ON THE NONLINEAR STABILITY OF SHEAR FLOWS AND VORTICES

ALEXANDRU D. IONESCU AND HAO JIA

ABSTRACT

In this article we present some of the main ideas in our recent work on the asymptotic stability of shear flows and vortices among solutions of the Euler equations in two dimensions. More precisely, we discuss the following results:

(1) a theorem on the nonlinear asymptotic stability of a large class of shear flows $(b(y), 0)$ in the finite channel $\mathbb{T} \times [0, 1]$, defined by strictly increasing Gevrey smooth functions b, which are linear outside a compact subset of the interval $(0, 1)$ and satisfy suitable spectral conditions;

(2) a theorem on the nonlinear asymptotic stability of point vortex solutions of the Euler equation in $\mathbb{R}^2$;

(3) heuristic analysis showing that the mechanism of inviscid damping is unlikely to work to produce global solutions of the α-generalized SQG equation in two dimensions, for any parameter $\alpha > 0$.

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1. INTRODUCTION

In this paper we present some of our recent results on the asymptotic stability of solutions of the two-dimensional incompressible Euler equation. More precisely, we consider solutions $u : [0, \infty) \times \mathcal{D} \to \mathbb{R}^2$ of the equation

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0, \tag{1.1}$$

where the domain $\mathcal{D}$ is either the entire plane $\mathcal{D} = \mathbb{R}^2$ or the finite channel $\mathcal{D} = \mathbb{T} \times [0, 1]$. Letting $\omega := -\partial_y u^x + \partial_x u^y$ denote the vorticity field, equation (1.1) can be written as

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi), \quad \Delta \psi = \omega. \tag{1.2}$$

In the case of the finite channel $\mathcal{D} = \mathbb{T} \times [0, 1]$, we impose also the boundary conditions

$$\psi(x, 0) \equiv 0, \quad \psi(x, 1) \equiv C_0, \tag{1.3}$$

where C_0 is a constant preserved by the flow.

The two-dimensional incompressible Euler equation is globally well posed for smooth initial data, by the classical result of Wolibner [48]. The long-time behavior of general solutions is, however, very difficult to understand, due to the lack of a global relaxation mechanism. A more realistic goal is to study the global nonlinear dynamics of solutions that are close to steady states of the 2D Euler equation. Coherent structures, such as shear flows and vortices, are particularly important in the study of the 2D Euler equation, since numerical simulations and physical experiments, such as those of [2,3,9,21,34,35,40,41], show that they tend to form dynamically and become the dominant feature of solutions.

The main topic in this article is the study of asymptotic stability of shear flows and vortices. This is a classical subject and a fundamental problem in hydrodynamics. Early investigations were started by Rayleigh [38], Kelvin [27], Orr [37], Taylor [44], among many others, with a focus on mode stability. More detailed understanding of general spectral properties and suitable linear decay estimates were obtained later, see, for example, [8,10,17,42]. In the direction of nonlinear results, Arnold [1] proved a general stability theorem, using the energy method, but this does not give asymptotic information on the global dynamics.

The full nonlinear asymptotic stability problem has only been investigated in recent years, starting with the work of Bedrossian–Masmoudi [7], who proved nonlinear stability in the simplest case of perturbations of the Couette flow, i.e., showing that small perturbations of the Couette flow on the infinite cylinder $\mathbb{T} \times \mathbb{R}$ converge weakly to nearby shear flows. This result was extended by the authors [23] to the finite channel $\mathbb{T} \times [0, 1]$, in order to be able to consider solutions with finite energy. In [24] the authors also proved asymptotic stability of point vortex solutions in $\mathbb{R}^2$, showing that small perturbations converge to a radial profile, and the position of the point vortex stabilizes rapidly at the center of the final radial profile. Finally, in [25] the authors proved nonlinear asymptotic stability of a large family of monotonic shear flows (a similar theorem was proved slightly later and independently by Masmoudi–Zhao [33]). In this article we discuss the main ideas of our papers [23–25].

The linearized equations around other stationary solutions were also investigated intensely in the last few years, and linear inviscid damping and decay was proved in many cases of physical interest, see, for example, [4, 15, 20, 45–47, 49, 50]. However, it also became clear that there are major conceptual difficulties in passing from linear to nonlinear stability, such as the presence of "resonant times" in the nonlinear problem, which require refined Fourier analysis techniques, and the fact that the final state of the flow is determined dynamically by the global evolution and cannot be described in terms of the initial data.

Nonlinear inviscid damping is a very subtle mechanism of stability that has only been established in 2 dimensions and for Euler-type equations. In fact, the heuristic analysis we present in Section 3 of this article suggests that this mechanism fails to produce global solutions of the α-generalized SQG equation in 2 dimensions, for any parameter $\alpha > 0$.

The Euler equations can also be viewed as the limiting case of the Navier–Stokes equations with small viscosity $\nu > 0$. In the presence of viscosity, one can have much more robust stability results, both in 2 and 3 dimensions, for initial data that is sufficiently small relative to ν. See the recent papers [5, 6, 12, 18] and references therein.

We note also that the problem of nonlinear inviscid damping is connected to the well-known Landau damping effect for the Vlasov–Poisson equations. We refer to the celebrated work of Mouhot–Villani [36] for the physical background and more references.

1.1. Monotonic shear flows

We consider a perturbative regime for the Euler equation (1.1), with velocity field given by $(b(y), 0)) + u(x, y)$ and vorticity given by $-b'(y) + \omega$. We define the Gevrey spaces $\mathcal{G}^{\lambda,s}(\mathbb{T} \times \mathbb{R})$ as the spaces of L^2 functions f on $\mathbb{T} \times \mathbb{R}$ defined by the norm

$$\|f\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times\mathbb{R})} := \big\|e^{\lambda\langle k,\xi\rangle^s} \tilde{f}(k,\xi)\big\|_{L^2_{k,\xi}} < \infty, \quad s \in (0,1], \lambda > 0. \tag{1.4}$$

In the above $(k, \xi) \in \mathbb{Z} \times \mathbb{R}$ and $\tilde{f}$ denotes the Fourier transform of f in (x, y). More generally, for any interval $I \subseteq \mathbb{R}$ we define the Gevrey spaces $\mathcal{G}^{\lambda,s}(\mathbb{T} \times I)$ by

$$\|f\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times I)} := \|Ef\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times\mathbb{R})}, \tag{1.5}$$

where $Ef(x) := f(x)$ if $x \in I$ and $Ef(x) := 0$ if $x \notin I$. The use of Gevrey spaces is necessary in the context of inviscid damping, mainly due to loss of regularity during the flow.

We will assume that the background shear flow $b \in C^\infty(\mathbb{R})$ satisfies the following:

(A) For some $\vartheta_0 \in (0, 1/10]$ and $\beta_0 > 0$

$$\vartheta_0 \le b'(y) \le 1/\vartheta_0 \quad \text{for } y \in [0,1] \quad \text{and} \quad b''(y) \equiv 0 \quad \text{for } y \notin [2\vartheta_0, 1-2\vartheta_0], \tag{1.6}$$

and

$$\|b\|_{L^\infty(0,1)} + \big\|b''\big\|_{\mathcal{G}^{\beta_0,1/2}} \le 1/\vartheta_0. \tag{1.7}$$

(B) The associated linear operator $L_k : L^2(0,1) \to L^2(0,1)$, $k \in \mathbb{Z}\backslash\{0\}$, given by

$$L_k f = b(y)f - b''(y)\varphi_k, \quad \text{where } \partial_y^2\varphi_k - k^2\varphi_k = f,\ \varphi_k(0) = \varphi_k(1) = 0, \tag{1.8}$$

has no discrete eigenvalues and, therefore, by the general theory of Fredholm operators, the spectrum of L_k is purely continuous spectrum $[b(0), b(1)]$ for all $k \in \mathbb{Z}\backslash\{0\}$.

For any function $H = H(x, y) : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, let $\langle H \rangle = \langle H \rangle(y)$ denote the average of H in x. Our main result in [25] is the following theorem:

Theorem 1.1. *Assume that $\beta_0, \vartheta_0 > 0$ and b satisfies the assumptions above. Then there are constants $\beta_1 > 0$ and $\bar{\varepsilon} > 0$ such that the following statement is true:*

Assume that ω_0 has compact support in $\mathbb{T} \times [2\vartheta_0, 1 - 2\vartheta_0]$, and satisfies

$$\|\omega_0\|_{\mathcal{G}^{\beta_0,1/2}(\mathbb{T}\times\mathbb{R})} = \varepsilon \le \bar{\varepsilon}, \qquad \int_{\mathbb{T}} \omega_0(x, y)\, dx = 0 \quad \text{for any } y \in [0, 1]. \tag{1.9}$$

Let $\omega : [0, \infty) \times \mathbb{T} \times [0, 1] \to \mathbb{R}$ denote the global smooth solution to the Euler equation

$$\begin{cases} \partial_t \omega + b(y)\partial_x \omega - b''(y)\partial_x \psi + u \cdot \nabla\omega = 0, \\ u = (u^x, u^y) = (-\partial_y \psi, \partial_x \psi), \quad \Delta\psi = \omega, \quad \psi(t, x, 0) = \psi(t, x, 1) = 0, \end{cases} \tag{1.10}$$

with initial data ω_0. Then we have the following conclusions:

(i) *For all $t \ge 0$, $\operatorname{supp}\omega(t) \subseteq \mathbb{T} \times [\vartheta_0, 1 - \vartheta_0]$.*

(ii) *There exists $F_\infty(x, y) \in \mathcal{G}^{\beta_1,1/2}$ with $\operatorname{supp} F_\infty \subseteq \mathbb{T} \times [\vartheta_0, 1 - \vartheta_0]$ such that*

$$\big\|\omega\big(t, x + tb(y) + \Phi(t, y), y\big) - F_\infty(x, y)\big\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times[0,1])} \lesssim_{\beta_0,\vartheta_0,\kappa} \frac{\varepsilon}{\langle t \rangle} \tag{1.11}$$

for all $t \ge 0$, where

$$\Phi(t, y) := \int_0^t \big\langle u^x \big\rangle(\tau, y)\, d\tau. \tag{1.12}$$

(iii) *We define the smooth functions $\psi_\infty, u_\infty : [0, 1] \to \mathbb{R}$ by*

$$\partial_y^2 \psi_\infty = \langle F_\infty \rangle, \quad \psi_\infty(0) = \psi_\infty(1) = 1, \quad u_\infty(y) := -\partial_y \psi_\infty(y). \tag{1.13}$$

Then the velocity field $u = (u^x, u^y)$ satisfies the bounds

$$\big\|\big\langle u^x \big\rangle(t, y) - u_\infty(y)\big\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times[0,1])} \lesssim \frac{\varepsilon}{\langle t \rangle^2} \tag{1.14}$$

and

$$\langle t \rangle \big\|u^x(t, x, y) - \big\langle u^x \big\rangle(t, y)\big\|_{L^\infty(\mathbb{T}\times[0,1])} + \langle t \rangle^2 \big\|u^y(t, x, y)\big\|_{L^\infty(\mathbb{T}\times[0,1])} \lesssim \varepsilon. \tag{1.15}$$

1.1.1. Remarks

The simplest case $b(y) = y$ (the Couette flow) was treated earlier in [7, 23]. We discuss now some of the assumptions and conclusions of our main theorem.

(1) Equation (1.10) for the vorticity deviation is equivalent to the original Euler equations (1.1)–(1.3). The condition $\int_{\mathbb{T}} \omega_0(x, y)\, dx = 0$ can be imposed without loss of generality, by replacing the shear flow $b(y)$ with the nearby shear flow $b(y) + \langle u_0^x \rangle(y)$.

(2) The assumption on the compact support of ω_0 is likely necessary to prove scattering in Gevrey spaces. Indeed, Zillinger [49] showed that scattering does not hold in high Sobolev spaces unless one assumes that the vorticity vanishes at high order at the boundary. This is due to what is called "boundary effect," which is not consistent with inviscid damping. Investigating the boundary effect in the context of asymptotic stability of Euler or Navier–Stokes equations is an interesting topic by itself, but we will not address it here.

(3) The assumption on the support of b'' is necessary to preserve the compact support of $\omega(t)$ in $\mathbb{T} \times [\vartheta_0, 1 - \vartheta_0]$, due to the nonlocal term $b''(y)\partial_x \psi$ in (1.10). Assumption (1.6) on the uniform monotonicity of the function b is also important for our proof, to ensure a uniform rate of inviscid damping. It is an open question to investigate what happens in the case of nonmonotone shear flows which are linearly stable, such as Kolmogorov or Poiseuille flows.

(4) There is a large class of shear flows b satisfying our assumptions, for instance, functions $b(y)$ satisfying $b'(y) \geq 1$ and $|b'''(y)| < 1$, $y \in [0, 1]$.

(5) The Gevrey regularity assumption (1.9) on the initial data ω_0 is likely sharp. See the recent construction of nonlinear instability of Deng–Masmoudi [16] for the Couette flow in slightly larger Gevrey spaces, and the more definitive counterexamples to inviscid damping in low Sobolev spaces by Lin–Zeng [30].

(6) The most important statement in Theorem 1.1 is (1.11), which provides strong control on the "profile" of the vorticity and from which the other statements follow easily. We note that the convergence (1.11) of the profile for vorticity holds in a slightly weaker Gevrey space, since $\beta_1 < \beta_0$. This is connected with the use of energy functionals with decreasing time-dependent weights to control the profile, and is a reflection of the phenomenon that "decay costs regularity" in inviscid damping.

(7) At the qualitative level, our main conclusion (1.11) shows that the vorticity ω converges weakly to the function $\langle F_\infty \rangle(y)$. This is consistent with a far-reaching conjecture regarding the long-time behavior of solutions of the $2D$ Euler equation, see [43], which predicts that for general generic solutions the vorticity field converges, as $t \to \infty$, weakly but not strongly in L^2_{loc} to a steady state. Proving such a conjecture for general solutions is, of course, well beyond the current PDE techniques, but the nonlinear asymptotic stability results we have so far in [7, 23–25] are consistent with this conjecture.

(8) One can gain some intuition and explain the more technical conclusions in Theorem 1.1 by examining a simple explicit case, corresponding to the Couette flow $b(y) = y$. In this case $b''(y) = 0$ and the linearization of the main equation (1.10) is

$$\partial_t \omega + y \partial_x \omega = 0, \tag{1.16}$$

which was studied by Orr in a pioneering work [37]. To simplify the discussion, assume that $x \in \mathbb{T}$, $y \in \mathbb{R}$ (to avoid the boundary issue which is not our main concern here). By direct calculation, we have $\omega(t, x, y) = \omega_0(x - yt, y)$. The stream function is given by

$\Delta\psi(t,x,y) = \omega(t,x,y)$ for $(x,y) \in \mathbb{T} \times \mathbb{R}$, so in the Fourier space we have the formulas

$$\widetilde{\omega}(t,k,\xi) = \widetilde{\omega_0}(k,\xi+kt), \quad \widetilde{\psi}(t,k,\xi) = -\frac{\widetilde{\omega_0}(k,\xi+kt)}{k^2+|\xi|^2}. \tag{1.17}$$

We remark that the conclusions in the full nonlinear Theorem 1.1 are consistent with these explicit formulas. Indeed, if ω_0 is smooth, so $\widetilde{\omega_0}(k,\xi)$ decays fast in (k,ξ), then:

(i) The main contribution comes from the frequencies $\xi = -kt + O(1)$, therefore $\widetilde{\psi}(t,k,\xi)$ decays like $|k|^{-2}\langle t\rangle^{-2}$ if $k \neq 0$. Similarly, since $u^x = -\partial_y\psi$ and $u^y = \partial_x\psi$, we see that $\widetilde{u^x}$ decays like $|k|^{-1}\langle t\rangle^{-1}$ and $\widetilde{u^y}$ decays like $|k|^{-1}\langle t\rangle^{-2}$, as claimed in (1.15).

(ii) It can be seen from (1.17) that the functions $\omega(t,x,y)$ and $\psi(t,x,y)$ are not uniformly smooth as $t \to \infty$, in the coordinates x, y. To identify smooth "profiles," we need to make changes of coordinates, i.e., we define

$$z = x - tv, \quad v = y, \quad F(t,z,v) = \omega(t,x,y), \quad \phi(t,z,v) = \psi(t,x,y). \tag{1.18}$$

Notice that $F(t,z,v) = \omega_0(z,v)$ (independent of t), while $\phi(t,z,v)$ is uniformly smooth for all t provided that ω_0 is smooth. Taking the Fourier transform in z, v, we have the formula

$$\widetilde{\phi}(t,k,\xi) = -\frac{\widetilde{\omega_0}(k,\xi)}{k^2+|\xi-kt|^2}. \tag{1.19}$$

An important observation of Orr is that for $k \neq 0$ and large ξ, the normalized stream function ϕ (as well as the velocity field) experiences a *transient growth* as t approaches the "critical time" $t_c = \xi/k$ before decaying to zero. This can be seen easily from the formula (1.19). This transient growth on the linearized level turns out to be crucial for the nonlinear analysis as well, and leads to the high regularity assumptions (Gevrey spaces) that are required for the nonlinear perturbation theory.

1.2. Point vortices

Vortices (radial functions) are stationary solutions of the Euler equation in $\mathbb{R}^2$ in vorticity formulation (1.2). The stability of vortices is a major open problem for 2D Euler equations, which is challenging even at the linear level as shown in **[4]** in the case of radially decreasing vortices.

In **[24]** we initiated the rigorous study of the full nonlinear asymptotic stability problem for vortices of the Euler equation in $\mathbb{R}^2$. We consider the simplest class of vortices, called *point vortices*, which are δ-functions centered at points in $\mathbb{R}^2$. Such solutions (and more generally the so called N-vortex solutions) are models of general solutions with vorticity concentrated sharply in small neighborhoods, and have been studied by many authors. See, for instance, the classical work of Kirchhoff **[28]**, C. C. Lin **[29]**, and the book of Majda–Bertozi **[31]** for more references.

