FORMATION OF SINGULARITIES IN NONLINEAR DISPERSIVE PDES

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ABSTRACT

This contribution addresses the problem of singularity formation in nonlinear dispersive equations. Despite significant progress made in the last 20 years, for most even simplest canonical models our understanding of the question is far from being complete. The aim of this note is to give a selection of results and open questions illustrating the present state of the problem in the context of some basic model equations, mostly of Schrödinger type, such as the semilinear Schrödinger and Schrödinger map equations, putting an emphasis on the role of solitons in the mechanisms of singularity formation.

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1. INTRODUCTION

Many physical processes involving nonlinear evolution of wave-like objects are modeled by semilinear Hamiltonian PDEs of dispersive type. Among the canonical examples are the nonlinear Schrödinger equation (NLS)

$$iu_t = -\Delta u + \mu|u|^{2p}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

and the nonlinear wave equation (NLW)

$$u_{tt} = \Delta u - \mu|u|^{2p}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1.1}$$

where $p > 0$ and $\mu \in \{-1, 1\}$. The equations are said to be focusing if $\mu = -1$ and defocusing if $\mu = 1$.

Other important examples are the wave and Schrödinger map equations. They are respectively the hyperbolic and Schrödinger analogues of the harmonic map heat flow, which is the gradient flow associated to the Dirichlet energy $\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx$ for maps u from $\mathbb{R}^d$ to an embedded Riemannian manifold[1] $M \subset \mathbb{R}^n$. We will limit ourselves to the case of $M = \mathbb{S}^2 \subset \mathbb{R}^3$, where the equations take a particular simple form:

$$u_{tt} = \Delta u + u\big(|\nabla u|^2 - |u_t|^2\big) \tag{1.2}$$

for wave maps,

$$u_t = u \times \Delta u$$

for Schrödinger maps, and

$$u_t = \Delta u + u|\nabla u|^2$$

for the heat flow. Here u is a map from $\mathbb{R}_t \times \mathbb{R}^d_x$ to $\mathbb{S}^2 \subset \mathbb{R}^3$.

Of course, the first question in the theory of such equations is the local well-posedness of the corresponding Cauchy problem (including existence, uniqueness, and continuous dependence of solutions on initial data). But once the local well-posedness is understood, which is often the case at least for the simplest models, the next step is to study the qualitative behavior of solutions, and in particular to answer the following questions:

- Do all maximal solutions exist globally in time or does finite time blow-up occur? If yes, for what classes of initial data?
- If the solution blows up in finite time, can one determine when, where, and how the singularities form?
- If the solution is global, can one determine its behavior as $t \to \infty$?

Despite substantial progress made in the last 20–30 years, a complete answer to most of these questions remains an open problem even for the relatively simple models. The general belief is that in nonlinear dispersive equations the linear dispersion tends to stabilize the dynamics, leading to a kind of universality in the long-time behavior: global solutions

1 For the Schrödinger map equation, one needs M to be a Kähler manifold.

are expected to decompose asymptotically as $t \to \infty$ into a sum of decoupled nonlinear bound states, such as solitary wave solutions, plus a radiation that disperses to zero, typically as a free linear wave. This prediction is known as the soliton resolution conjecture and is motivated by the theory of completely integrable equations such as the one-dimensional cubic NLS, KdV, and mKdV equations, for which this kind of behavior can be justified by means of the inverse scattering method at least for some classes of initial data. For the nonintegrable equations, this conjecture remains largely open. Most of the available results concern either the soliton-free dynamics typical for the defocusing nonlinearities or small data (usually below some threshold determined by the ground state of the equation), or the perturbative regimes near a single soliton or near a superposition of well-decoupled solitons. The only exceptions are wave-type models, such as the energy critical NLW equation ((1.1) with $p = \frac{2}{d-2}$, $d \geq 3$) and the energy critical wave maps into the two sphere ((1.2) with $d = 2$), for which satisfactory global results begin to emerge starting from the breakthrough work of Duyckaerts, Merle, and Kenig [25], where a resolution into solitons was established for all energy bounded radial solutions of the energy critical nonlinear wave equation in dimension 3 (see Section 5). For the Schrödinger-type models, such results are still out of reach. The only known global results in this setting correspond to a much weaker version of the soliton resolution conjecture as that proved by Tao [106] for the nonlinear Schrödinger equations with mass supercritical and energy subcritical nonlinearities in high dimensions. This weak version gives a decomposition of any global, energy bounded solution into a dispersive part that evolves according to the linear Schrödinger equation, a sum of decoupled pieces, each piece evolving (modulo the space translations) on a compact invariant set, and a remainder going to zero as $t \to \infty$ in the energy space, thus reducing the problem to a classification of solutions with a compact trajectory, the question which is largely open for the NLS equations.

