VARIATIONAL HOMOGENIZATION: OLD AND NEW

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ABSTRACT

This note is a summary of the contributions of the authors to the variational viewpoint on homogenization. After providing a broad context, two recent projects are discussed in detail: one concerning the large-scale behavior of quasicrystals, and the other involving phase transitions in periodically heterogeneous media.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 35B27; Secondary 35B25, 49S05, 49-06

KEYWORDS

Calculus of variations, periodic and quasiperiodic homogenization, phase transitions, PDE constrained optimization

INTERNATIONAL CONGRESS OF MATHEMATICIANS
2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 5, pp. 3724–3755
DOI 10.4171/ICM2022/129

1. INTRODUCTION

Homogenization, a subject with a long and rich history, deals with the macrobehavior of a medium as a large-scale average of its microscopic properties. The earliest investigations seeking such effective models, appear to go back to Maxwell [76], Lord Rayleigh [84], and others, around the start of the 20th century. For instance, in [84] Lord Rayleigh considers an arrangement of cylindrical rods of constant thermal conductivity in a rectangular array within an otherwise uniform medium. Assuming that the conductivity of the rods is significantly different from that of the background medium, the subject of homogenization addresses questions such as: on length-scales much larger than the period of the arrangement of the rods, can one approximate the heat distribution in the composite material, by instead studying an effective, homogeneous material? Remarkably, in [84] Lord Rayleigh discovers an explicit formula for the effective conductivity in the case of the above planar arrangement.

The study of homogenization has witnessed immense growth in the last half century, and continues to flourish. As it supplies tools for analysis of situations that involve multiple spatio-temporal scales, it is not surprising that homogenization plays an important role in such diverse fields as materials science [2, 88], fluid mechanics and mixing [54], climate modeling [35], biology [15, 16, 68], machine learning and data science [91]. The ubiquity of homogenization, on the one hand, and the intractability of direct computational approaches for large, multiscale problems, on the other, renders the analytical study of homogenization vitally important. The goal of this survey is to report progress, and the state-of-the-art, in one segment of this vast subject, focusing on the contributions of the authors to variational methods in homogenization. In particular, we do not discuss the recent burst of activity in stochastic homogenization [8, 69], applications of homogenization to study discrete and possibly random structures such as point clouds [89], optimal control theory and numerical analysis associated with homogenization [91].

The main thrust of this article is on variational methods. As a concrete example, we consider the benchmark problem in homogenization

$$\begin{cases} -\nabla \cdot \left(a\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \right) = 0 & \text{in } \Omega, \\ u_\varepsilon = g & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $a : \mathbb{R}^N \to (0, \infty)$ is a given periodic, measurable, bounded, uniformly elliptic, symmetric matrix field, $0 < \varepsilon \ll 1$ represents the length-scale of the heterogeneities, and $g \in L^2(\partial\Omega)$ is a given Dirichlet datum. Homogenization seeks to find an "effective" constant matrix $\overline{a}$ that is independent of the domain Ω and of the boundary condition g, such that the limit of solutions $\{u_\varepsilon\}_\varepsilon$ to (1.1) exists (call it u_0), and solves the "homogenized" partial differential equation (PDE)

$$\begin{cases} -\nabla \cdot \overline{a} \nabla u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It is, of course, of interest to also study quasiperiodic, or random choices of a. Early works that addressed the question of justification of the formal two-scale asymptotic expansion led to the development of important functional analytic tools that rely on the structure of the PDE. These include [14], methods of compensation compactness [87], G- and H-convergence [48], Bloch decomposition [34], among others. When the matrix field a is symmetric, the problem (1.1) has a variational formulation. Indeed, solutions to (1.1) are the unique minimizers to the sequence of variational problems

$$\min_{u|_{\partial\Omega}=g} E_\varepsilon(u) := \frac{1}{2}\int_\Omega \left\langle a\left(\frac{x}{\varepsilon}\right)\nabla u \cdot \nabla u\right\rangle dx. \tag{1.3}$$

The notion of Γ-convergence is well suited for the study of the $\varepsilon \to 0^+$ asymptotics of the energies E_ε in (1.3). This notion of convergence of a family of functionals defined on a Banach space was introduced by De Giorgi in 1975 (see [49]). As such, along with appropriate compactness, this scheme of convergence of functionals is the weakest notion that ensures that global minimizers of the approximating functional converge to a global minimizer of the limiting functional. In the example in (1.3) above, the limiting energy takes the form

$$E_0(u) := \frac{1}{2}\int_\Omega \langle \overline{a}\nabla u, \nabla u\rangle \, dx.$$

Γ-convergence is stable under continuous perturbations, and is therefore well adapted to the multiscale analysis of nonlinear problems that have variational structure. More crucially, it is sufficiently robust to allow for the limiting problem to be defined on a different space than the approximating problems (see Section 4 for an example). Being based on soft compactness and lower-semicontinuity arguments, approaches based on Γ-convergence are particularly well-suited when fine information that is uniform in the small parameter (such as a spectral gap) is difficult or even impossible to obtain. There is, however, a price to pay using Γ-convergence techniques in that the underlying arguments do not often yield rates of convergence.

2. AN OVERVIEW OF CONTRIBUTIONS TO HOMOGENIZATION

In [60], Fonseca and Francfort consider a quasistatic model aiming at understanding the interaction between damage and fracture. To prove that a certain incremental problem at a fixed time step is well posed, they state and use a homogenization conjecture (see [60, CONJECTURE 3.15]), known at the time to be true in some important convex examples (cf. [60, REMARK 3.16]). This conjecture was then proved to be true in the general convex case in [10], while the nonconvex case, including the quasiconvex one, remains open.

In [20], Fonseca, Bouchitté, and Mascarenhas introduced the so-called global method for relaxation, which is central to the study of minimization problems via the direct method in the calculus of variations. This method provides a unified pathway to identify the integral representation of the lower-semicontinuous envelope of certain functionals that naturally arise in several applications, such as in phase transitions, fracture mechanics, plasticity, and image segmentation. Moreover, as an application of their methodology, they address in [20, SECTION 4.3] a homogenization problem associated with integral energies coupling bulk and surface terms, which generalized several results in the literature, including [23].

In [24], Fonseca, Braides, and Francfort study dimension reduction problems for heterogeneous thin domains in the context of nonlinear elasticity. The domains considered are of the type

$$\Omega_\varepsilon := \big\{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega,\ |x_3| < \varepsilon h_\varepsilon(x')\big\},$$

where $\omega \subset \mathbb{R}^2$ is a bounded domain and h_ε is a smooth ε-dependent profile, while the elastic integral energy of the system involves a p-growth Carathéodory function $f_\varepsilon \equiv f_\varepsilon(x', x_3; \xi)$. As one of the main applications of their general asymptotic analysis, they consider the homogenization problem corresponding to the case where the profile h_ε is assumed to be periodic, and the elastic density f_ε is assumed to be independent of x_3 and periodic with respect to x', with the same period as h_ε. They obtain an integral representation for the effective energy on the middle section ω.

Another contribution to the study of minimization problems via the direct method in the calculus of variations is that of Fonseca and Müller in [63], where they address the study of lower semicontinuity and relaxation of functionals of the type

$$(u, v) \mapsto \int_\Omega f\big(x, u(x), v(x)\big)\,\mathrm{d}x,$$

where, for N, m, $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ is an open and bounded domain, $u : \Omega \to \mathbb{R}^m$, and $v : \Omega \to \mathbb{R}^d$ satisfies a partial differential constraint of the type $\mathcal{A}v = 0$. Here, $\mathcal{A}$ is a constant-coefficient linear partial differential operator of the form

$$\mathcal{A}v := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i} \quad \text{with } A^{(i)} \in \mathbb{R}^{l\times d} \text{ for all } i \in \{1, \dots, N\} \text{ and some } l \in \mathbb{N} \tag{2.1}$$

(see Section 3 for a more detailed description of these operators). In the literature, this context is nowadays referred as the $\mathcal{A}$-free setting. A typical example of such operators is $\mathcal{A} = \operatorname{curl}$, in which case $v = \nabla w$ for some potential w. In particular, for $w = u$ we are led to the so-called gradient case, where the integral energies take the form

$$u \mapsto \int_\Omega f\big(x, u(x), \nabla u(x)\big)\,\mathrm{d}x.$$

Though relevant in many applications, the curl case does not cover some important ones in which v must satisfy other linear partial differential constraints, such as Maxwell's equations in the case of electromagnetism, or, in the case of linear elasticity, v is the symmetric part of a gradient. Therefore, the $\mathcal{A}$-free fields setting offers a unified abstract approach to several of these PDE constraints.

In [25], besides further developing the analysis in [63], Fonseca, Braides, and Leoni address an homogenization problem in the $\mathcal{A}$-free setting. More precisely, they characterize the effective behavior of integrals energies of the form

$$v \mapsto \int_\Omega f\left(\frac{x}{\varepsilon}, v(x)\right)\mathrm{d}x \quad \text{subjected to } \mathcal{A}v = 0,$$

where $\varepsilon > 0$ is the usual homogenization small parameter, and the integrand f is periodic in the first variable and satisfies certain continuity, p-growth, and coercivity conditions.

The periodic homogenization result in [25] was generalized by Fonseca and Krömer [61] by working under weaker continuity assumptions and, most importantly, without assuming coercivity on f. Moreover, they extended the widely used two-scale convergence method (see [1, 82]) to the $\mathcal{A}$-free setting.

Also in the context of periodic homogenization in the general $\mathcal{A}$-free framework, Fonseca and Davoli consider in [42, 43] operators with variable coefficients, which is not a straightforward extension of the constant coefficient case. More precisely, these two papers are devoted to the study of the effective behavior, as $\varepsilon \to 0$, of integral energies of the form

$$v \mapsto \int_\Omega f\left(x, \frac{x}{\varepsilon^\alpha}, v(x)\right) \mathrm{d}x, \tag{2.2}$$

subject to periodically oscillating differential constraints of the type

$$\mathcal{A}_\varepsilon v := \sum_{i=1}^N A^i\left(\frac{\cdot}{\varepsilon^\beta}\right)\frac{\partial v}{\partial x_i} \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l) \tag{2.3}$$

or, in divergence form,

$$\mathcal{A}_\varepsilon v := \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(A^i\left(\frac{\cdot}{\varepsilon^\beta}\right)v\right) \to 0 \quad \text{strongly in } W^{-1,p}(\Omega;\mathbb{R}^l), \tag{2.4}$$

where $p \in (1, +\infty)$, $A^i(x) \in \mathbb{R}^{l\times d}$ for all $x \in \mathbb{R}^N$ and $i \in \{1, \dots, N\}$, $\alpha, \beta > 0$ are parameters, and f is assumed periodic in the second variable. Different asymptotic regimes are expected according to the ratio between α and β. The case in which $\beta > 0$ and $\alpha = 0$ with f independent of the first two variables ($f(x, y, \xi) \equiv f(\xi)$) is addressed in [43] under the $\mathcal{A}$-constraint (2.4). Also, Fonseca and Davoli consider in [43] the $\alpha > 0$ and $\beta > 0$ case under the $\mathcal{A}$-constraint (2.3). The remaining cases are announced in [42, 43] to be treated in forthcoming works.

