MACROSCOPIC LIMITS OF CHAOTIC EIGENFUNCTIONS

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ABSTRACT

We give an overview of the interplay between the behavior of high energy eigenfunctions of the Laplacian on a compact Riemannian manifold and the dynamical properties of the geodesic flow on that manifold. This includes the Quantum Ergodicity theorem, the Quantum Unique Ergodicity conjecture, entropy bounds, and uniform lower bounds on mass of eigenfunctions. The above results belong to the domain of *quantum chaos* and use *microlocal analysis*, which is a theory behind the classical/quantum, or particle/wave, correspondence in physics. We also discuss the toy model of quantum cat maps and the challenges it poses for Quantum Unique Ergodicity.

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1. INTRODUCTION

This article is an overview of some results on *macroscopic behavior of eigenstates in the high energy limit*. A typical model is given by Laplacian eigenfunctions:

$$-\Delta_g u_\lambda = \lambda^2 u_\lambda, \quad u_\lambda \in C^\infty(M), \quad \|u_\lambda\|_{L^2(M)} = 1.$$

Here we fix a compact connected Riemannian manifold without boundary (M, g) and denote by $\Delta_g \leq 0$ the corresponding Laplace–Beltrami operator. It will be convenient to denote the eigenvalue by λ^2, where $\lambda \geq 0$. The high-energy limit corresponds to taking $\lambda \to \infty$.

One way to study the macroscopic behavior of the eigenfunctions u_λ as $\lambda \to \infty$ is to look at weak limits of the probability measures $|u_\lambda|^2 \, d\,\mathrm{vol}_g$ where $d\,\mathrm{vol}_g$ is the volume measure on (M, g):

Definition 1. Let λ_j^2 be a sequence of eigenvalues of $-\Delta_g$ going to ∞. We say that the corresponding eigenfunctions u_{λ_j} converge weakly to some probability measure ν on M, if

$$\int_M a(x)\left|u_{\lambda_j}(x)\right|^2 d\,\mathrm{vol}_g(x) \to \int_M a(x)\, d\nu(x) \quad \text{as} \quad j \to \infty \tag{1.1}$$

for all test functions $a \in C^\infty(M)$.

Definition 1 can be interpreted in the context of quantum mechanics as follows. Consider a free quantum particle on the manifold M. Then the eigenfunctions u_λ are the wave functions of the *pure quantum states* of the particle. The left-hand side of (1.1) is the average value of the observable $a(x)$ for a given pure state; if we let a be the characteristic function of some set $\Omega \subset M$ then this expression is the probability of finding the quantum particle in Ω (this choice is only allowed if $\nu(\partial\Omega) = 0$). Taking $\lambda \to \infty$ gives the high-energy limit.

The statement (1.1) is macroscopic in nature because we first fix the observable a and then let the eigenvalue go to infinity. This is different from *microscopic* properties such as the breakthrough work of Logunov and Malinnikova on the area of the *nodal set* $\{x \in M \mid u_j(x) = 0\}$, see the review **[38]**. Ironically, the methods used in the macroscopic results described here are *microlocal* in nature (see Section 2 for a review), with the global geometry of M coming in the form of the long time behavior of the geodesic flow.

The results reviewed in this paper address the following fundamental question:

$$\text{For a given Riemannian manifold } (M, g)\text{, what can we say about the set of all weak limits of sequences of eigenfunctions?} \tag{1.2}$$

It turns out that the answer depends on the dynamical properties of the *geodesic flow* on (M, g). In particular:

- If (M, g) has *completely integrable* geodesic flow then there is a huge variety of possible weak limits. For example, if (M, g) is the round sphere, then there is a sequence of Gaussian beam eigenfunctions converging to the delta measure on any given closed geodesic (see Section 2.2 below).

- If the geodesic flow instead has *chaotic* behavior, more precisely it is ergodic with respect to the Liouville measure, then a density 1 sequence of eigenfunctions converges to the volume measure $d\,\mathrm{vol}_g / \mathrm{vol}_g(M)$. This statement, known as *Quantum Ergodicity*, is reviewed in Section 3.

- If the geodesic flow is *strongly chaotic*, more precisely it satisfies the Anosov property (i.e., it has a stable/unstable/flow decomposition), then the limiting measures have to be somewhat spread out. This comes in two forms: *entropy bounds* and *full support*. See Section 4 for a description of these results. The *Quantum Unique Ergodicity* conjecture states that in this setting any sequence of eigenfunctions converges to the volume measure; it is not known outside of arithmetic cases (see Section 4) and there are counterexamples in the related setting of quantum cat maps (see Section 5).

- Finally, there are several results in cases when the geodesic flow is ergodic but not Anosov, or it exhibits mixed chaotic/completely integrable behavior; see Section 3.

The present article focuses on the last three cases above, which are in the domain of *quantum chaos*. The general principle is that *chaotic behavior of the geodesic flow leads to chaotic/spread out macroscopic behavior of the eigenfunctions of the Laplacian*. See Figure 1 for a numerical illustration.

In particular, we will describe full support statements for weak limits – see Theorems 11 and 16 – proved in **[18–20]**. The key component is the *fractal uncertainty principle* first introduced by Dyatlov–Zahl **[21]** and proved by Bourgain–Dyatlov **[10]**. It originated in *open* quantum chaos, dealing with quantum systems where the underlying classical system allows for escape to infinity and has chaotic behavior. We refer to the reviews of the author **[15, 16]** for more on fractal uncertainty principle and its applications.

The above developments use *microlocal analysis*, which is a mathematical theory underlying the classical/quantum, or particle/wave, correspondence in physics. In particular, one typically obtains information on the *semiclassical measures*, which are probability measures μ on the cosphere bundle S^*M which are weak limits of sequences of eigenfunctions in a microlocal sense. These measures are sometimes called *microlocal lifts* of the weak limits, because the pushforward of μ to the base M is the weak limit of Definition 1. One of the advantages of these measures compared to the weak limits on M is that they are invariant under the geodesic flow. We give a brief review of microlocal analysis and semiclassical measures in Section 2 below.

