GLOBAL DYNAMICS AROUND AND AWAY FROM SOLITONS

KENJI NAKANISHI

ABSTRACT

This article reviews some results, as well as open questions, on global behavior of general solutions for nonlinear dispersive equations, with an emphasis on transitions of solutions around solitons with respect to time evolution and initial perturbation.

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1. INTRODUCTION

Nonlinear dispersive equations describe space-time evolution of waves in various physical phenomena, which are governed mainly by dispersion and nonlinear interactions of waves. A representative example is the nonlinear Schrödinger equation (NLS)

$$i\dot{u} - \Delta u = \lambda |u|^{p-1}u, \quad u(t,x) : \mathbb{R}^{1+d} \to \mathbb{C}, \tag{1.1}$$

where $d \in \mathbb{N}$, $p > 1$, and $\lambda \in \mathbb{R}$ are constants. Depending on the balance or competition between the dispersion and interaction, which differs equation by equation, as well as the initial data, the solutions of each equation exhibit a wide range of behavior in space-time. The three major types of solutions are

- scattering solutions which are dominated by dispersion—spreading waves with decaying amplitude;
- blow-up solutions which are dominated by nonlinearity—focusing waves with diverging amplitude;
- solitons for which dispersion and nonlinearity are in balance to keep a fixed shape of the wave.

Most of the equations are in the Hamiltonian form. For example, NLS may be written as

$$\dot{u} = iE_S'(u), \quad E_S(u) := \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{2} - \frac{\lambda |u|^{p+1}}{p+1} dx, \tag{1.2}$$

where $E_S'(u)$ denotes the Fréchet derivative. The Hamiltonian or energy E_S is well defined on $H^1(\mathbb{R}^d)$ if the nonlinear part is controlled by the Sobolev inequality, namely $d \le 2$ or $p + 1 \le \frac{2d}{d-2} =: 2^\star$. Then it is natural to consider solutions in the energy space $H^1(\mathbb{R}^d)$, where the energy $E_S(u)$ is conserved.

Nonlinear dispersive equations have been intensively studied since the late 20th century, so that we have by now a fair amount of knowledge on the fundamental questions from the PDE viewpoint, such as the unique existence of local solutions with wide range of regularity, of solutions with typical behavior, as well as their qualitative and quantitative properties, including asymptotic profiles.

In this century, there has been more progress in the study on *large solutions for long time*, in which the dispersion and nonlinearity have stronger and more complicated interplay, generating more diverse solutions. It is, however, in most cases too difficult to look at all general solutions and their long-time behavior, as there are so many possibilities while our method of analysis is still quite limited. Then the solitons are the natural first target to attack among all the solutions, as they are expected to indicate the balance or the threshold of dominance between the dispersion and nonlinearity. The *soliton resolution conjecture* has been the major slogan to promote this direction of study, which roughly asserts that: *Generic global solutions are asymptotic to a superposition of solitons getting away from each other and a dispersive decaying wave as $t \to \infty$.* In the case of NLS (for appropriate p), the asymptotic formula should take the form

$$u(t) - \sum_{n=1}^{N} e^{i\theta_n(t)}\varphi_n\big(x - c_n(t)\big) - v(t) \to 0, \tag{1.3}$$

in the energy space $H^1(\mathbb{R}^d)$ as $t \to \infty$, for some soliton profiles $\varphi_n \in H^1(\mathbb{R}^d)$ with some $\theta_n : \mathbb{R} \to \mathbb{R}$ and $c_n : \mathbb{R} \to \mathbb{R}^d$ satisfying $|c_m(t) - c_n(t)| \to \infty$ for $m \neq n$, and some dispersive component $v(t,x)$ solving the free equation $i\dot{v} = \Delta v$.

On the one hand, the conjecture is a natural extension from the case of completely integrable equations (e.g., $d = 1$ and $p = 3$ for NLS), where solitons are very stable and rigid: they are unchanged both by initial perturbation and collisions, up to a change of the parameters. The genericity condition in the conjecture is to eliminate some exceptional solutions, such as breathers, which appear already in the integrable case.

On the other hand, most of the nonlinear dispersive equations are not integrable, where most of solitons are unstable with respect to initial perturbations. Although this instability makes it more difficult to capture and maintain the solitons in reality and numerics, it does not diminish the importance of solitons in the study of global dynamics, especially regarding the role of a threshold. In fact, in the space of solutions or initial data, typically in the energy space, instability means that the soliton is a limit point of other types of solutions, while stability means that there is no nearby solution with much different behavior. Hence unstable solitons are naturally expected to play more distinct roles in classifying the other solutions. Even if the solitons are unstable, the threshold between different types of solutions should be clearly observed both in numerics and experiments, as one looks at a collection of solutions rather than the individual ones. Such structures among solutions may well be stable and robust with respect to perturbations of the equation, even if the behavior of each solution is changed.

Therefore, in studying the global dynamics, it is not sufficient to know merely that a soliton is unstable, we should investigate in which directions the instability appears, and in what types of behavior of solutions. In other words, we should look at *all solutions in a neighborhood of solitons*. Note that stability is an answer to this question, but instability (negation of stability) may not be a complete answer by itself. Determining the stability is, of course, the most important starting point, which has a vast amount of literature, but there has been more recent progress in getting to the next stage.

Instability means that some solutions starting nearby a soliton become eventually very different or far from the soliton in the solution space. While those solutions are still near the soliton, their behavior may be well approximated by the linearized operator, which is well described in terms of its spectrum. However, after the solutions go far away from the soliton, which is often the case, then the linearized operator tells little about their behavior. To see the essential features of those solutions, and thus the threshold nature of the unstable soliton, it is necessary to look at *those solutions after they get far from the soliton.* The recent research is getting also into this stage of study.

Also in practice, the solutions at $t = \infty$ do not have so much meaning, but the asymptotic descriptions as $t \to \infty$ should be regarded as an approximation for what happens in finite time. However, if the solitons are unstable, the asymptotic decomposition into them is useless by itself for a finite-time approximation, since unstable solitons may keep disappearing and appearing along the evolution. Hence we should look at the behavior of solutions *not only as $t \to \infty$, but also for all intermediate $t \in \mathbb{R}$*. The oscillatory scenario is an obstruction also in studying the asymptotic behavior, but the investigation for all time is even more demanding. Nevertheless, the recent research is getting also into this stage.