In [59], Fonseca, Ferreira, and Venkatraman initiated a similar research project to that of Fonseca and Davoli [42, 43] but, in contrast with the works mentioned above, outside of the periodic setting. In a nutshell, [59] addresses the effective behavior, as $\varepsilon \to 0$, of integral energies as in (2.2), with $\mathcal{A}$ as in (2.1), assuming a quasicrystalline assumption on the second variable of f in place of periodicity, which poses new challenges. We refer to Section 3 for a more detailed motivation and description of this work.

Next, we mention some authors' contributions concerning the gradient case, $\mathcal{A} = \operatorname{curl}$, or related cases. In [66], Fonseca and Zappale consider first and second order-derivatives in the multiscale case aimed at composites that may feature periodic properties at more than one microscale. The integral energies are of the form

$$u \mapsto \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, D^s u(x)\right) \mathrm{d}x,$$

where $s \in \{1, 2\}$, and f is assumed to be convex in the last variable and continuous. Besides considering general convex energies in the multiscale setting, one of the main novelties of [66] is the characterization of multiscale limits of second-order derivatives. Prior to [66], this characterization was only known for first-order derivatives.

Later, Fonseca and Baía [11] address the effective behavior, as $\varepsilon \to 0$, of integral energies of the form

$$u \mapsto \int_\Omega f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right) \mathrm{d}x$$

without assuming any convexity-type condition on f. This work extends those in the literature by not requiring uniform continuity in space.

In [57, 58], Fonseca and Ferreira revisit the multiscale framework in the case where f grows at most linearly. These studies fall within the realm of the space of functions of bounded variation, BV, and are aimed at identifying effective energies for composite materials in the presence of fracture or cracks. Precisely, they generalize in [58] the notion of two-scale convergence for sequences of Radon measures with finite total variation in [3] to the case of multiple periodic length scales of oscillations. The main result concerns the characterization of the multiscale limit of $\{(u_\varepsilon \mathcal{L}^N{}_{\lfloor\Omega}, Du_{\varepsilon\lfloor\Omega})\}_\varepsilon \subset \mathcal{M}(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega; \mathbb{R}^{d\times N})$ whenever $\{u_\varepsilon\}_\varepsilon$ is a bounded sequence in $\mathrm{BV}(\Omega; \mathbb{R}^d)$, where $\mathcal{M}(\Omega; \mathbb{R}^m)$ with $m \in \mathbb{N}$ is the Banach space of bounded Radon $\mathbb{R}^m$-valued measures, endowed with the total variation norm $|\cdot|$. This result requires considerable modifications of the single microscale case treated in [3], and is based on fine analytical and measure-theoretic arguments. Using this characterization, Fonseca and Ferreira treat in [57] multiscale homogenized problems in the space BV of functions of bounded variation of the form

$$\begin{aligned} u \mapsto & \int_\Omega f\left(\frac{x}{\varrho_1(\varepsilon)}, \dots, \frac{x}{\varrho_n(\varepsilon)}, \nabla u(x)\right) \mathrm{d}x \\ & + \int_\Omega f^\infty\left(\frac{x}{\varrho_1(\varepsilon)}, \dots, \frac{x}{\varrho_n(\varepsilon)}, \frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}(x)\right) \mathrm{d}\left|D^s u\right|(x) \end{aligned}$$

for $u \in \mathrm{BV}(\Omega; \mathbb{R}^d)$. Here, the distributional derivative of u, Du, is decomposed into its absolutely continuous part with respect to the N-dimensional Lebesgue measure, $\nabla u \mathcal{L}^N{}_{\lfloor\Omega}$, and its singular part, $D^s u$. Moreover, $f^\infty(y_1, \dots, y_n, \xi) := \limsup_{t\to\infty} f(y_1, \dots, y_n, t\xi)/t$ is the recession function of a function $f : \mathbb{R}^{nN} \times \mathbb{R}^{d\times N} \to \mathbb{R}$, separately periodic in the first n variables, and $\varrho_1, \dots, \varrho_n$ are positive functions on $(0, \infty)$, representing the length-scales, such that for all $i \in \{1, \dots, n\}$ and $j \in \{2, \dots, n\}$, $\lim_{\varepsilon\to 0} \varrho_i(\varepsilon) = 0$, $\lim_{\varepsilon\to 0} \varrho_j(\varepsilon)/\varrho_{j-1}(\varepsilon) = 0$. In the case of one microscale, Fonseca and Ferreira recover the result in [3] under more general conditions, as well as the results in [19, 45]. For two or more microscales, they obtain new results in the literature.

In [28], Fonseca and Bufford extend to $L^1(\Omega)$ the paramount two-scale compactness property, which asserts that from every bounded sequence, one can extract a subsequence that two-scale converges, with the average over the periodic cell coinciding with the usual weak two-scale limit. This L^1-extension is obtained under an equiintegrability condition on the sequence, and is proved in [28] using three different approaches: an adaptation of the L^p case with $p > 1$, a measure-theoretic argument, and the periodic-unfolding method.

In [27], Fonseca, Bufford, and Davoli address a multiscale homogenization problem in the context of dimension reduction in nonlinear elasticity, aiming at characterizing effective energies for thin, elastic plate-type composites. The energies considered are of the form

$$u \mapsto \frac{1}{h} \int_{\Omega_h} f\left(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla u(x)\right) \mathrm{d}x =: F_h(u)$$

for $u \in W^{1,2}(\Omega_h; \mathbb{R}^3)$, where $\Omega_h := \omega \times (-\frac{h}{2}, \frac{h}{2}) \subset \mathbb{R}^2 \times \mathbb{R}$, $x = (x', x_3) \in \omega \times (-\frac{h}{2}, \frac{h}{2})$, $h > 0$, f is periodic in its first two arguments, and satisfies both common assumptions in nonlinear elasticity and a nondegeneracy condition in a neighborhood of the set of proper rotations. The main result in [27] concerns the characterization of the effective energy associated with the rescaled energies $\frac{1}{h^2} F_h(\cdot)$ depending on the values of

$$\gamma_1 := \lim_{h\to 0} \frac{h}{\varepsilon(h)} \quad \text{and} \quad \gamma_2 := \lim_{h\to 0} \frac{h}{\varepsilon^2(h)},$$

where $\lim_{h\to 0} \varepsilon(h) = \lim_{h\to 0} \varepsilon^2(h) = 0$. This rescaling of the energies corresponds to Kirchhoff's nonlinear bending theory for plates, and the values of γ_1 and γ_2 represent the relative ratios between the thickness parameter h and the two homogenization length-scales, ε and ε^2. These authors obtain different limit models depending on these ratios. Their results extend those in [72, 90] to the multiscale case, and a key and nontrivial step in [27] is the characterization of the three-scale limit of the sequence of linearized elastic stresses. Indeed, the presence of three scales increases the technicality of the problem in all scaling regimes.

Very recently, in [36, 37], Fonseca, Cristoferi, Hagerty, and Popovici study a variational model for fluid–fluid phase transitions with small scale heterogeneities in the case where the small heterogeneities are of the same order of the scale governing the phase transition, and characterized by a small parameter $\varepsilon > 0$. The main result is the limit behavior, as $\varepsilon \to 0$, of integral energies of the form

$$u \mapsto \int_{\Omega} \left[\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}, u(x)\right) + \varepsilon \left|\nabla u(x)\right|^2\right] \mathrm{d}x,$$

where $W : \mathbb{R}^N \times \mathbb{R}^d \to [0, +\infty)$ is a double-well potential that is periodic in the first variable and has two zeros. This limit behavior is given not by an isotropic interfacial energy as one might expect given the isotropy of the surface energy penalization, $\varepsilon|\nabla u|^2$, but instead it has an anisotropic interfacial energy. This anisotropy results from the intricate interaction between homogenization and the phase transitions, and is encoded in the limit cell problem. In [32], the authors study fine properties on the minimizers of the family of problems defining the asymptotic cell formula obtained in [36, 37]. They also obtain bounds for the limiting anisotropic surface tension in terms of the large-scale behavior of the distance function to hyperplanes in certain periodic Riemannian metrics. This work, along with a discussion of [36], is the content of Section 5.2.

3. HOMOGENIZATION OF QUASICRYSTALLINE FUNCTIONALS VIA TWO-SCALE-CUT-AND-PROJECT CONVERGENCE

The work [59] addresses a homogenization problem aimed at understanding composites with a quasicrystalline microstructure. Such composites have been playing a central role in materials science and other areas of engineering [9, 18, 53, 70, 73, 74, 81, 93]; for example, Al–Cu–Fe quasicrystalline materials in polymer-based composites have significantly shown to improve wear-resistance to volume loss, and a two-fold increase in the elastic moduli. The 2011 Nobel Prize in Chemistry was awarded to Dan Shechtman for the striking discovery of quasicrystals, which was announced in the early 1980s.

A key feature of a quasicrystalline structure is that its properties are ordered but are neither periodic nor random. In particular, the mathematical study of quasicrystalline composites does not fit within the *classical* periodic homogenization theory, while almost-periodic and stochastic homogenization approaches do not take full advantage of the quasi-crystalline feature of the problem, often leading to asymptotic formulas that pose computational difficulties and are not stable under perturbations. Instead, in [59], a homogenization procedure based on the two-scale-cut-and-project convergence, introduced in [21] and recently revisited in [92], is adopted and further developed. This two-scale-cut-and-project homogenization procedure leads to a more tractable (even if higher-dimensional) cell problem.

To describe the problem and the results in [59], we first recall the cut-and-project method to model quasicrystals. This method was introduced by de Bruijn [44] and further developed by Duneau and Katz [52], and extends Penrose's ideas of aperiodic tilings of the plane [83] to higher dimensions (also see [21]). Roughly speaking, we can model an N-dimensional quasicrystalline patterns by cutting periodic tilings in an m-dimensional space, with $m > N$, through an N-dimensional subspace with irrational slope. To be precise, given an N-dimensional quasicrystal R and representing by $\sigma_R : \mathbb{R}^n \to \mathbb{R}$ a constitutive property of R, we can find $m \in \mathbb{N}$, with $m > N$, a Y^m-periodic function $\sigma : \mathbb{R}^m \to \mathbb{R}$ with $Y^m \subset \mathbb{R}^m$ a parallelotope, and a linear map $\boldsymbol{R} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\sigma_R(x) = \sigma(\boldsymbol{R}x). \tag{3.1}$$

In the homogenization literature, the structural condition (3.1) is referred to as quasi-periodicity [30, 75]. We refer to [21, 59] for relevant examples of such linear maps $\boldsymbol{R}$. Here, and in the sequel, we do not distinguish the linear map from its associated matrix in $\mathbb{R}^{m\times N}$, and denote both by $\boldsymbol{R}$. Also, we do not distinguish between the transpose matrix and the adjoint of $\boldsymbol{R}$, and denote both by $\boldsymbol{R}^*$.