2. SEMICLASSICAL MEASURES

Let us write the left-hand side of (1.1) as

$$\int_M a(x)\left|u_{\lambda_j}(x)\right|^2 d\,\mathrm{vol}_g(x) = \langle \mathbf{M}_a u_{\lambda_j}, u_{\lambda_j}\rangle_{L^2(M)}$$

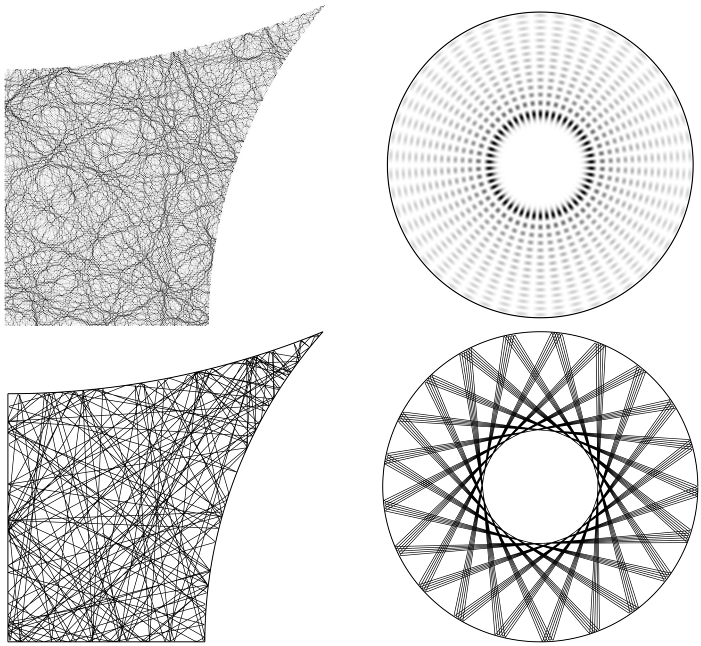

FIGURE 1
(Top) Typical eigenfunctions (with Dirichlet boundary conditions) for two planar domains. The picture on the left (courtesy of Alex Barnett, see [7] and [8] for a description of the method used and for a numerical investigation of Quantum Ergodicity) shows equidistribution, i.e., convergence to the volume measure in the sense of Definition 1. The picture on the right (where the domain is a disk) shows the lack of equidistribution, with the limiting measure supported in an annulus. This difference in quantum behavior is related to the different behavior of the billiard-ball flows on the two domains (which replace geodesic flows in this setting). (Bottom) Two typical billiard-ball trajectories on the domains in question. On the left we see ergodicity (equidistribution of the trajectory for long time), and on the right we see completely integrable behavior.

where $\mathbf{M}_a : L^2(M) \to L^2(M)$ is the multiplication operator by $a \in C^\infty(M)$. To define semiclassical measures, we will allow for more general operators in place of $\mathbf{M}_a$. These operators are obtained by a *quantization procedure*, which maps each smooth compactly supported function a on the cotangent bundle T^*M to an operator on $L^2(M)$ depending on the small number $h > 0$ called the semiclassical parameter:

$$a \in C_c^\infty(T^*M) \quad \mapsto \quad \mathrm{Op}_h(a) : L^2(M) \to L^2(M), \quad 0 < h \ll 1. \tag{2.1}$$

2.1. Semiclassical quantization

We briefly recall several basic principles of semiclassical quantization referring to the books of Zworski [49] and Dyatlov–Zworski [22, APPENDIX E] for the full presentation and pointers to the vast literature on the subject:

- The function a, often called the *symbol* of the operator $\mathrm{Op}_h(a)$, is defined on the cotangent bundle T^*M, whose points we typically denote by (x, ξ) where $x \in M$ and $\xi \in T_x^*M$. The canonical symplectic form on T^*M induces the *Poisson bracket*

$$\{f, g\} := \partial_\xi f \cdot \partial_x g - \partial_x f \cdot \partial_\xi g, \quad f, g \in C^\infty(T^*M).$$

In physical terms, this corresponds to using Hamiltonian mechanics for the "classical" side of the classical/quantum correspondence, where x is the position variable and ξ is the momentum variable.

- One can work with a broader class of smooth symbols a, where the compact support requirement is changed to growth conditions on the derivatives of a as $\xi \to \infty$. The resulting operators act on (semiclassical) Sobolev spaces, see, e.g.. **[22, §E.1.8]**.

- If $a(x, \xi) = a(x)$ is a function of x only, then

$$\mathrm{Op}_h(a) = \mathbf{M}_a \tag{2.2}$$

is the corresponding multiplication operator.

- If $a(x, \xi)$ is linear in ξ, that is, $a(x, \xi) = \langle \xi, X_x \rangle$ for some vector field $X \in C^\infty(M; TM)$, then, up to lower-order terms, the operator $\mathrm{Op}_h(a)$ is a rescaled differentiation operator along X,

$$\mathrm{Op}_h(a)u(x) = -ihXu(x) + \mathcal{O}(h). \tag{2.3}$$

This explains why a should be a function on the cotangent bundle T^*M: linear functions on the fibers of T^*M correspond to vector fields on M. (Quantization procedures do not depend on the choice of a Riemannian metric on M.)

- If $u \in C^\infty(M)$ oscillates at some frequency R, then differentiating u along a vector field X increases its magnitude by about R. One takeaway from (2.3) is that $\mathrm{Op}_h(a)u$ has roughly the same size as u if the function u oscillates at frequencies $\sim h^{-1}$. Thus we treat the semiclassical parameter h as the *effective wavelength* of oscillations of the functions to which we will apply $\mathrm{Op}_h(a)$. We will apply $\mathrm{Op}_h(a)$ to an eigenfunction u_λ, which oscillates at frequency $\sim \lambda$, so we will make the choice

$$h := \lambda^{-1}. \tag{2.4}$$

- If $M = \mathbb{R}^n$ and $a(x, \xi) = a(\xi)$ is a function of ξ only, then $\mathrm{Op}_h(a)$ is a Fourier multiplier,

$$\widehat{\mathrm{Op}_h(a)u}(\xi) = a(h\xi)\hat{u}(\xi), \quad u \in \mathscr{S}(\mathbb{R}^n). \tag{2.5}$$

Thus in addition to being the momentum variable, we can interpret ξ as a Fourier/frequency variable.