In short, to investigate the global dynamics of nonlinear dispersive equations, it is desired to describe the solutions *for all time and for all initial data* in a neighborhood of unstable solitons. The main interest is on *transitions of behavior both in time evolution and for initial perturbations*. The purpose of this article is to review a few results in this direction, as well as open questions.

2. GROUND STATES AS THE DYNAMICAL THRESHOLD

Among all the solitons, the most important ones are those with the least energy, namely the ground states, as its energy is the necessary amount to produce the balance between the dispersion and nonlinearity. This article is mostly focused on the ground states and their variants, even though some of them will be called excited states. For a concrete explanation, we take the nonlinear Klein–Gordon equation (NLKG)

$$\ddot{u} - \Delta u + mu = |u|^{p-1}u, \quad u(t,x) : \mathbb{R}^{1+d} \to \mathbb{R}, \tag{2.1}$$

where $d \in \mathbb{N}$, $p > 1$, and $m \geq 0$ are constants. It is the Hamiltonian flow with the energy

$$E_K\big(\vec{u}(t)\big) := \int_{\mathbb{R}^d} \frac{|\dot{u}|^2 + |\nabla u|^2 + m|u|^2}{2} - \frac{|u|^{p+1}}{p+1} dx, \tag{2.2}$$

similar to NLS in (1.2); in the energy space

$$\vec{u}(t) := \big(u(t,x), \dot{u}(t,x)\big) \in \mathcal{H} := H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d). \tag{2.3}$$

In the case $m = 0$ of the nonlinear wave equation (NLW), H^1 should be replaced with the homogeneous Sobolev space $\dot{H}^1$. The ground state $Q \in H^2(\mathbb{R}^d)$ is a nontrivial stationary solution of

$$-\Delta Q + mQ = |Q|^{p-1}Q \tag{2.4}$$

with the least energy. Its study has a long history for the stationary equation and the evolution equations, including the NLS case and the heat equation. By the existence result of Strauss [56] and the uniqueness result of Kwong [37], the entire set of the ground states is $\{\pm Q(x-c)\}_{c \in \mathbb{R}^d}$ for a unique radial positive function $Q(x) = Q(|x|) > 0$. In the massless case (NLW), the Pohozaev identity [54] implies that the ground state exists if and only if $d \geq 3$ and the nonlinear power is $p + 1 = \frac{2d}{d-2} =: 2^\star$, namely the energy-critical exponent, and the ground state Q is the explicit Aubin–Talenti function [2,57], maximizing the Sobolev inequality for $\dot{H}^1(\mathbb{R}^d) \subset L^{2^\star}(\mathbb{R}^d)$.

2.1. Below the ground states

The instability of Q follows from its min–max property:

$$E_K(\vec{Q}) = \min_{\varphi \in H^1(\mathbb{R}^d)\setminus\{0\}} \max_{\lambda>0} E_K(\lambda\vec{\varphi})$$
$$= \min\{E_K(\vec{\varphi}) \mid \varphi \in H^1(\mathbb{R}^d) \setminus \{0\}, K(\varphi) = 0\}, \tag{2.5}$$

where $\vec{Q} := (Q, 0)$, and the Nehari functional [50] is defined by

$$K(\varphi) := \frac{d}{d\lambda}\bigg|_{\lambda=1} E_K(\lambda\vec{\varphi}) = \int_{\mathbb{R}^d} |\nabla u|^2 + m|u|^2 - |u|^{p+1} dx. \tag{2.6}$$

A similar characterization is given by using the dilation $\varphi(\lambda x)$, leading to Derrick's theorem [11]. Another option is the L^2-invariant scaling $\lambda^{d/2}\varphi(\lambda x)$, which yields the virial functional (see [28] for their relations including the dynamics). Thus the energy space below the ground state is split into two open sets,

$$\mathcal{H}_< := \{\varphi \in \mathcal{H} \mid E_K(\varphi) < E_K(\vec{Q})\} = \mathcal{H}_<^+ \cup \mathcal{H}_<^-,$$
$$\mathcal{H}_<^+ := \{\varphi \in \mathcal{H}_< \mid K(\varphi_1) \geq 0\}, \quad \mathcal{H}_<^- := \{\varphi \in \mathcal{H}_< \mid K(\varphi_1) < 0\}. \tag{2.7}$$

It is easy to see that $\mathcal{H}_<^+$ is bounded and $\mathcal{H}_<^-$ is unbounded. Since $E_K(\vec{u})$ is conserved and $\mathcal{H}_<^\pm$ are separated from each other, both regions $\mathcal{H}_<^\pm$ are invariant with respect to the NLKG flow. Then all the solutions in $\mathcal{H}_<^+$ are global in time as soon as the Cauchy problem is locally well posed in $\mathcal{H}$ with a uniform lower bound on the existence time (which is the case for $p + 1 < 2^\star$ by Ginibre–Velo [25]).

Payne–Sattinger [52] proved (in the bounded domain case) that all solutions in $\mathcal{H}_<^-$ blow up in finite time for NLKG, as well as for the heat equation. Thus all the solutions with the energy below the ground state are split into the cases of global existence and blow-up, as two disjoint open sets in $\mathcal{H}$, which are distinguished by the initial data explicitly by sign $K(u(0))$. The openness means that both properties are stable, and the ground states are the joint boundary of the two regions

$$\{\pm Q(x - c)\}_{c\in\mathbb{R}^d} = \overline{\mathcal{H}_<^+} \cap \overline{\mathcal{H}_<^-}. \tag{2.8}$$

More recently, Kenig–Merle [31, 32] proved this type of dichotomy in the energy-critical case $p + 1 = 2^\star$ ($d \geq 3$) for NLW, as well as for NLS (in the radial case), proving moreover that all the solutions in $\mathcal{H}_<^+$ are scattering as $t \to \pm\infty$, namely

$$\lim_{t\to\pm\infty} \|\vec{u}(t) - \vec{v}^\pm(t)\|_{\mathcal{H}} = 0 \tag{2.9}$$

for some $v^\pm$ solving the free equation $\ddot{v} - \Delta v = 0$. Their method has been applied to many other equations, including NLKG [28, 33] and NLS [12, 13, 21, 27, 34, 35] with the energy-(sub)critical and mass-(super)critical power, namely for $2 + \frac{4}{d} \leq p + 1 \leq 2^\star$.