To state our main conclusions, consider solutions of the form

$$\text{vorticity field} = \kappa\,\delta\big(P(t)\big) + \omega, \quad \text{velocity field} = \nabla^{\perp}\Delta^{-1}\delta\big(P(t)\big) + u, \tag{1.20}$$

where $\kappa \in \mathbb{R}\backslash\{0\}$ is the strength of the point vortex, $\delta(P(t))$ is the Dirac mass centered at $P(t) = (P_1(t), P_2(t)) \in \mathbb{R}^2$. We assume that $P(t)$ is not in the support of ω, which will be satisfied as part of our analysis. Then the perturbation ω satisfies the equation

$$\partial_t\omega + U\cdot\nabla\omega + u\cdot\nabla\omega = 0, \quad \text{for } (x,y,t) \in \mathbb{R}^2\times[0,\infty), \tag{1.21}$$

where

$$U = \nabla^{\perp}\Delta^{-1}\delta\big(P(t)\big) = \frac{\kappa}{2\pi}\nabla^{\perp}\log\big|(x,y) - P(t)\big|. \tag{1.22}$$

The velocity field u and the stream function ψ are determined through

$$\begin{aligned} &u = \nabla^{\perp}\psi = (-\partial_y\psi, \partial_x\psi),\\ &\Delta\psi = \omega, \quad \lim_{|(x,y)|\to\infty}\Big\{\psi(x,y) - \frac{c_0}{2\pi}\log\big|(x,y)\big|\Big\} = 0, \end{aligned} \tag{1.23}$$

where

$$c_0 := \int_{\mathbb{R}^2}\omega(t,x,y)dxdy \tag{1.24}$$

is a constant preserved by the flow for all times (as long as the support of $\omega(t)$ is away from $P(t)$). In addition, the center $P(t)$ satisfies the transport ODE

$$P'(t) = \nabla^{\perp}\psi\big(t, P(t)\big). \tag{1.25}$$

Equations (1.21)–(1.25) can be derived rigorously when the vortex $P(t)$ lies outside of the support of $\omega(t)$, see, for example, [32]. In our case, this support condition is propagated dynamically by the flow, as a consequence of the proof of stability.

In [24] we prove axisymmetrization around a point vortex. More precisely, we prove that small, Gevrey smooth, and compactly supported perturbations symmetrize around the point vortex whose location changes in time and converges fast as $t \to \infty$.

Theorem 1.2. *Assume that $\kappa \in \mathbb{R}\backslash\{0\}$, $\lambda \in (0,\infty)$, $M \in (1,\infty)$, and $\omega_0 \in C_0^{\infty}(\mathbb{R}^2)$ satisfies the support property* $\operatorname{supp}\omega_0 \subseteq \{x \in \mathbb{R}^2 : |x| \in [1/M, M]\}$. *Assume that*

$$\int_{\mathbb{R}^2} e^{\lambda\langle\xi,\eta\rangle^{1/2}}\big|\widetilde{\omega_0}(\xi,\eta)\big|^2\,d\xi d\eta \le \varepsilon^2, \tag{1.26}$$

for a sufficiently small constant $\varepsilon \le \varepsilon(\kappa, M, \lambda)$, where $\widetilde{\omega_0}$ denotes the Fourier transform of ω_0. Then there is a unique smooth global solution (ω, P) of the system (1.21)–(1.25) such that $P(t)$ stays outside of the support of $\omega(t)$ for all $t \ge 0$. Moreover,

$$\big|P(t) - P_{\infty}\big| \lesssim \varepsilon\, e^{-c\langle t\rangle^{1/2}} \quad \textit{for all } t \ge 0, \tag{1.27}$$

for some $P_{\infty} \in \mathbb{R}^2$ and $c = c(\kappa, M, \lambda) > 0$, and the vorticity $\omega(t)$ converges weakly to a Gevrey-2 regular function $\omega_{\infty} \in C^{\infty}(\mathbb{R}^2)$ which is radial with respect to P_{∞}, as $t \to \infty$.

1.2.1. Adapted polar coordinates and precise results

To understand the mechanism of convergence in Theorem 1.2, we need to analyze the Euler equations in the polar coordinates, recentered around the moving point vortex $P(t)$. Let

$$(x, y) = P(t) + r(\cos\theta, \sin\theta). \tag{1.28}$$

In (r, θ) coordinates, we set the functions $u'_r, u'_\theta, \psi', \omega'$ as follows:

$$\begin{aligned} &\omega'(t,\theta,r) = \omega(t,x,y), \quad \psi'(t,\theta,r) = \psi(t,x,y), \\ &u'_r(t,\theta,r)e_r + u'_\theta(t,\theta,r)e_\theta = u(t,x,y), \end{aligned} \tag{1.29}$$

where $e_r := (\cos\theta, \sin\theta)$, $e_\theta := (-\sin\theta, \cos\theta)$. Equation (1.21) can be rewritten as

$$\partial_t\omega' - \big(P'(t), e_r\big)\partial_r\omega' - \frac{1}{r}\big(P'(t), e_\theta\big)\partial_\theta\omega' + \frac{\kappa}{2\pi r^2}\partial_\theta\omega' - \frac{\partial_\theta\psi'\partial_r\omega' - \partial_r\psi'\partial_\theta\omega'}{r} = 0, \tag{1.30}$$

where the stream function $\psi'(t, \theta, r)$ can be calculated through

$$\partial_r^2\psi' + \frac{1}{r}\partial_r\psi' + \frac{1}{r^2}\partial_\theta^2\psi' = \omega', \quad \lim_{r\to\infty}\left\{\psi'(t,r,\theta) - \frac{c_0}{2\pi}\log r\right\} = 0. \tag{1.31}$$

In the above,

$$P'(t) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}(\sin\theta, -\cos\theta)\,\omega'(t,\theta,r)d\theta dr, \tag{1.32}$$

and $(P'(t), e_r)$, $(P'(t), e_\theta)$ denote the scalar products between the vectors $P'(t)$, e_r, and e_θ. The velocity field (u'_θ, u'_r) can be calculated according to the formulas

$$u'_\theta(t,\theta,r) = \partial_r\psi', \quad u'_r(t,\theta,r) = -(1/r)\partial_\theta\psi'. \tag{1.33}$$

The following theorem is the full quantitative version of our main result in [24]:

Theorem 1.3. *Assume that $\beta_0, \vartheta_0 \in (0, 1/8]$, $\kappa \in (0, \infty)$, and assume ω'_0 is smooth initial data, satisfying the support condition* $\operatorname{supp}\omega'_0 \subseteq \mathbb{T}\times[\vartheta_0, 1/\vartheta_0]$ *and the smallness condition*

$$\|\omega'_0\|_{\mathcal{G}^{\beta_0,1/2}(\mathbb{T}\times\mathbb{R})} = \varepsilon \le \overline{\varepsilon}, \tag{1.34}$$

where $\overline{\varepsilon} = \overline{\varepsilon}(\beta_0, \vartheta_0, \kappa) > 0$ is sufficiently small and the Gevrey spaces $\mathcal{G}^{\beta_0,1/2}(\mathbb{T}\times\mathbb{R})$ are defined as in (1.4). We have the following conclusions:

(i) (global regularity) *There exist $\beta_1 = \beta_1(\beta_0, \vartheta_0, \kappa) > 0$ and a unique global solution $\omega' \in C([0,\infty) : \mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times\mathbb{R}))$ of the system (1.30)–(1.32) with initial data $\omega'(0) = \omega'_0$ such that* $\operatorname{supp}\omega'(t) \subseteq \mathbb{T}\times[\vartheta_0/2, 2/\vartheta_0]$ *and $|P(t)| < \vartheta_0/100$ for any $t \in [0,\infty)$.*

(ii) (asymptotic stability) *There exist $\Omega_\infty \in \mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times\mathbb{R})$ and $P_\infty = (P^1_\infty, P^2_\infty) \in \mathbb{R}^2$ with* $\operatorname{supp}\Omega_\infty \subseteq \mathbb{T}\times[\vartheta_0/2, 2/\vartheta_0]$ *and $|P_\infty| \le \vartheta_0/100$ such that*

$$\big\|\omega'\big(t, \theta + \kappa t/(2\pi r^2) + \Phi(t,r), r\big) - \Omega_\infty(\theta, r)\big\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times\mathbb{R})} \lesssim \varepsilon\langle t\rangle^{-1}, \tag{1.35}$$

$$\big|P(t) - P_\infty\big| \lesssim \varepsilon\, e^{-\beta_1 t^{1/2}}, \tag{1.36}$$

for any $t \geq 0$. Here

$$\Phi(t,r) := \int_0^t \frac{\langle u'_\theta \rangle(\tau,r)}{r}\, d\tau = \int_0^t \frac{\langle \partial_r \psi' \rangle(\tau,r)}{r} d\tau. \tag{1.37}$$

(iii) (control of the velocity field) *The velocity field u' satisfies the asymptotic bounds*

$$\big\| \langle u'_\theta \rangle(t,r) - u'_\infty(r) \big\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{R})} \lesssim \varepsilon \langle t \rangle^{-2}, \tag{1.38}$$

$$\langle t \rangle \big\| u'_\theta(t,\theta,r) - \langle u'_\theta \rangle(t,r) \big\|_{L^\infty(\mathbb{T}\times\mathbb{R})} + \langle t \rangle^2 \big\| u'_r(t,\theta,r) \big\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \lesssim \varepsilon, \tag{1.39}$$

where the function $u'_\infty \in \mathcal{G}^{\beta_1,1/2}(\mathbb{R})$ is defined by

$$\partial_r\big(r u'_\infty(r)\big) = r\Omega_\infty(r), \quad u'_\infty(r) = \begin{cases} 0 & \text{if } r \leq \vartheta_0/2, \\ c_0/(2\pi) & \text{if } r \geq 2/\vartheta_0. \end{cases}$$

