VARIETY OF FRACTIONAL LAPLACIANS

ALEXANDER I. NAZAROV

ABSTRACT

This paper is a survey of recent results on comparison of various fractional Laplacians.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 47A63; Secondary 35A23, 47G20

KEYWORDS

Fractional Laplacians, nonlocal differential operators

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 5, pp. 3842–3853
DOI 10.4171/ICM2022/34

Fractional Laplacians (FLs for brevity) and equations with them have been actively studied in last decades throughout the world in various fields of mathematics (analysis, partial differential equations, the theory of random processes) and its applications (in physics, biology). Hundreds of articles have been written on this topic. Note that the study of such operators and equations is complicated not only by the fact of nonlocality itself, but also by the existence of several nonequivalent definitions of a fractional Laplacian.

Historically, the first FL was the fractional Laplacian of order $s > 0$ in $\mathbb{R}^n$ defined (say, on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$) as

$$(-\Delta)^s u := \mathcal{F}^{-1}\big(|\xi|^{2s}\mathcal{F}u(\xi)\big),$$

where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i\langle\xi,x\rangle}u(x)\,dx.$$

For $s \in (0,1)$, the following relation holds:

$$(-\Delta)^s u(x) = c_{n,s}\cdot \text{V.P.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,$$

where

$$c_{n,s} = \frac{2^{2s}s}{\pi^{\frac{n}{2}}}\frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}.$$

We recall the definitions of the classical Sobolev–Slobodetskii spaces in $\mathbb{R}^n$ (see **[21, SUBSECTION 2.3.3]** or **[7]**),

$$H^s(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n) : \big(1+|\xi|^2\big)^{\frac{s}{2}}\mathcal{F}u(\xi) \in L_2(\mathbb{R}^n)\big\},$$

and corresponding spaces in a (say, Lipschitz and bounded) domain Ω (see **[21, SUBSECTION 4.2.1]** and **[21, SUBSECTION 4.3.2]**),

$$H^s(\Omega) = \big\{u|_\Omega : u \in H^s(\mathbb{R}^n)\big\};\quad \tilde{H}^s(\Omega) = \big\{u \in H^s(\mathbb{R}^n) : \text{supp}(u) \subset \overline{\Omega}\big\}.$$

Notice that the quadratic form of $(-\Delta)^s$ is naturally defined on $H^s(\mathbb{R}^n)$ by[1]

$$\big((-\Delta)^s u, u\big) = \int_{\mathbb{R}^n}|\xi|^{2s}\big|\mathcal{F}u(\xi)\big|^2\,d\xi, \tag{1}$$

and define the *restricted Dirichlet* FL as the positive self-adjoint operator with quadratic form (see, e.g., **[1, CHAP. 10]**)

$$Q_s^{\text{DR}}[u] \equiv \big((-\Delta_\Omega)^s_{\text{DR}}u, u\big) := \big((-\Delta)^s u, u\big);\quad \text{Dom}(Q_s^{\text{DR}}) = \tilde{H}^s(\Omega).$$

Remark 1. For $s \in (0,1)$, the following relation evidently holds:

$$Q_s^{\text{DR}}[u] = \frac{c_{n,s}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

1 As usual, we denote by $(\cdot,\cdot)$ the duality generated by the scalar product in L_2.

Notice that for $s \in (0,1)$ one can also define the *restricted Neumann* (or *regional*) FL by the quadratic form

$$Q_s^{\rm NR}[u] := \frac{c_{n,s}}{2} \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy; \quad {\rm Dom}(Q_s^{\rm NR}) = H^s(\Omega).$$

For some "intermediate" fractional Laplacians of this type, see, e.g., [16] and the references therein.

Now we turn to a different type of FLs, namely, to the spectral ones. Recall that the *spectral Dirichlet and Neumann* FLs are the sth powers of conventional Dirichlet and Neumann Laplacian in the sense of spectral theory. In a Lipschitz bounded domain Ω, they can be defined as the positive self-adjoint operators with quadratic forms

$$Q_s^{\rm DSp}[u] \equiv \big((-\Delta_\Omega)^s_{\rm DSp} u, u\big) := \sum_{j=1}^{\infty} \lambda_j^s \big|(u,\varphi_j)\big|^2, \tag{2}$$

$$Q_s^{\rm NSp}[u] \equiv \big((-\Delta_\Omega)^s_{\rm NSp} u, u\big) := \sum_{j=0}^{\infty} \mu_j^s \big|(u,\psi_j)\big|^2, \tag{3}$$

where λ_j, φ_j and μ_j, ψ_j are eigenvalues and (normalized) eigenfunctions of the Dirichlet and Neumann Laplacian in Ω, respectively. Notice that $\mu_0 = 0$ and $\psi_0 \equiv$ const.

For $s \in (0,1)$, the domains of these quadratic forms are

$${\rm Dom}(Q_s^{\rm DSp}) = \tilde H^s(\Omega); \quad {\rm Dom}(Q_s^{\rm NSp}) = H^s(\Omega).$$

For $s > 1$, the domains of spectral quadratic forms are more complicated. However, the following relations hold ([21, THEOREM 1.17.1/1] and [21, THEOREM 4.3.2/1]; see also [12, LEMMA 1] and [14, LEMMA 2]):

$$\tilde H^s(\Omega) = {\rm Dom}(Q_s^{\rm DSp}), \quad 0 < s < \frac{3}{2}; \quad \tilde H^s(\Omega) \subsetneq {\rm Dom}(Q_s^{\rm DSp}), \quad s \ge \frac{3}{2};$$

$$\tilde H^s(\Omega) = {\rm Dom}(Q_s^{\rm NSp}), \quad 0 < s < \frac{1}{2}; \quad \tilde H^s(\Omega) \subsetneq {\rm Dom}(Q_s^{\rm NSp}), \quad s \ge \frac{1}{2}.$$

It follows from the well-known Heinz inequality ([10]; see also [1, §10.4]) that for $u \in \tilde H^s(\Omega)$, $s \in (0,1)$, the following inequality holds:

$$Q_s^{\rm DSp}[u] \ge Q_s^{\rm NSp}[u]. \tag{4}$$

On the other hand, the inequality $Q_s^{\rm DR}[u] \ge Q_s^{\rm NR}[u]$ for $u \in \tilde H^s(\Omega)$, $s \in (0,1)$, is trivial.

