow at once from Theorem 4. This very well could be true; however, this is still an open problem, which seems to be beyond reach of existing methods.

In fact, the decay rate of $\Delta_n(\Phi_\lambda)$ is very closely related to a problem in Diophantine approximation, which is the separation between the elements of the set

$$\mathcal{E}^{(n)} := \{\eta : P(\eta) = 0 \text{ for some polynomial } P \in \mathcal{P}^{(n)}\},$$

where $\mathcal{P}^{(n)}$ is the set of polynomials of degree at most $n - 1$ with coefficients $-1, 0, 1$. As it will be clear from what follows, the set

$$\mathcal{E} := \bigcup_n \mathcal{E}^{(n)} \cap (0, 1)$$

is precisely the set of parameters for which Φ_λ contains exact overlaps.

We begin our discussion of the proof of Theorem 7 by explaining the connection between the behavior of $\Delta_n(\Phi_\lambda)$ and the separation properties of the sets $\mathcal{E}^{(n)}$ following Hochman [23, QUESTION 1.10]. This can be formalized as follows.

Lemma 9. *If it is true that the elements of $\mathcal{E}^{(n)}$ are separated by at least C^{-n} for some constant C for all n, then the exponential separation property holds for the IFS Φ_λ whenever it lacks exact overlaps.*

Sketch of proof. Fix some $\varepsilon > 0$ and assume $\lambda \in (\varepsilon, 1 - \varepsilon)$. We first observe that if $\Delta_n(\Phi_\lambda) < C^{-n}$ for some $C = C(\varepsilon)$, then there is some $\eta \in \mathcal{E}^{(n)}$ with

$$|\lambda - \eta| < \Delta_n(\Phi_\lambda)^\alpha$$

for some $\alpha = \alpha(\varepsilon) > 0$. This follows from the fact that the translation component of an n-fold composition of the maps in Φ_λ in some order is a polynomial in λ of degree at most $n - 1$ with coefficients $0, 1$. This means that $\Delta_n(\Phi_\lambda) = P(\lambda)$ for some $P \in \mathscr{P}^{(n)}$ that also depends on λ. To complete the proof of our observation, we need to argue that the only way $P(\lambda)$ can be very small is if λ is close to a root of P. For more details, see **[52, LEMMA 5.2]**.

Now suppose that λ is such that $\Delta_n(\Phi_\lambda) < C_2^{-2n/\alpha}$ for some n, where α is as in the previous paragraph and C_2 is the constant C in the assumption about the separation between the elements of $\mathcal{E}_n$. Then there is η_n such that $|\lambda - \eta_n| < C_2^{-2n}$. If Φ_λ contains no exact overlaps, then $\lambda \notin \mathcal{E}$ so $\lambda \neq \eta_n$. Now we take the smallest integer $\tilde{n} > n$ such that $|\lambda - \eta_n| > C_2^{-2\tilde{n}}$. It follows by the assumed separation property on $\mathcal{E}^{(\tilde{n})}$ that there is no $\eta_{\tilde{n}} \in \mathcal{E}^{(\tilde{n})}$ with $|\lambda - \eta_{\tilde{n}}| < C_2^{-2\tilde{n}}$. This means that $\Delta_{\tilde{n}}(\Phi_\lambda) \geq C_2^{-2\tilde{n}/\alpha}$, and the exponential separation property follows. ∎

It is not known whether or not the elements of $\mathcal{E}^{(n)}$ are exponentially separated. The best lower bound known for the minimal distance of the elements of $\mathcal{E}^{(n)}$ is $\exp(-Cn \log n)$ for some constant C (one could take, e.g., $C = 4$), which is due to Mahler **[33]**. This yields via the argument in the proof of Lemma 9 that for all λ such that Φ_λ contains no exact overlaps, there are infinitely many values of n with

$$\Delta_n(\Phi_\lambda) \geq \exp(-Cn \log n) \tag{5}$$

for some (other) constant C.

One may wonder if this weaker separation condition could be used in a refined form of Hochman's argument in place of exponential separation. This has been done in **[8]**, however, the argument requires that there are several values of n sufficiently close to each other such that the separation (5) holds. Such a condition can be satisfied if we assume that λ is not approximated too closely by elements of $\mathcal{E}^{(n)}$. Indeed, in the above argument the size of $\tilde{n}$ is controlled by the distance between λ and $\mathcal{E}^{(n)}$. More precisely, the following was proved in **[8]**.

Theorem 10 (Beruillard, Varjú). *Let $\lambda \in (1/2, 1)$ be such that Conjecture 3 does not hold for Φ_λ. Then there is $\delta > 0$ and there are infinitely many values of n such that there is $\eta_n \in \mathcal{E}^{(n)} \cap (1/2, 1)$ with*

$$|\lambda - \eta_n| < \exp(-n^{100}),$$
$$\dim \nu_{\eta_n} < 1 - \delta.$$

The exponent 100 can be replaced by any other number, or even by a slowly growing function of n, see **[8]** for details. This result along with Theorem 4 are major ingredients in the proof of Theorem 7. Given some $\lambda \in (1/2, 1)$ such that Φ_λ lacks exact overlaps, it can be shown that λ has only finitely many approximants η_n as in the conclusion of Theorem 10 or else Φ_λ satisfies the exponential separation property. In either case, Conjecture 3 follows for Φ_λ from one of Theorems 4 or 10.