1.2.2. Remarks

(1) We notice the similarities between Theorem 1.1 (in the Couette case $b(y) = y$) and Theorem 1.3. In the point vortex case, the inviscid damping is generated by the term $\frac{\kappa}{2\pi r^2}\partial_\theta \omega'$ in (1.30), $\kappa \neq 0$. Indeed, at the linearized level, equation (1.30) is

$$\partial_t \omega^{\rm lin} + \frac{\kappa}{2\pi r^2}\partial_\theta \omega^{\rm lin} = 0, \tag{1.40}$$

with the explicit solution

$$\omega^{\rm lin}(t,\theta,r) = \omega_0^{\rm lin}\big(\theta - \kappa t/(2\pi r^2), r\big). \tag{1.41}$$

Using now (1.31), we can express $\psi_k^{\rm lin}$, $k \in \mathbb{Z}\backslash\{0\}$, as

$$\psi_k^{\rm lin}(t,r) = \int_{\mathbb{R}} G_k(r,\rho)\omega_{0,k}^{\rm lin}(\rho) e^{-ik\kappa t/(2\pi\rho^2)}\, d\rho, \tag{1.42}$$

where $\psi_k^{\rm lin}$ and $\omega_{0,k}^{\rm lin}$ denote the kth Fourier modes of the functions $\psi^{\rm lin}$ and $\omega_0^{\rm lin}$ in θ and G_k is the associated Green function for the operator $\partial_r^2 + \partial_r/r - k^2/r^2$. These formulas and integration by parts in ρ lead to pointwise decay in time for the velocity field $u^{\rm lin} = (u_\theta^{\rm lin}, u_r^{\rm lin}) = (\partial_r \psi^{\rm lin}, -\partial_\theta \psi^{\rm lin}/r)$, consistent with the bounds (1.39). In other words, the main conclusions of Theorem 1.3 can be verified for the linearized flow as a consequence of the explicit formulas (1.41)–(1.42), as expected.

(2) The main difference between Theorems 1.1 and 1.3 comes from the global shift caused by the movement of the vortex $P(t)$. It is very important to prove that the point vortex stabilizes rapidly, according to (1.36), which gives just the right amount of decay to compensate for the loss of regularity caused by changes of variables and mixing.

(3) Finally, we note that the assumption that the point vortex lies outside the support of the perturbation is necessary for inviscid damping in Gevrey spaces. This is analogous to the "boundary effect" discussed earlier in the context of shear flows.

1.3. Organization

The rest of this paper is organized as follows: in Section 2 we discuss the main ideas in the proofs of Theorems 1.1 and 1.3. In Section 3 we discuss the limitations of the mechanism of inviscid damping, showing that it cannot be used to prove global regularity of solutions of the generalized SQG equations.

2. MAIN IDEAS

In this section we discuss some of the main ideas involved in the proofs of Theorems 1.1 and 1.3. Most of our discussion will be focused on the harder case of general monotonic shear flows, but some of the key ideas apply also in the case of point vortices.

2.1. Renormalization and the new equations

We introduce now a nonlinear change of variables and define the main quantities we need to control uniformly in time. We need to unwind the transportation in x. Assume that $\omega : [0,T] \times \mathbb{T} \times [0,1]$ is a sufficiently smooth solution of the system (1.10),

$$\begin{aligned} &\partial_t \omega + b(y)\partial_x \omega - b''(y)\partial_x \psi + u\cdot\nabla\omega = 0, \\ &(u^x, u^y) = (-\partial_y \psi, \partial_x \psi), \quad \Delta\psi = \omega, \quad \psi(t,x,1) = \psi(t,x,0) = 0, \end{aligned} \tag{2.1}$$

which is supported in $\mathbb{T} \times [\vartheta_0, 1-\vartheta_0]$ at all times $t \in [0,T]$, satisfying $\|\langle\omega\rangle(t)\|_{H^{10}} \ll 1$. We make the nonlinear change of variables

$$v = b(y) + \frac{1}{t}\int_0^t \langle u^x\rangle(\tau, y)\,d\tau, \quad z = x - tv. \tag{2.2}$$

The point of this change of variables is to eliminate two of the nondecaying terms in the evolution equation in (2.1), namely the terms $b(y)\partial_x\omega$ and $\langle u^x\rangle\partial_x\omega$. The change of variable $y \to v$ is crucial for our analysis, and it allows us to link the renormalized stream function ϕ to the profile F using the elliptic equation (2.7). The point is that this equation has constant coefficients at the linear level, so it is compatible with Fourier analysis.

Then we define the functions

$$F(t,z,v) := \omega(t,x,y), \quad \phi(t,z,v) := \psi(t,x,y), \tag{2.3}$$

$$V'(t,v) := \partial_y v(t,y), \quad V''(t,v) := \partial_{yy} v(t,y), \quad \dot{V}(t,v) := \partial_t v(t,y), \tag{2.4}$$

$$B'(t,v) := \partial_y b(y), \quad B''(t,v) := \partial_{yy} b(y). \tag{2.5}$$

The evolution equation in (2.1) becomes

$$\partial_t F - B''\partial_z\phi - V'\partial_v P_{\neq 0}\phi\,\partial_z F + (\dot{V} + V'\partial_z\phi)\,\partial_v F = 0, \tag{2.6}$$

where $P_{\neq 0}$ is projection off the zero mode, $P_{\neq 0}H(t,z,v) = H(t,z,v) - \langle H\rangle(t,v)$. The renormalized vorticity ϕ satisfies the elliptic-type equation

$$\partial_z^2\phi + (V')^2(\partial_v - t\partial_z)^2\phi + V''(\partial_v - t\partial_z)\phi = F, \tag{2.7}$$

The functions V', V'', B', B'', $\dot V$ also satisfy suitable evolution or elliptic equations in the new variables (t,v), which can be derived from (2.1) and the definitions, such as

$$\partial_t B'(t,v) + \dot V \partial_v B'(t,v) = \partial_t B''(t,v) + \dot V \partial_v B''(t,v) = 0, \tag{2.8}$$

$$\partial_t (V' - B') + \dot V \partial_v (V' - B') = \mathcal{H}/t, \tag{2.9}$$

$$\partial_t \mathcal{H} + \dot V \partial_v \mathcal{H} = -\mathcal{H}/t - V' \langle \partial_v P_{\neq 0}\phi \, \partial_z F\rangle + V' \langle \partial_z \phi \, \partial_v F\rangle, \tag{2.10}$$

where

$$\mathcal{H}(t,v) := tV'(t,v)\partial_v \dot V(t,v) = B'(t,v) - V'(t,v) - \langle F\rangle(t,v). \tag{2.11}$$

Equations (2.6)–(2.11) are the main equations we analyze in our proof.

2.2. Energy functionals and imbalanced weights

The main idea is to control the regularity of F for all $t \geq 0$, as well as other quantities such as ϕ, V', V'', B', B'', $\dot V$, using a bootstrap argument involving energy functionals and space-time norms. These norms depend on families of weights $A_k(t,\xi)$, $A_{NR}(t,\xi)$, $A_R(t,\xi)$, $k \in \mathbb{Z}$, $\xi \in \mathbb{R}$, which have to be designed carefully to control the nonlinearities.

To identify the main issue and motivate the choice of weights, assume first that F and ϕ satisfy the simplified closed system

$$\partial_t F - \partial_v P_{\neq 0}\phi \, \partial_z F = 0, \quad \partial_z^2 \phi + (\partial_v - t\partial_z)^2 \phi = F, \tag{2.12}$$

for $(z,v,t) \in \mathbb{T} \times \mathbb{R} \times [0,\infty)$. Compared to the original equations (2.6)–(2.7), we assume that $b'' \equiv 0$ (the Couette flow) and keep only one nonlinear term, the "reaction term" $\partial_v P_{\neq 0}\phi \cdot \partial_z F$. We would like to control, uniformly in time, an energy functional of the form

$$\mathcal{E}(t) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} A_k^2(t,\xi) \big|\tilde F(t,k,\xi)\big|^2 \, d\xi, \tag{2.13}$$

where $\tilde F$ denotes the spacial Fourier transform of F, for a suitable weight $A_k(t,\xi)$ which decreases in t. Let $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ and notice that

$$\widetilde{\partial_v P_{\neq 0}\phi}(t,k,\xi) = -\frac{i\xi}{k^2} \frac{\tilde F(t,k,\xi)}{1 + |t - \xi/k|^2} \mathbf{1}_{\mathbb{Z}^*}(k). \tag{2.14}$$

When $|\xi| \gg k^2$, the factor ξ/k^2 in (2.14) indicates a loss of one full derivative in v in the resonant region $\{(t,k,\xi) : |t - \xi/k| \ll |\xi|/k^2,\ k^2 \ll |\xi|\}$. This is a major obstruction to proving stability, which cannot be removed by standard symmetrization techniques.