In general, there are multiple choices for m, σ, and $\boldsymbol{R}$ (see [21]). However, the homogenization analysis in this cut-and-project setting does not depend on $\boldsymbol{R}$ provided it satisfies the following diophantine condition

$$\boldsymbol{R}^* k \neq 0 \quad \text{for all } k \in \mathbb{Z}^m \backslash \{0\}, \tag{3.2}$$

where $\boldsymbol{R}^*$ denotes the transpose of $\boldsymbol{R}$. This condition implies that some entries of $\boldsymbol{R}$ must be irrational, justifying the expression irrational slope used above.

In [59], we address the homogenization problem of characterizing the asymptotic behavior, as $\varepsilon \to 0^+$, of integral energies of the form

$$F_\varepsilon(u) := \int_\Omega f_R\left(x, \frac{x}{\varepsilon}, u(x)\right) \mathrm{d}x \tag{3.3}$$

for $u \in L^p(\Omega;\mathbb{R}^d)$ satisfying $\mathcal{A}u = 0$, where $p \in (1,\infty)$ and

$$\mathcal{A}u := \sum_{i=1}^{N} A^{(i)} \frac{\partial u}{\partial x_i} \quad \text{with } A^{(i)} \in \mathbb{R}^{l\times d} \text{ for all } i \in \{1,\dots,N\}.$$

The precise meaning of the preceding condition $\mathcal{A}u = 0$, in which case we say that u is $\mathcal{A}$-free, is by duality, i.e.,

$$\int_\Omega u \cdot \mathcal{A}^*\phi \,\mathrm{d}x = 0$$

for all $\phi \in C_c^1(\Omega;\mathbb{R}^l)$, where $\mathcal{A}^*$ is the formal adjoint of $\mathcal{A}$ with $\mathcal{A}^*\phi := -\sum_{i=1}^N (A^{(i)})^T \frac{\partial \phi}{\partial x_i}$.

As usual within studies involving $\mathcal{A}$-free vector fields, we assume that $\mathcal{A}$ satisfies the constant-rank property [63,80,87]; that is, there exists $r \in \mathbb{N}$ such that for all $w \in \mathbb{R}^n \setminus \{0\}$, we have

$$\operatorname{rank} \mathbb{A}(w) = r, \tag{3.4}$$

where $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^{l\times d}$ denotes the symbol of $\mathcal{A}$, and is defined by $\mathbb{A}(w) := \sum_{i=1}^N A^{(i)} w_i$ for $w \in \mathbb{R}^n$.

A key step to study the asymptotic behavior of the integral energies in (3.3) via the two-scale-cut-and-project convergence is the characterization of the two-scale-cut-and-project limits (or, for brevity, $\boldsymbol{R}$-two-scale limits) associated with L^p-bounded sequences of $\mathcal{A}$-free vector fields. As we mentioned before, this method has the benefit of taking full advantage of the quasicrystalline feature of the problem and, in contrast with the random homogenization case, leads to a simple and more tractable cell formula (see (3.13) below). Before stating our main homogenization result associated with the integral energies in (3.3), Theorem 3.9 below, we revise the main definitions and results regarding the cut-and-project-two-scale convergence obtained in [59], which are of interest on their own.

The notion of $\boldsymbol{R}$-two-scale convergence was introduced in [21] (also see [92]) as an extension of the usual notion of two-scale convergence [1,82] to enable the study of composites whose underlying microstructure has a quasicrystalline feature. In [21,92], the authors consider sequences in L^2 and their arguments are based on Fourier analysis, relying heavily on Parseval's and Plancherel's identities. Also, in [21] the authors characterize the $\boldsymbol{R}$-two-scale limit of bounded sequences in $W^{1,2}$, while in [92] the authors characterize the limit associated with bounded sequences in L^2 that are divergence-free or curl-free. In [59], besides generalizing these results to the more general setting of L^p with $p \in (1,\infty)$, we provide a unified approach to all the previous cases by considering bounded sequences in L^p that are $\mathcal{A}$-free, in the spirit of [61] concerning the periodic case.

We start by introduction the definition of $\boldsymbol{R}$-two-scale convergence. In what follows, we assume that ε takes values on an arbitrary sequence of positive numbers that converges to zero. Moreover, we use the subscript # within function spaces to highlight an underlying periodicity, in which case the domain indicates the periodicity cell. For instance, $C_\#(Y^m) = \{u \in C(\mathbb{R}^m) : u \text{ is } Y^m\text{-periodic}\}$ and, for a parallelotope in $\mathbb{R}^n$, $\Pi \subset \mathbb{R}^n$, $L^p_\#(\Pi) = \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : u \text{ is } \Pi\text{-periodic}\}$. Also, given a Lebesgue measurable set $B \subset \mathbb{R}^k$, with $k \in \mathbb{N}$, we use the average notation $\fint_B \cdot$ in place of $\frac{1}{\mathcal{L}^k(B)} \int_B \cdot$, where $\mathcal{L}^k(B)$ denotes the k-dimensional Lebesgue measure of B.

Definition 3.1 ($\boldsymbol{R}$-two-scale convergence). A sequence $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega; \mathbb{R}^k)$ is said to $\boldsymbol{R}$-two-scale converge to a function $u \in L^p(\Omega \times Y^m; \mathbb{R}^k)$, and we write $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u$ if for all $\varphi \in L^{p'}(\Omega; C_\#(Y^m; \mathbb{R}^k))$ we have

$$\lim_{\varepsilon \to 0^+} \int_\Omega u_\varepsilon(x) \cdot \varphi\left(x, \frac{\boldsymbol{R}x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \fint_{Y^m} u(x,y) \cdot \varphi(x,y)\, \mathrm{d}x\, \mathrm{d}y. \tag{3.5}$$

The next proposition states some basic properties of $\boldsymbol{R}$-two-scale convergence in $L^p(\Omega; \mathbb{R}^k)$, and we refer to [59, REMARKS 3.2 AND 3.3 AND PROPOSITIONS 3.4 AND 3.5] for its proof.

Proposition 3.2. *Let $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega; \mathbb{R}^k)$, $u \in L^p(\Omega \times Y^m; \mathbb{R}^k)$, and $\bar{u} \in L^p(\Omega; \mathbb{R}^k)$. Then,*

(i) *(uniqueness of $\boldsymbol{R}$-two-scale limits) There exists at most a function $\tilde{u} \in L^p(\Omega \times Y^m; \mathbb{R}^k)$ such that $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} \tilde{u}$.*

(ii) *(on the test functions) If $\{u_\varepsilon\}_\varepsilon$ is bounded in $L^p(\Omega; \mathbb{R}^k)$, then $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u$ if and only if (3.5) holds for all $\varphi \in C_c^\infty(\Omega; C_\#^\infty(Y^m; \mathbb{R}^k))$.*

(iii) *($\boldsymbol{R}$-two-scale and weak limits) If $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} u$, then $u_\varepsilon \rightharpoonup \bar{u}_0$ weakly in $L^p(\Omega; \mathbb{R}^k)$, where $\bar{u}_0(\cdot) := \fint_{Y^m} u(\cdot, y)\, \mathrm{d}y$. In particular, $\{u_\varepsilon\}_\varepsilon$ is bounded in $L^p(\Omega; \mathbb{R}^k)$.*

(iv) *($\boldsymbol{R}$-two-scale and strong limits) If $u_\varepsilon \to \bar{u}$ in $L^p(\Omega; \mathbb{R}^k)$, then $u_\varepsilon \xrightarrow{\boldsymbol{R}\text{-}2sc} \bar{u}$.*

The next proposition provides an important example of sequences that $\boldsymbol{R}$-two-scale converge, which is at the core of several homogenization results using the $\boldsymbol{R}$-two-scale convergence. In particular, it is used to prove the compactness property with respect to the $\boldsymbol{R}$-two-scale convergence stated in Proposition 3.4.

Proposition 3.3. *Let $\psi \in L^1(\Omega; C_\#(Y^m; \mathbb{R}^k))$, and assume that $\boldsymbol{R}$ satisfies (3.2). Then $\{\psi(\cdot, \frac{\boldsymbol{R}\cdot}{\varepsilon})\}_\varepsilon$ is an equiintegrable sequence in $L^1(\Omega; \mathbb{R}^k)$ such that*

$$\left\| \psi\left(\cdot, \frac{\boldsymbol{R}\,\cdot}{\varepsilon}\right) \right\|_{L^1(\Omega;\mathbb{R}^k)} \leqslant \|\psi\|_{L^1(\Omega; C_\#(Y^m;\mathbb{R}^k))} = \int_\Omega \sup_{y \in Y^m} |\psi(x,y)|\, \mathrm{d}x \tag{3.6}$$

and

$$\lim_{\varepsilon \to 0^+} \int_\Omega \psi\left(x, \frac{\boldsymbol{R}x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \fint_{Y^m} \psi(x,y)\, \mathrm{d}x\, \mathrm{d}y. \tag{3.7}$$

In particular, if $\psi \in L^p(\Omega; C_\#(Y^m; \mathbb{R}^k))$, then $\{\psi(\cdot, \frac{\boldsymbol{R}\cdot}{\varepsilon})\}_\varepsilon$ is a p-equiintegrable sequence in $L^p(\Omega; \mathbb{R}^k)$ that $\boldsymbol{R}$-two-scale converges to ψ.

The proof of Proposition 3.3 can be found in [59, PROPOSITION 3.7 AND COROLLARY 3.8], while the proof of the following compactness result can be found in [59, PROPOSITION 3.9].

Proposition 3.4. *Let $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega; \mathbb{R}^k)$ be a bounded sequence, and assume that $\boldsymbol{R}$ satisfies (3.2). Then, there exist a subsequence $\varepsilon' \preceq \varepsilon$ and a function $u \in L^p(\Omega \times Y^m; \mathbb{R}^k)$ such that $u_{\varepsilon'} \xrightarrow{\boldsymbol{R}\text{-}2sc} u$.*

As shown in [21, REMARK 2.8], this compactness property may fail in the case in which $\boldsymbol{R}$ does not satisfy (3.2).

To characterize the $\boldsymbol{R}$-two-scale limits associated with L^p-bounded sequences of $\mathcal{A}$-free vector fields, we recall below the notion of $(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free vector fields introduced in [59] (see [59, DEFINITION 3.7 AND REMARK 3.6]).

Definition 3.5 ($(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free fields). Let $w \in L^p(\Omega; L^p_\#(Y^m; \mathbb{R}^d))$, and define $\bar{w}_0 \in L^p(\Omega; \mathbb{R}^d)$ and $\bar{w}_1 \in L^p(\Omega; L^p_\#(Y^m; \mathbb{R}^d))$ by setting $\bar{w}_0 := \fint_{Y^m} w(\cdot, y)\,\mathrm{d}y$ and $\bar{w}_1 := w - \bar{w}_0$. We say that w is $(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free if the two following conditions hold:

$$\text{(i)} \quad \text{for all } \phi \in C^1_c(\Omega; \mathbb{R}^l), \text{ we have } \int_\Omega w_0 \cdot \mathcal{A}^* \phi \,\mathrm{d}x = 0, \tag{3.8}$$

$$\text{(ii)} \quad \text{for a.e. } x \in \Omega \text{ and for all } \psi \in C^1_\#(Y^m; \mathbb{R}^l),$$

$$\text{we have } \int_{Y^m} \bar{w}_1(x, y) \cdot \mathcal{A}^*_{\boldsymbol{R}} \psi(y)\,\mathrm{d}y = 0, \tag{3.9}$$

where

$$\mathcal{A}^* := -\sum_{i=1}^N (A^{(i)})^T \frac{\partial}{\partial x_i} \quad \text{and} \quad \mathcal{A}^*_{\boldsymbol{R}} := -\sum_{i=1}^N \sum_{m=1}^m (A^{(i)})^T \boldsymbol{R}_{mi} \frac{\partial}{\partial y_m}.$$

For brevity, we write $\mathcal{A}\bar{w}_0 = 0$ and $\mathcal{A}^y_{\boldsymbol{R}^*}\bar{w}_1 = 0$ to mean (i) and (ii), respectively.