Below we provide a wide generalization and sharpening of (4). To this end, we recall the basic facts on the generalized harmonic extensions related to fractional Laplacians of the order $\sigma \in (0,1)$ and of the *negative* order $-\sigma \in (-1,0)$.

It was known long ago that the square root of Laplacian is related to the harmonic extension and to the Dirichlet-to-Neumann map. In the breakthrough paper [4], the FL $(-\Delta)^\sigma$ (and therefore $(-\Delta_\Omega)^\sigma_{\rm DR}$) for any $\sigma \in (0,1)$ was related to the *generalized harmonic extension* and to the generalized Dirichlet-to-Neumann map.

Namely, let $u \in \tilde H^\sigma(\Omega)$. Then there exists a unique solution $w_\sigma^{\rm DR}(x,y)$ of the boundary value problem in the half-space

$$-{\rm div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \quad w|_{y=0} = u,$$

with finite energy (weighted Dirichlet integral)

$$\mathcal{E}^{\mathrm{R}}_\sigma(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} \big|\nabla w(x,y)\big|^2 \, dx dy,$$

and the relation

$$(-\Delta_\Omega)^\sigma_{\mathrm{DR}} u(x) = -C_\sigma \cdot \lim_{y\to 0^+} y^{1-2\sigma} \partial_y w^{\mathrm{DR}}_\sigma(x,y) \tag{5}$$

with

$$C_\sigma = \frac{4^\sigma \Gamma(1+\sigma)}{\Gamma(1-\sigma)}$$

holds in the sense of distributions in Ω and pointwise at every point of smoothness of u. Moreover, the function $w^{\mathrm{DR}}_\sigma(x,y)$ minimizes $\mathcal{E}^{\mathrm{R}}_\sigma$ over the set

$$\mathcal{W}^{\mathrm{DR}}_\sigma(u) = \big\{ w(x,y) : \mathcal{E}^{\mathrm{R}}_\sigma(w) < \infty, \ w|_{y=0} = u \big\},$$

and the following equality holds:

$$Q^{\mathrm{DR}}_\sigma[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}^{\mathrm{R}}_\sigma(w^{\mathrm{DR}}_\sigma). \tag{6}$$

In [20] this approach was substantially generalized. In particular, for $u \in \tilde{H}^\sigma(\Omega)$ (for $u \in H^\sigma(\Omega)$) there is a unique solution of the boundary value problem in the half-cylinder

$$-\operatorname{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in } \Omega \times \mathbb{R}_+; \quad w|_{y=0} = u,$$

satisfying, respectively, the Dirichlet or the Neumann boundary condition on the lateral surface of the half-cylinder and having finite energy

$$\mathcal{E}^{\mathrm{Sp}}_\sigma(w) = \int_0^\infty \int_\Omega y^{1-2\sigma} \big|\nabla w(x,y)\big|^2 \, dx dy.$$

Denote these solutions $w^{\mathrm{DSp}}_\sigma(x,y)$ and $w^{\mathrm{NSp}}_\sigma(x,y)$, respectively. The following relations hold in the sense of distributions in Ω and pointwise at every point of smoothness of u:

$$(-\Delta_\Omega)^\sigma_{\mathrm{DSp}} u(x) = -C_\sigma \cdot \lim_{y\to 0^+} y^{1-2\sigma} \partial_y w^{\mathrm{DSp}}_\sigma(x,y), \tag{7}$$

$$(-\Delta_\Omega)^\sigma_{\mathrm{NSp}} u(x) = -C_\sigma \cdot \lim_{y\to 0^+} y^{1-2\sigma} \partial_y w^{\mathrm{NSp}}_\sigma(x,y). \tag{8}$$

Moreover, these solutions minimize $\mathcal{E}^{\mathrm{Sp}}_\sigma$ over the sets

$$\mathcal{W}^{\mathrm{DSp}}_{\sigma,\Omega}(u) = \big\{ w(x,y) : \mathcal{E}^{\mathrm{Sp}}_\sigma(w) < \infty, \ w|_{y=0} = u, \ w|_{x\in\partial\Omega} = 0 \big\},$$
$$\mathcal{W}^{\mathrm{NSp}}_{\sigma,\Omega}(u) = \big\{ w(x,y) : \mathcal{E}^{\mathrm{Sp}}_\sigma(w) < \infty, \ w|_{y=0} = u \big\},$$

respectively, and the following equalities hold:

$$Q^{\mathrm{DSp}}_\sigma[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}^{\mathrm{Sp}}_\sigma(w^{\mathrm{DSp}}_\sigma); \quad Q^{\mathrm{NSp}}_\sigma[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}^{\mathrm{Sp}}_\sigma(w^{\mathrm{NSp}}_\sigma). \tag{9}$$