- For general manifolds M, one cannot define a quantization procedure canonically: a typical construction involves piecing together quantizations on copies of $\mathbb{R}^n$ using coordinate charts, see, e.g., [22, §E.1.7]. However, different choices of coordinate charts, etc., will give the same operator modulo lower-order terms $\mathcal{O}(h)$.

Several items above allude to "lower-order terms." We will consider the operators $\mathrm{Op}_h(a)$ in the *semiclassical limit* $h \to 0$ and will often have remainders of the form $\mathcal{O}(h)$, etc., which are operators on $C^\infty(M)$. (More generally, semiclassical analysis gives asymptotic expansions in powers of h with the remainder being $\mathcal{O}(h^N)$ for any N.) This is understood as follows: if the symbols involved are compactly supported in T^*M, then the remainders are bounded in norm as operators on L^2 (with constants in $\mathcal{O}(\bullet)$, of course, independent of h). For more general symbols, one has to take correct semiclassical Sobolev spaces and we skip these details here. We note that in the basic version of semiclassical calculus used in this section, the symbol a does not depend on h, which reflects the macroscopic nature of the results presented below.

Semiclassical quantization has several fundamental algebraic and analytic properties; once these are proved, one can use it as a black box without caring too much for the precise definition of $\mathrm{Op}_h(a)$. Of particular importance are the product, adjoint, and commutator rules:

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + \mathcal{O}(h), \tag{2.6}$$

$$\mathrm{Op}_h(a)^* = \mathrm{Op}_h(\bar{a}) + \mathcal{O}(h), \tag{2.7}$$

$$\big[\mathrm{Op}_h(a), \mathrm{Op}_h(b)\big] = -ih\,\mathrm{Op}_h\big(\{a,b\}\big) + \mathcal{O}(h^2), \tag{2.8}$$

and the L^2 boundedness statement: if $a \in C_c^\infty(T^*M)$ then $\|\mathrm{Op}_h(a)\|_{L^2\to L^2}$ is bounded uniformly in h.

2.2. Semiclassical measures for eigenfunctions

We can now introduce the main object of study in this article, which are semiclassical measures associated to high frequency sequences of eigenfunctions of the Laplacian. Semiclassical measures were originally introduced independently by Gérard [27] and Lions–Paul [37]. We refer to [49, CHAPTER 5] for a detailed treatment.

Following (2.4), we write the eigenvalue as h^{-2} where h is small. Let (M, g) be a Riemannian manifold and consider a sequence of Laplacian eigenfunctions:

$$-\Delta_g u_j = h_j^{-2} u_j, \quad h_j \to 0, \quad u_j \in C^\infty(M), \quad \|u_j\|_{L^2} = 1.$$

Definition 2. We say that the sequence u_j converges semiclassically to a finite Borel measure μ on the cotangent bundle T^*M, if

$$\big\langle \mathrm{Op}_{h_j}(a) u_j, u_j \big\rangle_{L^2} \to \int_{T^*M} a(x,\xi)\, d\mu(x,\xi) \quad \text{as} \quad j \to \infty \tag{2.9}$$

for all test functions $a \in C_c^\infty(T^*M)$. A measure μ on T^*M is called a *semiclassical measure* if it is the limit of some sequence of Laplacian eigenfunctions.

The statement (2.9) actually applies to a broader class of symbols a with polynomial growth as $\xi \to \infty$. By (2.2), if $a(x,\xi) = a(x)$ depends only on the position variable x, then the left-hand side of (2.9) is the integral $\int_M a|u_j|^2\, d\operatorname{vol}_g$. Comparing (2.9) with (1.1), we see that if u_j converges semiclassically to μ, then it converges weakly to the pushforward of μ to the base M. Thus we can think of semiclassical measures as (microlocal) lifts of the weak limits of Definition 1.

A quantum-mechanical interpretation of semiclassical measures is as follows: if $a \in C^\infty(T^*M)$ is a *classical observable* (a function of position and momentum) then $\operatorname{Op}_h(a)$ is the corresponding *quantum observable* and the expression $\langle \operatorname{Op}_h(a)u, u\rangle_{L^2}$ is the average value of the observable a on the quantum particle with wave function u. Thus (2.9) gives macroscopic information on the concentration of the particle in both position and momentum in the high-energy limit. Recalling (2.5), we can also interpret semiclassical measures as capturing the concentration of u_j simultaneously in the position and frequency.

One important property of Definition 2 is the presence of compactness: any sequence of eigenfunctions has a subsequence converging semiclassically to some measure; see **[49, THEOREM 5.2]** and **[22, THEOREM E.42]**. Other basic properties of semiclassical measures are summarized in the following

Proposition 3. *Let μ be a semiclassical measure for a Riemannian manifold (M, g). Then:*

- *μ is a probability measure;*
- *μ is supported on the cosphere bundle*

$$S^*M := \{(x,\xi) \in T^*M : |\xi|_g = 1\};$$

- *μ is invariant under the geodesic flow*

$$\varphi^t : S^*M \to S^*M.$$

Here the geodesic flow is naturally a flow on the sphere bundle SM, which is identified with S^*M using the metric g.

We give a sketch of the proof of Proposition 3 to show how the fundamental properties (2.6)–(2.8) can be used. The first claim follows by taking $a = 1$ in (2.9), in which case $\operatorname{Op}_h(a)$ is the identity operator. To see the second claim, we use that the semiclassically rescaled Laplacian $-h^2\Delta_g$ is a quantization of the quadratic function $|\xi|_g^2$ (giving the square of the length of the cotangent vector $\xi \in T_x^*M$ with respect to the metric g), so

$$P(h) := -h^2\Delta_g - 1 = \operatorname{Op}_h\big(|\xi|_g^2 - 1\big) + \mathcal{O}(h), \quad P(h_j)u_j = 0.$$

Now if $a \in C_c^\infty(T^*M)$ vanishes on S^*M, we can write $a = b(|\xi|_g^2 - 1)$ for some $b \in C_c^\infty(T^*M)$. By the product rule (2.6),

$$\operatorname{Op}_{h_j}(a)u_j = \operatorname{Op}_{h_j}(b)P(h_j)u_j + \mathcal{O}(h_j) = \mathcal{O}(h_j),$$

which by (2.9) gives $\int_{T^*M} a\, d\mu = 0$. Since this is true for any a vanishing on S^*M, we see that $\operatorname{supp}\mu \subset S^*M$ as needed.