In the case of finite time blow-up, even less is known. For the Schrödinger-type equations, the theory is still on the level of searching for possible blow-up mechanisms and studying their stability. Below we give a selection of corresponding results. The choice of the results is unavoidably related to those aspects of the problem which are most familiar to the author, the list of the references is by no means complete.

2. OVERVIEW OF THE WELL-POSEDNESS THEORY FOR THE NLS EQUATION

Consider the nonlinear Schrödinger equation on $\mathbb{R}^d$:

$$iu_t = -\Delta u + \mu|u|^{2p}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad p > 0, \quad \mu \in \{-1, 1\}, \tag{2.1}$$

with initial condition

$$u|_{t=0} = u_0 \in H^s(\mathbb{R}^d). \tag{2.2}$$

The solutions to (2.1), (2.2) satisfy formally mass, energy, and momentum conservation laws:

$$M\big(u(t)\big) \equiv \int_{\mathbb{R}^d} \big|u(t,x)\big|^2 dx = M(u_0),$$
$$E\big(u(t)\big) \equiv \int_{\mathbb{R}^d} \bigg(\big|\nabla u(t,x)\big|^2 + \frac{\mu}{p+1}\big|u(t,x)\big|^{2p+2}\bigg) dx = E(u_0),$$
$$P\big(u(t)\big) \equiv \operatorname{Im} \int_{\mathbb{R}^d} \overline{u(t,x)} \nabla u(t,x) dx = P(u_0).$$

The NLS equation is invariant with respect to time translations, spatial translations and rotations, and phase rotations. A less evident symmetry is the invariance under Galilei transformations, $u(t,x) \mapsto e^{-i\frac{|v|^2}{4}t + i\frac{v}{2}\cdot x} u(t, x - vt)$, $v \in \mathbb{R}^d$. In the case of $p = \frac{2}{d}$, there is an additional symmetry

$$u(t,x) \mapsto \frac{1}{|t|^{\frac{d}{2}}} e^{i\frac{|x|^2}{4t}} u\bigg(-\frac{1}{t}, \frac{x}{t}\bigg), \quad t \neq 0, \tag{2.3}$$

called pseudoconformal symmetry.

The NLS equation (2.1) is also invariant with respect to the scaling, $u(t,x) \mapsto \lambda^{\frac{1}{p}} u(\lambda^2 t, \lambda x)$, $\lambda > 0$, that preserves the homogeneous Sobolev norm $\|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$ with $s_c = \frac{d}{2} - \frac{1}{p}$. This defines a notion of criticality: the Cauchy problem (2.1), (2.2) is said to be subcritical if $s > s_c$, critical if $s = s_c$, and supercritical if $s < s_c$. As we will see below, the notion of criticality plays a fundamental role in the well-posedness theory of (2.1). Of a particular interest are the mass critical case $s_c = 0$ and the energy critical case $s_c = 1$ when the critical regularity coincides with one of the conservation laws.

The local well-poseness of the NLS equation is well understood (see, e.g., [10, 105] and the references therein). The Cauchy problem (2.1), (2.2) is locally well-posed in H^s for[2] $s \geq \max\{0, s_c\}$, and typically, also in $\dot{H}^{s_c}$ if $s_c \geq 0$. In the latter case the solutions arising from $\dot{H}^{s_c}$ small initial data are global and scatter both forward and backward in time (i.e., converge to a linear solution as $t \to \pm\infty$).

In the subcritical case $s > s_c$, the lifespan of the solutions admits a lower bound depending only on the H^s norm of initial data,[3] which in a standard way implies that the solution of (2.1), (2.2) is either global or its H^s norm becomes unbounded in finite time. By the mass and energy conservation, this ensures global well-posedness in H^s for $s \geq 0$ in the mass subcritical range $p < \frac{2}{d}$ independently of the sign of μ, and in H^s with $s \geq 1$ in the defocusing energy subcritical case $p < \frac{2}{d-2}$.

The critical well-posedness ($s = s_c \geq 0$) is more subtle. In this case the lifespan of solutions given by the local theory depends on the profile of the initial data, not only on its $\dot{H}^{s_c}$ norm. In the defocusing case, however, typically the uniform boundedness of the solution in $\dot{H}^{s_c}$ on its maximal interval of existence implies that the solution is global and scatters. In particular, one has global well-posedness and scattering in $\dot{H}^{s_c}$ for the defocusing

2 In the case when p is not an integer, one has also to assume that s is compatible with the smoothness of the nonlinear term.