2.2. Above the threshold

Although the dichotomy into scattering and blow-up is a very simple, explicit, and complete classification of the global dynamics, there seems to be no intrinsic reason in the equations to restrict the solutions below the ground states $E_K(\vec{u}) < E_K(\vec{Q})$, as those ground states are not local extrema, but rather saddle points for the energy. It also seems impossible to impose such a strict condition (inequality) both in numerical experiments and in physical ones. It is therefore more natural to impose a condition of the form

$$E_K(\vec{u}) < E_K(\vec{Q}) + \varepsilon \tag{2.10}$$

for some small $\varepsilon > 0$, which includes in particular a full neighborhood of the ground states.

As soon as the energy is above the ground state, however, the topological separation is lost between $K(u) > 0$ and $K(u) < 0$, or between the scattering and blow-up, which enables a transition between different types of behavior. One may expect that this transition could make the global dynamics very complicated and even chaotic, as it can possibly happen for many times. Nakanishi–Schlag [47, 49] showed that it is not the case, and the complication remains minimal for small $\varepsilon > 0$, at least for NLKG with $d = p = 3$, which has been extended to NLS and NLW in [36, 48]. That is because the transition is allowed only for one time for each solution from the scattering region to the blow-up region (or vice versa), taking place only in a small neighborhood of the ground states, and described well by the linearized equation around the ground states. The behavior of solutions away from the ground states is essentially the same as in the case below the ground state, in the sense that both the scattering and blow-up are characterized by monotonicity of the virial identity. Thus all the solutions with $E_K(\vec{u}) < E_K(\vec{Q}) + \varepsilon$ are classified into $9 = 3 \times 3$ sets of global behavior, depending whether it is scattering, blowing-up, or asymptotic to the ground states in $t > 0$ and $t < 0$. In the simple case of NLKG with $p = d = 3$ under the radial symmetry, the classification reads as follows. For any $\varphi \in \mathcal{H}$ and $X \subset \mathcal{H}$, let $\varphi^\dagger := (\varphi_1, -\varphi_2)$ and $X^\dagger := \{\varphi^\dagger \mid \varphi \in X\}$ denote the time inversion.

Theorem 2.1 ([47]). *Let $p = d = 3$, $m > 0$, and*

$$\mathcal{H}_{\varepsilon,r} := \{\varphi(x) = \varphi(|x|) \in \mathcal{H} \mid E_K(\varphi) < E_K(\vec{Q}) + \varepsilon\} \tag{2.11}$$

for $\varepsilon > 0$. If $\varepsilon > 0$ is small enough, then there is a C^1-manifold $\mathcal{M} \subset \mathcal{H}_{\varepsilon,r}$ of codimension 1 with the following properties: $\mathcal{H}_{\varepsilon,r} \setminus (\mathcal{M} \cup -\mathcal{M})$ is a union of two domains $\mathcal{S}$ and $\mathcal{B}$. Let u be any solution of (2.1) with $\vec{u}(0) \in \mathcal{H}_{\varepsilon,r}$. If $\vec{u}(0) \in \mathcal{S}$, then u is scattering as $t \to \infty$. If $\vec{u}(0) \in \mathcal{B}$, then u blows up in finite time for $t > 0$. If $\vec{u}(0) \in \mathcal{M}$, then $u - Q$ is scattering as $t \to \infty$. Moreover, $\mathcal{M}$ and $\mathcal{M}^\dagger$ intersect transversely, while $\mathcal{M}^\dagger \cap (-\mathcal{M}) = \emptyset$.

The transversal intersection of $\mathcal{M} \cap \mathcal{M}^\dagger$ implies that all the $9 = 3 \times 3$ combinations of behavior in $t > 0$ and $t < 0$ are nonempty. The above result clarifies the important role of the center-stable manifold $\mathcal{M}$ and the center-unstable manifold $\mathcal{M}^\dagger$ of the ground states $\pm Q$, which had been constructed by Bates–Jones [3], while the scattering on the manifolds to the ground states had been established by Schlag [55] and Beceanu [4] for NLS. The key ingredient for the above classification is the fact that the transition cannot happen more than once, which is called the one-pass theorem. It may be regarded as a small perturbation from the threshold dynamics on $E_K(\vec{u}) = E_K(\vec{Q})$, in particular the nonexistence of a homoclinic orbit for the ground states, which had been established by Duyckaerts–Merle [19] for the energy-critical NLS, and extended to other cases [18, 20]. More precisely, the one-pass theorem prohibits solutions from reentry into a small neighborhood of the ground states after escaping from there. If we distinguish between the positive ground states Q and the negative $-Q$, then the number of classification sets is $14 = 4 \times 4 - 2$, as follows:

$$\begin{aligned} &\mathcal{S}^\dagger \cap \mathcal{S}, \quad \mathcal{B}^\dagger \cap \mathcal{B}, \quad \pm(\mathcal{M}^\dagger \cap \mathcal{M}), \quad (\pm\mathcal{M}^\dagger) \cap \mathcal{S}, \quad (\pm\mathcal{M}^\dagger) \cap \mathcal{B}, \\ &\mathcal{S}^\dagger \cap (\pm\mathcal{M}), \quad \mathcal{B}^\dagger \cap (\pm\mathcal{M}), \quad \mathcal{S}^\dagger \cap \mathcal{B}, \quad \mathcal{B}^\dagger \cap \mathcal{S}, \end{aligned} \tag{2.12}$$

The subtraction of -2 from 4×4 is due to the absence of $(\pm\mathcal{M}^\dagger) \cap (\mp\mathcal{M})$, namely connecting orbits between Q and $-Q$, which is also precluded by the one-pass theorem.