Next, we state our main result regarding the characterization of the limits of bounded sequences in L^p that are $\mathcal{A}$-free.

Theorem 3.6. *Let $\boldsymbol{R} \in \mathbb{R}^{m\times n}$ satisfy (3.2). A function $u \in L^p(\Omega \times Y^m; \mathbb{R}^d)$ is the $\boldsymbol{R}$-two-scale limit of an $\mathcal{A}$-free sequence $\{u_\varepsilon\}_\varepsilon \subset L^p(\Omega; \mathbb{R}^d)$ if and only if u is $(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free in the sense of Definition 3.5, that is,*

$$\mathcal{A}\bar{u}_0 = 0 \quad \textit{and} \quad \mathcal{A}^y_{\boldsymbol{R}^*}\bar{u}_1 = 0 \tag{3.10}$$

in the sense of (3.8) and (3.9), respectively, where $\bar{u}_0 := \fint_{Y^m} u(\cdot, y)\,\mathrm{d}y$ and $\bar{u}_1 := u - \bar{u}_0$.

The proof of Theorem 3.6 in [59] uses similar arguments to those in [61] concerning the periodic case (see [61, THEOREM 2.12]). The sufficient part in Theorem 3.6, which guarantees that (3.10) fully characterizes the $\boldsymbol{R}$-two-scale limits, is new in the literature even for $p = 2$ and $\mathcal{A} :=$ curl or $\mathcal{A} :=$ div treated in [21,92]. Furthermore, in [59, SECTION 5], we give an alternative proof of Theorem 3.6 for the $\mathcal{A} :=$ curl case using arguments based on Fourier analysis that differ from those in [21,92] because Parseval's and Plancherel's identities do not hold for $p \neq 2$. This alternative proof provides the equivalent alterative characterization for the $\boldsymbol{R}$-two-scale limit of bounded sequences in $W^{1,p}$ in Theorem 3.7 below, and we believe it provides useful arguments to study homogenization problems involving quasicrystalline functionals in the $\mathcal{A} :=$ curl case.

Theorem 3.7. *Let $\boldsymbol{R} \in \mathbb{R}^{m\times n}$ satisfy (3.2) and let $Y^m \subset \mathbb{R}^m$ be a parallelotope. Then, a function $v \in L^p(\Omega \times Y^m; \mathbb{R}^n)$ is the $\boldsymbol{R}$-two-scale limit of a sequence $\{\nabla v_\varepsilon\}_\varepsilon$ with $\{v_\varepsilon\}_\varepsilon$ bounded in $W^{1,p}(\Omega)$ if and only if there exist $v_0 \in W^{1,p}(\Omega)$ and $v_1 \in L^p(\Omega; \mathcal{G}^p_{\boldsymbol{R}})$ such that*

$$v = \nabla v_0 + v_1,$$

where

$$\mathcal{G}^p_{\boldsymbol{R}} := \left\{w \in L^p_\#(Y^m;\mathbb{R}^n) : \hat{w}_k = \lambda_k \boldsymbol{R}^* k \text{ for some } \{\lambda_k\}_{k\in\mathbb{Z}^m} \subset \mathbb{C} \text{ with } \lambda_0 = 0\right\} \quad (3.11)$$

with $\hat{w}_k := \fint_{Y^m} w(y) e^{-2\pi i k\cdot y}\,\mathrm{d}y$, $k \in \mathbb{Z}^m$, *denoting the Fourier coefficients of* w.

Remark 3.8. We recall that if $u_\varepsilon \in L^p(\Omega;\mathbb{R}^n)$ is curl-free in $\mathbb{R}^n$ with Ω simply connected, then there exists $v_\varepsilon \in W^{1,p}(\Omega)$ such that $u_\varepsilon = \nabla v_\varepsilon$. Thus, in terms of the notations in the two previous results with $d = N$, we have $\bar{u}_0 = \nabla v_0$ and $\bar{u}_1 = v_1$. In particular, (3.11) provides an alternative characterization of $\mathcal{A}_{\boldsymbol{R}^*}$- and $\mathcal{A}^y_{\boldsymbol{R}^*}$-free vector fields (see Definition 3.5) in the $\mathcal{A} :=$ curl case (also see [59, REMARK 5.7] for a more detailed analysis).

Finally, we state the main homogenization result in [59] associated with the integral energies in (3.3), proved under the following assumptions on the Lagrangian, $f_R : \Omega \times \mathbb{R}^n \times \mathbb{R}^d \to [0,\infty)$:

(H1) (quasicrystallinity) there exist $m \in \mathbb{N}$, with $m > N$, a matrix $\boldsymbol{R} \in \mathbb{R}^{m\times n}$ satisfying (3.2), and a continuous function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0,\infty)$ such that the function $f(x,\cdot,\xi)$ is Y^m-periodic for each $(x,\xi) \in \Omega \times \mathbb{R}^d$, with Y^m denoting a parallelotope in $\mathbb{R}^m$, and

$$f_R(x,z,\xi) = f(x,\boldsymbol{R}z,\xi)$$

for all $(x,z,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^d$.

(H2) (growth) there exist $p \in (1,\infty)$ and $C > 0$ such that

$$0 \leqslant f_R(x,z,\xi) \leqslant C\left(1 + |\xi|^p\right)$$

for all $(x,z,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^d$.

For the proof in [59] of the Γ-liminf inequality in Theorem 3.9 below, we require, in addition,

(H3) (convexity) for all $(x,y) \in \Omega \times \mathbb{R}^m$, the function $\xi \mapsto f(x,y,\xi)$ is convex and C^1.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, let $f_R : \Omega \times \mathbb{R}^n \times \mathbb{R}^d \to [0,\infty)$ be a function satisfying (H1)–(H3), let F_ε be the functional introduced in (3.3), and assume that (3.4) holds. Then, the sequence $\{F_\varepsilon\}_\varepsilon$ Γ-converges on $\mathcal{U}_{\mathcal{A}} := \{u \in L^p(\Omega;\mathbb{R}^d) : \mathcal{A}u = 0\}$ as $\varepsilon \to 0^+$, with respect to the weak topology in $L^p(\Omega;\mathbb{R}^d)$, to the functional $\mathcal{F}_{\mathrm{hom}}$ defined, for $u \in \mathcal{U}_{\mathcal{A}}$, by*

$$\mathcal{F}_{\mathrm{hom}}(u) := \inf_{w\in\mathcal{W}_{\mathcal{A}}} \int_\Omega \fint_{Y^m} f\big(x,y,u(x)+w(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$\mathcal{W}_{\mathcal{A}} := \Big\{w \in L^p\big(\Omega; L^p_\#(Y^m;\mathbb{R}^d)\big) : w \text{ is } \big(\mathcal{A},\mathcal{A}^y_{\boldsymbol{R}^*}\big)\text{-free in the sense of Definition 3.5}, \text{ with } \int_{Y^m} w(\cdot,y)\,\mathrm{d}y = 0\Big\}. \quad (3.12)$$

Precisely, given an arbitrary sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ *converging to* 0*, the following pair of statements holds:*

(1) *(Γ-liminf inequality) Let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{\mathcal{A}}$ *be a sequence such that* $u_n \rightharpoonup u$ *in* $L^p(\Omega;\mathbb{R}^d)$ *for some* $u \in L^p(\Omega;\mathbb{R}^d)$*. Then,* $u \in \mathcal{U}_{\mathcal{A}}$ *and*

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) \geqslant \mathcal{F}_{\mathrm{hom}}(u).$$

(2) *(recovery sequence) For every* $u \in \mathcal{U}_{\mathcal{A}}$*, there exists sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{\mathcal{A}}$ *such that* $u_n \rightharpoonup u$ *in* $L^p(\Omega;\mathbb{R}^d)$ *and*

$$\limsup_{n\to\infty} F_{\varepsilon_n}(u_n) \leqslant \mathcal{F}_{\mathrm{hom}}(u).$$

Moreover, for all $u \in \mathcal{U}_{\mathcal{A}}$*, we have*

$$\mathcal{F}_{\mathrm{hom}}(u) = \int_\Omega f_{\mathrm{hom}}\big(x, u(x)\big)\,\mathrm{d}x,$$

where

$$f_{\mathrm{hom}}(x,\xi) := \inf_{v\in\mathcal{V}_{\mathcal{A}}} \fint_{Y^m} f\big(x, y, \xi + v(y)\big)\,\mathrm{d}y \tag{3.13}$$

with

$$\mathcal{V}_{\mathcal{A}} := \left\{v \in L^p_\#(Y^m;\mathbb{R}^d) : v \text{ is } \mathcal{A}_{\boldsymbol{R}^*}\text{-free in the sense of (3.9) and } \int_{Y^m} v(y)\,\mathrm{d}y = 0\right\}. \tag{3.14}$$

Remark 3.10 (On the hypotheses of Theorem 3.9, cf. [59, REMARK 1.2]). (i) In the homogenization literature, measurability of f with respect to the fast-variable is often preferred over continuity. As we discuss in [59, SECTION 2], measurability of f_R requires, in general, Borel-measurability of f. A common approach to deal with lack of continuity is to combine periodicity with Scorza–Dragoni's-type results that, up to a set of small measure, allow reducing the problem to the continuity setting. Here, however, we cannot use such an argument because a set of small m-dimensional Lebesgue measure, the ambient space for the fast variable in terms of (the periodic function) f, may not have small N-dimensional Lebesgue, the ambient space for the fast variable in terms of (the quasicrystalline function) f_R. (ii) The nonconvex case raises nontrivial difficulties in the quasicrystalline setting, and will be the subject of a forthcoming work. (iii) In the Sobolev setting, homogenization of integral energies of the form (3.3) under *nonperiodic* assumptions was undertaken in [22, 41, 71] in the $\mathcal{A} :=$ curl case, assuming coercivity. Within the quasicrystalline framework, Theorem 3.9 extends these results to the general $\mathcal{A}$-free setting and without coercivity.

The proof in [59] of Theorem 3.9, which we sketch next, is based on Γ-convergence and on two-scale convergence adapted to the quasicrystalline setting, also called two-scale-cut-and-project convergence.

Proof of Theorem 3.9. We refer to [59] for a detailed proof of the assertions in Theorem 3.9. Here, we only present a sketch of the proof. Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be an arbitrary sequence converging to 0.