3 In fact, typically one has a slightly stronger result including the persistence of regularity: if $u_0 \in H^{s'}$ with $s' > s$, then the solution stays in $H^{s'}$ as long as it exists in H^s.

energy critical ($s_c = 1$) and mass critical ($s_c = 0$) NLS equations. In the energy critical case, this was proved by Bourgain [6], Grillakis [34], Tao [104] for spherically symmetric initial data, and by Colliander, Keel, Staffilani, Takaoka, and Tao [12], Ryckman and Visan [98], and Visan [111] for general data. We also refer to the seminal paper of Kenig and Merle [48] where the powerful concentration compactness/rigidity method was introduced. The corresponding result for the mass critical NLS was proved by Killip, Tao, Visan, and Zhang [50,53,109] in the case of spherically symmetric initial data, and by Dodson for general initial data, see [17] and the references therein. In the energy supercritical case $s_c > 1$, the fact that the $\dot{H}^{s_c}$ bounds imply global existence and scattering was established by Killip and Visan [51] in dimension $d \geq 5$. We also refer to Miao, Murphy, and Zheng [84] for the case of $d = 4$. Similar results hold in the mass supercritical, energy subcritical range, see, e.g., Kenig and Merle [49], although in this case unconditional global existence and scattering in $\dot{H}^{s_c}$ is expected, see Dodson [19] for some partial results in this direction as well as for the history of the problem. In the energy supercritical case, finite time blow-up may occur. This has been recently proved by Merle, Raphaël, Rodnianski, and Szeftel [79], see Section 6.

For the focusing NLS, the picture is different. On the one hand, large initial data may lead to finite time blow-up as soon as $0 \leq s_c$, and, on the other hand, if $s_c \leq 1$, the equation admits solitary wave solutions, which shows that even for global in time solutions scattering may not occur.

The existence of finite time blow-up for the focusing NLS with $p \geq \frac{2}{d}$ follows from the virial identity [33]:

$$\frac{d^2}{dt^2} \int |x|^2 \left|u(x,t)\right|^2 dx = 8E(u) - \frac{4(dp-2)}{p+1} \int_{\mathbb{R}^d} \left|u(t,x)\right|^{2p+2} dx,$$

which holds for finite variance $H^1 \cap H^{s_c}$ solutions of (2.1), and shows that if $E(u) < 0$, then the solution breaks down in finite time. Note that in the mass critical case, any blow-up solution is trivially bounded in the critical Sobolev space. For the energy critical NLS, one also might have blow-up solutions with bounded $\dot{H}^1$ norm. In the case $0 < s_c < 1$, the situation is different: solutions that stay bounded in $\dot{H}^{s_c}$ are expected to be global. In the radial case this property was proved by Merle and Raphaël [76].

As mentioned above, the focusing NLS with $s_c \leq 1$ admits a family of solitary wave solutions. In the energy subcritical range $0 < p < \frac{2}{d-2}$, they have the form

$$u(t,x) = e^{i(\omega - \frac{v^2}{4})t + i\frac{v}{2}\cdot x} Q_\omega(x - vt),$$

where $v \in \mathbb{R}^d$, $\omega > 0$, and the profile Q_ω solves the elliptic equation

$$-\Delta Q_\omega + \omega Q_\omega - |Q_\omega|^{2p} Q_\omega = 0,$$

which after the rescaling $Q_\omega(y) = \omega^{\frac{1}{2p}} Q(\omega^{\frac{1}{2}} x)$ takes the form

$$-\Delta Q + Q - |Q|^{2p} Q = 0. \tag{2.4}$$

For any $0 < p < \frac{2}{d-2}$, there is a unique positive radially symmetric H^1 solution to this equation, called ground state (see, e.g., [10,105] and the references therein); the ground state solution is smooth and exponentially decaying. In the one-dimensional case, the ground state is explicit, namely $Q(x) = \frac{(p+1)^{\frac{1}{2p}}}{\cosh^{\frac{1}{p}}(px)}$. If $d \geq 2$, equation (2.4) has also radial H^1 solutions which change sign (called exited states).

In the energy critical case, solitary wave solutions are given by

$$u(t,x) = e^{-i\frac{v^2}{4}t + i\frac{v}{2}\cdot x} W(x - vt),$$

where $v \in \mathbb{R}^d$ and W is a stationary solution, satisfying

$$\Delta W + |W|^{\frac{4}{d-2}} W = 0, \quad W \in \dot{H}^1(\mathbb{R}^d). \tag{2.5}$$

The radial solutions of this elliptic equation are completely classified: they are of the form $W_{\alpha,\lambda}(x) = e^{i\alpha}\lambda^{\frac{d-2}{2}} W(\lambda x)$, where

$$W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}. \tag{2.6}$$

3. MASS CRITICAL FOCUSING NLS

In this section we consider the mass critical NLS

$$\begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{d}} u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = u_0. \end{cases} \tag{3.1}$$

3.1. Global existence and scattering below the ground state

For $p \geq \frac{2}{d}$, the local well-posedness theory ensures global existence and scattering for initial data with small $\dot{H}^{s_c}$ norm. Typically, in the range $0 \leq s_c \leq 1$, this smallness can be related to the ground state of the problem. In the case of the mass critical NLS, one has:

Theorem 3.1 (Global existence and scattering below the ground state). *For any $u_0 \in L^2(\mathbb{R}^d)$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the solution to (3.1) is global and scatters forward and backward in time (that is, there exist $u_-, u_+ \in L^2$ such that $\|u(t) - e^{i\Delta t}u_\pm\|_{L^2} \to 0$ as $t \to \pm\infty$).*

This result has a long history. In the case of H^1 solutions, the global existence follows from the variational characterization of the mass critical ground state proved by M. Weinstein in [112]:

$$\|f\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} \leq \frac{d+2}{d}\left(\frac{\|f\|_{L^2}^{\frac{4}{d}}}{\|Q\|_{L^2}^{\frac{4}{d}}}\right)\|\nabla f\|_{L^2}^2, \quad \forall f \in H^1, \tag{3.2}$$

the equality being achieved if and only if $f(x) = zQ(\lambda(x-a))$ for some $z \in \mathbb{C}$, $\lambda > 0$ and $a \in \mathbb{R}^d$. Inequality (3.2) shows that the H^1 norm of the solutions is controlled by their mass and energy as soon as $M(u) < M(Q)$. Global existence and scattering for L^2 data with finite invariance is also classical, see, e.g., [10]. The general L^2 result is much more difficult and has been proved only recently, see Killip, Tao, Visan, Zhang [50,53] and Dodson [16].

Applying the pseudoconformal transformation to the soliton $e^{it}Q$ gives an explicit blow-up solution

$$S(t) = \frac{1}{t^{\frac{d}{2}}} e^{i\frac{|x|^2}{4t} - i\frac{1}{t}} Q\left(\frac{x}{t}\right) \tag{3.3}$$

that has the same mass as the ground state Q. Thus the bound $M(u) < M(Q)$ is optimal not only for scattering but also for global existence.

The minimal mass dynamics is also well understood. In [72] Merle proved that up to the symmetries of the equation, $S(t)$ is the only H^1 minimal mass finite time blow-up solution, see also Hmidi, Keraani [40] for a simplified proof. By the pseudoconformal invariance, this result also implies that any global minimal mass nonscattering H^1 solution with finite variance is a ground state solitary wave. The L^2 case was studied by Dodson [20, 21] who proved:

Theorem 3.2 (Threshold dynamics, Dodson [20, 21]). *Let $1 \le d \le 15$ and consider $u_0 \in L^2(\mathbb{R}^d)$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Then either the solution of (3.1) is global and scatters as $t \to \pm\infty$ or it coincides with $e^{it}Q$ up to the symmetries of the equation (including the pseudoconformal symmetry).*

We next turn to the case $M(u) > M(Q)$. In this case the virial identity ensures the existence of a large set of initial data leading to finite time blow-up both forward and backward in time, but gives no information on the structure of the singularity, and for general large mass data little is known in this direction. Essentially, only two general results are available. First, one has the following lower bound on the blow-up rate of H^s solutions, which is a direct consequence of the scaling invariance of the problem: if u_0 is in H^s with $s > 0$, such that the corresponding solution u blows up in finite time $T > 0$, then

$$\|u(t)\|_{\dot{H}^s} \ge \frac{C(u_0)}{(T-t)^{\frac{s}{2}}}, \quad \forall t \in [0, T[.$$

Second, it is known that any blow-up solution concentrates at the blow-up time at least the mass of the ground state. This is a consequence of Theorem 3.1. We refer to [50] and to the references therein for the precise statements and for the history of the L^2 concentration results. For masses slightly above the critical mass, more can be done. We discuss the corresponding results in the next subsection.

3.2. Near ground state blow-up dynamics

In this subsection, we focus on the H^1 blow-up solutions with mass slightly above the critical mass:

$$u_0 \in H^1(\mathbb{R}^d), \quad \|Q\|_{L^2} < \|u_0\|_{L^2} \le \|Q\|_{L^2} + \alpha, \quad 0 < \alpha \ll 1. \tag{3.4}$$

The mass and energy conservation, together with the variational characterization of the ground state (3.2), ensures that near the blow-up time these solutions behave as a modulated ground state, admitting a decomposition of the following form:

$$u(t,x) = \lambda^{\frac{d}{2}}(t) e^{i\mu(t)}\big(Q(z) + r(t,z)\big), \quad z = \lambda(t)\big(x - q(t)\big),$$