The key original idea of Bedrossian–Masmoudi [7] is to use *imbalanced weights* $A_k(t,\xi)$ to absorb this derivative loss, taking advantage of the favorable structure of the nonlinearity that does not allow for contributions to the resonant region to come from bilinear interactions of small frequencies and frequencies in the resonant region (due to the factor $\partial_z F$ in the reaction term). More precisely, the weights have to satisfy the unusual property

$$\frac{A_\ell(t,\eta)}{A_k(t,\xi)} \approx \left|\frac{\eta}{\ell^2}\right| \frac{1}{1 + |t - \eta/\ell|}, \tag{2.15}$$

when $k \neq \ell, \ell \neq 0, \xi = \eta + O(1), k = \ell + O(1)$, and $1 + |t - \eta/\ell| \ll |\eta|/\ell^2$. In addition, these weights have to decrease in time, in the quantitative form,

$$-\frac{\partial_t A_\ell(t,\xi)}{A_\ell(t,\xi)} \gtrsim \frac{1}{\langle t - \xi/k\rangle}, \tag{2.16}$$

if $k \in \mathbb{Z}\backslash\{0\}$, $\langle t - \xi/k\rangle \lesssim |\xi|/k^2$, and $|\ell| \leq \langle\xi\rangle$, in order to be able to control some of the nonlinear terms using the Cauchy–Kowalevski terms coming from time differentiation of the energy functional $\mathcal{E}$. This leads to loss of regularity of the profile F during the evolution, which is the price to pay to prove nonlinear decay of the stream function ϕ.

2.2.1. The weights A_{NR}, A_R, A_k

For the sake of completeness, we summarize here the construction of our main imbalanced weights A_R, A_{NR}, A_k in [23–25]. Given $\delta_0 = \delta_0(\beta_0, \vartheta_0) > 0$, we define first the decreasing function $\lambda : [0,\infty) \to [\delta_0, 3\delta_0/2]$ by

$$\lambda(0) = \frac{3}{2}\delta_0, \quad \lambda'(t) = -\frac{\delta_0\sigma_0^2}{\langle t\rangle^{1+\sigma_0}}, \tag{2.17}$$

for small positive constant σ_0 (say $\sigma_0 = 0.01$). Then we define

$$A_R(t,\xi) := \frac{e^{\lambda(t)\langle\xi\rangle^{1/2}}}{b_R(t,\xi)} e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}, \quad A_{NR}(t,\xi) := \frac{e^{\lambda(t)\langle\xi\rangle^{1/2}}}{b_{NR}(t,\xi)} e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}, \tag{2.18}$$

$$A_k(t,\xi) := e^{\lambda(t)\langle k,\xi\rangle^{1/2}} \left(\frac{e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}}{b_k(t,\xi)} + e^{\sqrt{\delta}|k|^{1/2}}\right), \tag{2.19}$$

where $\delta > 0$ is a small constant and $k \in \mathbb{Z}$.

To construct the main functions b_k, b_{NR}, b_R that appear in (2.18)–(2.19), we start by defining two functions $w_{NR}, w_R : [0,\infty) \times \mathbb{R} \to [0,1]$, which distinguish between resonant and nonresonant regions and play a key role in the analysis. Resonance is measured in terms of the size of the denominators $\langle t - \xi/k\rangle$, which appear in formula (2.14). The intervals $I_{k,\eta}$ defined below, where this factor is small are called "resonant" intervals. Notice the imbalance in (2.24) between the weights $w_R(t,\eta)$ and $w_{NR}(t,\eta)$, especially around the center of the resonant intervals, consistent with the loss of derivative discussed earlier.

Assume that $\delta > 0$ is small, $\delta \ll \delta_0$. For $|\eta| \leq \delta^{-10}$, we define simply

$$w_{NR}(t,\eta) := 1, \quad w_R(t,\eta) := 1. \tag{2.20}$$

For $\eta > \delta^{-10}$, we define $k_0(\eta) := \lfloor\sqrt{\delta^3\eta}\rfloor$. For $l \in \{1,\ldots,k_0(\eta)\}$, we define

$$t_{l,\eta} := \frac{1}{2}\left(\frac{\eta}{l+1} + \frac{\eta}{l}\right), \quad t_{0,\eta} := 2\eta, \quad I_{l,\eta} := [t_{l,\eta}, t_{l-1,\eta}]. \tag{2.21}$$

Notice that $|I_{l,\eta}| \approx \eta/l^2$ and

$$\delta^{-3/2}\sqrt{\eta}/2 \leq t_{k_0(\eta),\eta} \leq \cdots \leq t_{l,\eta} \leq \eta/l \leq t_{l-1,\eta} \leq \cdots \leq t_{0,\eta} = 2\eta.$$

We define

$$w_{NR}(t,\eta) := 1, \; w_R(t,\eta) := 1 \quad \text{if } t \geq t_{0,\eta} = 2\eta. \tag{2.22}$$

Then we define, for $k \in \{1, \dots, k_0(\eta)\}$,

$$w_{NR}(t,\eta) := \begin{cases} \left(\frac{1+\delta^2|t-\eta/k|}{1+\delta^2|t_{k-1,\eta}-\eta/k|}\right)^{\delta_0} w_{NR}(t_{k-1,\eta},\eta) & \text{if } t \in [\eta/k, t_{k-1,\eta}], \\ \left(\frac{1}{1+\delta^2|t-\eta/k|}\right)^{1+\delta_0} w_{NR}(\eta/k,\eta) & \text{if } t \in [t_{k,\eta}, \eta/k]. \end{cases} \tag{2.23}$$

We define also the weight w_R by the formula

$$w_R(t,\eta) := \begin{cases} w_{NR}(t,\eta)\frac{1+\delta^2|t-\eta/k|}{1+\delta^2\eta/(8k^2)} & \text{if } |t-\eta/k| \le \eta/(8k^2), \\ w_{NR}(t,\eta) & \text{if } t \in I_{k,\eta},\ |t-\eta/k| \ge \eta/(8k^2), \end{cases} \tag{2.24}$$

for any $k \in \{1, \dots, k_0(\eta)\}$ and notice that for $t \in I_{k,\eta}$,

$$\frac{\partial_t w_{NR}(t,\eta)}{w_{NR}(t,\eta)} \approx \frac{\partial_t w_R(t,\eta)}{w_R(t,\eta)} \approx \frac{\delta^2}{1+\delta^2|t-\eta/k|}. \tag{2.25}$$

For small values of $t = (1-\beta)t_{k_0(\eta),\eta}$, $\beta \in [0,1]$, we define w_{NR} and w_R by the formulas

$$w_{NR}(t,\eta) = w_R(t,\eta) := \left(e^{-\delta\sqrt{\eta}}\right)^{\beta} w_{NR}(t_{k_0(\eta),\eta},\eta)^{1-\beta}. \tag{2.26}$$

If $\eta < -\delta^{-10}$, then we define $w_R(t,\eta) := w_R(t,|\eta|)$, $w_{NR}(t,\eta) := w_{NR}(t,|\eta|)$, and $I_{k,\eta} := I_{-k,-\eta}$. To summarize, the resonant intervals $I_{k,\eta}$ are defined for $(k,\eta) \in \mathbb{Z}\times\mathbb{R}$ satisfying $|\eta| > \delta^{-10}$, $1 \le |k| \le \sqrt{\delta^3|\eta|}$, and $\eta/k > 0$.

Finally, we define the weights $w_k(t,\eta)$ by the formula

$$w_k(t,\eta) := \begin{cases} w_{NR}(t,\eta) & \text{if } t \notin I_{k,\eta}, \\ w_R(t,\eta) & \text{if } t \in I_{k,\eta}. \end{cases} \tag{2.27}$$

If particular, $w_k(t,\eta) = w_{NR}(t,\eta)$ unless $|\eta| > \delta^{-10}$, $1 \le |k| \le \sqrt{\delta^3|\eta|}$, $\eta/k > 0$, and $t \in I_{k,\eta}$.

The functions w_{NR}, w_R, and w_k have the right size but lack optimal smoothness in the frequency parameter η, mainly due to the jump discontinuities of the function $k_0(\eta)$. This smoothness is important in symmetrization arguments (energy control of the transport terms) and in commutator arguments. To correct this problem, we fix $\varphi : \mathbb{R} \to [0,1]$ an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$, and let $d_0 := \int_{\mathbb{R}} \varphi(x)\,dx$. For $k \in \mathbb{Z}$ and $Y \in \{NR, R, k\}$, let

$$\begin{aligned} b_Y(t,\xi) &:= \int_{\mathbb{R}} w_Y(t,\rho)\varphi\left(\frac{\xi-\rho}{L_{\delta'}(t,\xi)}\right)\frac{1}{d_0 L_{\delta'}(t,\xi)}\,d\rho, \\ L_{\delta'}(t,\xi) &:= 1 + \frac{\delta'\langle\xi\rangle}{\langle\xi\rangle^{1/2}+\delta' t}, \qquad \delta' \in [0,1]. \end{aligned} \tag{2.28}$$

The length $L_{\delta'}(t,\xi)$ in (2.28) is chosen to optimize the smoothness in ξ of the functions $b_Y(t,\cdot)$, while not changing significantly the size of the weights. The parameter δ' is fixed sufficiently small, depending only on δ.