The last claim is also simple to prove: if $b \in C_c^\infty(T^*M)$ is arbitrary, then

$$0 = \langle [P(h_j), \mathrm{Op}_{h_j}(b)] u_j, u_j \rangle_{L^2} = -i h_j \langle \mathrm{Op}_{h_j}(\{|\xi|_g^2, b\}) u_j, u_j \rangle_{L^2} + \mathcal{O}(h_j^2).$$

Here the first equality follows from the fact that $P(h_j)u_j = 0$ and $P(h_j)$ is self-adjoint; the second one uses the commutator rule (2.8). Now (2.9) shows that the Poisson bracket $\{|\xi|_g^2, b\}$ integrates to 0 with respect to μ. But the Hamiltonian flow of $|\xi|_g^2/2$, restricted to S^*M, is the geodesic flow φ^t, so we get

$$\int_{S^*M} \partial_t|_{t=0}(b \circ \varphi^t)\, d\mu = 0 \quad \text{for all } b \in C_c^\infty(T^*M),$$

from which it follows that $\int_{S^*M} b \circ \varphi^t \, d\mu$ is independent of t and thus μ is invariant under the flow φ^t.

We now give the microlocal formulation of the question (1.2) asked at the beginning of the article:

$$\text{For a given Riemannian manifold } (M, g)\text{, what can we say about the set of all semiclassical measures?} \tag{2.10}$$

The general expectation is that

- when the geodesic flow on (M, g) is "predictable," i.e., completely integrable, there are semiclassical measures which can concentrate on small flow-invariant sets;
- on the other hand, when the geodesic flow on (M, g) has chaotic behavior, semiclassical measures have to be more "spread out."

One of the results supporting the first point above is the following theorem of Jakobson–Zelditch **[33]**: if M is the round sphere then *any* measure satisfying the conclusions of Proposition 3 is a semiclassical measure. See also the work of Studnia **[46]** and Arnaiz–Macià **[6]** in the related case of the quantum harmonic oscillator.

The rest of this article presents various results which support the second point above, in particular giving several ways of defining chaotic behavior of the geodesic flow and the way in which a measure is "spread out."

3. ERGODIC SYSTEMS

We first describe what happens under a "mildly chaotic" assumption on the geodesic flow $\varphi^t : S^*M \to S^*M$, namely that it is *ergodic* with respect to the Liouville measure. Here the Liouville measure $\mu_L = c\, d\,\mathrm{vol}_g(x)\, dS(\xi)$ is a natural flow-invariant probability measure on S^*M, with dS denoting the volume measure on the sphere S_x^*M corresponding to g and c some constant. By definition, the flow φ^t is ergodic with respect to μ_L if every φ^t-invariant Borel subset $\Omega \subset S^*M$ has $\mu_L(\Omega) = 0$ or $\mu_L(\Omega) = 1$.

We say that a sequence of eigenfunctions u_j *equidistributes* if it converges to μ_L in the sense of Definition 2, that is, in the high-energy limit the probability of finding the corresponding quantum particle in a set becomes proportional to the volume of this set. A central

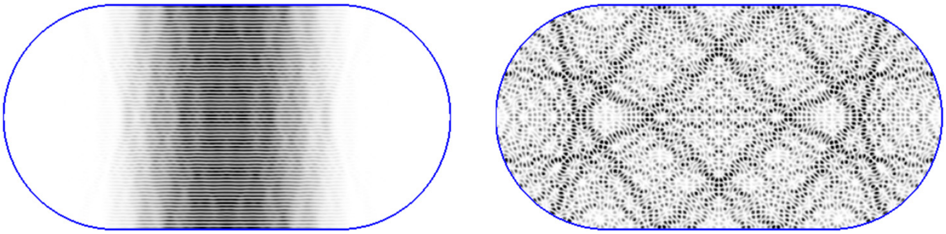

FIGURE 2
Two Dirichlet eigenfunctions for a Bunimovich stadium, courtesy of Alex Barnett (see the caption to Figure 1): the right one shows equidistribution, but the left one does not. Quantum Ergodicity implies that most eigenfunctions look from afar like that on the right.

result in quantum chaos is the following Quantum Ergodicity theorem of Shnirelman [44], Zelditch [47], and Colin de Verdière [14], which states that when the geodesic flow is ergodic, most eigenfunctions equidistribute:

Theorem 4. *Assume that the geodesic flow is ergodic with respect to the Liouville measure. Then for any choice of orthonormal basis of eigenfunctions* $\{u_k\}$ *there exists a density* 1 *subsequence* u_{k_j} *which converges semiclassically to* μ_L *in the sense of Definition* 2.

See [49, CHAPTER 15] and the review of Dyatlov [17] for more recent expositions of the proof. The version of Theorem 4 for compact manifolds with boundary was proved by Gérard–Leichtnam [28] for convex domains in $\mathbb{R}^n$ with $W^{2,\infty}$ boundaries and Zelditch–Zworski [48] for compact Riemannian manifolds with piecewise C^∞ boundaries. In this setting one imposes (Dirichlet or Neumann) boundary conditions on the eigenfunctions, and the geodesic flow is naturally replaced by the billiard-ball flow (reflecting off the boundary). See Figures 1 and 2 for numerical illustrations.

A natural question is whether the entire sequence of eigenfunctions equidistributes, i.e., whether μ_L is the *only* semiclassical measure. For general manifolds with ergodic classical flows this is not always true, as proved by Hassell [32]. In particular, for the case of the Bunimovich stadium shown on Figure 2, the paper [32] shows that for almost every choice of the parameter of the stadium (i.e., the aspect ratio of its central rectangle) there exist semiclassical measures which are not the Liouville measure.