Before we discuss the details of how this can be done, a further remark about Theorem 10 is in order. We have seen that if $\Delta_n(\Phi_\lambda) < C^{-n}$ for some n and λ with an appropriate

constant C, then λ is approximated by some $\eta \in \mathcal{E}^{(n)}$. However, we claim some additional properties of this η in Theorem 10, most importantly that $\dim \nu_\eta < 1 - \delta$. Now we indicate how this can be deduced. This leads us to a somewhat lengthy digression; however, it also gives us the opportunity to introduce several concepts and ideas that will be needed later on.

Already in Theorem 4, the exponential separation property can be relaxed (see **[23, THEOREMS 1.3 AND 1.4]**). Instead of assuming $\Delta_n(\Phi) > C^{-n}$, it is enough to know that there are not too many pairs of n-fold compositions of maps in Φ whose translation components are closer than C^{-n}. Likewise in the proof of Theorem 10, we work with a similarly relaxed version of (5).

To properly quantify this, we use entropy. Let X be a bounded real valued random variable and let $r \in \mathbf{R}_{>0}$. The entropy of X at scale r is defined as

$$H(X; r) = H\big(\lfloor r^{-1} X \rfloor\big),$$

where $H(\cdot)$ on the right is Shannon entropy. This is the entropy of X with respect to a partition of $\mathbf{R}$ into consecutive intervals of length r. The choice of this partition is not canonical, and we obtain different values of $H(X; r)$ by translating X. There are advantages of averaging over translations of X in the definition of $H(X; r)$, as it is done, e.g., in **[8, 53]** and subsequent papers; however, we ignore this point here for the sake of simplicity.

By definition, $\Delta_n(\Phi_\lambda) > r$ implies that the points in the support of $\sum_{j=0}^{n-1} \xi_j \lambda^j$ are separated by a distance of at least r, hence

$$H\left(\sum_{j=0}^{n-1} \xi_j \lambda^j; r\right) = \log(2) \cdot n.$$

In the proof of Theorem 10, instead of working with lower bounds on $\Delta_n(\Phi_\lambda)$ like (5), we work with bounds of the type

$$H\left(\sum_{j=0}^{n-1} \xi_j \lambda^j; r\right) \geq \beta n \tag{6}$$

with suitable β and r.

Now consider some $\lambda > 1/2$ that lacks the approximations η_n as described in the conclusion in Theorem 10. We discuss how this assumption can be used to show that bounds of the type (6) hold for suitably many different values of n. Using such bounds and arguments based on Hochman's proof of Theorem 4, which we do not discuss in this paper, it can be shown that $\dim \nu_\lambda = 1$ proving (the contrapositive of) Theorem 10.

In short, the failure of (6) with a suitably small r implies that λ can be approximated by some $\eta_n \in \mathcal{E}^{(n)}$ such that Φ_{η_n} has enough exact overlaps to force $\dim \nu_{\eta_n} \leq \beta / \log \eta_n^{-1}$.

We give some more details. For every pair of numbers x_1, x_2 in the support of $\sum_{j=0}^{n-1} \xi_j \lambda^j$ such that $|x_1 - x_2| \leq r$, there is a polynomial $P \in \mathcal{P}^{(n)}$ such that

$$|x_1 - x_2| = \big|P(\lambda)\big| \leq r.$$

As we have already seen, all such polynomials have a root near λ provided $r < C^{-n}$ for a suitable constant C. If $r < \exp(-Cn \log n)$ for another suitable C, then all the roots obtained

this way as (x_1, x_2) goes over all pairs of points in the support of $\sum_{j=0}^{n-1} \xi_j \lambda^j$ that are at distance not more than r can be shown to coincide. This follows from Mahler's aforementioned bound on the separation of elements in $\mathcal{E}^{(n)}$. For an alternative argument, see [**8**, **SECTION 3**].

Now it follows that if

$$H\left(\sum_{j=0}^{n-1} \xi_j \lambda^j; r\right) < \log(2) \cdot n$$

for some $r < \exp(-Cn \log n)$, then there is some $\eta_n \in \mathcal{E}^{(n)}$ close to λ (the common root of the polynomials discussed in the previous paragraph) such that

$$H\left(\sum_{j=0}^{n-1} \xi_j \eta_n^j\right) \leq H\left(\sum_{j=0}^{n-1} \xi_j \lambda^j; r\right). \tag{7}$$

Notice that on the left there is no designated scale, so $H(\cdot)$ stands for Shannon entropy there. Provided $H(\sum_{j=0}^{n-1} \xi_j \lambda^j; r)$ is sufficiently small, this can be turned into a bound on $\dim \nu_{\eta_n}$ with the help of Theorem 8. Indeed, combining our observations, we see that

$$H\left(\sum_{j=0}^{n-1} \xi_j \lambda^j; r\right) \leq \beta n$$

implies

$$\dim \nu_{\eta_n} \leq \frac{h(\Phi_{\eta_n})}{\log \eta_n^{-1}} \leq \frac{H(\sum_{j=0}^{n-1} \xi_j \eta_n^j)}{n \log \eta_n^{-1}} \leq \frac{H(\sum_{j=0}^{n-1} \xi_j \lambda^j; r)}{n \log \eta_n^{-1}} \leq \frac{\beta}{\log \eta_n^{-1}}.$$

By the assumption that λ lacks the approximations as in the conclusion of Theorem 10, we conclude $|\lambda - \eta_n| > \exp(-n^{100})$. As we have already discussed, this implies that we can find an $\tilde{n}$ not larger than n^{100} such that even (5) holds with $\tilde{n}$ in place of n. This provides a sufficiently plentiful supply of numbers n such that at least a bound of the type (6) holds.