These definitions can be used to prove the key properties (2.15)–(2.16), as well as many other properties needed in the nonlinear analysis. We notice also that

$$\begin{aligned} e^{\lambda(t)\langle\xi\rangle^{1/2}} &\le A_{NR}(t,\xi) \le A_R(t,\xi) \le e^{\lambda(t)\langle\xi\rangle^{1/2}} e^{2\sqrt{\delta}\langle\xi\rangle^{1/2}}, \\ e^{\lambda(t)\langle k,\xi\rangle^{1/2}} &\le A_k(t,\xi) \le 2e^{\lambda(t)\langle k,\xi\rangle^{1/2}} e^{2\sqrt{\delta}\langle k,\xi\rangle^{1/2}}, \end{aligned} \tag{2.29}$$

for any $k \in \mathbb{Z}$, $t \geq 0$, and $\xi \in \mathbb{R}$. Finally, to prove commutator estimates in the context of our problem, we need to know that the weights vary sufficiently slowly in ξ. In our case the weights satisfy the key inequalities

$$\left|A_k(t,\xi) - A_k(t,\eta)\right| \lesssim \left[\frac{C(\delta)}{\langle k,\xi\rangle^{1/2}} + \sqrt{\delta}\right] \max\left\{A_k(t,\xi), A_k(t,\eta)\right\} \tag{2.30}$$

if $\langle \xi - \eta\rangle \lesssim 1 \ll \min\{\langle k,\xi\rangle, \langle k,\eta\rangle\}$. Such bounds are suitable to control the commutators by letting δ small enough, due to the gain of $\sqrt{\delta}$ at large frequencies.

2.3. The auxiliary nonlinear profile

In the case of general shear flows, an essential new difficulty that is not present in the Couette case, is the additional linear term $B''\partial_z\phi$ in (2.6). This linear term cannot be treated as a perturbation if b'' is not assumed small. On the linearized level, one can understand the evolution by using spectral analysis, especially the regularity analysis of generalized eigenfunctions corresponding to the continuous spectrum. However, it is still a challenge to combine the linear spectral analysis with the more sophisticated Fourier analysis tools needed for controlling the nonlinearity. We deal with this basic issue in two steps: first, we define an auxiliary nonlinear profile $F^*(t)$ given by

$$F^*(t,z,v) = F(t,z,v) - \int_0^t B''(0,v)\partial_z\phi'(s,z,v)\,ds. \tag{2.31}$$

Thus F^* takes into account the linear effect accumulated up to time t and can be bounded perturbatively, using the methods outlined in the previous subsection. The function ϕ' is a small but crucial modification of ϕ, defined as the unique solution to the elliptic equation

$$\begin{aligned} &\partial_z^2\phi' + (B_0')^2(\partial_v - t\partial_z)^2\phi' + B_0''(\partial_v - t\partial_z)\phi' = F,\\ &\phi'\big(t,b(0)\big) = \phi'\big(t,b(1)\big) = 0, \end{aligned} \tag{2.32}$$

on $\mathbb{T} \times [b(0), b(1)]$. This equation is obtained by freezing the coefficients of the main elliptic equation (2.7) at time $t = 0$ to gain additional smoothness.

On a heuristic level, we expect that the full evolution of F consists of two contributions: the main, linear evolution that changes the size of the profile most significantly, and a small but rough (compared with the linear evolution) nonlinear correction. We can view (2.31) as a bounded linear transformation in both space and time from F to F^* which takes into account the bulk linear evolution. The key point is that this transformation can be inverted to get bounds on the full profile F from bounds on F^*.

2.4. Control of the full profile

We still need to recover the bounds on F and the improved bounds on $F - F^*$. This is a critical step where we need to use our main spectral assumption and the precise estimates on the linearized flow. To link $F - F^*$ with the linearized flow, we define an auxiliary function ϕ^*, which can be approximately viewed as a stream function associated with F^*, and set $g = F - F^*$, $\varphi := \phi' - \phi^*$. The functions g and φ satisfy the inhomogeneous linear system with trivial initial data

$$\begin{aligned} &\partial_t g - B_0''(v)\partial_z\varphi = H, \quad g(0,z,v) = 0, \\ &B_0'(v)^2(\partial_v - t\partial_z)^2\varphi + B_0''(v)(\partial_v - t\partial_z)\varphi + \partial_z^2\varphi = g(t,z,v), \end{aligned} \tag{2.33}$$

where $(t,z,v) \in [0,\infty) \times \mathbb{T} \times [b(0),b(1)]$. The functions $B_0'(v) = B'(0,v)$ and $B_0''(v) = B''(0,v)$ are time-independent, very smooth, and can be expressed in terms of the original shear flow b. The source term H is given by $H = B_0''(v)\partial_z\phi^*$.

The function ϕ^* is determined by the auxiliary profile F^*. Since we have already proved quadratic bounds on the profile F^*, we can use elliptic estimates to prove quadratic bounds on ϕ^*, and then on the source term H. Therefore, we can think of (2.33) as a linear inhomogeneous system with trivial initial data, and adapt the linear theory to our situation.

Decomposing in modes, conjugating by e^{-ikvt}, and using Duhamel's formula, we can further reduce to the study of the homogeneous initial-value problem

$$\begin{aligned} &\partial_t g_k + ikvg_k - ikB_0''\varphi_k = 0, \quad g_k(0,v) = X_k(v)e^{-ikav}, \\ &(B_0')^2\partial_v^2\varphi_k + B_0''(v)\partial_v\varphi_k - k^2\varphi_k = g_k, \quad \varphi_k\big(b(0)\big) = \varphi_k\big(b(1)\big) = 0. \end{aligned} \tag{2.34}$$

for $(t,v) \in [0,\infty) \times [b(0),b(1)]$, where $k \in \mathbb{Z} \setminus \{0\}$ and $a \in \mathbb{R}$.

2.5. Analysis of the linearized flow

Equation (2.34) was analyzed, at least when $a = 0$, by Wei–Zhang–Zhao in **[45]** and by the second author in **[26]**. We follow the approach in **[26]**. The main idea is to use the spectral representation formula and reduce the analysis of the linearized flow to the analysis of generalized eigenfunctions corresponding to the continuous spectrum.

More precisely, using general spectral theory, we can express the stream function as an oscillatory integral of the spectral density function (which depends both on the physical and the spectral variables). As a consequence, given data X_k smooth and satisfying $\operatorname{supp} X_k \subseteq [b(\vartheta_0), b(1-\vartheta_0)]$ we find a representation formula

$$\begin{aligned} \widetilde{g_k}(t,\xi) &= \widetilde{X_k}(\xi + kt + ka) \\ &\quad + ik\int_0^t\int_{\mathbb{R}} \widetilde{B_0''}(\zeta)\widetilde{\Pi_k'}(\xi + kt - \zeta - k\tau, \xi + kt - \zeta, a)\,d\zeta\,d\tau \end{aligned} \tag{2.35}$$

for the solution g_k of the linear evolution equation (2.34), where $\Pi_k'(\xi,\eta,a)$ can be expressed in terms of a family of generalized eigenfunctions. As proved in **[26]**, these eigenfunctions cannot be calculated explicitly, but can be estimated very precisely in the Fourier space,

$$\big\|\big(|k| + |\xi|\big)W_k(\eta + ka)\widetilde{\Pi_k'}(\xi,\eta,a)\big\|_{L^2_{\xi,\eta}} \lesssim_\delta \big\|W_k(\eta)\widetilde{X_k}(\eta)\big\|_{L^2_\eta}, \tag{2.36}$$

for any $a \in \mathbb{R}$, for a large family of weights W_k that satisfy a slow variation property similar to (2.30). This leads to suitable control on the functions $g_k = F_k - F_k^*$, which allows us to close the bootstrap argument.

2.6. Energy functionals and the bootstrap proposition

We are now ready to summarize our main argument: given a solution $\omega : [0,T] \times \mathbb{T} \times [0,1] \to \mathbb{R}$ of equation (2.1), we define first the functions $F, \phi, V', V'', \dot{V}, B', B'', \mathcal{H}$ as in (2.3)–(2.5) and (2.11). To construct useful energy functionals, we need to modify the functions V', B', B'' which are not "small," so we define the new variables

$$B_0'(v) := B'(0,v) = (\partial_y b)\big(b^{-1}(v)\big), \quad B_0''(v) := B''(0,v) = (\partial_y^2 b)\big(b^{-1}(v)\big),$$
$$V_*' := V' - B_0', \quad B_*' := B' - B_0', \quad B_*'' := B'' - B_0''. \tag{2.37}$$

Our main goal is to control the functions F and ϕ. For this we need to consider two auxiliary functions F^* and ϕ', defined as in (2.31)–(2.32). Then we define the renormalized elliptic profiles

$$\Theta(t,z,v) := \big(\partial_z^2 + (\partial_v - t\partial_z)^2\big)\big(\Psi(v)\,\phi(t,z,v)\big),$$
$$\Theta^*(t,z,v) := \big(\partial_z^2 + (\partial_v - t\partial_z)^2\big)\big(\Psi(v)\,\big(\phi(t,z,v) - \phi'(t,z,v)\big)\big), \tag{2.38}$$

where $\Psi : \mathbb{R} \to [0,1]$ is a Gevrey class cut-off function, satisfying

$$\|e^{\langle\xi\rangle^{3/4}}\widetilde{\Psi}(\xi)\|_{L^\infty} \lesssim 1,$$
$$\operatorname{supp}\Psi \subseteq \big[b(\vartheta_0/4), b(1-\vartheta_0/4)\big], \quad \Psi \equiv 1 \text{ in } \big[b(\vartheta_0/3), b(1-\vartheta_0/3)\big]. \tag{2.39}$$