Now we set $s = -\sigma \in (-1, 0)$. The operators $(-\Delta_\Omega)^{-\sigma}_{\mathrm{DR}}$, $(-\Delta_\Omega)^{-\sigma}_{\mathrm{DSp}}$, and $(-\Delta_\Omega)^{-\sigma}_{\mathrm{NSp}}$ are defined by corresponding quadratic forms (1)–(3)[2] with domains

$$\mathrm{Dom}(Q^{\mathrm{DR}}_{-\sigma}) = \begin{cases} \tilde{H}^{-\sigma}(\Omega) & \text{if either } n \geq 2 \text{ or } \sigma < \frac{1}{2}; \\ \{u \in \tilde{H}^{-\sigma}(\Omega) : (u, \mathbf{1}) = 0\} & \text{if } n = 1 \text{ and } \sigma \geq \frac{1}{2}; \end{cases}$$

$$\mathrm{Dom}(Q^{\mathrm{DSp}}_{-\sigma}) = H^{-\sigma}(\Omega); \quad \mathrm{Dom}(Q^{\mathrm{NSp}}_{-\sigma}) = \big\{u \in \tilde{H}^{-\sigma}(\Omega) : (u, \mathbf{1}) = 0\big\}.$$

The first two equalities were proved in [14, LEMMA 1]; the third follows from [21, THEOREM 2.10.5/1]. We notice that $(-\Delta_\Omega)^{-\sigma}_{\mathrm{NSp}} u$ is defined up to an additive constant which can be naturally fixed by assumption $((-\Delta_\Omega)^{-\sigma}_{\mathrm{NSp}} u, \mathbf{1}) = 0$.

Remark 2. By [21, THEOREMS 4.3.2/1 AND 2.10.5/1], for $0 < \sigma \leq \frac{1}{2}$ we have $\tilde{H}^{-\sigma}(\Omega) \subseteq H^{-\sigma}(\Omega)$ (even $\tilde{H}^{-\sigma}(\Omega) = H^{-\sigma}(\Omega)$ if $0 < \sigma < \frac{1}{2}$) whereas in the case $\frac{1}{2} < \sigma < 1$, $H^{-\sigma}(\Omega)$ is a subspace of $\tilde{H}^{-\sigma}(\Omega)$. However, in the latter case we can consider an arbitrary $f \in \mathrm{Dom}(Q^{\mathrm{DR}}_{-\sigma})$ as a functional on $H^{\sigma}(\Omega)$, put $\tilde{f} = f|_{\tilde{H}^{-\sigma}(\Omega)} \in \mathrm{Dom}(Q^{\mathrm{DSp}}_{-\sigma})$ and define $Q^{\mathrm{DSp}}_{-\sigma}[f] := Q^{\mathrm{DSp}}_{-\sigma}[\tilde{f}]$.

Next, we connect FLs of the negative order with the generalized Neumann-to-Dirichlet map. It was done in [5] for the spectral Dirichlet FL and in [3] for the FL in $\mathbb{R}^n$ (and therefore for the restricted Dirichlet FL). Variational characterization of these operators was given in [14]. The spectral Neumann FL was considered in [17].

Let $u \in \tilde{H}^{-\sigma}(\Omega)$ (for $n = 1$ and $\sigma \geq \frac{1}{2}$ assume in addition that $(u, \mathbf{1}) = 0$). We consider the problem[3]

$$\tilde{\mathcal{E}}^{\mathrm{R}}_{-\sigma}(w) := \mathcal{E}^{\mathrm{R}}_{\sigma}(w) - 2(u, w|_{y=0}) \to \min \tag{10}$$

on the set $\mathcal{W}^{\mathrm{DR}}_{-\sigma}$, that is, the closure of smooth functions on $\mathbb{R}^n \times \bar{\mathbb{R}}_+$ with bounded support, with respect to $\mathcal{E}^{\mathrm{R}}_{\sigma}(\cdot)$.

If $n > 2\sigma$ (this is a restriction only for $n = 1$) then the minimizer is determined uniquely. Denote it by $w^{\mathrm{DR}}_{-\sigma}(x, y)$. Then (5) and (6) imply

$$(-\Delta_\Omega)^{-\sigma}_{\mathrm{DR}} u(x) = \frac{2\sigma}{C_\sigma} w^{\mathrm{DR}}_{-\sigma}(x, 0); \quad Q^{\mathrm{DR}}_{-\sigma}[u] = -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}^{\mathrm{R}}_{-\sigma}(w^{\mathrm{DR}}_{-\sigma}) \tag{11}$$

(the first relation holds for a.a. $x \in \Omega$).

In case $n = 1 \leq 2\sigma$, the minimizer $w^{\mathrm{DR}}_{-\sigma}(x, y)$ is defined up to an additive constant. However, by assumption $(u, \mathbf{1}) = 0$, the functional $\tilde{\mathcal{E}}^{\mathrm{R}}_{-\sigma}(w^{\mathrm{DR}}_{-\sigma})$ does not depend on the choice of the constant, and the second relation in (11) holds. The first equality in (11) also holds if we choose the constant such that $w^{\mathrm{DR}}_{-\sigma}(x, 0) \to 0$ as $|x| \to \infty$.

Notice that the function $w^{\mathrm{DR}}_{-\sigma}$ solves the Neumann problem in the half-space

$$-\mathrm{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \qquad \lim_{y \to 0^+} y^{1-2\sigma} \partial_y w = -u$$

(the boundary condition holds in the sense of distributions). So, we can consider $(-\Delta_\Omega)^{-\sigma}_{\mathrm{DR}}$ as the Neumann-to-Dirichlet map, and (10) gives the "dual" variational characterization of negative restricted Dirichlet FL.

---

2 We emphasize that $(-\Delta_\Omega)^{-\sigma}_{\mathrm{DR}}$ is not the inverse to $(-\Delta_\Omega)^{\sigma}_{\mathrm{DR}}$.