2.3. Higher energy

The next question is what if the energy is much bigger than the ground state, namely $E_K(\vec{u}) > E_K(\vec{Q}) + \varepsilon$. Actually, the more general statement of the above result in the nonradial case [49], taking account of the Lorentz invariance and conserved momentum $P(\vec{u}) := \int_{\mathbb{R}^d} \dot{u}\nabla u dx$, is in a bigger region

$$\left[E_K(\vec{u})^2 - \left|P(\vec{u})\right|^2\right]^{1/2} < E_K(\vec{Q}) + \varepsilon \quad \left([z]^\alpha := |z|^{\alpha-1}z\right), \tag{2.13}$$

which includes the ground state solitons of any traveling speed (slower than the light), but the classification is essentially the same as above. The main interest is how and where the dynamics change essentially.

There are at least two obvious candidates for the next energy level. One is the other stationary solutions, namely the excited state, and the other is multisolitons. Note that the excited states have at least twice the ground state energy $E_K(\vec{u}) > 2E_K(\vec{Q})$, because they have to be sign-changing due to the uniqueness of positive solutions by Gigas–Ni–Nirenberg [24] and Kwong [37], then both the positive and negative parts must have more energy than $E_K(\vec{Q})$ due to the characterization (2.5). On the other hand, if a solution u is asymptotic to a sum of $N \in \mathbb{N}$ ground states moving away from each other, as in the soliton resolution conjecture, then $E_K(\vec{u}) \ge NE_K(\vec{Q})$, where the equality $E_K(\vec{u}) = NE_K(\vec{Q})$ happens only if the asymptotic speeds of the solitons are all zero. Asymptotic multisolitons were constructed by Martel–Merle [39] for NLS with positive speeds in the stable case, which has been extended to the unstable case [6], as well as to NLKG [9], and with zero speed for NLS [51] and NLKG [1]. Therefore, in view of the soliton resolution conjecture, it is natural to expect that the classification in Theorem 2.1 should extend up to $E_K(\vec{u}) < 2E_K(\vec{Q})$, at least concerning the asymptotic behavior.

If one looks at the full-time dynamics, however, there is another candidate for an essential change of the dynamics. It is a heteroclinic orbit connecting the two distinct ground states Q and $-Q$ in a weak sense: more precisely, a solution u satisfying

$$\lim_{t\to\pm\infty} \left\|\vec{u}(t) \pm \vec{Q} - \vec{v}_\pm(t)\right\|_{\mathcal{H}} = 0 \tag{2.14}$$

for some free solutions $v_\pm$, which may be called *heteroscattering*. The one-pass theorem precludes such solutions for $E_K(\vec{u}) < E_K(\vec{Q}) + \varepsilon$, which is $\mathcal{M}^\dagger \cap (-\mathcal{M}) = \varnothing$, but it is not difficult to construct such a solution with the energy close to $2E_K(\vec{Q})$, by superposing in space-time two heteroscattering solutions, from Q to 0, and from 0 to $-Q$, respectively.

A simple numerical experiment indicates that such solutions appear at a much lower energy level than $2E_K(\vec{Q})$. Then it seems natural to conjecture that there is a threshold energy $E^* \in (E_K(\vec{Q}), 2E_K(\vec{Q}))$ such that for $E_K(\vec{u}) < E^*$ the 14-set classification (2.12) is valid, while for $E_K(\vec{u}) > E^*$ there are solutions satisfying (2.14), increasing the number of solution sets to $4 \times 4 = 16$ at least. Related questions are if there is heteroscattering between Q and $-Q$ with the minimal energy E^*, and what the complete classification of dynamics is for $E_K(\vec{u}) < E^* + \varepsilon$, or for $E_K(\vec{u}) < 2E_K(\vec{Q})$.

A remarkably successful method to go to higher energy is the channel-of-energy by Duyckaerts–Kenig–Merle [16], which settled the soliton resolution conjecture for the energy-critical NLW in the radial 3D case, including the blow-up solutions with a bounded energy norm. It has recently been extended to the higher odd dimensions [17], as well as to the 4D case [15] and to the wave maps under rotational symmetry. Without the rotational symmetry, there are also similar results [14] along time sequences. It seems, however, that this method depends heavily on the special property of the wave equation, that is, the single speed of propagation, while dispersive equations in general have wide ranges of group velocity. It is a challenging and important problem to extend the method or find a similar one for the more dispersive equations such as NLKG and NLS.

3. TRANSITION BETWEEN SOLITONS

Since solitons are the key junctions of global dynamics for nonlinear dispersive equations, it is an important problem to understand the behavior of the solutions migrating from a neighborhood of one soliton to another. In fact, when the equation has both stable and unstable solitons, it is generally expected that solutions starting near the unstable ones will get away from them and eventually approach some of the stable ones. However, the conservation laws prohibit the solutions to approach the latter solitons in the energy norm, unless the two solitons happen to be close to each other in the conserved quantities. In general, the approach should be only in the weak or local topology, where the excessive energy is radiated away in a dispersive wave component.

This type of transition from one soliton to another should happen also between unstable solitons. Trying to include such a behavior into a classification as above seems still a bit too ambitious, as the complete classification for NLKG or NLS is yet much below all the excited state solitons. However, we can make a model problem by adding some spatial inhomogeneity, which is an easy way to create standing waves. Specifically, the nonlinear Schrödinger equation with a potential (NLSP)

$$i\dot{u} - \Delta u + Vu = |u|^2 u, \quad u(t,x) : \mathbb{R}^{1+3} \to \mathbb{R}, \tag{3.1}$$

is a good model to consider a classification of solutions including two different solitons, both stable and unstable ones. The standing waves for NLSP are solutions in the form $u(t,x) = e^{-it\omega}\varphi(x)$ for some $\omega \in \mathbb{R}$, for which the equation is reduced to