with $\lambda(t) \sim \|\nabla u(t)\|_{L^2}$, $\|r(t)\|_{H^1} \ll 1$. Although giving no information on the blow-up rate $\lambda(t)$ and on the blow-up location $q(t)$, this variational result is conceptually important, showing that the blow-up profiles arising from initial data (3.4) are close to the ground state, and thus providing a starting point for their perturbative analysis. Such analysis was initiated in [90] where we considered the one-dimensional mass critical NLS with even initial data of the form $u_0 = Q + \eta_0$, $\|\eta_0\|_{H^1} + \|x\eta_0\|_{L^2} \ll 1$, and showed that for an open set of initial perturbations η_0 the corresponding solution u blows up in finite time $T > 0$ with the following asymptotic behavior as $t \to T$:

$$u(t,x) = e^{i\mu(t)}\lambda^{\frac{1}{2}}(t)\big(Q\big(\lambda(t)x\big) + r\big(t, \lambda(t)x\big)\big), \quad \big\|r(t)\big\|_{H^1} \ll 1, \quad \big\|r(t)\big\|_{L^\infty} = o(1),$$
$$\lambda(t) = \left(\frac{\ln|\ln(T-t)|}{2\pi(T-t)}\right)^{1/2}\big(1 + o(1)\big). \tag{3.5}$$

The existence of a stable blow-up regime with the log-log blow up rate (3.5) was predicted by numerical computations and formal arguments in a number of works, see, e.g., Landman, LeMesurier, Papanicolaou, Sulem, and Sulem [62, 64], Smirnov and Fraiman [99], Sulem and Sulem [102] and the references therein. In the H^1 setting the log-log blow-up regime was studied in details by Merle and Raphaël. Assuming some coercivity property of the linearization around Q, they proved the following (see [66,73–75,93], and the references therein).

Theorem 3.3 (Merle, Raphaël). (i) *Any solution arising from initial data* (3.4) *and blowing up in finite time T admits a representation of the form*

$$u(t) = e^{i\mu(t)}\lambda^{\frac{d}{2}}(t)Q\big(\lambda(t)\big(\cdot - q(t)\big)\big) + u^* + o_{L^2}(1), \quad t \to T,$$

with $\lim_{t\to T}\lambda(t) = +\infty$, $\lim_{t\to T} q(t) = q^* \in \mathbb{R}^d$, *and* $u^* \in L^2(\mathbb{R}^d)$. *Furthermore, one of the following alternatives holds:*

- *either the blow-up rate $\lambda(t)$ satisfies the log-log law* (3.5) *and then the limiting profile u^* does not belong to*[4] H^1,
- *or*
$$\lambda(t) \geq \frac{c(u_0)}{T-t} \tag{3.6}$$
and then $u^* \in H^1$.

(ii) *The set of initial data satisfying* (3.4) *and such that the corresponding solution blows up in finite time in the log-log regime* (3.5) *is open in H^1 and contains the initial data* (3.4) *with $E(u_0) \leq 0$.*

The coercivity property required in Theorem 3.3 was proved in dimension 1 in [73] and checked numerically for $2 \leq d \leq 4$ in [30], and for $5 \leq d \leq 10$ in [113].

4 More precisely, in this case one has $\int_{|x-q(t)|\leq R}|u^*(x)|^2dx \sim \frac{C}{(\ln|\ln R|)^2}$, as $R \to 0$.

The log-log blow-up regime of Theorem 3.3 is known to remain stable under H^s perturbations of initial data for all $s > 0$, see Colliander and Raphaël [13]. An interesting open question is wether this stability persists in the L^2 setting.

The set of initial data (3.4) such that the corresponding solution blows up satisfying (3.6) is nonempty. A large class of solutions with the pseudoconformal blow-up rate $\|\nabla u(t)\|_{L^2} \sim \frac{1}{T-t}$ was constructed by Bourgain and Wang [7] in dimensions 1 and 2, starting from perturbations of the minimal blow-up solution (3.3) by smooth rapidly decaying limiting profiles u^* vanishing at zero to a sufficiently large order, see also Krieger and Schlag [58]. In [82] Merle, Raphaël, and Szeftel proved that the Bourgain–Wang solutions with slightly supercritical mass are unstable, belonging to the boundary of the H^1 open sets of global solutions that scatter both forward and backward in time, and solutions that blow up in finite time in the log-log regime. It is not known whether blow-up solutions with a blow-up rate strictly greater than the pseudoconformal rate exist in the regime (3.4). For larger masses, an example of such solutions was constructed by Martel and Raphaël [68] in the 2D case. The solutions of [68] are obtained by considering the interaction of K solitary waves concentrated at the vertices of a K-sided regular polygon and showing that this leads to a $KM(Q)$-mass solution blowing up at infinity with the rate $\|\nabla u(t)\|_{L^2} \sim \ln t$ as $t \to +\infty$, and thus, after applying the pseudoconformal transformation (2.3), to a solution that blows up as $t \to 0$ with the rate $\|\nabla u(t)\|_{L^2} \sim \frac{|\ln t|}{t}$.

We also refer to [66, 67] and references therein for the results on the near soliton blow-up dynamics for the mass-critical gKdV equation where the picture is more complete.