3 Notice that by the result of [4] the duality $(u, w|_{y=0})$ is well defined.

In a similar way we provide the "dual" variational characterization of the operators $(-\Delta_\Omega)^{-\sigma}_{\mathrm{DSp}}$ and $(-\Delta_\Omega)^{-\sigma}_{\mathrm{NSp}}$. Namely, let $u \in \tilde{H}^{-\sigma}(\Omega)$ (for the spectral Neumann operator assume in addition that $(u, \mathbf{1}) = 0$). Consider the problem

$$\tilde{\mathcal{E}}^{\mathrm{Sp}}_{-\sigma}(w) = \mathcal{E}^{\mathrm{Sp}}_{\sigma}(w) - 2(u, w|_{y=0}) \to \min$$

respectively on the sets

$$\mathcal{W}^{\mathrm{DSp}}_{-\sigma,\Omega} = \{w(x,y) : \mathcal{E}^{\mathrm{Sp}}_{\sigma}(w) < \infty,\ w|_{x\in\partial\Omega} = 0\},$$
$$\mathcal{W}^{\mathrm{NSp}}_{-\sigma,\Omega} = \{w(x,y) : \mathcal{E}^{\mathrm{Sp}}_{\sigma}(w) < \infty\}.$$

Denote corresponding minimizers $w^{\mathrm{DSp}}_{-\sigma}(x,y)$ and $w^{\mathrm{NSp}}_{-\sigma}(x,y)$, respectively[4]. Then (7)–(9) imply

$$Q^{\mathrm{DSp}}_{-\sigma}[u] = -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}^{\mathrm{Sp}}_{-\sigma}(w^{\mathrm{DSp}}_{-\sigma}); \quad (-\Delta_\Omega)^{-\sigma}_{\mathrm{DSp}} u(x) = \frac{2\sigma}{C_\sigma} w^{\mathrm{DSp}}_{-\sigma}(x,0); \tag{12}$$
$$Q^{\mathrm{NSp}}_{-\sigma}[u] = -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}^{\mathrm{Sp}}_{-\sigma}(w^{\mathrm{NSp}}_{-\sigma}); \quad (-\Delta_\Omega)^{-\sigma}_{\mathrm{NSp}} u(x) = \frac{2\sigma}{C_\sigma} w^{\mathrm{NSp}}_{-\sigma}(x,0) \tag{13}$$

(the second equalities in (12) and (13) hold for a.a. $x \in \Omega$; in the latter case, we should choose the additive constant such that $w^{\mathrm{NSp}}_{-\sigma}(x,y) \to 0$ as $y \to +\infty$).

Also the functions $w^{\mathrm{DSp}}_{-\sigma}$ and $w^{\mathrm{NSp}}_{-\sigma}$ solve the boundary value problem in the half-cylinder

$$-\operatorname{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in } \Omega \times \mathbb{R}_+; \qquad \lim_{y\to 0^+} y^{1-2\sigma}\partial_y w = -u$$

with the Dirichlet or the Neumann boundary condition on the lateral surface $\partial\Omega\times\mathbb{R}_+$, respectively (the Neumann boundary condition on the bottom holds in the sense of distributions).

Now we are in a position to formulate the first group of our main results, namely, the comparison of various FLs in the sense of quadratic forms. These statements were proved in [12, THEOREM 2], [14, THEOREM 1], and [17, THEOREM 3] (for some partial results see also [6, 9, 19]).

Theorem 3. *Let $s > -1$ and $s \notin \mathbb{N}_0$. Suppose that*[5] $u \in \tilde{H}^s(\Omega)$, $u \not\equiv 0$. *Then the following relations hold:*

$$Q^{\mathrm{DSp}}_s[u] > Q^{\mathrm{DR}}_s[u] > Q^{\mathrm{NSp}}_s[u], \quad \text{if } s \in (2k, 2k+1),\ k \in \mathbb{N}_0; \tag{14}$$
$$Q^{\mathrm{DSp}}_s[u] < Q^{\mathrm{DR}}_s[u] < Q^{\mathrm{NSp}}_s[u], \quad \text{if } s \in (2k-1, 2k),\ k \in \mathbb{N}_0. \tag{15}$$

Proof. We prove the theorem in three steps.

1. Let $s \in (0,1)$. Notice that we can assume any function $w \in \mathcal{W}^{\mathrm{DSp}}_{s,\Omega}(u)$ to be extended by zero to $(\mathbb{R}^n \setminus \Omega) \times \mathbb{R}_+$. Then evidently

$$\mathcal{W}^{\mathrm{DSp}}_{s,\Omega}(u) \subset \mathcal{W}^{\mathrm{DR}}_{s}(u) \quad \text{and} \quad \mathcal{E}^{\mathrm{Sp}}_s = \mathcal{E}^{\mathrm{R}}_s|_{\mathcal{W}^{\mathrm{DSp}}_{s,\Omega}(u)}.$$

---

4 Notice that $w^{\mathrm{NSp}}_{-\sigma}(x,y)$ is defined up to an additive constant. By assumption $(u,\mathbf{1}) = 0$, the functional $\tilde{\mathcal{E}}^{\mathrm{Sp}}_{-\sigma}(w^{\mathrm{NSp}}_{-\sigma})$ does not depend on the choice of the constant.

5 We assume in addition that $(u,\mathbf{1}) = 0$ in two cases:

(1) for the left inequality in (15), if $n = 1$ and $s \le -\frac{1}{2}$;

(2) for the right inequality in (15), if $s < 0$.

Therefore, formulae (6) and (9) provide

$$Q_s^{\rm DSp}[u] = \frac{C_s}{2s} \cdot \min_{w \in \mathcal{W}_{s,\Omega}^{\rm DSp}(u)} \mathcal{E}_s^{\rm DSp}(w) \geq \frac{C_s}{2s} \cdot \min_{w \in \mathcal{W}_s^{\rm DR}(u)} \mathcal{E}_s^{\rm DR}(w) = Q_s^{\rm DR}[u],$$

and the first inequality in (14) follows with the "$\geq$" sign.