We return to the proof of Theorem 7. We suppose to the contrary that $\lambda \in (1/2, 1)$ is a counterexample to Conjecture 3. By Theorem 10, there are infinitely many approximants η_n to λ satisfying the conclusion of that theorem. We fix such an η_n corresponding to a suitably large n.

By virtue of (4), we have $h(\Phi_{\eta_n}) \leq (1 - \delta) \log \eta_n$. Our next step is to convert this information to something that is easier to exploit with the methods of Diophantine Approximation. We introduce a definition for this purpose. The Mahler measure of an algebraic number η with minimal polynomial $a_d(x - \eta^{(1)}) \cdots (x - \eta^{(d)}) \in \mathbf{Z}[x]$ is defined as

$$M(\eta) = |a_d| \prod_{j=1}^{d} \max\left(1, \left|\eta^{(j)}\right|\right),$$

i.e., it is the product of the absolute values of the leading coefficient and the roots outside the unit disk. This quantity is widely used in number theory as a measure of the "complexity" of η. Notice that if $\eta \in \mathbf{Q}$, then $M(\eta)$ is the maximum of the absolute values of the numerator and the denominator of η.

Breuillard and Varjú [9] found a connection between the entropy rate and the Mahler measure. A form of this most suited for the proof of Theorem 7 is the following.

Theorem 11 (Breuillard, Varjú). *For any $h \in (0, \log 2)$, there is a number $C(h)$ such that $h(\Phi_\eta) \leq h$ implies $M(\eta) < C(h)$ for all algebraic numbers η.*

See [54, **THEOREM 9**] for the details of how this follows from the technical results of [9].

Using this theorem, we conclude that $M(\eta_n) < C$ for a constant C that only depends on λ, but not on n. Furthermore, recall that we have $|\lambda - \eta_n| < \exp(-n^{100})$. Now we use the following, which follows easily from a more general result of Mignotte [34].

Theorem 12 (Mignotte). *Let η be an algebraic number of degree at most n. Let $\tilde{n} > n(\log n)^2$ be an integer, and let $\widetilde{\eta} \neq \eta \in \mathcal{E}^{(\tilde{n})}$. Then there is an absolute constant C, such that*

$$|\eta - \widetilde{\eta}| \geq C^{-\tilde{n}} M(\eta)^{-2\tilde{n}}.$$

We finish our discussion of the proof of Theorem 7. Thanks to the approximation of λ by η_n this theorem acts as a substitute for the separation condition between elements of $\mathcal{E}^{(\tilde{n})}$ in the proof of Lemma 9, and we can conclude that $\Delta_{\tilde{n}}(\Phi_\lambda) > C^{-\tilde{n}}$ for a suitable choice of $\tilde{n}$ for some C independent of n. Now we are in a position to apply Theorem 4 to show that Conjecture 3 holds for λ, which is our desired contradiction proving Theorem 7.

The original argument in [54] used an alternative variant of Theorem 12, which was deduced from an observation of Garsia [20] and a transversality argument of Solomyak [49]. It was pointed out by Vesselin Dimitrov that the transversality argument can be replaced by a simpler version based on Jensen's formula. This has the advantage that it is applicable in greater generality. See [41, **LEMMATA 2.3 AND 4.6**] for details.

3. FAILURE OF EXPONENTIAL SEPARATION

As we discussed in the previous section, it is not known whether Bernoulli convolutions without exact overlaps satisfy the exponential separation property. However, they are known to satisfy a slightly weaker lower bound on Δ_n, and this played an important role in the proof of Conjecture 3 for this class of IFS's.

On the other hand, there are some IFS's without exact overlaps for which it is known that the exponential separation property fails, and moreover, Δ_n converges to 0 in an arbitrarily fast prescribed way.

Theorem 13 (Baker; Bárány, Käenmäki). *Let $(\eta_n) \subset \mathbf{R}_{>0}$. Then there is an IFS Φ without exact overlaps such that $\Delta_n(\Phi) \leq \eta_n$ for all n.*

The first examples of such IFSs were given by Baker [4] in the form

$$\left\{ x \mapsto \frac{x}{2}, x \mapsto \frac{x+1}{2}, x \mapsto \frac{x+s}{2}, x \mapsto \frac{x+t}{2}, x \mapsto \frac{x+1+s}{2}, x \mapsto \frac{x+1+t}{2} \right\}$$

for suitable choices of the parameters t, s, and by Bárány, Käenmäki [5] in the form

$$\{ x \mapsto \lambda x, x \mapsto \lambda x + 1, x \mapsto \lambda x + t \}$$

for suitable choices of λ, t. Baker's example was modified by Chen [11], who disposed of the last two maps and replaced the denominator 2 by an arbitrary real algebraic number not smaller than 2. These constructions were further extended by Baker [3].

In what follows we give a heuristic argument to show why such IFSs with very small separation may be expected to exist. Our purpose (due to limitation of space) is not to give insight to the proofs of Theorem 13, which are based on a variety of tools, such as continued fraction expansions in [4] and the transversality method in [5]. Instead, we just aim to highlight the difference between families of IFSs depending on a single parameter, such as Bernoulli convolutions, or the examples covered by Corollary 6, and families depending on more than one parameter, which will be discussed in the next two sections.

Let

$$\Phi_{x,y} = \{\varphi_{i,x,y} : i \in \Lambda\}$$

be a family of IFS's (smoothly) depending on two parameters. Let $n \in \mathbf{Z}_{>0}$, and we write $\Gamma^{(n)}$ for the collection of curves in the parameter space, which arise as the solution sets of equations of the form

$$\varphi_{i_1,x,y} \circ \cdots \circ \varphi_{i_n,x,y} = \varphi_{\tilde{i}_1,x,y} \circ \cdots \circ \varphi_{\tilde{i}_n,x,y}$$

in (x, y) where $i_1, \ldots, i_n$ and $\tilde{i}_1, \ldots, \tilde{i}_n$ are two distinct sequences of indices in Λ. Note that the union of all these curves is the set of all parameter points for which the IFS contains exact overlaps.