The regimes discussed above correspond to a single point finite time blow-up. The examples of multipoint blow-up solutions can be also constructed using as building blocks either the explicit blow-up solution (3.3) (see Merle [71]), or the log-log blow-up solutions of Theorem 3.3 (see Fan [28]). The general conjecture is that for any finite time blow-up solution, the singular set is given by a finite number of points, each point concentrating at least the mass of the ground state, see, e.g., [74].

4. MASS SUPERCRITICAL, ENERGY SUBCRITICAL NLS

In this section we discuss briefly the known blow-up regimes for the focusing NLS in the range $\frac{2}{d} < p < \frac{2}{d-2}$.

4.1. Self-similar blow-up

Numerical simulations and formal arguments (see, e.g., Sulem and Sulem [102] and the references therein) strongly suggest the existence of stable blow-up solutions of the following self-similar form:

$$u(t,x) \approx \frac{1}{(2b(T-t))^{\frac{1}{2p}}} e^{-i\frac{1}{2b}\ln(T-t)} V\left(\frac{x}{(2b(T-t))^{\frac{1}{2}}}\right), \quad b > 0. \tag{4.1}$$

Substituting this ansatz into the NLS equation leads to the following elliptic equation for the profile $V(y)$:

$$-\Delta V + V - ib\left(\frac{1}{p} + y \cdot \nabla\right)V - |V|^{2p}V = 0, \quad y \in \mathbb{R}^d. \tag{4.2}$$

It is expected that for a discrete set of values of b, this equation admits nontrivial zero-energy radial solutions, although these solutions fail to belong to $\dot{H}^{s_c}$ (in accordance with the growth of the $\dot{H}^{s_c}$ norm proved in [75]) due to a slow decay at infinity, $V(y) \sim \frac{C}{|y|^{\frac{1}{p}+\frac{i}{b}}}$, as $|y| \to \infty$. Thus, to obtain, say, H^1 solutions, one has to view (4.1) as a local approximation near the blow-up point ($x = 0$), and then to extend it to the region $|x| \gg \sqrt{T-t}$ by a well-localized time-independent profile, smooth away from the origin and behaving as $\frac{C}{|x|^{\frac{1}{p}+\frac{i}{b}}}$ near the origin.

Rigorous results justifying the above self-similar blow-up scenario are currently available only in the case $0 < s_c \ll 1$ where bifurcation-type arguments starting from the mass critical case can be used, see Merle, Raphaël, and Szeftel [81], Bahri, Martel, and Raphaël [2].

4.2. Standing sphere and contracting sphere blow-up solutions

In addition to the self-similar blow-up (4.1), which is expected to be generic, two other blow-up regimes are known for the mass supercritical NLS. The first is given by the so-called standing sphere blow-up solutions discovered by Raphaël [94] in the context of the two-dimensional quintic NLS and later on generalized to the quintic NLS in higher dimensions $d \geq 3$ by Raphaël and Szeftel [96]. Standing sphere blow-up solutions are radial, stable in their symmetry class solutions that blow up in finite time on a fixed sphere in the 1D log-log regime. The heuristic behind these solutions is that in the radial setting the quintic NLS takes the following form:

$$iu_t = -\partial_r^2 u - \frac{d-1}{r}\partial_r u - |u|^4 u, \quad r = |x|,$$

where for solutions concentrated near a fixed sphere $r = r_0 > 0$ the second term on the right-hand side can be viewed as a lower order term. Thus, one can expect the dynamics to be governed by the one-dimensional quintic NLS,

$$iu_t = -\partial_r^2 u - |u|^4 u,$$

for which one has a stable log-log blow-up regime. The above idea of reduction to a lower dimensional mass critical NLS was adapted to the 3D cylindrically symmetric cubic NLS by Holmer and Roudenko [43] and Zwiers [114], yielding the existence of finite time blow-up solutions concentrating on a fixed circle in the 2D log-log regime. These are the only known examples of blow-up solutions with a nontrivial blow-up set.

Another blow-up scenario occurs in the range $d \geq 2$, $\frac{2}{d} < p < 5$. In this case there exist radial solutions, called contracting sphere blow-up solutions, that blow up in finite time by concentration of the corresponding 1D ground state on a sphere of radius[5] $\sim t^{\frac{\alpha}{1+\alpha}}$ at the

5 With blow-up time set to $t = 0$.

rate $\sim t^{\frac{1}{1+\alpha}}$ with $\alpha = \frac{2-p}{p(d-1)}$:

$$u(t,x) \approx e^{i\theta(t)+iv(t)r/2}\lambda(t)Q\big(\lambda(t)\big(r-q(t)\big)\big), \quad r = |x|,$$

where

$$q(t) \sim t^{\frac{\alpha}{\alpha+1}}, \quad \lambda(t) \sim t^{-\frac{1}{\alpha+1}}, \quad v(t) \sim t^{-\frac{1}{\alpha+1}}, \quad \theta \sim t^{\frac{\alpha-1}{\alpha+1}},$$

and $Q(y) = \frac{(p+1)^{\frac{1}{2p}}}{\cosh^{\frac{1}{p}}(py)}$. The contracting sphere blow-up was predicted numerically and heuristically in [29, 42], see also the references therein. Rigorously, the existence of contracting sphere blow-up solutions was proved for the 3D cubic NLS in [41], and in the range $d \geq 2$, $0 < s_c < 1$, $p < 5$ by Merle, Raphaël, and Szeftel in [83].