To complete the proof, we observe that for $u \not\equiv 0$ the corresponding extension $w_s^{\rm DSp}$ (extended by zero) cannot be a solution of the homogeneous equation in the whole half-space $\mathbb{R}^n \times \mathbb{R}_+$ since such a solution should be analytic in the half-space. Thus, it cannot provide $\min_{w \in \mathcal{W}_s^{\rm DR}(u)} \mathcal{E}_s^{\rm DR}(w)$.

Since $w_s^{\rm DR}|_{\Omega \times \mathbb{R}_+} \in \mathcal{W}_{s,\Omega}^{\rm NSp}(u)$, the proof of the second inequality in (14) is even simpler.

2. Now let $s = -\sigma \in (-1, 0)$. We again extend functions in $\mathcal{W}_{-\sigma,\Omega}^{\rm DSp}$ by zero and obtain

$$\mathcal{W}_{-\sigma,\Omega}^{\rm DSp} \subset \mathcal{W}_{-\sigma}^{\rm DR} \quad \text{and} \quad \tilde{\mathcal{E}}_{-\sigma}^{\rm Sp} = \tilde{\mathcal{E}}_{-\sigma}^{\rm R}|_{\mathcal{W}_{-\sigma,\Omega}^{\rm DSp}}.$$

Therefore, formulae (11) and (12) provide

$$Q_{s,\Omega}^{\rm DSp}[u] = -\frac{2\sigma}{C_\sigma} \cdot \min_{w \in \mathcal{W}_{-\sigma,\Omega}^{\rm DSp}} \tilde{\mathcal{E}}_{-\sigma}^{\rm Sp}(w) \leq -\frac{2\sigma}{C_\sigma} \cdot \min_{w \in \mathcal{W}_{-\sigma}^{\rm DR}} \tilde{\mathcal{E}}_{-\sigma}^{\rm R}(w) = Q_s^{\rm DR}[u],$$

and the left part in (15) follows with the "$\leq$" sign. To complete the proof, we repeat the argument of the first part. The proof of the right part is similar.

3. Now let $s > 1$, $s \notin \mathbb{N}$. We put $k = \lfloor \frac{s+1}{2} \rfloor$ and define for $u \in \tilde{H}^s(\Omega)$,

$$v = (-\Delta)^k u \in \tilde{H}^{s-2k}(\Omega), \quad s - 2k \in (-1, 0) \cup (0, 1).$$

Note that $v \not\equiv 0$ if $u \not\equiv 0$, and

$$(v, \mathbf{1}) = \mathcal{F} v(0) = |\xi|^{2k} \mathcal{F} u(\xi)|_{\xi=0} = 0.$$

Then we have

$$Q_{s,\Omega}^{\rm DSp}[u] = Q_{s-2k,\Omega}^{\rm DSp}[v], \quad Q_s^{\rm DR}[u] = Q_{s-2k}^{\rm DR}[u], \quad Q_s^{\rm NSp}[u] = Q_{s-2k}^{\rm NSp}[u],$$

and the conclusion follows from steps 1 and 2. ■

The second group of our results is related to the pointwise comparison of FLs. These statements were proved in [12, THEOREM 1], [14, THEOREM 3], and [17, THEOREM 4] (a partial result can be found in [8]).

Theorem 4. A. *Let $s \in (0, 1)$, and let $u \in \tilde{H}^s(\Omega)$, $u \geq 0$, $u \not\equiv 0$. Then the following relation holds in the sense of distributions:*

$$(-\Delta_\Omega)_{\rm DSp}^s u > (-\Delta_\Omega)_{\rm DR}^s u \quad \textit{in } \Omega. \tag{16}$$

B. *Let $s \in (-1, 0)$ for $n \geq 2$, and let $s \in (-\frac{1}{2}, 0)$ for $n = 1$. Suppose that $u \in \tilde{H}^s(\Omega)$, $u \geq 0$ in the sense of distributions, $u \not\equiv 0$. Then the following relation holds:*

$$(-\Delta_\Omega)_{\rm DSp}^s u < (-\Delta_\Omega)_{\rm DR}^s u \quad \textit{in } \Omega. \tag{17}$$

C. *Suppose that* Ω *is convex. Let* $s \in (0,1)$*, and let* $u \in \tilde{H}^s(\Omega)$*,* $u \geq 0$*,* $u \not\equiv 0$*. Then the following relation holds in the sense of distributions:*

$$(-\Delta_\Omega)^s_{\mathrm{DR}} u > (-\Delta_\Omega)^s_{\mathrm{NSp}} u \quad \text{in } \Omega. \tag{18}$$

Proof. A. We introduce the function

$$W_s(x,y) := w_s^{\mathrm{DR}}(x,y) - w_s^{\mathrm{DSp}}(x,y).$$

Note that formulae (5) and (7) imply

$$(-\Delta_\Omega)^s_{\mathrm{DSp}} u - (-\Delta_\Omega)^s_{\mathrm{DR}} u = C_\sigma \cdot \lim_{y\to 0^+} y^{1-2s} \partial_y W_s(x,y) \tag{19}$$

in the sense of distributions.

By the strong maximum principle, the assumptions $u \geq 0$, $u \not\equiv 0$ imply that $w_s^{\mathrm{DR}} > 0$ in $\mathbb{R}^n \times \mathbb{R}_+$. Thus, $w_s^{\mathrm{DR}} > w_s^{\mathrm{DSp}}$ at $\partial\Omega \times \mathbb{R}_+$ and, again by the strong maximum principle, $W_s > 0$ in $\Omega \times \mathbb{R}_+$.