$$-\Delta u + \omega u + Vu = |u|^2 u. \tag{3.2}$$

More precisely, let $V : \mathbb{R}^3 \to \mathbb{R}$ be "nice" enough, e.g., a radial Schwartz function, such that the linear Schrödinger operator $-\Delta + V$ has only one eigenvalue, denoted by $e_0 < 0$. Let $\phi_0 \in H^2(\mathbb{R}^3)$ be the corresponding eigenfunction or the ground state of $-\Delta + V$, normalized in $L^2(\mathbb{R}^3)$. Let E_V be the Hamiltonian of NLSP, defined by

$$E_V(u) := E_S(u) + \int_{\mathbb{R}^3} \frac{V|u|^2}{2} dx. \tag{3.3}$$

Then one may construct two different types of standing waves for NLSP. One family is generated from the linear ground state ϕ_0 by bifurcation, which is small and stable with negative energy in the asymptotic form (see [26])

$$\begin{aligned} &\Phi[z] = z\big[\phi_0 + O\big(|z|^2\big)\big] \quad \text{in } H^1(\mathbb{R}^3), \quad \omega[z] = e_0 + O\big(|z|^2\big), \\ &E_V\big(\Phi[z]\big) = e_0|z|^2/2 + O\big(|z|^4\big), \quad M\big(\Phi[z]\big) = |z|^2/2 + O\big(|z|^4\big), \end{aligned} \tag{3.4}$$

with a small parameter $z = (\Phi[z]|\phi_0) \in \mathbb{C}$, where

$$M(u) := \int_{\mathbb{R}^d} \frac{|u|^2}{2} dx \tag{3.5}$$

denotes the conserved mass. The other family is generated from the scaling limit of the ground state Q of NLS ($V = 0$), which is large and unstable with positive energy, in the asymptotic form (see [45])

$$\begin{aligned} &\Psi[\zeta] = \zeta Q(|\zeta|x) + O\big(|\zeta|^{-3/2}\big) \quad \text{in } H^1(\mathbb{R}^3), \quad \omega[\zeta] = |\zeta|^2, \\ &E_V\big(\Psi[\zeta]\big) = |\zeta| E_S(Q) + O\big(|\zeta|^{-1}\big), \quad M\big(\Psi[\zeta]\big) = |\zeta|^{-1} M(Q) + O\big(|\zeta|^{-3}\big), \end{aligned} \tag{3.6}$$

with a large parameter $\zeta \in \mathbb{C}$. Since both the families converge to 0 in $L^2(\mathbb{R}^3)$ in the limits $z \to 0$ and $\zeta \to \infty$, respectively, the asymptotic regimes are contained in the small mass region $M(u) \ll 1$. We can prove that for each fixed $M(u) \ll 1$, there is a unique $|z| \ll 1$ such that $\Phi[z]$ are the least energy standing waves, namely the ground states for the prescribed mass $M(u)$, and also there is a unique $|\zeta| \gg 1$ such that $\Psi[\zeta]$ are the second least energy ones, or the first excited states.

Actually, both of them are the ground state solutions of (3.2) for the corresponding $\omega > 0$, which may be obtained by the min–max variational argument. From the dynamical viewpoint, however, it seems more appropriate to compare them in terms of the energy and mass, without fixing parameter ω, which is not intrinsic in the equation.

Gustafson–Nakanishi–Tsai [26] proved the scattering to the ground states Φ for small solutions in $H^1(\mathbb{R}^3)$, that is,

$$\lim_{t \to \pm\infty} \big\| u(t) - \Phi\big[z(t)\big] - v_\pm(t) \big\|_{H^1} = 0, \tag{3.7}$$

for some free solutions $v_\pm$ and some function $z : \mathbb{R} \to \mathbb{C}$ with convergent $|z(t)|$ as $t \to \pm\infty$. The exceptional case $|z(t)| \to 0$ is also included. This result has been extended by Nakanishi [45, 46] to the energy slightly above the first excited solitons Ψ under the radial symmetry restriction (which was not imposed in [26]), with a classification of the global dynamics similar to Theorem 2.1, or more closely to the NLS case in [48].

More precisely, let $\mathcal{E}_V(\mu) = E_V(\Psi[\zeta])$ be the first excited energy for the mass $M(\Psi[\zeta]) = \mu$. Then for sufficiently small $\varepsilon > 0$, all the radial solutions with

$$M(u) < \varepsilon, \quad E_V(u) \leq \mathcal{E}_V\big(M(u)\big) + \varepsilon/M(u), \tag{3.8}$$

are classified into $9 = 3 \times 3$ sets characterized by their behavior in $t > 0$ and $t < 0$, either scattering to the ground states Φ as in (3.7), blowing-up, or staying around the excited states Ψ. Moreover, the solutions in the last case make the center-stable manifold of Ψ for $t > 0$, and the center-unstable manifold for $t < 0$.

Note that the restriction $M(u) < \varepsilon$ may be removed if $V = 0$, trivially by using the scaling invariance. Then the above result is reduced to [48], except that the scattering to the excited states is not established in [45]. The same problem with the defocusing nonlinearity

$$i\dot{u} - \Delta u + Vu = -|u|^2 u, \tag{3.9}$$

was also studied in [46], for which all solutions in $H^1(\mathbb{R}^3)$ with small mass scatter to the ground states Φ as $t \to \pm\infty$, while there is no other standing wave in $H^1(\mathbb{R}^3)$.

3.1. Threshold dynamics

As mentioned above, the scattering to the first excited states Ψ remains to be proven on the center-stable manifold. This is mainly because of the lack of complete information on the spectrum of the linearized operator. It is a highly nontrivial problem even without the potential, which was solved by Marzuola–Simpson [43] by a computer-assisted proof.

In the nonradial case, however, a notable difference appears from the case without the potential, where Beceanu [4] proved the scattering to the solitons generated by the Galilei and translation invariance from the ground state Q. Both invariances are destroyed by the potential, and thus the only remaining soliton is fixed at the origin, provided that the potential has a simple shape, e.g., $V(x) = ae^{-|x|^2}$ with some constant $a < 0$. Then the natural conjecture on the dynamics on the center-stable manifold of Ψ is the following:

(1) For $E_V(u) < \mathcal{E}_0(M(u))$, all solutions on the manifold scatter to Ψ.