The key difference between this setting and a family depending on a single parameter is that exact overlaps occur along curves in the parameter space rather than at isolated points. These curves may intersect each other, and then there is no separation between them, which rules out the arguments presented for the proof of Corollary 6 and later in Section 2.

We now give the heuristic suggesting the existence of the IFS's claimed in Theorem 13. We give a recursive construction. After the kth step, we will have a sequence $n_1, \ldots, n_k \in \mathbf{Z}_{\geq 1}$, a sequence $\gamma_1, \ldots, \gamma_k$, where γ_j is a segment of a curve in $\Gamma^{(n_j)}$, and a sequence $\delta_1, \ldots, \delta_{k-1} \in \mathbf{R}_{>0}$. These will satisfy the property that γ_k is contained in the δ_j neighborhood of γ_j for all $j < k$.

We begin the process by setting γ_1 to be any segment (of positive length) of a curve in $\Gamma^{(1)}$. Suppose now that $\gamma_1, \ldots, \gamma_k$ and $\delta_1, \ldots, \delta_{k-1}$ are given for some $k \geq 1$. We choose a curve $\tilde{\gamma}_{k+1} \in \Gamma^{(n_{k+1})}$ for some $n_{k+1} > n_k$ that intersects γ_k. The existence of such a curve is plausible, but requires proof, and this is why this construction is only a heuristic. We observe that $\Delta_n(\Phi_{x,y}) = 0$ for all $n \geq n_k$ and $(x, y) \in \gamma_k$. By continuity, there is a choice of δ_k so that $\Delta_n(\Phi_{x,y}) \leq \eta_n$ holds for all $n \in [n_k, n_{k+1})$ and (x, y) in the δ_k neighborhood of γ_k. Finally, we set γ_{k+1} to be a suitable segment of $\tilde{\gamma}_{k+1}$ contained in the δ_j neighborhood of γ_j for all $j \leq k$.

It is immediate from the construction that there is a point (x, y) which is contained in the (closed) δ_k neighborhood of γ_k for all k, and that $\Delta_n(\Phi_{x,y}) \leq \eta_n$ for all n.

With a small modification of the construction, we can ensure that $\Phi_{x,y}$ contains no exact overlaps for the resulting parameter point (x, y). Indeed, observe that $\bigcup \Gamma^{(n)}$ is

a countable set, and let $\gamma_1^*, \gamma_2^*, \ldots$ be an enumeration of it. In the construction, we have considerable liberty in choosing the curve segment γ_k so we can make sure that it does not intersect γ_k^*. (This requires, in particular, that we choose $\widetilde{\gamma}_k$ not to coincide with γ_k^*. The possibility of this is again plausible, but requires proof.) Then in the next step of the construction, we can ensure that δ_k is chosen to be sufficiently small so that γ_k^* is entirely outside the δ_k neighborhood of γ_k. This way we can ensure that the resulting parameter point (x, y) at the end of the process is not contained in γ_k^* for any k, and hence $\Phi_{x,y}$ is without exact overlaps.

4. IFSS WITH ALGEBRAIC CONTRACTION FACTORS

In this section we discuss the following result of Rapaport [39].

Theorem 14 (Rapaport). *Conjectures 2 and 3 hold for all IFSs in which all contraction parameters are algebraic numbers.*

This is a far reaching common generalization of Hochman's Corollaries 5 and 6. We discuss some of the main ideas in the special case of the family of IFSs

$$\Phi_{s,t} = \left\{ x \mapsto \frac{x}{3}, x \mapsto \frac{x}{3} + 1, x \mapsto \frac{x}{3} + s, x \mapsto \frac{x}{3} + t \right\}$$

with uniform probability weights. This is perhaps the simplest family not contained in the results of Hochman, and as was shown by Chen (see Section 3), this family contains IFSs without exact overlaps that fail the exponential separation property (in a very strong sense).

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables taking the values $0, 1, s, t$ with equal probabilities. As we discussed in Section 2, the exponential separation property can be relaxed in Hochman's results. Instead of a lower bound on Δ_n, it suffices to have bounds of the form

$$H\left(\sum_{j=0}^{n-1} \xi_j \cdot 3^{-j}; C^{-n} \right) \geq (\log 3 - \varepsilon_n)n \tag{8}$$

for infinitely many values of n with some constant C and a sequence $\varepsilon_n \to 0$. (See Section 2 for the definition of this notation.)

Theorem 14 is proved by verifying condition (8). With this aim in mind, we examine what happens when (8) fails for some n, C and ε_n. We write $\mathcal{L}^{(n)}$ for the family of (inhomogeneous) linear forms of the form $a_1 \cdot 1 + a_2 Y_1 + a_3 Y_2$, where each a_i is a sum of a subset of the numbers $1, 3^{-1}, \ldots, 3^{-n+1}$ and each term 3^j is allowed in at most one of the a_i. This definition is designed so that the values taken by the random variable $\sum_{j=0}^{n-1} \xi_j \cdot 3^{-j}$ are precisely the values of the linear forms in $\mathcal{L}^{(n)}$ evaluated at s and t.