Both the standing sphere and contracting sphere blow-up are L^2-concentration mechanisms in a contrast to the self-similar collapse where no mass concentration occurs.

5. TYPE II BLOW-UP IN THE ENERGY CRITICAL MODELS

In the last 15–20 years there have been significant developments in the study of the blow-up phenomenon for the energy critical equations and, more specifically, in the study of energy bounded blow-up solutions (the so-called type II blow-up), including their constructions and in some cases their classification. Below we review some of these developments.

5.1. Blow-up for Schrödinger maps from $\mathbb{R}^2$ to $\mathbb{S}^2$

Consider the Schrödinger flow for maps from $\mathbb{R}^2$ to $\mathbb{S}^2$:

$$\begin{aligned} &u_t = u \times \Delta u, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \\ &u|_{t=0} = u_0, \end{aligned} \tag{5.1}$$

where $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x)) \in \mathbb{S}^2 \subset \mathbb{R}^3$. Equation (5.1) is a special case of the Landau–Lifshitz equation

$$u_t = a_1 u \times \Delta u + a_2\big(\Delta u + |\nabla u|^2 u\big), \quad a_1 \in \mathbb{R}, \quad a_2 \geq 0, \tag{5.2}$$

arising in the theory of ferromagnetism. In the case $a_1 = 0$, $a_2 = 1$, one recovers the harmonic map heat flow.

Schrödinger map equation (5.1) conserves the energy

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^2} dx |\nabla u|^2. \tag{5.3}$$

The problem is energy critical since both the equation (5.1) and energy (5.3) are invariant with respect to the scaling $u(t,x) \to u(\lambda^2 t, \lambda x)$, $\lambda \in \mathbb{R}_+$.

The local/global well-posedness of (5.1) has been extensively studied. Local existence for smooth initial data goes back to Sulem, Sulem, and Bardos [103], see also McGahagan [70]. The case of small data of low regularity was studied in several works. Global existence for equivariant small energy initial data was established by Chang, Shatah, and

Uhlenbeck in [11]. We recall that a map $u : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ is called equivariant if it has the form

$$u(x) = e^{m\theta R} v(r), \quad v : \mathbb{R}_+ \to \mathbb{S}^2 \subset \mathbb{R}^3, \tag{5.4}$$

for some $m \in \mathbb{Z}$. Here (r, θ) are the polar coordinates in $\mathbb{R}^2$, $x_1 + ix_2 = e^{i\theta} r$, and R is the generator of the horizontal rotations,

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

or equivalently, $Ru = \mathbf{k} \times u$, $\mathbf{k} = (0, 0, 1)$. The m-equivariance is preserved by the Schrödinger flow (5.1). Global existence and scattering for general small energy initial data was proved by Bejenaru, Ionescu, Kenig, and Tataru in [3]. For large data, such a result cannot hold because of the existence of a rich family of finite energy stationary solutions, i.e., harmonic maps. The lowest energy giving rise to a nontrivial harmonic map is 4π, the corresponding harmonic map is, up to the symmetries, the stereographic projection

$$\phi_1(x) = e^{\theta R} Q(r), \quad Q = \left(\frac{2r}{r^2+1}, 0, \frac{r^2-1}{r^2+1} \right). \tag{5.5}$$

The stereographic projection is a member of a family of equivariant harmonic maps ϕ_m, $m \in \mathbb{Z}^+$:

$$\begin{aligned} \phi_m(x) &= e^{m\theta R} Q^m(r), \quad Q^m = (h_1^m, 0, h_3^m) \in \mathbb{S}^2, \\ h_1^m(r) &= \frac{2r^m}{r^{2m}+1}, \quad h_3^m(r) = \frac{r^{2m}-1}{r^{2m}+1}. \end{aligned} \tag{5.6}$$

All these maps are minimizers of the energy in their homotopy class. Namely, one has

$$\deg \phi_m = m, \quad \mathcal{E}(\phi_m) = 4\pi m.$$

The general threshold conjecture is that global existence and scattering hold for all zero homotopy data with energies $\mathcal{E}(u) < 8\pi$. The corresponding result for the wave map equation (1.2) follows from the works of Sterbenz and Tataru [100], where the wave maps from $\mathbb{R}^2$ into general compact target manifold were considered, and it was shown that any smooth finite energy solution to the wave map equation is either global and scatters in a suitable sense or concentrates a nontrivial harmonic map at its maximal time of existence; see Lawrie and Oh [63]. Under symmetry reductions, the results of this type were obtained earlier, see, e.g., Struwe [101] and the references therein. To the best of author's knowledge, for the Schrödinger map equation (5.1) in full generality, the threshold conjecture is open. In the equivariant setting, global existence and scattering for initial data with $\mathcal{E}(u_0) < 4\pi$ was proved by Bejenaru, Ionescu, Kenig, and Tataru [4].