(2) For $E_V(u) = \mathcal{E}_0(M(u))$, there are solutions with the asymptotic behavior

$$\lim_{t\to\infty} \|u(t) - e^{-i\theta(t)} Q_\omega\big(x - c(t)\big)\|_{H^1} = 0 \tag{3.10}$$

for some $\theta : \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}^3$ satisfying $\dot{\theta} \to \omega$, $|c| \to \infty$, and $\dot{c} \to 0$ as $t \to \infty$, where Q_ω is the ground state of (3.2) with $V = 0$ for some $\omega > 0$ satisfying $M(Q_\omega) = M(u)$. The other solutions on the manifold scatter to Ψ.

(3) For $E_V(u) > \mathcal{E}_0(M(u))$, there are also scatterings into a sum of the Galilei transforms of some Q_ω and the ground states Φ.

In short, the solutions on the center-stable manifold scatter either to the excited states Ψ trapped by the potential at $x = 0$, or to the ground state solitons without potential escaping to $|x| \to \infty$. The threshold between the two cases is the solitons escaping to $|x| \to \infty$ but with the zero asymptotic speed, for which the minimal energy is as in the case (2).

The scattering to Ψ for the solutions initially away from $x = 0$ requires the attractive force of the potential, which may be derived by the Newtonian approximation, but it is valid only on a finite-time interval. Extending it to $t \to \infty$ requires the dissipative effect by the radiation of dispersive waves. Such a scattering result was established by Gang–Sigal [23] in the case of stable solitons for initial data close to the origin. The scattering described above in the case of (3) is also complicated as it contains three different components. Such a scattering result was established by Cuccagna–Maeda [10], also in the stable case for initial data that are already escaping. Classifying all the solutions on the manifold may well require more ideas than the combination of those results.

3.2. Higher mass

Another problem is to extend the classification to $M(u) > \varepsilon$. This sounds plausible at least in the simple defocusing case, where Φ may be extended to all mass as the unique energy minimizers. However, the argument in [45, 46] does not simply extend, because it relies heavily on the smallness of Φ, as well as on $M(u)$, to control all the interactions with the ground states, especially during the concentration–compactness argument for the dispersive component. In the focusing case, the problem does not seem easy even for the smaller potentials, e.g., $V(x) = -ae^{-|x|^2}$ with $0 < a \ll 1$ such that $-\Delta + V > 0$. In this case, there are no small solitons and so the ground states are the perturbations of Q_ω for all mass. Hence it is natural to expect that the same results as in Kenig–Merle [31] (or Holmer–Roudenko [27] for the cubic NLS), and in Nakanishi–Schlag [48] should hold without the small mass condition. It may an option to rely on the stability of the threshold structure with respect to the change of equation (here by the parameter a), including the case of bigger a.

4. TRANSITION BETWEEN MULTISOLITONS

In view of the soliton resolution conjecture, it is an important and necessary step in the study of global dynamics to understand the behavior of solutions migrating between neighborhoods of multisolitons, where the neighborhood may be in the weaker sense as in the previous section. Obviously, this is an even harder problem, so it seems natural to seek for simpler models which admit similar dynamics. The nonlinear Klein–Gordon equation with the damping term

$$\ddot{u} + 2\alpha\dot{u} - \Delta u + u = |u|^{p-1}u, \quad u(t,x) : \mathbb{R}^{1+d} \to \mathbb{R}, \tag{4.1}$$

for some constants $\alpha > 0$, $p > 1$, turns out to be a good model. In fact, Burq–Raugel–Schlag [5] proved the soliton resolution conjecture for all radial solutions in the energy space for the energy-subcritical power $p + 1 < 2^\star$. In this case, solutions asymptotic to solitons are those exponentially converging to some (radial) stationary solutions. The major difference from the conservative NLKG comes from the energy decay

$$\partial_t E_K(\vec{u}) = -2\alpha\|\dot{u}\|_{L^2}^2, \tag{4.2}$$

which makes the analysis much simpler, both in the linear and nonlinear parts. The stable and unstable manifolds had been constructed much earlier by Keller [30] around general stationary solutions. The soliton resolution along time sequences had been established by Feireisl [22] without the radial restriction (but for smaller p), as a consequence of the concentration–compactness due to Lions [38] for the stationary problem. The soliton resolution in the general case takes, as $t \to \infty$, the form of

$$\vec{u}(t) = \sum_{n=1}^{N} \vec{\varphi}_n\big(x - c_n(t)\big) + o(1) \quad \text{in } \mathcal{H}, \tag{4.3}$$

where φ_n are some stationary solutions and $c_n : [0, \infty) \to \mathbb{R}^d$ are some functions satisfying $|c_m - c_n| \to \infty$ for each $m \neq n$. The existence of such solutions with polygonal symmetry was also proven by Feireisl [22]. This allows us to discuss the dynamics around, away, and between multisolitons, as a model case for the more difficult conservative case (NLKG).

More recently, Côte–Martel–Yuan–Zhao [8] characterized the set of asymptotic 2-solitons consisting of the ground state Q of NLKG, namely

$$\vec{u}(t) = \vec{Q}\big(x - c_1(t)\big) - \vec{Q}\big(x - c_2(t)\big) + o(1) \quad \text{in } \mathcal{H} \quad (t \to \infty) \tag{4.4}$$

as a manifold with codimension 2 in the energy space $\mathcal{H}$, together with the asymptotic formula for $c_n(t)$, as well as nonexistence of similar solutions with the same sign on $\vec{Q}$.

Moreover, Côte–Martel–Yuan [7] proved the soliton resolution conjecture in the 1D case without any restriction in the energy space. That is, for any initial data in $\mathcal{H}$, the solution either blows up in finite time, or is asymptotic to a form

$$\vec{u}(t) = \pm \sum_{n=1}^{N} (-1)^n \vec{Q}\big(x - c_n(t)\big) + o(1), \tag{4.5}$$

in $\mathcal{H}$, for some $N \in \mathbb{Z}$ with $c_n - c_{n-1} \to \infty$ as $t \to \infty$. The existence of such solutions for every N is also proven in [7]. To the best of the author's knowledge, this is the first and only result so far of soliton resolution in the entire energy space with no restriction for the full limit $t \to \infty$ that contains moving solitons, provided that the damping is acceptable for the conjecture.