Step 1. Fix $u \in \mathcal{U}_{\mathcal{A}}$ and assume that $w \in \mathcal{W}_{\mathcal{A}} \cap C^1(\overline{\Omega}; C^1_{\#}(Y^m; \mathbb{R}^d))$. For $(x, y) \in \Omega \times Y^m$, define

$$\psi(x, y) := f\big(x, y, u(x) + w(x, y)\big).$$

Using (H1), (H2), the continuity of f, and the regularity of w, we conclude that $\psi \in L^1(\Omega; C_{\#}(Y^m))$. Then, by Proposition 3.3, we have

$$\lim_{n\to\infty} \int_\Omega f_R\left(x, \frac{x}{\varepsilon_n}, w_n(x)\right) \mathrm{d}x = \int_\Omega \fint_{Y^m} f\big(x, y, u(x) + w(x, y)\big) \,\mathrm{d}y \,\mathrm{d}x, \tag{3.15}$$

where, for $x \in \Omega$,

$$w_n(x) := u(x) + w\left(x, \frac{\boldsymbol{R}x}{\varepsilon_n}\right).$$

It can be checked that

$$\begin{aligned} &\{w_n\}_{n\in\mathbb{N}} \quad \text{is a } p\text{-equiintegrable sequence in } L^p(\Omega; \mathbb{R}^d), \\ &w_n \rightharpoonup u \text{ weakly in } L^p(\Omega; \mathbb{R}^d), \quad \mathcal{A}w_n \to 0 \text{ in } W^{-1,p}(\Omega; \mathbb{R}^l). \end{aligned}$$

Then, using an $\mathcal{A}$-free periodic extension lemma established in **[63, LEMMA 2.15]** (also see **[61, LEMMA 2.8]** and **[59, LEMMA 2.3]**), we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^d)$ such that

$$\{u_n\}_{n\in\mathbb{N}} \text{ is } p\text{-equiintegrable}, \quad \mathcal{A}u_n = 0 \text{ in } L^p(\Omega; \mathbb{R}^l), \quad u_n - w_n \to 0 \text{ in } L^p(\Omega; \mathbb{R}^d). \tag{3.16}$$

In particular, $u_n \rightharpoonup u$ weakly in $L^p(\Omega; \mathbb{R}^d)$. Moreover, from (3.15) and a continuity-type result for f_R under (3.16) proved in **[59, LEMMA 4.2]**, we have

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega f_R\left(x, \frac{x}{\varepsilon_n}, u_n(x)\right) \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega f_R\left(x, \frac{x}{\varepsilon_n}, w_n(x)\right) \mathrm{d}x \\ &= \int_\Omega \fint_{Y^m} f\big(x, y, u(x) + w(x, y)\big) \,\mathrm{d}y \,\mathrm{d}x. \end{aligned}$$

Using the preceding arguments and a density argument, we can show that for each $\delta > 0$, $u \in \mathcal{U}_{\mathcal{A}}$, and $w \in \mathcal{W}_{\mathcal{A}}$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{\mathcal{A}}$ such that $u_n \rightharpoonup u$ weakly in $L^p(\Omega; \mathbb{R}^d)$ as $n \to \infty$, and

$$\lim_{n\to\infty} \int_\Omega f_R\left(x, \frac{x}{\varepsilon_n}, u_n(x)\right) \mathrm{d}x \leqslant \int_\Omega \fint_{Y^m} f\big(x, y, u(x) + w(x, y)\big) \,\mathrm{d}y \,\mathrm{d}x + \delta. \tag{3.17}$$

Hence, taking the infimum over $w \in \mathcal{W}_{\mathcal{A}}$ first, and then letting $\delta \to 0$ in (3.17), we get

$$\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u) \leqslant \mathcal{F}_{\text{hom}}(u),$$

where

$$\begin{aligned} \Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u) := \inf\Big\{ &\limsup_{n\to\infty} F_{\varepsilon_n}(u_\varepsilon) : u_n \rightharpoonup u \text{ in } L^p(\Omega; \mathbb{R}^d) \text{ as } n \to \infty, \\ &\mathcal{A}u_n = 0 \text{ for all } n \in \mathbb{N}\Big\}. \end{aligned}$$

Step 2. Here, we prove the Γ-liminf inequality. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{\mathcal{A}}$ be a sequence such that $u_n \rightharpoonup u$ in $L^p(\Omega; \mathbb{R}^d)$ for some $u \in L^p(\Omega; \mathbb{R}^d)$.

Because $u_n \in \mathcal{U}_{\mathcal{A}}$ for all $n \in \mathbb{N}$ and convergence $u_n \rightharpoonup u$ in $L^p(\Omega; \mathbb{R}^d)$, we have $u \in \mathcal{U}_{\mathcal{A}}$. Moreover, by the sufficient part in Theorem 3.6 and by Proposition 3.2, we have

$u_n \xrightarrow{\boldsymbol{R}\text{-}2sc} v$ for a vector-field v that is $(\mathcal{A}, \mathcal{A}^y_{\boldsymbol{R}^*})$-free in the sense of Definition 3.5, with $\int_{Y^m} v(\cdot, y)\,\mathrm{d}y = u(\cdot)$. In particular, we have the decomposition

$$v = u + v_1, \quad v_1 \in L^p\big(\Omega; L^p_\#(Y^m; \mathbb{R}^d)\big), \quad \mathcal{A}^y_{\boldsymbol{R}^*} v_1 = 0, \quad \int_{Y^m} v_1(\cdot, y)\,\mathrm{d}y = 0.$$

Let $\{\psi_j\}_{j\in\mathbb{N}} \subset C_c(\Omega; C_\#(Y^m; \mathbb{R}^l))$ be a sequence converging to v in $L^p(\Omega \times Y^m; \mathbb{R}^d)$ and pointwise in $\Omega \times Y^m$. By (H3), we have, for all $n,\ j \in \mathbb{N}$,

$$f\left(x, \frac{\boldsymbol{R}x}{\varepsilon_n}, u_n(x)\right) \geqslant f\left(x, \frac{\boldsymbol{R}x}{\varepsilon_n}, \psi_j\left(x, \frac{\boldsymbol{R}x}{\varepsilon_n}\right)\right) + \frac{\partial f}{\partial \xi}\left(x, \frac{\boldsymbol{R}x}{\varepsilon}, \psi_j\left(x, \frac{\boldsymbol{R}x}{\varepsilon}\right)\right) \cdot \left(u_n(x) - \psi_j\left(x, \frac{\boldsymbol{R}x}{\varepsilon}\right)\right).$$

Integrating this estimate over Ω and passing to the limit as $n \to \infty$, Proposition 3.3 and (H2)–(H3) yield

$$\begin{aligned} \liminf_{n\to\infty} F_{\varepsilon_n}(u_n) &= \liminf_{n\to\infty} \int_\Omega f\left(x, \frac{\boldsymbol{R}x}{\varepsilon_n}, u_n(x)\right)\mathrm{d}x \\ &\geqslant \int_\Omega \fint_{Y^m} f\big(x, y, \psi_j(x, y)\big)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_\Omega \fint_{Y^m} \frac{\partial f}{\partial \xi}\big(x, y, \psi_j(x, y)\big) \cdot \big(v(x, y) - \psi_j(x, y)\big)\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{3.18}$$

for all $j \in \mathbb{N}$. Letting $j \to \infty$ in this inequality, we obtain from Fatou's lemma and (H1) that

$$\begin{aligned} \liminf_{n\to\infty} F_{\varepsilon_n}(u_n) &\geqslant \int_\Omega \fint_{Y^m} f\big(x, y, v(x, y)\big)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_\Omega \fint_{Y^m} f\big(x, y, u(x) + v_1(x, y)\big)\,\mathrm{d}y\,\mathrm{d}x \\ &\geqslant \inf_{w\in\mathcal{W}_\mathcal{A}} \int_\Omega \fint_{Y^m} f\big(x, y, u(x) + w(x, y)\big)\,\mathrm{d}x\,\mathrm{d}y = \mathcal{F}_{\mathrm{hom}}(u). \end{aligned}$$

Step 3. From Steps 1 and 2, we conclude that for all $u \in \mathcal{U}_\mathcal{A}$, we have

$$\mathcal{F}_{\mathrm{hom}}(u) = \Gamma\text{-}\liminf_{n\to\infty} F_{\varepsilon_n}(u) = \Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u), \tag{3.19}$$

where

$$\Gamma\text{-}\liminf_{n\to\infty} F_{\varepsilon_n}(u) := \inf\Big\{\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) : u_n \rightharpoonup u \text{ in } L^p(\Omega; \mathbb{R}^d) \text{ as } n \to \infty,\ \mathcal{A}u_n = 0 \text{ for all } n \in \mathbb{N}\Big\}.$$

Formula (3.19) asserts that $\{F_\varepsilon\}_\varepsilon$ Γ-converges as $\varepsilon \to 0^+$, with respect to the weak topology in $L^p(\Omega; \mathbb{R}^d)$, to $\mathcal{F}_{\mathrm{hom}}$ on $\mathcal{U}_\mathcal{A}$, and is equivalent to proving that both the Γ-liminf inequality and the recovery sequence properties in Theorem 3.9 hold (see **[39]**).

Step 4. Fix $u \in \mathcal{U}_\mathcal{A}$, and let $w \in \mathcal{W}_\mathcal{A}$. It can be checked that

$$x \in \Omega \mapsto f_{\mathrm{hom}}\big(x, u(x)\big) \tag{3.20}$$

is a measurable map. Moreover, for a.e. $x \in \Omega$, we have $w(x, \cdot) \in \mathcal{V}_\mathcal{A}$. Thus, for a.e. $x \in \Omega$,

$$\inf_{v\in\mathcal{V}_\mathcal{A}} \fint_{Y^m} f\big(x, y, u(x) + v(y)\big)\,\mathrm{d}y \leqslant \fint_{Y^m} f\big(x, y, u(x) + w(x, y)\big)\,\mathrm{d}y.$$

Integrating this estimate over Ω, and then taking the infimum over $w \in \mathcal{W}_{\mathcal{A}}$, we conclude that

$$\int_{\Omega} f_{\hom}(x, u(x))\,\mathrm{d}x \leqslant \mathcal{F}_{\hom}(u).$$

The proof of the converse inequality makes use of a measurable selection criterion proved in [61, LEMMA 3.10] (also see [31]), and we refer to [59, PROPOSITION 4.6] for the details. ∎

4. PHASE TRANSITIONS IN HETEROGENEOUS MEDIA

Heterogeneous media abound in nature, ranging from biological tissues [68] to geological formations [4]. An essential thermodynamic feature of such systems is phase transitions. The presence of heterogeneities during phase transformations is, in general, expected to lead to complex interactions such as pinning and depinning phenomena of interfacial structures, and stick–slip behaviors for possibly anisotropic interface motion [17]. In [36], Fonseca, Cristoferi, Hagerty, and Popovici initiate a project to understand the interaction of the dynamics of phase transitions with heterogeneities. Further progress is made in [32], and the goal of this section is to outline these developments.

The study of pattern formation in equilibrium configurations phase separation is an extremely complex phenomenon, which has attracted the interest of many mathematicians. In the case of homogeneous substances, variational models such as the Modica–Mortola functional (see [78,79,86]) and its vectorial (see [12,65]), anisotropic (see [13,64]), and non-isothermal variants (see [38]), have been proven capable of describing the stable configurations observed in experiments. For composite materials, it has been realized experimentally (see [17]) that the microscopic scale heterogeneities can affect the macroscopic equilibrium configurations, as well as the dynamics of interfaces. Therefore, physics requires the mathematical models to include these microscopic effects.