After changing of the variable $t = y^{2s}$, the function W_s satisfies the following relations:

$$\Delta_x W_s(x, t^{\frac{1}{2s}}) + 4s^2 t^{\frac{2s-1}{s}} \partial^2_{tt} W_s(x, t^{\frac{1}{2s}}) = 0 \quad \text{in } \Omega \times \mathbb{R}_+; \quad W_s|_{t=0} = 0. \tag{20}$$

The differential operator in (20) satisfies the assumptions of the boundary point lemma **[11]** at any point $(x,0) \in \Omega \times \{0\}$. Therefore, we have for any $x \in \Omega$,

$$\liminf_{y\to 0^+} y^{1-2s} \partial_y W_s(x,y) = 2s \liminf_{t\to 0^+} \frac{W_s(x, t^{\frac{1}{2s}})}{t} > 0.$$

This gives (16) in view of (19).

B. Put $\sigma = -s \in (0,1)$ and consider extensions $w_{-\sigma}^{\mathrm{DR}}$ and $w_{-\sigma}^{\mathrm{DSp}}$. Making the change of the variable $t = y^{2\sigma}$, we rewrite the boundary value problem for $w_{-\sigma}^{\mathrm{DR}}(x, t^{\frac{1}{2\sigma}})$ as follows:

$$\Delta_x w_{-\sigma}^{\mathrm{DR}} + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}} \partial^2_{tt} w_{-\sigma}^{\mathrm{DR}} = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \quad \partial_t w_{-\sigma}^{\mathrm{DR}}|_{t=0} = -\frac{u}{2\sigma}. \tag{21}$$

Since $w_{-\sigma}^{\mathrm{DR}}$ vanishes at infinity, $w_{-\sigma}^{\mathrm{DR}}(x, t^{\frac{1}{2\sigma}}) > 0$ for $t > 0$ by the maximum principle.

Further, the function $w_{-\sigma}^{\mathrm{DSp}}(x, t^{\frac{1}{2\sigma}})$ satisfies the equalities in (21) for $x \in \Omega$. Since $w_{-\sigma}^{\mathrm{DSp}}|_{x\in\partial\Omega} = 0$, we infer that the function

$$\hat{W}_s(x,t) := w_{-\sigma}^{\mathrm{DR}}(x, t^{\frac{1}{2\sigma}}) - w_{-\sigma}^{\mathrm{DSp}}(x, t^{\frac{1}{2\sigma}})$$

verifies the following relations:

$$\Delta_x \hat{W}_s + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}} \partial^2_{tt} \hat{W}_s = 0 \quad \text{in } \Omega \times \mathbb{R}_+; \quad \partial_t \hat{W}_s|_{t=0} = 0; \quad \hat{W}_s|_{x\in\partial\Omega} > 0.$$

Now the boundary point lemma **[11]** implies $\hat{W}_s(x,0) > 0$, which gives (17) in view of (11) and (12).

C. This statement is more complicated and requires the representation formulae for w_s^{DR} and w_s^{NSp}, see **[4]** and **[20]**, respectively:

$$w_s^{\mathrm{DR}}(x,y) = \mathrm{const} \cdot \int_{\mathbb{R}^n} \frac{y^{2s} u(z)\, dz}{(|x-z|^2 + y^2)^{\frac{n+2s}{2}}};$$

$$w_s^{\mathrm{NSp}}(x,y) = \sum_{j=0}^{\infty} (u,\psi_j)_{L_2(\Omega)} \cdot \mathcal{Q}_s(y\sqrt{\mu_j})\psi_j(x), \quad \mathcal{Q}_s(\tau) = \frac{2^{1-s}\tau^s}{\Gamma(s)}\mathcal{K}_s(\tau)$$

(here $\mathcal{K}_s(\tau)$ stands for the modified Bessel function of the second kind).

First of all, these formulae imply for $u \ge 0$, $u \not\equiv 0$ that

$$\lim_{y\to+\infty} w_s^{\mathrm{DR}}(x,y) = 0; \qquad \lim_{y\to+\infty} w_s^{\mathrm{NSp}}(x,y) = (u,\psi_0)_{L_2(\Omega)} \cdot \psi_0(x) > 0;$$

the second relation follows from the asymptotic behavior (see, e.g., [20, (3.7)])

$$\mathcal{K}_s(\tau) \sim \Gamma(s)2^{s-1}\tau^{-s}, \quad \text{as } \tau \to 0;$$

$$\mathcal{K}_s(\tau) \sim \left(\frac{\pi}{2\tau}\right)^{\frac{1}{2}} e^{-\tau}\big(1 + O(\tau^{-1})\big), \quad \text{as } \tau \to +\infty.$$

Next, for $x \in \partial\Omega$ we derive by convexity of Ω that

$$\partial_{\mathbf{n}} w_s^{\mathrm{DR}}(x,y) = \mathrm{const} \cdot \int_{\mathbb{R}^n} \frac{y^{2s}\langle (z-x),\mathbf{n}\rangle u(z)\,dz}{(|x-z|^2+y^2)^{\frac{n+2s+2}{2}}} < 0.$$

Thus, the difference $\tilde{W}_s(x,y) = w_s^{\mathrm{NSp}}(x,y) - w_s^{\mathrm{DR}}(x,y)$ has the following properties in the half-cylinder $\Omega \times \mathbb{R}_+$:

$$-\operatorname{div}(y^{1-2s}\nabla\tilde{W}_s) = 0; \quad \tilde{W}_s|_{y=0} = 0; \quad \tilde{W}_s|_{y=\infty} > 0; \quad \partial_{\mathbf{n}}\tilde{W}_s|_{x\in\partial\Omega} > 0.$$

By the strong maximum principle, $\tilde{W}_s > 0$ in $\Omega \times \mathbb{R}_+$. Finally, we apply again the boundary point lemma [11] to the function $\tilde{W}_s(x, t^{\frac{1}{2s}})$ and obtain for $x \in \Omega$,

$$\liminf_{y\to 0^+} y^{1-2s}\partial_y\tilde{W}_s(x,y) = 2s \liminf_{t\to 0^+} \frac{\tilde{W}_s(x,t^{\frac{1}{2s}})}{t} > 0.$$

This gives (18) in view of (5) and (8). ■

Notice that for nonconvex domains, the relation (18) does not hold in general. We provide a corresponding counterexample.