Then it is natural to ask the questions raised in the first section, namely the global dynamics in the full neighborhood of such solutions for all $t \geq 0$. In particular, it is a good place to investigate the migration between different numbers of multisolitons. Ishizuka–Nakanishi [29] considered the simplest case, namely a neighborhood of 2-solitons and transition to 1-solitons, and established a classification into 5 sets of different behavior. To state the precise result, some notation is needed. Let

$$L := -\Delta + 1 - pQ^{p-1} \tag{4.6}$$

be the linearized operator for the static NLKG around the ground state Q, and let $\rho \in H^2(\mathbb{R}^d)$ be its normalized ground state with $L\rho = -\nu^2 \rho$ for some constant $\nu > 0$. Define operators acting on $\mathcal{H}$ in the matrix form

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{L}_\alpha := \begin{pmatrix} L & 2\alpha \\ 0 & 1 \end{pmatrix}. \tag{4.7}$$

Then the linearization of (4.1) around $\vec{Q}$ is written as $\partial_t \vec{u} = J\mathcal{L}_\alpha \vec{u}$. The damped linearized operator $J\mathcal{L}_\alpha$ has eigenfunctions of the form

$$\nu^\pm := -\alpha \pm \sqrt{\nu^2 + \alpha^2}, \quad Y^\pm := (1, \nu^\pm)\rho \implies J\mathcal{L}_\alpha Y^\pm = \nu^\pm Y^\pm. \tag{4.8}$$

For any $z = (z_1, z_2) \in (\mathbb{R}^d)^2$, let $\mathcal{H}_\perp(z) \subset \mathcal{H}$ be the energy subspace defined by

$$\mathcal{H}_\perp(z) := \{\varphi \in \mathcal{H} \mid \langle J\varphi | Y^-(x - z_k)\rangle = 0 \ (k = 1, 2)\}, \tag{4.9}$$

where $\langle \cdot | \cdot \rangle$ denotes the inner product of $(L^2(\mathbb{R}^d))^2$. Then it is easy to see that for $|z_1 - z_2| \gg 1$ (depending on $\alpha > 0$), the energy space is decomposed into a direct sum

$$\mathcal{H} = \mathbb{R}Y^+(x - z_1) \oplus \mathbb{R}Y^+(x - z_2) \oplus \mathcal{H}_\perp(z). \tag{4.10}$$

Let $\mathcal{H}_\perp(z; \delta) := \{\varphi \in \mathcal{H}_\perp(z) \mid \|\varphi\|_{\mathcal{H}} < \delta\}$ be the open ball in the subspace. Then

Theorem 4.1 ([29]). *For any $d \in \mathbb{N}$, $\alpha > 0$ and $p \in (2, 2^\star - 1)$, there is a small $\delta > 0$ such that for any $z \in (\mathbb{R}^d)^2$ satisfying $|z_1 - z_2| > 1/\delta$, there are two Lipschitz functions $G_1, G_2 : (-\delta, \delta) \times \mathcal{H}_\perp(z; \delta) \to (-\delta, \delta)$ with the following properties. For any $h_1, h_2 \in (-\delta, \delta)$ and any $\varphi \in \mathcal{H}_\perp(z; \delta)$, let u be the solution of (4.1) with the initial data*

$$\vec{u}(0) = \sum_{n=1,2} (-1)^n [\vec{Q} + h_n Y^+](x - z_n) + \varphi. \tag{4.11}$$

Then its global behavior is classified by the initial data as follows. Let $n^* := 3 - n$.

(1) *If $h_n < G_n(h_{n^*}, \varphi)$ for both $n = 1, 2$, then u is global with $\|\vec{u}(t)\|_{\mathcal{H}} \to 0$ as $t \to \infty$; we have the global decaying case.*

(2) *If $h_n = G_n(h_{n^*}, \varphi)$ and $h_{n^*} < G_{n^*}(h_n, \varphi)$ for one of $n = 1, 2$, then u is global with $\vec{u}(t) \to (-1)^n \vec{Q}(x - z_\infty)$ in $\mathcal{H}$, for some $z_\infty \in \mathbb{R}^d$, as $t \to \infty$; this is the 1-soliton case with $(-1)^n Q$.*

(3) *If $h_n = G_n(h_{n^*}, \varphi)$ for both $n = 1, 2$, then u is global with*

$$\vec{u}(t) + \vec{Q}(x - z_1(t)) - \vec{Q}(x - z_2(t)) \to 0$$

in $\mathcal{H}$, for some $z_n : [0, \infty) \to \mathbb{R}^d$ satisfying $|z_1(t) - z_2(t)| \to \infty$, as $t \to \infty$; this is the 2-soliton case.

(4) *Otherwise, u blows up in finite time.*

The 2-soliton case (3) may be characterized as $h = G_0(\varphi)$ by another Lipschitz function $G_0 : \mathcal{H}_\perp(z; \delta) \to (-\delta, \delta)^2$.

Moreover, we obtain a full-time description for all those solutions. In particular, in the 1-soliton case (2), the soliton component starting from $(-1)^{n^*} \vec{Q}(x - z_{n^*})$ decays due to the instability, while the other component from $(-1)^n \vec{Q}(x - z_n)$ remains for all time, moving in space and eventually converging to $(-1)^n \vec{Q}(x - z_\infty)$.

The above classification of dynamics is for all initial data in a small neighborhood of any superposition of $\pm\vec{Q}$ with sufficient distance from each other. For each sign of $\pm\vec{Q}$, there is a Lipschitz manifold of codimension 1 consisting of solutions convergent to $\pm\vec{Q}$, translated in space. The two manifolds are joined together at their boundary by the manifold of codimension 2 consisting of solutions asymptotic to 2-solitons, moving away from each other. The connected union of those three manifolds separates the rest of the neighborhood into the open set of global decaying solutions and the open set of blow-up solutions.