Another natural question is what happens when the classical flow has *mixed* behavior, e.g., S^*M is the union of two flow-invariant sets of positive Lebesgue measure such that the flow is ergodic on one of them and completely integrable on the other. *Percival's Conjecture* claims that this mixed behavior translates to macroscopic behavior of eigenfunctions, namely one can split any orthonormal basis of eigenfunctions into three parts: one of them equidistributes in the ergodic region, another has semiclassical measures supported in the completely integrable region, and the remaining part has density 0. A version of this conjecture for mushroom billiards was proved by Gomes in his thesis [29, 30]; see also the earlier work of Galkowski [26] and Rivière [41].

4. STRONGLY CHAOTIC SYSTEMS

We now describe what is known when the geodesic flow on M is assumed to be strongly chaotic. The latter assumption is understood in the sense of the following *Anosov property*:

Definition 5. Let (M, g) be a compact Riemannian manifold without boundary. We say that the geodesic flow $\varphi^t : S^*M \to S^*M$ has the Anosov property if there exists a flow/unstable/stable decomposition of the tangent spaces

$$T_\rho(S^*M) = E_0(\rho) \oplus E_u(\rho) \oplus E_s(\rho), \quad \rho \in S^*M,$$

where E_0 is the one-dimensional space spanned by the generator of the flow, while E_u, E_s depend continuously on ρ, are invariant under the flow φ^t, and satisfy the exponential decay condition for some $\theta > 0$:

$$\left|d\varphi^t(\rho)v\right| \le Ce^{-\theta|t|}|v|, \quad \begin{cases} v \in E_u(\rho), & t \le 0, \\ v \in E_s(\rho), & t \ge 0. \end{cases}$$

A large family of manifolds with Anosov geodesic flows is given by compact Riemannian manifolds of negative sectional curvature, see the book of Anosov [5]. An important special case is given by *hyperbolic surfaces*, which are compact, oriented Riemannian manifolds of dimension 2 with Gauss curvature identically equal to -1. See Figure 3 for a numerical illustration.

The Anosov property implies that the geodesic flow is ergodic with respect to the Liouville measure, so Quantum Ergodicity applies to give that most eigenfunctions equidis-

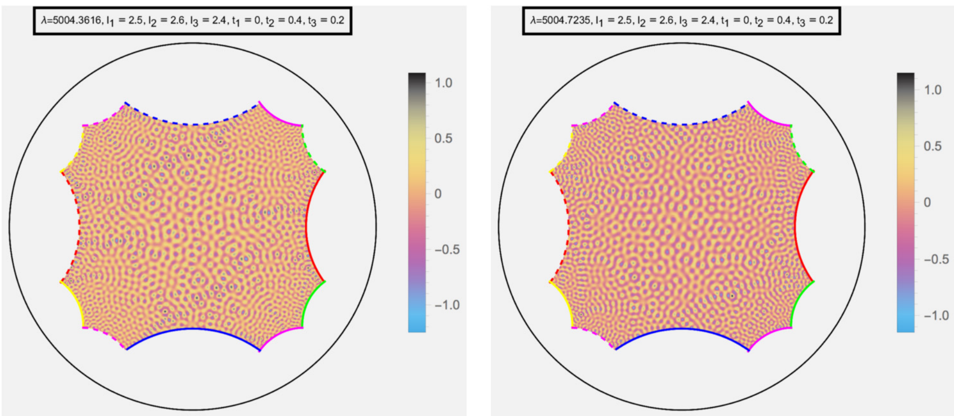

FIGURE 3

Two Laplacian eigenfunctions on a hyperbolic surface, courtesy of Alex Strohmaier (see Strohmaier–Uski [45]). Here we view the surface as a quotient of the hyperbolic plane by a group of isometries, or equivalently as the result of gluing together appropriate sides of the pictured fundamental domain. On a microscopic level the two eigenfunctions look different, but the macroscopic features are the same – both show equidistribution.

tribute. The major open question is the following *Quantum Unique Ergodicity* conjecture which claims equidistribution for the entire sequence of eigenfunctions:

Conjecture 6. *Assume that (M, g) is a compact Riemannian manifold with Anosov geodesic flow. Then μ_L is the only semiclassical measure.*

Conjecture 6 was originally stated by Rudnick–Sarnak [42] in the context of hyperbolic surfaces. It is known in the special case of *arithmetic* hyperbolic surfaces, which have additional symmetries commuting with the Laplacian, called Hecke operators, and we consider a joint basis of eigenfunctions of the Laplacian and a Hecke operator; see Lindenstrauss [36] and Brooks–Lindenstrauss [13]. In general, in spite of significant partial progress described below, the conjecture is open. One of the issues with a potential proof is that Quantum Unique Ergodicity fails in the related setting of quantum cat maps; see Theorem 14 below.

4.1. Entropy bounds

A major step towards Quantum Unique Ergodicity (Conjecture 6) are *entropy bounds*, originating in the work of Anantharaman [1]:

Theorem 7. *Assume that the geodesic flow on (M, g) has the Anosov property. Then any semiclassical measure μ has positive Kolmogorov–Sinai entropy, $\mathbf{h}_{\mathrm{KS}}(\mu) > 0$.*

Here the Kolmogorov–Sinai entropy $\mathbf{h}_{\mathrm{KS}}(\mu)$ is a nonnegative number associated to each flow-invariant measure μ; roughly speaking, it expresses the complexity of the flow from the point of view of that measure, and is one way to measure how "spread out" the measure is—measures which are more concentrated have lower entropy, and measures which are more spread out have higher entropy. Theorem 7 in particular implies the following conjecture of Colin de Verdière [14]:

$$\begin{gathered}\text{On a hyperbolic surface, no semiclassical measure}\\ \text{can be supported on a closed geodesic}\end{gathered} \tag{4.1}$$

since the entropy of a measure supported on a closed geodesic is zero.

The lower bound on entropy in Theorem 7 is in general complicated. However, in the case of hyperbolic (i.e., constant negative curvature) manifolds Anantharaman–Nonnenmacher [3] gave the following easy to state bound:

Theorem 8. *Assume that (M, g) is an n-dimensional hyperbolic manifold. Then any semiclassical measure μ satisfies*

$$\mathbf{h}_{\mathrm{KS}}(\mu) \geq \tfrac{n-1}{2}. \tag{4.2}$$

We remark that the Liouville measure in this setting has entropy $n - 1$, so (4.2) in some sense excludes "half" of all invariant measures as possible semiclassical measures. For other entropy(-type) bounds, see the works of Anantharaman–Koch–Nonnenmacher [2], Rivière [39, 40], and Anantharaman–Silberman [4].