In this paper, we consider a variational approach to the study of phase transitions in heterogeneous media in the case where the scale of the heterogeneities is the same as those at which the phase transitions phenomenon takes place. In particular, we study a Modica–Mortola like phase field model where the heterogeneities are modeled by oscillations in the potential. To be precise, let $d, N \geqslant 1$, fix an open bounded set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary and, for $\varepsilon > 0$, define the energy $\mathcal{F}_\varepsilon : H^1(\Omega;\mathbb{R}^d) \to [0,\infty]$ as

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}, u(x)\right) + \varepsilon \left|\nabla u(x)\right|^2 \right] \mathrm{d}x. \tag{4.1}$$

Here $u \in H^1(\Omega;\mathbb{R}^d)$ represents the phase field variable. The assumptions that the double-well potential $W : \mathbb{R}^N \times \mathbb{R}^d \to [0,\infty)$ has to satisfy differ according to the questions addressed, and therefore we will present them in each section.

We are interested in understanding what is the sharp interface limit as the parameter $\varepsilon \to 0$. Local minimizers of this limit under a mass constraint will describe equilibrium configurations.

Previous investigations on models related to the one considered in this paper have been undertaken by several authors. In particular, in [6] (see also [5]) Ansini, Braides, and Chiadò Piat considered the case where oscillations are in the forcing term $f(\nabla u)$ (which generalizes $|\nabla u|^2$), while in [50] and [51] by Dirr, Lucia, and Novaga investigated the interaction of the fluid with a periodic mean zero external field. Moreover, in [26], Braides and Zeppieri studied the Γ expansion of the scalar one-dimensional case, allowing the zeros of the potential to jump in a specific way. Finally, the case of higher-order derivatives is examined in [67] by Francfort and Müller.

5. PHASE FIELD MODEL

In this section, we present the results obtained in [32, 33, 36, 37].

5.1. Sharp interface limit

In order to study the sharp interface limit of the energy (4.1), we assume that the double-well potential $W : \mathbb{R}^N \times \mathbb{R}^d \to [0, \infty)$ satisfies the following properties:

(A1) For all $p \in \mathbb{R}^d$, $x \mapsto W(x, p)$ is Q-periodic, where $Q := (-1/2, 1/2)^N$;

(A2) W is a Carathéodory function, i.e.,

 (i) for all $p \in \mathbb{R}^d$, the function $x \mapsto W(x, p)$ is measurable,

 (ii) for a.e. $x \in Q$, the function $p \mapsto W(x, p)$ is continuous;

(A3) There exist $z_1, z_2 \in \mathbb{R}^d$ such that, for a.e. $x \in Q$, $W(x, p) = 0$ if and only if $p \in \{z_1, z_2\}$,

(A4) There exists a continuous function $\widetilde{W} : \mathbb{R}^d \to [0, \infty)$, vanishing only at $p = z_1$ and at $p = z_2$, such that $\widetilde{W}(p) \leqslant W(x, p)$ for a.e. $x \in Q$;

(A5) There exist $C > 0$ and $q \geqslant 2$ such that

$$\frac{1}{C}|p|^q - C \leqslant W(x, p) \leqslant C\big(1 + |p|^q\big)$$

for a.e. $x \in Q$ and all $p \in \mathbb{R}^d$.

Remark 5.1. Assumption (A2)(i) above is the strongest we can ask when modeling periodic inclusions of different materials. Indeed, when each cell Q is composed of k different inclusions of materials each in a region $E_1, \dots, E_k \subset Q$, the potential W takes the form

$$W(x, p) := \sum_{i=1}^{k} W_i(p)\chi_{E_i}(x),$$

where $W_i : \mathbb{R}^d \to [0, \infty)$ are continuous functions with quadratic growth at infinity and such that $W_i(p) = 0$ if and only if $p \in \{z_1, z_2\}$. Therefore the function W in the first variable is, in general, only measurable. Moreover, the continuity of W in the second variable, the nondegeneracy of the potential (A4), and the growth at infinity in the second variable (A5) are compatible with what is usually assumed in the physical literature.

The limiting functional will be an interfacial energy whose energy density is defined via a cell formula as follows.

Definition 5.2. For $\nu \in \mathbb{S}^{N-1}$, let $u_{0,\nu} : \mathbb{R}^N \to \mathbb{R}^d$ be the function

$$u_{0,\nu}(x) := \begin{cases} z_1 & \text{if } x \cdot \nu \leqslant 0, \\ z_2 & \text{if } x \cdot \nu > 0, \end{cases}$$

and denote by $\mathcal{Q}_\nu$ the family of cubes centered at the origin with unit length sides and having two faces orthogonal to ν. For $T > 0$, $Q_\nu \in \mathcal{Q}_\nu$, and $\rho \in C_c^\infty(B(0,1))$ with $\int_{\mathbb{R}^N} \rho(x)\mathrm{d}x = 1$, where $B(0,1)$ is the unit ball in $\mathbb{R}^N$, consider the class of functions

$$\mathcal{C}(\rho, Q_\nu, T) := \big\{u \in H^1(TQ_\nu; \mathbb{R}^d) : u = u_{0,\nu} * \rho \text{ on } \partial(TQ_\nu)\big\}.$$

We define the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ as

$$\sigma(\nu) := \lim_{T\to\infty} g(\nu, T),$$

where, for each $\nu \in \mathbb{S}^{N-1}$ and $T > 0$,

$$g(\nu, T) := \frac{1}{T^{N-1}} \inf\left\{\int_{TQ_\nu} \big[W\big(y, u(y)\big) + |\nabla u|^2\big]\mathrm{d}y : Q_\nu \in \mathcal{Q}_\nu,\ u \in \mathcal{C}(\rho, Q_\nu, T)\right\}.$$

The main properties of the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ that are relevant for our study are collected in the following result. For the proof, see **[36, LEMMA 4.1, REMARK 4.2, LEMMA 4.3, PROPOSITION 4.4]**.

Lemma 5.3. *The following hold:*

(i) *For every $\nu \in \mathbb{S}^{N-1}$, the quantity $\sigma(\nu)$ is well defined and finite;*

(ii) *The value of $\sigma(\nu)$ does not depend on the choice of the mollifier ρ;*

(iii) *The map $\nu \mapsto \sigma(\nu)$ is upper semicontinuous on $\mathbb{S}^{N-1}$;*

(iv) *The infimum in the definition of $g(\nu, T)$ may be taken with respect to one fixed cube $Q_\nu \in \mathcal{Q}_\nu$, i.e., given $\nu \in \mathbb{S}^{N-1}$, for any $Q_\nu \in \mathcal{Q}_\nu$ it holds*

$$\sigma(\nu) = \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ_\nu} \big[W\big(y, u(y)\big) + |\nabla u|^2\big]\mathrm{d}y : u \in \mathcal{C}(\rho, Q_\nu, T)\right\}.$$

We are now in position to introduce the limiting functional.

Definition 5.4. Define the functional $\mathcal{F}_0 : L^1(\Omega; \mathbb{R}^d) \to [0,\infty]$ as

$$\mathcal{F}_0(u) := \begin{cases} \int_{\partial^* A} \sigma(\nu_A(x))\, \mathrm{d}\mathcal{H}^{N-1}(x) & \text{if } u \in \mathrm{BV}(\Omega; \{z_1, z_2\}), \\ +\infty & \text{else}, \end{cases} \tag{5.1}$$

where $A := \{u = z_1\}$, and $\nu_A(x)$ denotes the measure theoretic external unit normal to the reduced boundary $\partial^* A$ of A at the point x.

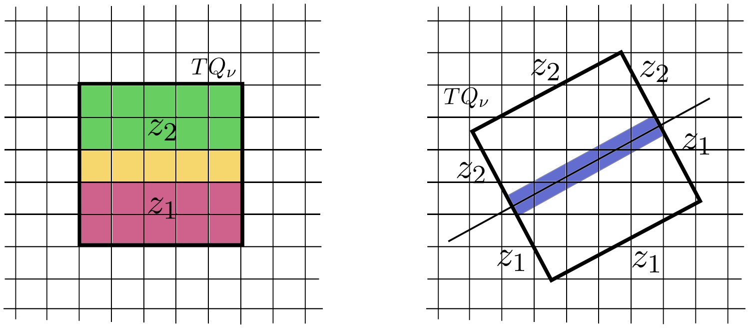

FIGURE 1

The source of anistropy for the limiting functional. If $\nu_A(x)$ is oriented with a direction of periodicity of W, the (local) recovery sequence would simply be obtained by using a rescaled version of the recovery sequence for $\sigma(\nu_A(x))$ in each yellow cube and by setting z_1 in the green region, and z_2 in the pink one. If, instead, $\nu_A(x)$ is not oriented with a direction of periodicity of W, the above procedure does not guarantee that we recover the desired energy, since the energy of such functions is *not* the sum of the energy of each cube.

Remark 5.5. Note that by Lemma 5.3 (i), it holds that $\mathcal{F}_0(u) < \infty$ for all $u \in \mathrm{BV}(\Omega;\{z_1,z_2\})$, and, by Lemma 5.3 (ii), the definition does not depend on the choice of the mollifier ρ.

Theorem 5.6. *Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ be a sequence such that $\varepsilon_n \to 0^+$ as $n \to \infty$. Assume that (A1), (A2), (A3), (A4), and (A5) hold.*

(i) *If $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ is such that*

$$\sup_{n\in\mathbb{N}} \mathcal{F}_{\varepsilon_n}(u_n) < +\infty$$

then, up to a subsequence (not relabeled), $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, for some function $u \in \mathrm{BV}(\Omega;\{z_1,z_2\})$.

(ii) *The functional $\mathcal{F}_0$ is the Γ-limit in the L^1 topology of the family of functionals $\{\mathcal{F}_{\varepsilon_n}\}_{n\in\mathbb{N}}$.*

Remark 5.7. The most interesting aspect of the above result is the anisotropic character of the limiting functional. This might come as a surprise since the initial functional $\mathcal{F}_\varepsilon$ is isotropic in its penalization of gradients, but there is a hidden anisotropy: the possible mismatch between the directions of periodicity of W and the local orientation of the limiting interface $\partial^* A$ (see Figure 1).