Our bootstrap argument is based on controlling simultaneously energy functionals and space-time integrals. For this we need carefully chosen weights A_{NR}, A_R, and A_k, defined as in Section 2.2.1. Let $\dot{A}_Y(t,\xi) := (\partial_t A_Y)(t,\xi) \le 0$, $Y \in \{NR, R, k\}$, and define, for any $t \in [0,T]$,

$$\mathcal{E}_f(t) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} A_k^2(t,\xi)\big|\tilde{f}(t,k,\xi)\big|^2\,d\xi, \quad f \in \{F, F^*\},$$
$$\mathcal{B}_f(t) := \int_1^t \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \big|\dot{A}_k(s,\xi)\big|A_k(s,\xi)\big|\tilde{f}(s,k,\xi)\big|^2\,d\xi ds, \tag{2.40}$$

$$\mathcal{E}_{F-F^*}(t) := \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}} \big(1 + \langle k,\xi\rangle/\langle t\rangle\big)A_k^2(t,\xi)\big|\widetilde{(F-F^*)}(t,k,\xi)\big|^2\,d\xi,$$
$$\mathcal{B}_{F-F^*}(t) := \int_1^t \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}} \big(1 + \langle k,\xi\rangle/\langle s\rangle\big)\big|\dot{A}_k(s,\xi)\big|A_k(s,\xi)\big|\widetilde{(F-F^*)}(s,k,\xi)\big|^2\,d\xi ds, \tag{2.41}$$

$$\mathcal{E}_\Phi(t) := \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}} A_k^2(t,\xi)\frac{|k|^2\langle t\rangle^2}{|\xi|^2 + |k|^2\langle t\rangle^2}\big|\widetilde{\Phi}(t,k,\xi)\big|^2\,d\xi, \quad \Phi \in \{\Theta, \Theta^*\},$$
$$\mathcal{B}_\Phi(t) := \int_1^t \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}} \big|\dot{A}_k(s,\xi)\big|A_k(s,\xi)\frac{|k|^2\langle s\rangle^2}{|\xi|^2 + |k|^2\langle s\rangle^2}\big|\widetilde{\Phi}(s,k,\xi)\big|^2\,d\xi ds, \tag{2.42}$$

$$\mathcal{E}_g(t) := \int_{\mathbb{R}} A_R^2(t,\xi)\big|\tilde{g}(t,\xi)\big|^2\,d\xi, \quad g \in \{V_*', B_*', B_*''\},$$
$$\mathcal{B}_g(t) := \int_1^t \int_{\mathbb{R}} \big|\dot{A}_R(s,\xi)\big|A_R(s,\xi)\big|\tilde{g}(s,\xi)\big|^2\,d\xi ds, \tag{2.43}$$

$$\mathcal{E}_{\mathcal{H}}(t) := \mathcal{K}^2 \int_{\mathbb{R}} A_{NR}^2(t,\xi)\big(\langle t\rangle/\langle\xi\rangle\big)^{3/2}\big|\tilde{\mathcal{H}}(t,\xi)\big|^2\,d\xi,$$
$$\mathcal{B}_{\mathcal{H}}(t) := \mathcal{K}^2 \int_1^t \int_{\mathbb{R}} \big|\dot{A}_{NR}(s,\xi)\big| A_{NR}(s,\xi)\big(\langle s\rangle/\langle\xi\rangle\big)^{3/2}\big|\tilde{\mathcal{H}}(s,\xi)\big|^2\,d\xi ds, \tag{2.44}$$

where $\mathbb{Z}^* := \mathbb{Z}\setminus\{0\}$ and $\mathcal{K} \geq 1$ is a large constant that depends only on δ.

Our main bootstrap proposition is the following:

Proposition 2.1. *Assume $T \geq 1$ and $\omega \in C([0,T] : \mathcal{G}^{2\delta_0,1/2})$ is a sufficiently smooth solution of the system (2.1), with the property that $\omega(t)$ is supported in $\mathbb{T}\times[\vartheta_0, 1-\vartheta_0]$ and that $\|\langle\omega\rangle(t)\|_{H^{10}} \ll 1$ for all $t \in [0,T]$. Define F, F^*, Θ, Θ^* B'_*, B''_*, V'_*, $\mathcal{H}$ as above. Assume that ε_1 is sufficiently small (depending on δ),*

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,V'_*,B'_*,B''_*,\mathcal{H}\}} \mathcal{E}_g(t) \leq \varepsilon_1^3 \quad \textit{for any } t \in [0,1], \tag{2.45}$$

and

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,V'_*,B'_*,B''_*,\mathcal{H}\}} \big[\mathcal{E}_g(t) + \mathcal{B}_g(t)\big] \leq \varepsilon_1^2 \quad \textit{for any } t \in [1,T]. \tag{2.46}$$

Then for any $t \in [1,T]$, we have the improved bounds

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,V'_*,B'_*,B''_*,\mathcal{H}\}} \big[\mathcal{E}_g(t) + \mathcal{B}_g(t)\big] \leq \varepsilon_1^2/2. \tag{2.47}$$

Moreover, we also have the stronger bounds for $t \in [1,T]$, namely

$$\sum_{g\in\{F,\Theta\}} \big[\mathcal{E}_g(t) + \mathcal{B}_g(t)\big] \lesssim_\delta \varepsilon_1^3. \tag{2.48}$$

This proposition is the main ingredient in the proof of Theorem 1.1 in [25]. Its proof is based on implementing the steps outlined in Sections 2.2–2.5. It is important to control not only the main variables F, Θ, F^* and Θ^*, but also the variables V'_*, B'_*, and B''_* which are connected to the change of variables $y \to v$. These variables appear in many nonlinear terms, so it is important to control their smoothness precisely, as part of a combined bootstrap argument, in a way that is consistent with the smoothness of the functions F and Θ.

The function $\mathcal{H}$ plays a different role, as it is the only variable that decays in time and encodes the convergence of the system as $t \to \infty$. This function decays at a rate of $\langle t\rangle^{-3/4}$, in a weaker topology, which shows that the function $\partial_v \dot{V}$ decays fast at an integrable rate of $\langle t\rangle^{-7/4}$, again in a weaker topology. We remark also that the bootstrap control on the variable $F - F^*$ is slightly stronger than on the variables F and F^* separately, which is needed to compensate for the lack of symmetry in some of the transport terms.

3. AN UNSTABLE MODEL: THE GENERALIZED SQG EQUATION

We consider now the generalized surface quasigeostrophic equations (gSQG)

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, & (t,x) \in [0,T)\times\mathcal{D}, \\ u = -\nabla^\perp(-\Delta)^{-1+\alpha/2}\theta, \end{cases} \tag{3.1}$$

where $\alpha \in [0,2]$ and $\mathcal{D}$ is a domain in $\mathbb{R}^2$. The case $\alpha = 1$ corresponds to the surface quasi-geostrophic (SQG) equation, introduced by Constantin–Majda–Tabak [13] as a model for the full 3D Euler equations. Notice that the case $\alpha = 0$ corresponds to the 2D incompressible Euler equations and the case $\alpha = 2$ produces stationary solutions.

These are the so-called active scalar equations, which have been analyzed extensively both in the setting of smooth solutions θ and in the setting of the so-called α-patches, which are solutions for which θ is a step function. The local regularity theory is generally well understood: as expected, suitable initial data lead to local in time unique solutions that propagate the regularity of the initial data, both in the smooth and the patch setting (see, for example, [13,19,22,39] for regularity results of this type).

The construction of nontrivial global solutions for the gSQG equations is a very challenging open problem for all parameters $\alpha \in (0,2)$, both in the smooth and in the patch case (the construction of solutions that blow up in finite time is also a challenging open problem, but we will not discuss it here). In fact, the only known nonstationary global solutions of finite energy, both in the smooth and the patch setting, are special rotating solutions, periodic in time. See the recent work [11] for the construction of such solutions in the harder smooth case, and more references. See also [14] for the construction of a stable class of global solutions in the patch case, using the mechanism of dispersion, but which have infinite energy.

It is tempting to try to use the mechanism of inviscid damping to construct families of nontrivial global solutions of the gSQG equations, at least for some parameters $\alpha \in (0,2)$, by perturbing around stationary solutions. The easiest would be to perturb around shear flows on the finite channel domain $\mathcal{D} = \mathbb{T} \times [0,1]$, in particular around the Couette flow corresponding to $\theta(t,x,y) = -1$. The fractional Laplacian $(-\Delta)^{-1+\alpha/2}$ on the domain $\mathcal{D} = \mathbb{T} \times [0,1]$ can be defined using explicit spectral theory. The vorticity deviation $\omega = \theta + 1 : [0,T] \times \mathbb{T} \times [0,1] \to \mathbb{R}$ satisfies the system

$$\begin{aligned} &\partial_t \omega + \partial_y a(y) \partial_x \omega - \partial_y \psi \partial_x \omega + \partial_x \psi \partial_y \omega = 0, \\ &\psi = -(-\Delta)^{-1+\alpha/2} \omega, \quad \psi(t,x,1) = \psi(t,x,0) = 0, \end{aligned} \tag{3.2}$$

where $a = a(y)$ is given by $(-\partial_y^2)^{1-\alpha/2} a(y) = -1$, $a(0) = a(1) = 0$. Notice that if $\alpha = 0$ this is the same as the Euler equation (2.1) for the Couette flow $b(y) = y - 1/2$, as expected.