We write $\mathcal{L}^{(n)} - \mathcal{L}^{(n)}$ for the set of linear forms that can be written as the difference of two elements of $\mathcal{L}^{(n)}$. We also fix some parameter point (s_0, t_0) such that the IFS lacks exact overlaps. We consider pairs of elements in the support of $\sum_{j=0}^{n-1} \xi_j \cdot 3^{-j}$ that are at distance no more than C^{-n}. Then for any such pair, there corresponds a linear form $L \in \mathcal{L}^{(n)} - \mathcal{L}^{(n)}$ such that $|L(s_0, t_0)| \leq C^{-n}$. We write $\mathcal{A}^{(n)}$ for the collection of linear

forms in $\mathcal{L}^{(n)} - \mathcal{L}^{(n)}$ that arise in this way. (This definition depends on C, s_0 and t_0, which we suppress in our notation.)

Let n be such that (8) fails (for some choice of ε_n and C). We distinguish two cases depending on the rank of $\mathcal{A}^{(n)}$. The first case arises when there are at least two linearly independent forms in $\mathcal{A}^{(n)}$, and the second case is when the elements of $\mathcal{A}^{(n)}$ are all scalar multiples of each other.

In the first case, we take two linearly independent $L_1, L_2 \in \mathcal{L}^{(n)} - \mathcal{L}^{(n)}$. Provided C is sufficiently large, the lines determined by L_1 and L_2 cannot be parallel. Indeed, if that was the case, their distance would be a rational number with denominator bounded by an exponential in n, which we can force to be 0 by taking C sufficiently large. Since the lines are not parallel, we can solve the equations

$$L_1(s_n, t_n) = 0,$$
$$L_2(s_n, t_n) = 0,$$

and find that their solution (s_n, t_n) is a pair of rational numbers with denominators bounded by an exponential in n. Moreover, the distance of (s_n, t_n) from (s_0, t_0) will be an arbitrarily small exponential in n if C is chosen sufficiently large.

The points (s_n, t_n) have the same repellency property as those in the proof of Corollary 6. We discuss next how to show that the second case, that is when the elements of $\mathcal{A}^{(n)}$ are proportional, arises for only finitely many values of n. Then the argument for Corollary 6 can be carried over to prove (8).

We begin by extending the definition of entropy rates. Let ℓ be a line in $\mathbf{R}^2$ (that does not necessarily contain 0). We denote by $Y_\ell^{(n)}$ the random $\ell \to \mathbf{R}$ function $(s, t) \mapsto \sum_{j=0}^{n-1} \xi_j(s, t) \cdot 3^{-j}$. We define the entropy rate of the line ℓ by

$$h(\ell) := \lim_{n \to \infty} \frac{H(Y_\ell^{(n)})}{n}.$$

Here $H(Y_\ell^{(n)})$ stands for the Shannon entropy of $Y_\ell^{(n)}$, which is a random element taking finitely many values. It can be shown that $H(Y_\ell^{(n)})$ is subadditive, hence the limit exists and is equal to the infimum. The quantity $h(\ell)$ measures the amount of exact overlaps that occur simultaneously for all parameter points $(s, t) \in \ell$.

Now suppose that the second case occurs for some n in our above discussion, that is the linear forms in $\mathcal{A}^{(n)}$ are proportional. Let ℓ be the line on which all elements of $\mathcal{A}^{(n)}$ vanish. It is immediate from the definition of $\mathcal{A}^{(n)}$ that

$$H(Y_\ell^{(n)}) \leq H\left(\sum_{j=0}^n \xi_j 3^{-j}; C^{-n}\right).$$

Supposing

$$H\left(\sum_{j=0}^n \xi_j 3^{-j}; C^{-n}\right) \leq (\log 3 - \varepsilon)n \tag{9}$$

for some $\varepsilon > 0$, we can conclude

$$h(\ell) \leq \log 3 - \varepsilon.$$

In light of all this, the next proposition—implicit in [39]—implies that the second case and (9) for some fixed $\varepsilon > 0$ may occur for only finitely many n's.

Proposition 15. *Let (s_0, t_0) be some parameters such that the IFS Φ_{s_0,t_0} contains no exact overlaps. Fix some $\varepsilon > 0$. Then there is a neighborhood of (s_0, t_0) that is not intersected by any lines ℓ with $h(\ell) \leq \log 3 - \varepsilon$.*

We end this section by discussing the proof of this result. Suppose to the contrary that the result is false, that is, there is a sequence $\ell_1, \ell_2, \ldots$ of lines passing closer and closer to (s_0, t_0) with $h(\ell_n) < \log 3 - \varepsilon$. We suppose as we may that the lines ℓ_n converge (in any reasonable topology) to a line ℓ_∞. We also suppose for simplicity that none of $\ell_1, \ell_2, \ldots, \ell_\infty$ is parallel to either of the s or t axes, and none of them goes through the origin.

We associate a self-similar measure in $\mathbf{R}^2$ to each line ℓ_j. For $j = 1, 2, \ldots, \infty$, let σ_j and τ_j be the unique numbers such that ℓ_j is spanned by $(\sigma_j, 0)$ and $(0, \tau_j)$. For $\sigma, \tau \in \mathbf{R}$, we define the IFS

$$\Psi_{\sigma,\tau} := \left\{ (x, y) \mapsto \left(\frac{x}{3}, \frac{y}{3} \right), (x, y) \mapsto \left(\frac{x}{3} + 1, \frac{y}{3} + 1 \right), \right.$$
$$\left. (x, y) \mapsto \left(\frac{x}{3} + \sigma, \frac{y}{3} \right), (x, y) \mapsto \left(\frac{x}{3}, \frac{y}{3} + \tau \right) \right\},$$

and write $\nu_{\sigma,\tau}$ for the associated self-similar measure (with equal probability weights).