The constant in the bound (4.2) matches (in the case of surfaces) the counterexamples for quantum cat maps given in Theorem 14 below. Thus an important milestone on the way to Quantum Unique Ergodicity would be to prove the following:

Conjecture 9. *Let μ be a semiclassical measure on an n-dimensional hyperbolic manifold (M, g). Then $\mathbf{h}_{\mathrm{KS}}(\mu) > \frac{n-1}{2}$.*

We conclude this subsection with another conjecture which would go a long way towards Quantum Unique Ergodicity but does not exclude the counterexample of Theorem 14:

Conjecture 10. *Let μ be a semiclassical measure on a compact manifold (M, g) with Anosov geodesic flow. Then we have $\mu = \alpha\mu_L + (1-\alpha)\mu'$ for some $\alpha \in (0, 1]$, where μ_L is the Liouville measure and μ' is some probability measure on S^*M.*

4.2. Full support property

Another way to characterize how much a measure μ is "spread out" is by looking at its support, $\operatorname{supp}\mu \subset S^*M$. For surfaces with Anosov geodesic flows, Dyatlov–Jin [19] (in the hyperbolic case) and Dyatlov–Jin–Nonnenmacher [20] (in the general case) showed that the support of every semiclassical measure is the entire S^*M:

Theorem 11. *Let μ be a semiclassical measure on a compact surface (M, g) with Anosov geodesic flow. Then $\operatorname{supp}\mu = S^*M$, that is, $\mu(U) > 0$ for every nonempty open set $U \subset S^*M$.*

Theorem 11 and entropy bounds give different restrictions on the set of possible semiclassical measures. On the one hand (assuming (M, g) is a hyperbolic surface for simplicity), the entropy bound (4.2) implies that the Hausdorff dimension of $\operatorname{supp}\mu$ is at least 2, but there exist flow-invariant measures supported on proper subsets of S^*M of dimension arbitrarily close to 3. On the other hand, there exist measures which have full support and small entropy: one can, for example, take a convex combination of the Liouville measure and a measure supported on a closed geodesic.

The key new ingredient in the proof of Theorem 11 is the *fractal uncertainty principle* of Bourgain–Dyatlov [10]. We state the following version appearing in [20]:

Theorem 12. *Let $\nu, h \in (0, 1)$ and assume that $X, Y \subset \mathbb{R}$ are ν-porous up to scale h, namely for any interval $I \subset \mathbb{R}$ of length $|I| \in [h, 1]$, there exists a subinterval $J \subset I$ of length $|J| = \nu|I|$ such that $X \cap J = \emptyset$ (and similarly for Y). Then there exist constants $C, \beta > 0$ depending only on ν such that for all $f \in L^2(\mathbb{R})$,*

$$\operatorname{supp}\hat{f} \subset h^{-1}Y \quad \Rightarrow \quad \|\mathbf{1}_X f\|_{L^2(\mathbb{R})} \le Ch^\beta \|f\|_{L^2(\mathbb{R})}. \tag{4.3}$$

One should think of the parameter ν in Theorem 12 as fixed and h as going to 0. The sets X, Y can depend on h as long as they are ν-porous; a basic example is given by $\frac{h}{10}$-neighborhoods of some sets which are porous up to scale 0 (e.g., Cantor sets). The estimate (4.3) can be interpreted as follows: if a function f lives in the (semiclassically rescaled) frequency space in a porous set Y, then only a small part of the L^2-mass of f can concentrate on the porous set X. We refer the reader to the review **[15]** for more details.

The proof of Theorem 11 can be roughly summarized as follows (restricting to the case of hyperbolic surfaces for simplicity): assume that a sequence of eigenfunctions $\{u_j\}$ converges semiclassically to a measure μ such that $\mu(\mathcal{U}) = 0$ for some nonempty open set $\mathcal{U} \subset S^*M$. Using microlocal methods, one can show that u_j is in a certain sense concentrated on both of the sets

$$\Omega_\pm(h_j) := \{\rho \in S^*M \mid \varphi^{\mp t}(\rho) \notin \mathcal{U} \text{ for all } t \in [0, \log(1/h_j)]\}$$

of geodesics which do not cross the set $\mathcal{U}$ in the future or in the past for time $\log(1/h_j)$. Here one can barely make sense of localization in the position–frequency space on each of the sets $\Omega_\pm(h_j)$, i.e., construct operators $A_\pm$ which localize to these sets and write $u_j = A_+u_j + o(1) = A_-u_j + o(1)$. However, the sets $\Omega_\pm(h)$ have porous structure (see Figure 5 below for the related case of quantum cat maps), and one can use the Fractal Uncertainty Principle to show that $\|A_+A_-\|_{L^2\to L^2} = o(1)$, giving a contradiction. We refer to **[15]** for a detailed exposition of the proof.

Theorem 11 only applies to surfaces because the Fractal Uncertainty Principle is only known for subsets of $\mathbb{R}$. A naïve generalization of Theorem 12 to higher dimensions is false: for example, the sets

$$X = [0, h/10] \times [0, 1], \quad Y = [0, 1] \times [0, h/10] \subset \mathbb{R}^2$$

are both $\frac{1}{10}$-porous up to scale h (where we replace intervals by balls in the definition of porosity), but they do not satisfy an estimate of type (4.3): the Fourier transform of the indicator function of $h^{-1}Y$ has large L^2 mass on X. (See **[16, §6]** for a more detailed discussion.) However, this does not translate to a counterexample for semiclassical measures, leaving the door open for the following:

Conjecture 13. *Let μ be a semiclassical measure on a compact manifold (M, g) with Anosov geodesic flow. Then* $\operatorname{supp}\mu = S^*M$.

An analog of Conjecture 13 is known for certain quantum cat maps, see Theorem 16 below.