At first glance it seems plausible to adapt the ideas described in Sections 2.1–2.2 to prove global regularity of the system (3.2), at least for some $\alpha > 0$ small. One can still perform a nonlinear change of variables and derive a system of equations for a profile F, as in Section 2.1. A simplified version of this system is the closed equation

$$\partial_t F - \partial_v P_{\neq 0} \Phi \, \partial_z F = 0, \quad \widetilde{P_{\neq 0} \Phi}(t,k,\xi) = \frac{\tilde{F}(t,k,\xi)}{[k^2 + (\xi - tk)^2]^{1-\alpha/2}} \mathbf{1}_{\mathbb{Z}^*}(k) \tag{3.3}$$

for the smooth function $F : [0,T] \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, which is analogous to the simplified equation (2.12) considered in Section 2.2.

Surprisingly, our analysis (in collaboration also with Javier Gómez-Serrano) reveals that the system (3.3) is unstable, for any $\alpha > 0$. To see this, let

$$\begin{aligned} \mathcal{E}_F(t) &:= \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}} W_k^2(t,\xi) \big|\tilde{F}(t,k,\xi)\big|^2 \, d\xi, \\ \mathcal{B}_F(t) &:= \sum_{k\in\mathbb{Z}^*} \int_0^t \int_{\mathbb{R}} \big|\dot{W}_k(s,\xi)\big| W_k(s,\xi) \big|\tilde{F}(s,k,\xi)\big|^2 \, d\xi, \end{aligned} \tag{3.4}$$

where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. We will show below that it is not possible to find a family of weights W_k, decreasing in t and compatible with nonlinear analysis, for which one could control the energy functional $\mathcal{E}_F$ for uniformly all times.

Indeed, we calculate

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_F(t) &= \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}} 2\dot{W}_k(t,\xi) W_k(t,\xi) \big|\tilde{F}(t,k,\xi)\big|^2 \, d\xi \\ &\quad + 2\Re \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}} W_k^2(t,\xi) \partial_t \tilde{F}(t,k,\xi) \overline{\tilde{F}(t,k,\xi)} \, d\xi. \end{aligned} \tag{3.5}$$

Therefore, since $\partial_t W_k \le 0$, for any $t \in [1,T]$, we have

$$\mathcal{E}_F(t) + 2\mathcal{B}_F(t) = \mathcal{E}_F(0) + \int_0^t \Big\{ 2\Re \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}} W_k^2(s,\xi) \partial_s \tilde{F}(s,k,\xi) \overline{\tilde{F}(s,k,\xi)} \, d\xi \Big\} ds. \tag{3.6}$$

Using equation (3.3), the cubic term on the right-hand side of (3.6) is equal to

$$\begin{aligned} &C \left| 2\Re \left\{ \sum_{k,\ell\in\mathbb{Z}^*} \int_0^t \int_{\mathbb{R}^2} W_k^2(s,\xi) i\eta \widetilde{\Phi}(s,\ell,\eta) i(k-\ell) \tilde{F}(s,k-\ell,\xi-\eta) \overline{\tilde{F}(s,k,\xi)} \, d\xi d\eta ds \right\} \right| \\ &= C \left| 2\Re \left\{ \sum_{k,\ell\in\mathbb{Z}^*} \int_0^t \int_{\mathbb{R}^2} W_k^2(s,\xi) \frac{\eta \tilde{F}(s,\ell,\eta) \overline{\tilde{F}(s,k,\xi)}}{[\ell^2 + (\eta - s\ell)^2]^{1-\alpha/2}} (k-\ell) \right.\right. \\ &\qquad \left.\left. \times \tilde{F}(s,k-\ell,\xi-\eta) \, d\xi d\eta ds \right\} \right| \\ &= C \left| \sum_{k,\ell\in\mathbb{Z}^*} \int_0^t \int_{\mathbb{R}^2} \left[\frac{\eta W_k^2(s,\xi)}{[\ell^2 + (\eta - s\ell)^2]^{1-\alpha/2}} - \frac{\xi W_\ell^2(s,\eta)}{[k^2 + (\xi - sk)^2]^{1-\alpha/2}} \right] \right. \\ &\qquad \left. \times \tilde{F}(s,\ell,\eta) \overline{\tilde{F}(s,k,\xi)} (k-\ell) \tilde{F}(s,k-\ell,\xi-\eta) \, d\xi d\eta ds \right|, \end{aligned} \tag{3.7}$$

where in the last identity we use symmetrization in (k,ξ) and (ℓ,η), based on the fact that F is real-valued.

We restrict ourselves to the range

$$\xi, \eta = N + O(1), \quad k = 2, \ell = 1, \tag{3.8}$$

where N is very large. This corresponds to the main "reaction term" in the original equation (3.3), where the frequency of Φ in the nonlinearity is large and the frequency of F is small.

To estimate the right-hand side of (3.6) using the bulk term $\mathcal{B}_F$ defined in (3.4), we need that the weights satisfy the inequality

$$\left|\frac{\eta W_2^2(s,\xi)}{[1+(\eta-s)^2]^{1-\alpha/2}}-\frac{\xi W_1^2(s,\eta)}{[4+(\xi-2s)^2]^{1-\alpha/2}}\right| \lesssim \sqrt{|\dot{W}_2(s,\xi)|W_2(s,\xi)}\sqrt{|\dot{W}_1(s,\eta)|W_1(s,\eta)}, \tag{3.9}$$

for all $\xi,\eta = N + O(1)$ and $s \in [0,\infty)$.

Assume that we are further restricting to a neighborhood of the largest resonant time $s = N + O(1)$. We notice that in this case the two terms on the left-hand side of (3.9) cannot have a meaningful cancelation because the denominator of the first term varies uniformly between 1 and C if ξ and s are fixed and $\eta = N + O(1)$, while all the other numerators and denominators vary much less. So we would need

$$\frac{\eta W_2^2(s,\xi)}{[1+(\eta-s)^2]^{1-\alpha/2}}+\frac{\xi W_1^2(s,\eta)}{[4+(\xi-2s)^2]^{1-\alpha/2}} \lesssim \sqrt{|\dot{W}_2(s,\xi)|W_2(s,\xi)}\sqrt{|\dot{W}_1(s,\eta)|W_1(s,\eta)},$$

for all $\xi,\eta,s = N + O(1)$. In other words, the symmetrization performed in (3.7) does not help in the resonant case $\xi,\eta,s = N + O(1)$. In particular, for all $\eta, s = N + O(1)$,

$$N W_2^2(s,\eta) + N^{-1+\alpha} W_1^2(s,\eta) \lesssim W_2(s,\eta)W_1(s,\eta)\sqrt{\frac{|\dot{W}_2(s,\eta)|}{W_2(s,\eta)}\frac{|\dot{W}_1(s,\eta)|}{W_1(s,\eta)}}. \tag{3.10}$$

Using the mean inequality twice, this can only be satisfied if

$$N^{\alpha/2} \lesssim \frac{|\dot{W}_2(s,\eta)|}{W_2(s,\eta)} + \frac{|\dot{W}_1(s,\eta)|}{W_1(s,\eta)}, \quad \text{if } s,\eta = N + O(1). \tag{3.11}$$

Unfortunately, it is not possible to find suitable weights that satisfy a bound like (3.11), for any $\alpha > 0$. This is because the weights also need to satisfy basic bounds like

$$W_k(s,\xi) \approx W_k(s,\eta) \tag{3.12}$$

for any $s \in [0,\infty)$, $k \in \{1,2\}$, and $\xi,\eta \in \mathbb{R}$, $|\xi-\eta| \le 1$. These bounds are essential in order for the weights to be compatible with nonlinear analysis. Letting $W_k(s,\xi) = e^{\lambda_k(s,\xi)}$, $k \in \mathbb{Z}$, and $\lambda = \lambda_1 + \lambda_2$, it follows from (3.11)–(3.12) that $\lambda : [0,\infty) \times \mathbb{R} \to [0,\infty)$ is a decreasing function in s satisfying

$$\langle \eta \rangle^{\alpha/2} \lesssim \left|(\partial_s \lambda)(s,\eta)\right|, \quad \left|\lambda(s,\eta) - \lambda(s,\xi)\right| \lesssim 1 \tag{3.13}$$

if $\eta \gg 1$, $|\eta - s| \le 2$, and $|\xi - \eta| \le 2$. We use these inequalities with $s = \eta = N \gg 1$ and recall that $\alpha > 0$ to see that

$$\lambda(N-1, N-1) \ge \lambda(N,N) + cN^{\alpha/2}. \tag{3.14}$$

We can then apply this inductively to conclude that $\lambda(N-n, N-n) \ge \lambda(N,N) + cnN^{\alpha/2}$ for $n = 1,\dots,N/2$. In particular, $\lambda(N/2,N/2) \ge cN^{1+\alpha/2}$, which would force $\lambda(0,N/2) \ge cN^{1+\alpha/2}$ (since λ is decreasing in s). However, this is not compatible with the bounds (3.12) when $s = 0$, giving the final contradiction.

Notice that most of this argument applies in the Euler case $\alpha = 0$, except that (3.13) does not imply (3.14) (in fact, our weights A_k constructed in Section 2.2.1 satisfy (3.13) but not (3.14)). To summarize, these calculations show that the main construction used in the proof of global stability of the Couette flow for the 2D Euler equations does not extend to any more singular generalized SQG equations.

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ALEXANDRU D. IONESCU

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA, aionescu@math.princeton.edu

HAO JIA

Department of Mathematics, University of Minnesota, 206 Church St. S.E., Minneapolis, MN 55455, USA, jia@umn.edu