It is immediate from the definitions that the same exact overlaps occur for the random variables $Y_{\ell_j}^{(n)}$ as for the IFS $\Psi(\sigma_j, \tau_j)$. It follows that

$$h(\Psi_{\sigma_j,\tau_j}) = h(\ell_j) \leq \log 3 - \varepsilon$$

for $j < \infty$. Using this, it can be shown that

$$\dim \nu_{\sigma_j,\tau_j} \leq \frac{\log 3 - \varepsilon}{\log 3} = 1 - \varepsilon / \log 3.$$

It is a general phenomenon that the dimension of self-similar measures depends lower semicontinuously on the parameters, see, e.g., [16] for results of this type covering even self-affine measures. Using this, it follows that

$$\dim \nu_{\sigma_\infty,\tau_\infty} \leq 1 - \varepsilon / \log 3.$$

The proof of Proposition 15 is now finished by establishing a suitable analogue of Conjecture 3 for the IFSs $\Psi_{\sigma,\tau}$, which shows that Ψ_{σ_0,θ_0} and hence $\Phi_{s,t}$ for all $(s, t) \in \ell$ including (s_0, t_0) contains exact overlaps. This can be done along the lines of the proof of Corollary 5 discussed in Section 1 using a higher dimensional version of Hochman's theorem, which can be found in [22]. The crucial difference between the IFSs $\Phi_{s,t}$ and $\Psi_{\sigma,\tau}$ is that the ambient space is 2-dimensional for the latter and this matches the number of parameters. This means that exact overlaps occur at single points (as opposed to along lines), which have the required repellency property.

5. HOMOGENEOUS IFSS OF THREE MAPS

In this section, we discuss the IFSs

$$\Phi_{\lambda,t} = \big\{(x \mapsto \lambda x, x \mapsto \lambda x + 1, x \mapsto \lambda x + t)\big\}.$$

Rapaport and Varjú [41] made some partial progress towards extending the results for Bernoulli convolutions discussed in Section 2 to this setting and to some more general IFSs (see [41, SECTION 3]).

Before we can state these results, we need to introduce some relevant notation and terminology. We write $\mu_{\lambda,t}$ for the self-similar measure associated to the IFS $\Phi_{\lambda,t}$ and uniform probability weights. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random $\mathbf{R} \to \mathbf{R}$ functions taking the values $t \mapsto 0, t \mapsto 1$ and $t \mapsto t$ with equal probability. Let $U \subset (0,1) \times \mathbf{R}$, $n \in \mathbf{Z}_{\geq 0}$, and write $A_U^{(n)}$ for the random $U \to \mathbf{R}$ function

$$(\lambda, t) \mapsto \sum_{j=1}^{n} \xi_j(t) \lambda^j.$$

We define the entropy rate

$$h(U) := \lim_{n \to \infty} \frac{H(A_U^{(n)})}{n} = \inf \frac{H(A_U^{(n)})}{n}.$$

We abbreviate $A_{\{\lambda,t\}}^{(n)}$ as $A_{\lambda,t}^{(n)}$, and $h(\{\lambda,t\})$ as $h(\lambda,t)$. One should think about $h(\lambda,t)$ as a quantity expressing the amount of exact overlaps contained in the IFS $\Phi_{\lambda,t}$ and $h(U)$ aims to quantify the amount of exact overlaps occurring simultaneously for the parameter points in U.

We write $\mathcal{R}$ for the set of meromorphic functions on the unit disc that can be written as ratios of two power series with coefficients $-1, 0, 1$. We denote by Γ the set of curves $\gamma \subset (0,1) \times \mathbf{R}$ that are either of the following two forms:

- $\gamma = \{(\lambda, t) \in (0,1) \times \mathbf{R} : t = R(\lambda)\}$ for some $R \in \mathcal{R}$,

- $\gamma = \{(\lambda_0, t) : t \in \mathbf{R}\}$ for some fixed $\lambda_0 \in (0,1)$.

It can be shown that exact overlaps occur in the family of IFSs $\Phi_{\lambda,t}$ along finite unions of curves in Γ, but not all elements of Γ arises in this way.

The next result is an analogue of Theorem 10 in the setting of the IFS $\Phi_{\lambda,t}$.

Theorem 16 (Rapaport, Varjú). *Suppose that Conjecture 3 does not hold for the IFS $\Phi_{\lambda,t}$ for some choice of parameters λ and t. Then for every $\varepsilon > 0$ and $N \geq 1$, there exist $n \geq N$ and $(\eta, s) \in (0,1) \times \mathbf{R}$ such that*

(1) $|\lambda - \eta|, |t - s| \leq \exp(-n^{\varepsilon^{-1}})$,

(2) $\frac{1}{n \log \eta^{-1}} H(A_{\eta,s}^{(n)}) \leq \dim \mu_{\lambda,t} + \varepsilon$,

(3) $h(\gamma) \geq \min\{\log 3, \log \lambda^{-1}\} - \varepsilon$ *for all $\gamma \in \Gamma$ with $(\eta, s) \in \gamma$.*

Item (2) in the conclusion means that the IFS $\Phi_{\eta,s}$ contains enough overlaps after n iteration to force the dimension of $\mu_{\eta,s}$ below $\dim \mu_{\lambda,t} + \varepsilon$. Item (3) in the conclusion implies that not all of these exact overlaps occur along the same curve γ. From these properties it can be deduced in particular that η and s are algebraic numbers and roots of polynomials of low degree with small integer coefficients. (For a precise statement, see [41, **THEOREM 1.3**].) This yields a bound on the number of possible points that can arise as (η, s) in the conclusion and together with Item (1), this shows that the Hausdorff dimension of the set of exceptional parameters for which Conjecture 3 fails is 0. This improves Hochman's bound, which is 1, albeit that bound is given for the stronger notion of packing dimension, which may exceed the Hausdorff dimension.