We would like to comment on the main ideas behind the proof of Theorem 5.6. Compactness follows by using classical arguments (see [65]), since the nondegeneracy assumption (A4) allows reducing to the case of a nonoscillating potential

$$\mathcal{F}_{\varepsilon_n}(u_n) \geqslant \int_\Omega \left[\frac{1}{\varepsilon_n}\widetilde{W}\big(u_n(x)\big) + \varepsilon_n\left|\nabla_n u(x)\right|^2\right] \mathrm{d}x.$$

The liminf inequality (see [36, PROPOSITION 6.1]) is based on a standard blow-up argument (see [62]) at a point $x_0 \in \partial^* A$ to reduce to the case where the limiting function is $u_{0,\nu}$ and the domain is $Q_\nu \in \mathcal{Q}_\nu$, where $\nu = \nu_A(x_0)$. Then, a technical lemma (see [36, LEMMA 3.1]) in the spirit of De Giorgi's slicing method (see [46]) allows modifying the given sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(Q_\nu;\mathbb{R}^d)$ into a new sequence $\{v_n\}_{n\in\mathbb{N}} \subset H^1(Q_\nu;\mathbb{R}^d)$ with $v_n \to u_{0,\nu}$ in L^1 such that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \geqslant \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v_n),$$

and $v_n = \rho_n * u_{0,\nu}$ on ∂Q_ν, where $\rho_n(x) := \varepsilon_n^{-N}\rho(x/\varepsilon_n)$. The required inequality then follows by using a change of variable, and the definition of $\sigma(\nu)$ together with Lemma 5.3 (iv).

The main challenges are related to the proof of the limsup inequality (see [36, PROPOSITION 7.1]) for a function $u \in \mathrm{BV}(\Omega, \{a, b\})$, which requires new geometric arguments. The idea is first to prove the result for functions $u \in \mathrm{BV}(\Omega; \{a, b\})$, whose outer normals to the reduce boundary have rational coordinates, and then use the density of this class of functions in $\mathrm{BV}(\Omega; \{a, b\})$ together with Reshetnyak's upper semicontinuity theorem (by Lemma 5.3 (iii) the function $\nu \mapsto \sigma(\nu)$ is upper semicontinuous on $\mathbb{S}^{N-1}$) to conclude in the general case. In order to tackle the first step, we use a general strategy developed by De Giorgi, which can be seen as a sort of *reverse* blow-up argument: we consider the localized Γ-limsup as a map on Borel sets and we prove that it is indeed a Radon measure λ. This is done by using a simplification of the De Giorgi–Letta coincidence criterion for Borel measures (see [47]) by Dal Maso, Fonseca, and Leoni (see [40, COROLLARY 5.2]). Next, we show that λ is absolutely continuous with respect to the measure $\mu := \mathcal{H}^{N-1}\lfloor_{\partial^* A}$. The result follows by proving that the density of λ with respect to μ at a point $x_0 \in \partial^* A$ is bounded above by $\sigma(\nu_A(x_0))$. It is in this step that we exploit the fact that $\nu_A(x_0) \in \mathbb{S}^{N-1} \cap \mathbb{Q}^{N-1}$. Indeed, by using the fact that W is periodic (with a different period) also as a function on any cube Q whose faces are normal to directions in $\mathbb{S}^{N-1} \cap \mathbb{Q}^{N-1}$, we can estimate the energy of a configuration similar to that in Figure 1 on the left.

Remark 5.8. The strategy used to prove the above result is robust enough to be easily adapted to prove the analogous result when a mass constraint is enforced. Moreover, as a consequence of the Γ-limit result, we get that the function $\sigma : \mathbb{S}^{N-1} \to [0, \infty)$ is continuous, and its 1-homogeneous extension is convex.

The upshot of the foregoing result is that microscopic heterogeneities during phase transitions result in anisotropic surface tensions at the macroscopic level. Natural follow-up questions are:

(1) Beyond convexity, what can one say about the effective surface tension σ? What functions σ are attainable as effective surface tensions of phase transitions in periodic media?

(2) Considering the gradient flow dynamics of an energy as in (4.1), what are the $\varepsilon \to 0$ asymptotics? Does one indeed obtain a suitable weak formulation of anisotropic mean curvature flow, by analogy with the isotropic setting? Further-

more, what happens to the asymptotics of the gradient flow when the length-scales of homogenization and phase transitions differ?

In [32], we provide partial answers to the first question above, by relating it to a geometry problem. In the sequel, we assume the product form of the potential W:

$$W(y,\xi) := a(y)(1-u^2)^2, \quad y \in \mathbb{R}^N, u \in \mathbb{R}. \tag{5.2}$$

Here $a : \mathbb{R}^N \to \mathbb{R}$ is Q-periodic, and nondegenerate in the sense that

$$\theta \leqslant a(y) \leqslant \Theta, \quad y \in \mathbb{R}^N, \tag{5.3}$$

for some $0 < \theta < \Theta < \infty$. Note that assumptions (A1)–(A5) of Section 5.1 are satisfied with $z_1 = -1$, $z_2 = 1$, and $\widetilde{W}(p) := (1-p^2)^2$. The fact that u is scalar-valued is crucial for a number of the results proven in [32, 33] since we use arguments based on the maximum principle. However, this is not the case of all the results, and we will indicate this as appropriate.

5.2. Bounds on the anisotropic surface tension σ

5.2.1. A geometric framework

Consider the periodic Riemannian metric on $\mathbb{R}^N$ that is conformal to the Euclidean one, defined as follows: given points $x, y \in \mathbb{R}^N$, we set

$$d_{\sqrt{a}}(x,y) := \inf_{\gamma} \int_0^1 \sqrt{a(\gamma(t))}\left|\dot{\gamma}(t)\right| dt,$$

where the infimum is taken over Lipschitz continuous curves $\gamma : [0,1] \to \mathbb{R}^N$ such that $\gamma(0) = x$, $\gamma(1) = y$. It is easily seen that the formula defining $d_{\sqrt{a}}$ is independent of the parameterization of the competitor curves γ. Furthermore, standard arguments via the Hopf–Rinow theorem imply that $\mathbb{R}^N$ with the metric $d_{\sqrt{a}}$ is a complete metric space. Equivalently, geodesically complete: given any pair of points $x, y \in \mathbb{R}^N$ there exists a distance-minimizing geodesic joining them, whose length is equal to $d_{\sqrt{a}}(x,y)$ (see [86] for details). Now fix a direction $\nu \in \mathbb{S}^{N-1}$, and consider the plane Σ_ν through the origin with normal ν,

$$\Sigma_\nu := \left\{y \in \mathbb{R}^N : y \cdot \nu = 0\right\}.$$

Next, define the signed distance function in the $d_{\sqrt{a}}$-metric to the plane Σ_ν, via

$$h_\nu(y) := \operatorname{sgn}(y\cdot\nu) \inf_{z\in\Sigma_\nu} d_{\sqrt{a}}(y,z),$$

where the signum function is defined as

$$\operatorname{sgn}(t) := \begin{cases} 1, & t \geqslant 0, \\ -1, & t < 0. \end{cases}$$

It can be shown (see [32, LEMMA 2.2]) that h_ν is Lipschitz continuous, with

$$\left|\nabla h_\nu(y)\right| = \sqrt{a(y)} \quad \text{at a.e. } y \in \mathbb{R}^N. \tag{5.4}$$

These observations, together with (5.3), yield

$$\begin{aligned} \sqrt{\theta}(y\cdot\nu) &\leqslant h_\nu(y) \leqslant \sqrt{\Theta}(y\cdot\nu), \quad & y\cdot\nu \geqslant 0, \\ \sqrt{\Theta}(y\cdot\nu) &\leqslant h_\nu(y) \leqslant \sqrt{\theta}(y\cdot\nu), \quad & y\cdot\nu < 0. \end{aligned} \tag{5.5}$$

In order to explain the relationship that the $d_{\sqrt{a}}$-metric bears with the anisotropic surface tension σ, it is useful to revisit the case $a \equiv 1$, and the celebrated Modica–Mortola example. Then,

$$\sigma(\nu) = \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ_\nu} \left[W\big(u(y)\big) + |\nabla u|^2\right] : u \in \mathscr{C}(\rho, Q_\nu, T) \right\}.$$

Elementary algebraic manipulations that effectively boil down to completing the square, yield that the infimum above is asymptotically reached by the one-dimensional profile satisfying equipartition of energy. In the model case of (5.2), this entails that the optimal cost is achieved by the choice $u(y) = q \circ (y\cdot\nu)$, where $q := \tanh$. The associated cost is given by

$$\sigma(\nu) \equiv \sigma_0 := \int_{-\infty}^{\infty} \left[W\big(q\circ(y\cdot\nu)\big) + \left|\nabla\big(q\circ(y\cdot\nu)\big)\right|^2\right] d(y\cdot\nu) = 2\int_{-1}^{1} \sqrt{W(s)}\, ds. \tag{5.6}$$

To make the connection to the $\sqrt{a}$-metric, we begin by noting that when $a \equiv 1$ we have $h_\nu(y) \equiv y\cdot\nu$. Our main motivation, then, is to obtain a similar formula that is exact when a is nonconstant, or at least supplies reasonable bounds for the nonconstant $\nu \mapsto \sigma(\nu)$. We do so by encoding the heterogeneous effects of a into the geometry of the underlying space, i.e., by working in the $\sqrt{a}$-metric. We turn to making these comments precise.

Fix $\nu \in \mathbb{S}^{N-1}$. Then, the cell formula defining $\sigma(\nu)$, proven in [36,37] and specialized to our setting, reads (see Lemma 5.3 (iv))

$$\begin{aligned} \sigma(\nu) = \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\bigg\{ \int_{TQ_\nu} \left[a(y)W(u) + |\nabla u|^2\right] dy : u \in H^1(TQ_\nu), \\ u = \rho * u_{0,\nu} \text{ on } \partial(TQ_\nu) \bigg\}. \end{aligned}$$

Here, we recall that $u_{0,\nu}(y) := \operatorname{sgn}(y\cdot\nu)$ and ρ is any standard smooth normalized mollifier (it is shown in Lemma 5.3 (ii) that $\sigma(\nu)$ is independent of this choice). A preliminary step is to observe, by De Giorgi's slicing method (see [32, Lemma A.1]), that, equivalently,

$$\begin{aligned} \sigma(\nu) = \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\bigg\{ \int_{TQ_\nu} \left[a(y)W(u) + |\nabla u|^2\right] dy : u \in H^1(TQ_\nu), \\ u = q\circ h_\nu \text{ along } \partial(TQ_\nu) \bigg\}. \end{aligned} \tag{5.7}$$

For each fixed $T \gg 1$, by the direct method of the calculus of variations, the variational problem inside the limit has a minimizer. Such a minimizer is, perhaps, not unique, but for each T we select one, and call it u_T. We discuss various properties of u_T below in Section 5.2.2. In light of (5.7), it is clear by energy comparison that

$$\sigma(\nu) \leqslant \liminf_{T\to\infty} \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[a(y)W(q\circ h_\nu) + \left|\nabla(q\circ h_\nu)\right|^2\right] dy.$$