Example 5. Put temporarily $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 \cap \Omega_2 = \emptyset$. If $u \ge 0$ is a smooth function supported in Ω_1 then easily $(-\Delta_\Omega)^s_{\mathrm{NSp}} u \equiv 0$ in Ω_2. On the other hand, $w_s^{\mathrm{DR}}(x,y) > 0$ for all $x \in \mathbb{R}^n$, $y > 0$, and the boundary point lemma gives $(-\Delta_\Omega)^s_{\mathrm{DR}} u < 0$ in Ω_2. Now we join Ω_1 with Ω_2 by a small channel, and the inequality $(-\Delta_\Omega)^s_{\mathrm{DR}} u < (-\Delta_\Omega)^s_{\mathrm{NSp}} u$ in Ω_2 holds by continuity.

The last group of results in our survey is related to an obvious identity

$$(-\Delta u, u) = \int_\Omega |\nabla u|^2\,dx = \int_\Omega \big|\nabla|u|\big|^2\,dx = (-\Delta|u|,|u|), \quad u \in \tilde{H}^1(\Omega).$$

The following statement was proved in [13, THEOREM 3].[6]

6 The proof was given for the Dirichlet operators (restricted and spectral); however, it is mentioned in [22, PROPOSITION 1] that for the spectral Neumann FL the proof runs without changes.

Theorem 6. *Let $s \in (0,1)$. Then*

A. *For any $u \in \tilde{H}^s(\Omega)$, we have $|u| \in \tilde{H}^s(\Omega)$ and*

$$Q_s^{\mathrm{DR}}[u] \geq Q_s^{\mathrm{DR}}\big[|u|\big]; \quad Q_s^{\mathrm{DSp}}[u] \geq Q_s^{\mathrm{DSp}}\big[|u|\big];$$

B. *For any $u \in H^s(\Omega)$, we have $|u| \in H^s(\Omega)$ and*

$$Q_s^{\mathrm{NR}}[u] \geq Q_s^{\mathrm{NR}}\big[|u|\big]; \quad Q_s^{\mathrm{NSp}}[u] \geq Q_s^{\mathrm{NSp}}\big[|u|\big].$$

For a sign-changing u, all inequalities are strict.

Proof. For $s \in (0,1]$, the Nemytskii operator $u \mapsto |u|$ is a continuous transform of $H^s(\mathbb{R}^n)$ into itself, see, e.g., [18, THEOREM 5.5.2/3].

There are several proofs of the inequality for Q_s^{DR}; in particular, its representation in Remark 1 provides this inequality immediately. This proof works for Q_s^{NR} as well.

We show another proof that works also for spectral quadratic forms.

Let u be sign-changing. Consider the extension w_s^{DR} and notice that $|w_s^{\mathrm{DR}}| \in \mathcal{W}_s^{\mathrm{DR}}(|u|)$. Therefore,

$$\frac{2s}{C_s} \cdot Q_s^{\mathrm{DR}}\big[|u|\big] = \min_{w \in \mathcal{W}_s^{\mathrm{DR}}(|u|)} \mathcal{E}_s^{\mathrm{R}}(w) \leq \mathcal{E}_s^{\mathrm{R}}\big(|w_s^{\mathrm{DR}}|\big) = \mathcal{E}_s^{\mathrm{R}}(w_s^{\mathrm{DR}}) = \frac{2s}{C_s} \cdot Q_s^{\mathrm{DR}}[u].$$

Moreover, w_s^{DR} is sign-changing, so $|w_s^{\mathrm{DR}}|$ cannot be a solution of the homogeneous equation by the maximum principle and thus cannot be a minimizer for the energy. ■

What happens for $s > 1$? If $s \in (1, \frac{3}{2})$ then the operator $u \mapsto |u|$ is a bounded transform of $H^s(\mathbb{R}^n)$ into itself, see, e.g., [2, SECTION 4]. To the best of our knowledge, its continuity is still an open problem. Moreover, it is easy to show that the assumption $s < \frac{3}{2}$ cannot be improved, see, e.g., [15, EXAMPLE 1].

So, the question about the behavior of quadratic forms of FLs under the transform $u \mapsto |u|$ seems reasonable for $s \in (1, \frac{3}{2})$. The following statement was proved in [15].

Theorem 7. *Let $s \in (1, \frac{3}{2})$, and let $u \in \tilde{H}^s(\Omega)$ be sign-changing. Then*

$$Q_s^{\mathrm{DR}}[u] < Q_s^{\mathrm{DR}}\big[|u|\big]. \tag{22}$$

The sketch of proof. Define $u^{\pm} = \frac{1}{2}(|u| \pm u)$ and assume for a moment that u^+ and u^- are smooth and have disjoint supports. Then

$$Q_s^{\mathrm{DR}}\big[|u|\big] - Q_s^{\mathrm{DR}}[u] = 4\big((-\Delta_\Omega)^s_{\mathrm{DR}} u^+, u^-\big) = 4\big((-\Delta_\Omega)^{s-1}_{\mathrm{DR}} u^+, (-\Delta) u^-\big).$$

By Remark 1,

$$\begin{aligned}
&\big((-\Delta_\Omega)^{s-1}_{\mathrm{DR}} u^+, (-\Delta) u^-\big) \\
&\quad = \frac{c_{n,s-1}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u^+(x) - u^+(y))(-\Delta u^-(x) + \Delta u^-(y))}{|x-y|^{n+2s-2}}\, dx\, dy \\
&\quad = c_{n,s-1} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{u^+(x) \Delta u^-(y)}{|x-y|^{n+2s-2}}\, dx\, dy
\end{aligned}$$

(notice that $u^+(x) u^-(x) \equiv 0$).