It is still an open problem whether an analogue of Theorem 11 holds for the IFS $\Phi_{\lambda,t}$. One possible formulation is the following.

Question 17. Is it true that for all $\varepsilon > 0$, there is M such that the following holds? Let $(\lambda, t) \in (\varepsilon, 1 - \varepsilon) \times \mathbf{R}$ be such that $h(\lambda, t) \leq \min(\log 3, \log \lambda^{-1}) - \varepsilon$ and $h(\gamma) \geq \min(\log 3, \log \lambda^{-1}) - M^{-1}$ for all $\gamma \in \Gamma$ with $(\lambda, t) \in \gamma$. Then $M(\lambda) \leq M$.

We note that a condition about the entropy rate of curves passing through (λ, t) is necessary. Indeed, we have, for example, $h(\gamma) = \log 3 - (2/3) \log 2$ for the curve $\gamma = \{(\lambda, 1) : \lambda \in (0, 1)\}$, and hence $h(\lambda, 1) \leq \log 3 - (2/3) \log 2$ for all $\lambda \in (0, 1)$.

We also have the following conditional result towards Conjecture 3.

Theorem 18 (Rapaport, Varjú). *Suppose that the answer to Question 17 is affirmative. Then Conjecture 3 holds for the IFS $\Phi_{\lambda,t}$ with equal probability weights for all $\lambda \in (0, 1)$ and $t \in \mathbf{R}$.*

Using ideas from [9], one can answer Question 17 affirmatively if we restrict λ to be near 1. This allows for the following unconditional partial resolution of Conjecture 3.

Theorem 19 (Rapaport, Varjú). *Conjecture 3 holds for the IFS $\Phi_{\lambda,t}$ with equal probability weights for all $(\lambda, t) \in (2^{-2/3}, 1) \times \mathbf{R}$.*

The key new ingredient in the proof of Theorem 16 compared to that of Theorem 10 is the following result, whose role is similar to that of Proposition 15 in the proof of Theorem 14.

Proposition 20. *Let $(\lambda, t) \in (0, 1) \times \mathbf{R}$ be such that the IFS $\Phi_{\lambda,t}$ contains no exact overlaps. Then for all $h < \min(\log \lambda^{-1}, \log 3)$, there is a neighborhood of (λ, t) that is not intersected by a curve $\gamma \in \Gamma$ with $h(\gamma) \leq h$.*

The proof of this result like Proposition 15 is done by attaching suitable fractal objects to curves and relating their dimension to the entropy rates of the curves. Then the proposition is proved using lower semicontinuity of dimension and a limiting argument. The fractal measures used in the paper [41] are analogues of self-similar measures in function fields. A suitable notion of dimension is introduced for these objects and Hochman's theorem is generalized to this setting. The analogue of the exponential separation property is verified using an argument similar to that used in the proof of Corollary 6. An additional dif-

ficulty compared to the setting of Section 4 is caused by the fact that the curves in Γ are not necessarily lines and they may develop singularities, which complicates limiting arguments.

The proofs of Theorems 18 and 19 is complicated by the fact that like in the case of Bernoulli convolutions, the parameter points with exact overlaps have a weaker than exponential repellency property. To address this, an argument similar to that discussed at the end of Section 2 is used. This is the reason why we need to assume an affirmative answer to Question 17. The argument also requires a stronger form of Proposition 20 with a modified entropy rate. The precise statement requires some preparation. For this reason, we omit it and refer to **[41, PROPOSITION 2.4]**.

6. OTHER DEVELOPMENTS

We survey some recent results about aspects of self-similar measures other than their dimensions. Due to limitation of space, our discussion will be very brief.

6.1. Fourier decay

We first discuss Fourier decay of self-similar measures. Specifically, we discuss the following three properties:

- A measure μ on $\mathbf{R}$ is Rajchman if its Fourier transform vanishes at infinity, that is,
$$\lim_{|\xi| \to \infty} |\widehat{\mu}(\xi)| = 0.$$

- A measure μ on $\mathbf{R}$ has polylogarithmic Fourier decay if there is a constant $a > 0$ such that for all sufficiently large ξ, we have
$$|\widehat{\mu}(\xi)| < |\log|\xi||^{-a}.$$

- A measure μ on $\mathbf{R}$ has power Fourier decay if there is a constant $a > 0$ such that for all sufficiently large ξ, we have
$$|\widehat{\mu}(\xi)| < |\xi|^{-a}.$$

There are various motivations for studying these properties. The Rajchman property is closely related to an old subject in the theory of trigonometric series about the so-called sets of uniqueness and sets of multiplicity, see **[27]** for more. Fourier decay has also applications in metric Diophantine approximation. For example, polylogarithmic Fourier decay is sufficient to guarantee that almost all numbers with respect to the measure are normal in every bases. (In the case of self-similar measures on $\mathbf{R}$, even the Rajchman property is enough for this, see **[2, THEOREM 1.4]**.) Power decay is very useful in proving absolute continuity of the measure, which we discuss more in the next section.

Results about these properties of self-similar measures come in two flavors. In the first category, properties are proved for most self-similar measures in a parametric family, in the second the properties are proved for explicit self-similar measures, that is, the hypotheses of the results are testable in concrete examples.