5. QUANTUM CAT MAPS

We finally discuss *quantum cat maps*, which are toy models in quantum chaos with microlocal properties similar to Laplacians on hyperbolic manifolds (though the extensive research on them demonstrates that they are a "tough toy to crack"). They were originally introduced by Hannay and Berry in **[31]**. We start with two-dimensional quantum cat maps which are analogous to hyperbolic surfaces. These maps quantize toral automorphisms (a.k.a. "Arnold cat maps")

$$x \mapsto Ax \bmod \mathbb{Z}^2, \quad x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \tag{5.1}$$

where $A \in \mathrm{SL}(2, \mathbb{Z})$ is a 2×2 integer matrix with determinant 1. We make the assumption that A is *hyperbolic*, i.e., it has no eigenvalues on the unit circle. A basic example of such a matrix is

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5.2}$$

Quantizations of the map (5.1) are not operators on L^2 of a manifold, instead they are unitary $N \times N$ matrices, where the integer N is related to the semiclassical parameter h as follows:

$$2\pi N h = 1.$$

The semiclassical limit $h \to 0$ studied above now turns into the limit $N \to \infty$.

Before introducing quantizations of cat maps, we briefly discuss the adaptation of the quantization procedure (2.1) to this setting, which has the form

$$a \in C^\infty(\mathbb{T}^2) \quad \mapsto \quad \mathrm{Op}_N(a) : \mathbb{C}^N \to \mathbb{C}^N. \tag{5.3}$$

That is, functions on the 2-torus are quantized to $N \times N$ matrices. The quantization procedure also depends on a twist parameter $\theta \in \mathbb{T}^2$, but we suppress this in the notation. (If N is even, then we can always just take $\theta = 0$ in what follows.) See, for example, **[18, §2.2]** for more details.

Now, for $A \in \mathrm{SL}(2, \mathbb{Z})$, its quantization is a family of unitary $N \times N$ matrices $B_N : \mathbb{C}^N \to \mathbb{C}^N$ which satisfies the following *exact Egorov's theorem*:

$$B_N^{-1} \mathrm{Op}_N(a) B_N = \mathrm{Op}_N(a \circ A) \quad \text{for all } a \in C^\infty(\mathbb{T}^2). \tag{5.4}$$

Such B_N exists and is unique modulo multiplication by a unit length scalar. The statement (5.4) intertwines conjugation by B_N (corresponding to quantum evolution) with pullback by the map (5.1) (corresponding to classical evolution). It is analogous to Egorov's Theorem for Riemannian manifolds (see, e.g., **[49, THEOREM 15.2]**), which states that

$$e^{-ith\Delta_g/2} \mathrm{Op}_h(a) e^{ith\Delta_g/2} = \mathrm{Op}_h(a \circ \varphi^t) + \mathcal{O}(h)$$

where the geodesic flow $\varphi^t : S^*M \to S^*M$ is extended to T^*M as the Hamiltonian flow of $|\xi|_g^2/2$. Thus the quantum cat map B_N should be thought of as an analog of the Schrödinger propagator $e^{ith\Delta_g/2}$, eigenfunctions of B_N are analogous to Laplacian eigenfunctions, and the dynamics of the geodesic flow in this setting is replaced by the dynamics of the map (5.1).

Using the quantization (5.3), we can define similarly to (2.9) semiclassical measures associated to sequences of eigenfunctions

$$B_{N_j} u_j = \lambda_j u_j, \quad u_j \in \mathbb{C}^{N_j}, \quad \|u_j\|_{\ell^2} = 1, \quad N_j \to \infty.$$

These are probability measures on $\mathbb{T}^2$ which are invariant under the map (5.1) (as can be seen directly from Egorov's theorem (5.4)).

When the matrix A is hyperbolic, the map (5.2) is ergodic with respect to the Lebesgue measure on $\mathbb{T}^2$. Using this fact, Bouzouina–de Bièvre **[11]** showed Quantum Ergodicity in this setting: if we put together orthonormal bases of eigenfunctions of B_N

for all N, then there exists a density 1 subsequence of this sequence which converges to the Lebesgue measure.

On the other hand, Faure–Nonnenmacher–De Bièvre **[25]** showed that Quantum Unique Ergodicity fails for quantum cat maps:

Theorem 14. *Let $A \in \mathrm{SL}(2,\mathbb{Z})$ be a hyperbolic matrix. Fix any periodic trajectory $\gamma \subset \mathbb{T}^2$ of the map (5.1). Then there exists a sequence of eigenfunctions u_j of the quantum cat map B_{N_j}, for some $N_j \to \infty$, which converge semiclassically to the measure*

$$\tfrac{1}{2}\delta_\gamma + \tfrac{1}{2}\mu_L \tag{5.5}$$

where δ_γ is the delta probability measure on the trajectory γ and μ_L is the Lebesgue measure on $\mathbb{T}^2$.

We remark that the choice of N_j in Theorem 14 is highly special: one takes them so that the matrix A^{k_j} is the identity modulo $2N_j$ where k_j is very small, namely $k_j \sim \log N_j$. This implies that the quantum cat map B_{N_j} also has a short period, namely $B_{N_j}^{k_j}$ is a scalar. See the papers of Dyson–Falk **[23]** and Bonechi–De Bièvre **[9]** for more information on the periods of the cat map. A numerical illustration of Theorem 14 is given on Figure 4.