Since the supports of u^+ and u^- are disjoint, we can integrate by parts. Using the definition of $c_{n,s}$, we derive

$$\Delta_y \frac{c_{n,s-1}}{|x-y|^{n+2s-2}} = \frac{2s(n+2s-2)c_{n,s-1}}{|x-y|^{n+2s}} = -\frac{c_{n,s}}{|x-y|^{n+2s}}$$

and obtain

$$Q_s^{\mathrm{DR}}[|u|] - Q_s^{\mathrm{DR}}[u] = -4c_{n,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{u^+(x)u^-(y)}{|x-y|^{n+2s}}\, dxdy.$$

It remains to observe that $c_{n,s} < 0$ for $s \in (1,2)$, and (22) follows.

In the general case, the result was obtained in [15] using a quite nontrivial approximation procedure. ■

Conjecture 8. *For $s \in (1, \frac{3}{2})$, the inequalities similar to (22) should hold for spectral quadratic forms.*

ACKNOWLEDGMENTS

I am deeply grateful to my friend and coauthor Professor Roberta Musina for many years of cooperation. Indeed, without her, I could not even get started in this field.

FUNDING

This work was supported by RFBR grant 20-01-00630.

REFERENCES

[1] M. S. Birman and M. Z. Solomyak, *Spectral theory of self-adjoint operators in Hilbert space*. 2nd edn., revised and extended, Lan', St. Petersburg, 2010 (in Russian); English transl. of the 1st ed.: Math. Appl., Sov. Ser., 5, Kluwer, Dordrecht, 1987.

[2] G. Bourdaud and W. Sickel, Composition operators on function spaces with fractional order of smoothness. In *Harmonic analysis and nonlinear partial differential equations*, edited by T. Ozawa and M. Sugimoto, pp. 93–132, RIMS Kôkyûroku Bessatsu B26, Res. Inst. for Math. Sci, Kyoto, 2011.

[3] X. Cabré and Y. Sire, Nonlinear equations for fractional Laplacians. I: Regularity, maximum principles, and Hamiltonian estimates. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31** (2014), no. 1, 23–53.

[4] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* **32** (2007), no. 8, 1245–1260.

[5] A. Capella, J. Dávila, L. Dupaigne, and Y. Sire, Regularity of radial extremal solutions for some non-local semilinear equations. *Comm. Partial Differential Equations* **36** (2011), no. 8, 1353–1384.

[6] Z.-Q. Chen and R. Song, Two-sided eigenvalue estimates for subordinate processes in domains. *J. Funct. Anal.* **226** (2005), 90–113.

[7] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), no. 5, 521–573.

[8] M. M. Fall, Semilinear elliptic equations for the fractional Laplacian with Hardy potential. *Nonlinear Anal.* **193** (2020), 111311, 29 pp. arXiv:1109.5530v4, 2012.

[9] R. L. Frank and L. Geisinger, Refined semiclassical asymptotics for fractional powers of the Laplace operator. *J. Reine Angew. Math.* **712** (2016), 1–37.

[10] E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung. *Math. Ann.* **193** (1951), 415–438 (in German).

[11] L. I. Kamynin and B. N. Himčenko, Theorems of Giraud type for second order equations with a weakly degenerate non-negative characteristic part. *Sib. Math. J.* **18** (1977), 76–91.

[12] R. Musina and A. I. Nazarov, On fractional Laplacians. *Comm. Partial Differential Equations* **39** (2014), no. 9, 1780–1790.

[13] R. Musina and A. I. Nazarov, On the Sobolev and Hardy constants for the fractional Navier Laplacian. *Nonlinear Anal.* **121** (2015), 123–129.

[14] R. Musina and A. I. Nazarov, On fractional Laplacians–2. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33** (2016), no. 6, 1667–1673.

[15] R. Musina and A. I. Nazarov, A note on truncations in fractional Sobolev spaces. *Bull. Math. Sci.* **9** (2019), no. 1, 1950001, 7 pp.

[16] R. Musina and A. I. Nazarov, Strong maximum principles for fractional Laplacians. *Proc. Roy. Soc. Edinburgh Sect. A* **149** (2019), no. 5, 1223–1240.

[17] A. I. Nazarov, On comparison of fractional Laplacians. 2021, arXiv:2108.05416.

[18] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*. De Gruyter Ser. Nonlinear Anal. Appl. 3, de Gruyter, Berlin, 1996.

[19] R. Servadei and E. Valdinoci, On the spectrum of two different fractional operators. *Proc. Roy. Soc. Edinburgh Sect. A* **144** (2014), no. 4, 831–855.

[20] P. R. Stinga and J. L. Torrea, Extension problem and Harnack's inequality for some fractional operators. *Comm. Partial Differential Equations* **35** (2010), no. 11, 2092–2122.

[21] H. Triebel, *Interpolation theory, function spaces, differential operators*. Deutscher Verlag Wissensch, Berlin, 1978.

[22] N. S. Ustinov, On solvability of a semilinear problem with spectral Neumann Laplacian and critical right-hand side. *Algebra i Analiz* **33** (2021), no. 1, 194–212 (Russian).

ALEXANDER I. NAZAROV

PDMI RAS, Fontanka 27, St. Petersburg 191023, Russia, al.il.nazarov@gmail.com