The 2-soliton case of (3) was already established by Côte–Martel–Yuan–Zhao [8]. The above theorem extends the dynamics description to the full neighborhood. Note that the manifolds of 1-solitons in (2) are far from those constructed by Keller [30], or by any general method to construct local invariant manifolds, because the manifolds in the above theorem are in a neighborhood of 2-solitons. In other words, it describes the transition from the 2-solitons to the 1-solitons with respect to initial perturbations. In the proof, we also need to describe the transition in time for each initial data on the 1-soliton manifolds. The transition time tends to infinity as the initial data approaches the 2-soliton manifold, so the global dynamics is not at all uniform or continuous within the small neighborhood.

The structure or relation of those three manifolds is in the simplest form as one may expect, by a small perturbation in the energy space from the superposition of two ground states, each having one unstable direction. However, proving this is not so simple as it may appear, because we need to control the two unstable modes with the same eigenvalue, namely $Y^{\pm}(x - z_n(t))$ with $n = 1, 2$. The difficulty comes from the fact that the solitons are getting away from each other, but very slowly, namely $|z_1(t) - z_2(t)| \sim |\log t|$, and the soliton interactions are of order $O(1/t)$ and not integrable in time. In fact, this changes the growth order of the unstable modes from the linearized approximation, making the unstable dynamics far from the superposition of the 1-soliton case. The coupling of the two unstable modes could be even more complicated because the interaction can possibly change the direction of instability, too. It may be illustrated by a simple ODE model with a small parameter $\varepsilon \in \mathbb{R}$, namely

$$\frac{d}{dt}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \nu^+ & \varepsilon^2 e^{-t} \\ \varepsilon^2 e^{-t} & \nu^+ + \frac{\varepsilon}{1+t} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{4.12}$$

which mimics the linearized interaction of the two unstable modes $h_n(t)Y^+(x - c_n(t))$. It is easy to check for the above ODE that

$$\begin{aligned} &\varepsilon > 0, \quad \big(h_1(0), h_2(0)\big) = (1, 0) \implies \lim_{t\to\infty} h_2(t)/h_1(t) = \infty, \\ &\varepsilon < 0, \quad \big(h_1(0), h_2(0)\big) = (0, 1) \implies \lim_{t\to\infty} h_1(t)/h_2(t) = \infty, \end{aligned} \tag{4.13}$$

so the direction of $h(t) \in \mathbb{R}^2$ is completely changed by the interaction. If such a transfer were to happen for the 2-soliton interactions, then the structure of the neighborhood could be more complicated than the above result.

Fortunately, it is not the case because we can prove that nonintegrable interactions are essentially in the scalar part of the above matrix, and the remainder, namely the nonscalar part of the matrix, is uniformly integrable and small. This follows from the reflection symmetry of the equation and the 2-solitons, together with a detailed description of the behavior of solutions in the full neighborhood and all time.

4.1. 3-solitons and soliton merger

It is natural to expect a similar structure for more than 2 solitons ($N \geq 3$ in (4.5)), namely the joint boundary of manifolds with less solitons. However, to prove or disprove such a result seems to be fundamentally more difficult, as the full-time dynamics in the full neighborhood should include a new and more dramatic phenomenon, which may be called *soliton merger*. The distinction between $N \leq 2$ and $N \geq 3$ stems from the fact that the soliton interaction is attractive for the same sign and repulsive for the opposite sign. It is essential for the proof of the above result; 2-solitons with opposite signs are repelling each other as long as both of them exist.

If we start from a small neighborhood of the 3-soliton in the form of (4.5), then the situation is different. Even though the solitons initially have alternating signs, if the middle soliton is destroyed by the instability and the other two survive, then the remaining 2-solitons have the same sign and so start attracting each other. The result of Côte–Martel–Yuan **[7]** implies that they cannot remain to be 2-solitons, but the solution either blows-up, decays to 0, or is asymptotic to 1-soliton. The transition in the last case from 2-solitons to 1-soliton is very different from the case in Theorem 4.1. As the simplest case, consider the initial data with the even symmetry

$$\vec{u}(0) = \vec{Q}(x+c) + hY^+(x+c) + \vec{Q}(x-c) + hY^+(x-c) \tag{4.14}$$

with a small parameter $h \in \mathbb{R}$ and a large parameter $c > 1$. It is easy to see that if $0 < h \ll 1$ and $c \gg 1$ is large enough depending on h, then the solution u blows up, and similarly if $0 > h \gg -1$ with $c \gg 1$ then the solution is globally decaying to 0. Since both types of behavior are stable (in 1D), there must be some intermediate $h \in \mathbb{R}$ for a fixed large c such that the solution u converges to $\pm Q$. For such solutions, the even symmetry implies that both the soliton components from $x = \pm c$ are destroyed, but afterward another soliton emerges at $x = 0$. Because of the energy damping, the latter component has to absorb some energy, at least half of $E(\vec{Q})$ from each of the two destroyed solitons, before they are dissipated. This may be regarded as a sort of collision, but very far from the elastic ones in the completely integrable case.

Inelastic collisions have been studied for the generalized KdV by Mizumachi **[44]** and Martel–Merle **[40, 41]**, where the inelasticity is in a small radiation. For perturbation from the integrable NLS, Perelman **[53]** proved that the collision splits the smaller soliton into two pieces. For the energy-critical NLW in 5D, Martel–Merle **[42]** showed the existence of radiation after collision. The above phenomenon looks quite different also from those cases.

Describing the soliton merger and determining the manifold structure around the 3-solitons (or more) seem to be challenging problems. It does not look obvious even whether the merged soliton can take both signs $\pm Q$ or only one. Another question is whether there exists a similar solution in the conservative case such as NLKG. Those questions may be difficult also for numerical experiments because the merger requires some balance between the two dynamics of different orders, namely the exponential instability and the logarithmic movement of solitons.

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KENJI NAKANISHI

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan, kenji@kurims.kyoto-u.ac.jp