Towards proving the opposite bound, we introduce the function $\phi : \mathbb{R} \to \mathbb{R}$, by

$$\phi(z) := 2\int_0^z \sqrt{W(s)}\, ds.$$

This function plays a fundamental role in the Modica–Mortola analysis corresponding to $a \equiv 1$. For any $T \gg 1$, using (5.4) and completing squares, we find

$$\begin{aligned}
&\frac{1}{T^{N-1}}\int_{TQ_\nu} \big[a(y)W(u_T) + |\nabla u_T|^2\big]\, dy \\
&\quad = \frac{2}{T^{N-1}}\int_{TQ_\nu} \nabla h_\nu \cdot \sqrt{W(u_T)}\nabla u_T\, dy + \frac{1}{T^{N-1}}\int_{TQ_\nu} \big|\nabla u_T - \sqrt{W(u_T)}\nabla h_\nu\big|^2\, dy \\
&\quad \geqslant \frac{1}{T^{N-1}}\int_{TQ_\nu} \nabla h_\nu \cdot \nabla\big(\phi(u_T)\big)\, dy \\
&\quad = \frac{1}{T^{N-1}}\int_{TQ_\nu} \nabla h_\nu \cdot \nabla\big(\phi(q\circ h_\nu)\big)\, dy + \frac{1}{T^{N-1}}\int_{TQ_\nu} \nabla h_\nu \cdot \nabla\big(\phi(u_T) - \phi(q\circ h_\nu)\big)\, dy \\
&\quad = \frac{1}{T^{N-1}}\int_{TQ_\nu} |\nabla h_\nu|^2 \phi'(q\circ h_\nu) q'(h_\nu)\, dy \\
&\quad + \frac{1}{T^{N-1}}\int_{TQ_\nu} \nabla h_\nu \cdot \nabla\big(\phi(u_T) - \phi(q\circ h_\nu)\big)\, dy \\
&\quad = \frac{1}{T^{N-1}}\int_{TQ_\nu} 2a(y)W(q\circ h_\nu)\, dy + \frac{1}{T^{N-1}}\int_{TQ_\nu} \nabla h_\nu \cdot \nabla\big(\phi(u_T) - \phi(q\circ h_\nu)\big)\, dy \\
&\quad = \frac{1}{T^{N-1}}\int_{TQ_\nu} \big[\, a(y)W(q\circ h_\nu) + \big|\nabla(q\circ h_\nu)\big|^2\,\big]\, dy \\
&\qquad + \frac{1}{T^{N-1}}\int_{TQ_\nu} \nabla h_\nu \cdot \nabla\big(\phi(u_T) - \phi(q\circ h_\nu)\big)\, dy,
\end{aligned} \tag{5.8}$$

where in the last line we used the fact that the function $q \circ h_\nu$ achieves equipartition of energy. Indeed, by the definition of h_ν, we have

$$\big|\nabla(q\circ h_\nu)(y)\big|^2 = (q'\big(h_\nu(y)\big)^2 \big|\nabla h_\nu(y)\big|^2 = a(y)W(q\big(h_\nu(y)\big).$$

Provided we can control the error term

$$\limsup_{T\to\infty}\left|\frac{1}{T^{N-1}}\int_{TQ_\nu} \nabla h_\nu(y) \cdot \nabla\big(\phi(u_T) - \phi(q\circ h_\nu)\big)\, dy\right| := \lambda_0(\nu),$$

we observe that the test function $q \circ h_\nu$ gives two-sided bounds on $\sigma(\nu)$. Controlling the term λ_0 is complicated by the fact that it couples a product of weakly converging sequences (on expanding domains). Indeed, rescaling using $y = Tx$ in order to work in a fixed domain Q_ν, the two weakly converging factors making up the above product are

(1) the oscillatory factor: by (5.4) and (5.3), the term $\{\nabla h_\nu(T\,\cdot)\}_T$, which is bounded in L^∞, converges weakly-*; and

(2) the concentration factor: the terms $\nabla\phi(u_T(T\,\cdot))$ and $\nabla\phi(q\circ h_\nu(T\,\cdot)$ converge weakly-* to measures (see Section 5.2.2 for precise statements).

In particular, as one of the factors converges to a measure, standard tools such as compensated compactness, used traditionally to pass to the limit in products of weakly converging sequences, are unavailable, and we must control this term "by hand." In Section 5.2.2 below, we obtain fine information on the concentration effects, and in Section 5.2.3 we deduce partial results concerning the oscillatory effects. Finally, we put these together in Section 5.2.4, where we obtain bounds on $\lambda_0(\nu)$.

5.2.2. Structure of minimizers of the cell formula

For fixed $T \gg 1$, let $u_T \in C^2(TQ_\nu)$ (by elliptic regularity) a minimizer of the energy

$$\int_{TQ_\nu} \left[a(y)W(u) + |\nabla u|^2\right] dy,$$

among competitors that equal $q \circ h_\nu$ along the boundary $\partial(TQ_\nu)$, and set

$$v_T(x) := u_T(Tx), \quad x \in Q_\nu.$$

which minimizes the energy.

Lemma 5.9. *The functions v_T converge in L^1 to $u_{0,\nu} : Q_\nu \to \{\pm 1\}$.*

The proof of this lemma (see [32, Lemma 3.1]) is a nice application of the convexity of the one-homogeneous extension of σ (see Remark 5.8), using Jensen's inequality. The argument, without any changes, holds in the complete generality of the setting of [36] on the potential (vectorial, coupled, measurable dependence on the fast variable), and does not rely on the specific structure requested in (5.2). Combining Lemma 5.9 with the results of Caffarelli–Cordoba [29], we find that the level sets of v_T, for T sufficiently large, converge uniformly to $\Sigma_\nu \cap Q_\nu$.

Restricting ourselves to the scalar setting of (5.2), an argument using the strong maximum principle yields that, for all $T < \infty$, we have

$$-1 < u_T(y) < 1,$$

(see [32, Lemma 3.2]). In particular, $w_T := \frac{1}{\sqrt{2}} \tanh^{-1} u_T$ is well defined, finite, and smooth in TQ_ν. Further, the function w_T verifies the elliptic boundary value problem

$$\begin{cases} \Delta w_T = \frac{4}{\sqrt{2}} \tanh w_T(|\nabla w_T|^2 - a(y)), & y \in TQ_\nu, \\ w_T(y) = h_\nu(y), & y \in \partial(TQ_\nu). \end{cases}$$

Proposition 5.10. *Let w_T be as above, and let $T \gg 1$. There exist universal constants α_0 and $\eta_0 > 0$ such that the following holds:*

$$\begin{cases} \sqrt{\Theta}(y \cdot \nu) - \alpha_0 \geqslant w_T(y) \geqslant \sqrt{\theta}(y \cdot \nu) - \eta_0, & \textit{if } w_T(y) > 0, \\ -\sqrt{\theta}(y \cdot \nu) + \eta_0 \geqslant w_T(y) \geqslant -\sqrt{\Theta}(y \cdot \nu) + \alpha_0, & \textit{if } w_T(y) < 0. \end{cases} \tag{5.9}$$

Proposition 5.10 asserts that, up to universal constants, the function w_T satisfies exactly the same growth rates as the function h_ν, see (5.5). To prove Proposition 5.10, consider, for instance, the lower bound in the first of the two inequalities in (5.9). The main observation is that the function $y \mapsto \zeta_T(y) := \frac{y \cdot \nu}{w_T(y) + \eta_0}$ satisfies an elliptic PDE that verifies a maximum principle. The remaining inequalities follow from similar arguments, and we refer the reader to [32, Proposition 3.4] for details.

5.2.3. The planar metric problem

Our results on the distance function h_ν concern its large-scale behavior. The bounds on σ that we discuss in Section 5.2.4 below, depend solely on the large-scale behavior of the distance functions h_ν for which one can readily invoke efficient numerical algorithms, for example fast marching and sweeping methods [85].

A natural question concerns the large-scale homogenized behavior of h_ν, i.e., the characterization of the limit

$$\lim_{T\to\infty} \frac{h_\nu(Ty)}{T}, \quad y \in \mathbb{R}^N,$$

in a suitable topology of functions. We completely answer this question.

Theorem 5.11. *Let $\nu \in \mathbb{S}^{N-1}$. There exists a real number $c(\nu) \in [\sqrt{\theta}, \sqrt{\Theta}]$ such that for each $K \subseteq \mathbb{R}^N$ compact, we have*

$$\lim_{T\to\infty} \sup_{y\in K} \left| \frac{1}{T} h_\nu(Ty) - c(\nu)(y\cdot\nu) \right| = 0.$$

Moreover, for all compact subsets K of $\mathbb{R}^N \setminus \Sigma_\nu$, we have

$$\lim_{T\to\infty} \sup_{y\in K} \left| \frac{1}{T(y\cdot\nu)} h_\nu(Ty) - c(\nu) \right| = 0.$$

We can interpret Theorem 5.11 as a homogenization result for the eikonal equation in half-spaces. Indeed, it is well known (see, for example, [77]) that for each fixed $\nu \in \mathbb{S}^{N-1}$, the functions $k_T(y) := T^{-1}h_\nu(Ty)$ and $\ell(y) := c(\nu)(y\cdot\nu)$ are the unique viscosity solutions to

$$\begin{cases} |\nabla k_T| = \sqrt{a(Ty)} & \text{in } \{y\cdot\nu \geqslant 0\}, \\ k_T = 0 & \text{on } \Sigma_\nu, \end{cases} \quad \text{and} \quad \begin{cases} |\nabla \ell| = c(\nu) & \text{in } \{y\cdot\nu \geqslant 0\}, \\ \ell = 0 & \text{on } \Sigma_\nu. \end{cases} \tag{5.10}$$

Theorem 5.11 shows that viscosity solutions of the heterogeneous eikonal equations, i.e., k_T in (5.10), converge locally uniformly to ℓ. A viscous and stochastic version of these equations (termed the "planar metric problem") was introduced by Armstrong and Cardaliaguet [7], and studied by others [55, 56] in the context of stochastic homogenization of geometric flows. Small modifications of our arguments, in fact, yield homogenization theorems for first order Hamilton–Jacobi equations in almost periodic media in half-spaces, with Lipschitz dependence on the fast variable, and convex dependence on the gradient variable.

5.2.4. Bounds on the anisotropic surface tension

As explained in the string of inequalities (5.8), the function $q \circ h_\nu$ provides tight upper and lower bounds for the effective anisotropy $\sigma(\nu)$. To be precise, we have

Theorem 5.12. *Let $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ be the anisotropic surface energy as in (5.2). Let $q : \mathbb{R} \to \mathbb{R}$ be defined by*

$$q(z) := \tanh(z), \quad z \in \mathbb{R}.$$

For $\nu \in \mathbb{S}^{N-1}$, *define*

$$\underline{\lambda}(\nu) := \liminf_{T\to\infty} \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[a(y)W(q\circ h_\nu) + \left|\nabla(q\circ h_\nu)\right|^2\right] dy,$$
$$\overline{\lambda}(\nu) := \limsup_{T\to\infty} \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[a(y)W(q\circ h_\nu) + \left|\nabla(q\circ h_\nu)\right|^2\right] dy.$$

There exist $\Lambda_0 > 0$ *and* $\lambda_0 : \mathbb{S}^{N-1} \to [0, \Lambda_0]$ *such that*

$$\overline{\lambda}(\nu) - \lambda_0(\nu) \leqslant \sigma(\nu) \leqslant \underline{\lambda}(\nu).$$

We remark that in general, $\lambda_0(\nu)$ is never zero, unless $a \equiv 1$.

FUNDING

The research of Rita Ferreira was partially supported by baseline and start-up funds from King Abdullah University of Science and Technology (KAUST). The research of Irene Fonseca was partially funded by NSF grant DMS-19006238. The research of Raghav Venkatraman was partially funded by the Simons Foundation Postdoctoral Award, and an AMS-Simons Travel Award.

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