The entropy of the measure (5.5) is equal to half the entropy of the Lebesgue measure. This matches the constant in the entropy bound of Theorem 8. Since from the point of view of microlocal analysis quantum cat maps have similar properties to hyperbolic surfaces,

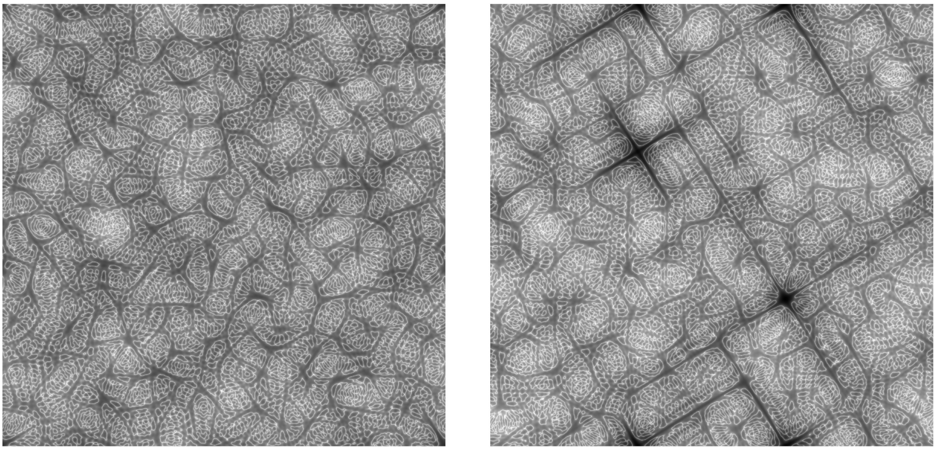

FIGURE 4
Phase space concentration for two eigenfunctions of the quantum cat map with A given by (5.2) and $N = 1292$. More specifically, we plot the absolute value of a smoothened out Wigner transform of the eigenfunction on the logarithmic scale (see, e.g., **[18, §2.2.5]**). On the left is a typical eigenfunction, showing equidistribution. On the right is a particular eigenfunction of the type constructed in **[25]**, corresponding to a measure of the type (5.5) featuring the closed trajectory $\{(\frac{1}{3},0),(\frac{2}{3},\frac{1}{3}),(\frac{2}{3},0),(\frac{1}{3},\frac{2}{3})\}$. The existence of such an eigenfunction relies on the careful choice of N: A^{18} is the identity matrix modulo $2N$.

significant new insights would be needed to show that a counterexample of the kind (5.5) cannot occur for hyperbolic surfaces.

Faure–Nonnenmacher **[24]** showed that the constant $\frac{1}{2}$ in (5.5) is sharp: the mass of the pure point part of any semiclassical measure for a quantum cat map is less than or equal to the mass of its Lebesgue part. Brooks **[12]** generalized this to a statement that the mass of lower entropy components of any semiclassical measure is less than or equal to the mass of higher entropy components; this in particular implies an entropy bound analogous to (4.2).

There is also an analogue of arithmetic Quantum Unique Ergodicity in the setting of cat maps: Kurlberg–Rudnick **[35]** introduced Hecke operators which commute with B_N and showed that any sequence of joint eigenfunctions of B_N and these operators converges to the Lebesgue measure. This does not contradict the counterexample of Theorem 14 since for the values of N_j chosen there, the map B_{N_j} has eigenvalues of high multiplicity.

We now discuss the recent results on support of semiclassical measures for cat maps, proved using the fractal uncertainty principle. For two-dimensional cat maps, Schwartz **[43]** showed the following:

Theorem 15. *Let μ be a semiclassical measure for a quantum cat map associated to some hyperbolic matrix $A \in \mathrm{SL}(2, \mathbb{Z})$. Then* $\operatorname{supp} \mu = \mathbb{T}^2$.

Similarly to Section 4.2, the proof uses that no function can be localized simultaneously on the two sets

$$\Omega_\pm(N) := \left\{\rho \in \mathbb{T}^2 \;\middle|\; A^{\mp j}(\rho) \notin \mathcal{U} \text{ for all } j = 0, \dots, \frac{\log N}{\log|\lambda_+|}\right\}$$

where λ_+ is the eigenvalue of A such that $|\lambda_+| > 1$. Here $\mathcal{U} \subset \mathbb{T}^2$ is some nonempty open set. See Figure 5.

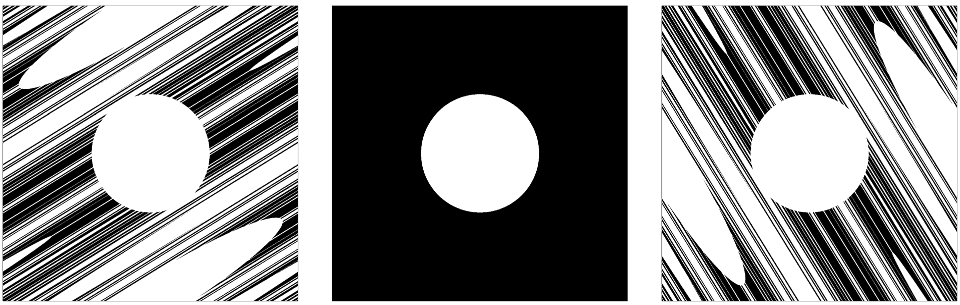

FIGURE 5
A set $\mathcal{U} \subset \mathbb{T}^2$ (center picture, in white) and the corresponding sets $\Omega_+(N)$, $\Omega_-(N)$ (left/right picture). The set $\Omega_+(N)$ is "smooth" in the unstable direction of the matrix A and porous in the stable direction, with the porosity constant depending only on $\mathcal{U}$. Same is true for $\Omega_-(N)$ but switching the roles of the stable/unstable directions. The fractal uncertainty principle of Theorem 12 can be used to show that no function can be localized on both $\Omega_+(N)$ and $\Omega_-(N)$.

We finally discuss the quantum cat map analog of the higher-dimensional Conjecture 13, by considering quantum cat maps associated to symplectic integer matrices $A \in \mathrm{Sp}(2n, \mathbb{Z})$. In this setting Dyatlov–Jézéquel [18] proved

Theorem 16. *Let μ be a semiclassical measure for a quantum cat map associated to a matrix $A \in \mathrm{Sp}(2n, \mathbb{Z})$ such that:*

- *A has a simple eigenvalue λ_+ such that all other eigenvalues satisfy $|\lambda| < \lambda_+$; and*
- *the characteristic polynomial of A is irreducible over the rationals.*

Then $\operatorname{supp} \mu = \mathbb{T}^{2n}$.

Here the first condition makes it possible to still use the one-dimensional Fractal Uncertainty Principle in the proof.

We remark that there are examples of semiclassical measures which do not have full support for some matrices A satisfying the first condition of Theorem 16 but not the second condition. In particular, there exist semiclassical measures supported on tori associated to any A-invariant rational Lagrangian subspace of $\mathbb{R}^{2n}$. See the work of Kelmer [34] and the discussion in [18, APPENDIX A].

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