We begin by discussing results in the first category. Erdős [14] proved that Bernoulli convolutions (see Section 2) have power Fourier decay for almost all choices of the parameter $\lambda \in (0, 1)$. His argument was revisited by Kahane [26] who showed that the exceptional set of parameters where the power decay fails is, in fact, of 0 Hausdorff dimension. This method was exposed in the survey [38], where the exponent a was also studied, and the term Erdős–Kahane argument was coined. Recently Solomyak [50] showed that nondegenerate self-similar measures on **R** have power Fourier decay if the vector of contraction parameters avoid an exceptional set of 0 Hausdorff dimension. See the references in [50, SECTION 1.1] and [51] for more recent applications of the Erdős–Kahane method.

The first results in the second category are also in the setting of Bernoulli convolutions. Erdős [13] proved that Bernoulli convolutions are not Rajchman when λ^{-1}, the reciprocal of the parameter, is a Pisot number, except when the probability weights are uniform and $\lambda = 1/2m$ for an odd integer m. Recall that a Pisot number is an algebraic integer all of whose Galois conjugates lie inside the complex unit disk. Salem [43] proved the converse of Erdős result by showing that Bernoulli convolutions are Rajchman when λ^{-1} is not Pisot.

The Rajchman property of general self-similar measures has been understood more recently. Sahlsten and Li [31] proved that self-similar measures are Rajchman whenever the semigroup generated by the contraction parameters is not lacunary, that is, it is not contained in $\{\lambda^n : n \in \mathbf{Z}_{\geq 0}\}$ for some n. Their work is based on a new method relying on renewal theory originating in [30]. See also [2], where this result is extended to self-conformal measures using a different method. The lacunary case was analyzed by Brémont [7], see also Varjú, Yu [55]. Finally, the problem was solved by Rapaport [40] for self-similar measures on $\mathbf{R}^d$.

For Bernoulli convolutions, polylogarithmic Fourier decay follows from a result of Bufetov and Solomyak [10, PROPOSITION 5.5] for algebraic parameters λ provided λ^{-1} is neither Salem nor Pisot, that is, it has another Galois conjugate outside the complex unit disk, see also [19]. Under a mild Diophantine condition for the contraction parameters, Sahlsten and Li [31] proved polylogarithmic Fourier decay for self-similar measures. Informally speaking, their condition requires that the semigroup generated by the contraction parameters is not approximated by lacunary semigroups in a suitable quantitative sense. See [2] for a similar result under a different Diophantine condition. Polylogarithmic Fourier decay was also established by Varjú and Yu [55] for certain self-similar measures in the lacunary case.

It is an important open problem to characterize which self-similar measures have power Fourier decay. Very little is known about this. See [12] for explicit examples of Bernoulli convolutions with power Fourier decay and [32] for results about self-similar measures on $\mathbf{R}^d$ for $d \geq 3$.

6.2. Absolute continuity

Let μ be a self-similar measure on **R** associated to an IFS with contraction factors $\{\lambda_i\}$ that contains no exact overlaps, and probability weights $\{p_i\}$. One may expect that μ is

not only of dimension 1 if

$$\frac{\sum p_i \log p_i^{-1}}{\sum p_i \log \lambda_i^{-1}} > 1, \tag{10}$$

as predicted by Conjecture 3, but it is also absolutely continuous. When there is equality in (10), the self-similar measure is almost always singular, see [37, THEOREM 1.1].

In general, this expectation is false. Simon and Vágó [48] showed that in some families of IFSs, there is a dense G^d set of parameters, which violate the above statement. See [36] for earlier related results in a different setting. However, it could still be true that (10) and the lack of exact overlaps imply absolute continuity for some families of self-similar measures, for example for Bernoulli convolutions.

Nevertheless, it is expected that self-similar measures are absolutely continuous for almost all choices of the parameters in parametric families when (10) holds. For Bernoulli convolutions, this was proved by Erdős for λ near 1, as a consequence of power Fourier decay with parameter $a > 1$. The result has been extended to the optimal range $\lambda \in [1/2, 1]$ by Solomyak [49] using the transversality method. See [6, 37, 38] and their references for further developments. Shmerkin [44] proved that the set of exceptional parameters in $[1/2, 1]$ that make the Bernoulli convolution singular is of Hausdorff dimension 0. His method is based on a result of his that the convolution of a measure of dimension 1 and another one with power Fourier decay is absolutely continuous. He used this in conjunction with Hochman's theorem and the Erdős-Kahane method. See [42, 46] and the references therein for further developments using this method.

Explicit examples of absolutely continuous self-similar measures are rare. The first examples were given by Garsia [20] as the Bernoulli convolutions with parameters of Mahler measure 2. See [12] for a generalization of this construction, and see [56] for an improvement on the regularity of the density function using Shmerkin's method. Varjú gave new examples of absolutely continuous Bernoulli convolutions in [53]. This paper relies on a similar method to Hochman's in a quantitatively refined form. A crucial point is that it requires the separation condition to hold at all sufficiently small scales rather than just at infinitely many of them. This restricts the method to algebraic parameters currently. A recent improvement was given by Kittle [28], who gave further new examples of absolutely continuous Bernoulli convolutions. While all the new examples in [53] are very close to 1, e.g., $1 - 10^{-50}$, this is not the case for [28], which includes, e.g., one near $0.799533\ldots$ The paper [28] also introduces a new tool to quantify the smoothness of measures at scales.

See [32] for results about absolute continuity of self-similar measures on $\mathbf{R}^d$ for $d \geq 3$.

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