Local existence for the Scrödinger map equation (5.1) in the energy space in the case of nontrivial homotopy equivariant initial data with energies slightly above the energy of the ground state,

$$u_0 \in \Sigma_m, \quad \mathcal{E}(u_0) \le 4\pi m + \varepsilon^2, \quad 0 < \varepsilon \ll 1, \tag{5.7}$$

where $\Sigma_m = \{u(x) = e^{m\theta R}v(r) \in \mathbb{S}^2 \subset \mathbb{R}^3 : E(u) < \infty, v(0) = -v(\infty) = -\mathbf{k}\}$, was established by Gustafson, Kang, and Tsai [36]. The conservation of energy, together with the inequality [35]

$$\mathrm{dist}^2_{\dot{H}^1}(u, \mathcal{S}_m) \lesssim E(u) - 4\pi m, \quad \forall \quad u \in \Sigma_m,$$

where $\mathcal{S}_m = \{e^{\alpha R}\phi_m(\lambda x), \alpha \in \mathbb{R}, \lambda > 0\}$, ensures that the Schrödinger maps with initial data (5.7) have the form $u(t,x) = e^{\alpha(t)R}\phi_m(\lambda(t)x) + O_{\dot{H}^1}(\varepsilon)$, which reduces the problem to understanding the behavior of the functions $\lambda(t)$ and $\alpha(t)$. It was shown by Gustafson, Kang, and Tsai [36], as well as Gustafson, Nakanishi, and Tsai [37], that if $m \geq 3$, then the initial data (5.7) lead to global solutions that, for all t, remain $\dot{H}^1$-close to the initial soliton $e^{\alpha(0)R}\phi_m(\lambda(0)x)$ and, furthermore, scatter as $t \to \infty$ to a nearby member of the family $\mathcal{S}_m$ (in fact, in [37] this is proved for the general Landau–Lifshitz equation (5.2)). The paper [37] treats also the case of $m = 2$ for the harmonic map heat flow under further restriction $v_2 = 0$, showing that global existence persists in this case while the stability may fail: as $t \to +\infty$, the solutions still converge to the family $\mathcal{S}_2$ but the evolution along this family described by the parameter $\lambda(t)$ does not necessarily converge or stay close to a particular harmonic map, more complicated asymptotics of $\lambda(t)$ occur as well. It is natural to expect similar behavior for the 2-equivariant Schrödinger maps. The case of $m = 1$ was studied by Bejenaru and Tataru in [5] where it was proved that ϕ_1 is unstable in $\dot{H}^1$ but stable within its equivariant class in some smaller space that includes H^1.

The question of existence of finite time blow-up for both the wave and Schrödinger maps from $\mathbb{R}^2$ to $\mathbb{S}^2$ has been a long standing problem. The bubbling results mentioned above show that for the wave maps the only possible scenario for singularity formation is by concentration of a nontrivial harmonic map. The first rigorous confirmations of this scenario were obtained by Krieger, Schlag, and Tataru [60], Rodnianski–Sterbenz [97], and Raphaël– Rodnianski [95]. Krieger, Schlag, and Tataru considered the equivariant wave maps (that is, the wave maps of the form (5.4) with $v(t,r) = (\sin\varphi(t,r), 0, \cos\varphi(t,r)))$ of corotation index 1 and showed that for any $\nu > \frac{1}{2}$ there exist initial data arbitrary close to ϕ_1 in $\dot{H}^1$, leading to finite time blow-up solutions of the form $u(t,x) \approx \phi_1(\lambda(t)x)$ with $\lambda(t) = (T-t)^{-1-\nu}$. The specificity of these solutions is that they are of finite Sobolev regularity depending on the blow-up rate. Namely, one has $u \in \dot{H}^{1+\nu-}$. The construction of [60] was extended to the whole range $\nu > 0$ by Gao and Krieger [32] ($\nu \leq 0$ is precluded by the concentration results of [100, 101]). Similar results were obtained for the focusing energy critical wave equation in dimension 3 by Krieger, Schlag, and Tataru [59, 61], see also [22] for the case of more exotic scales, and Donninger–Krieger [23] for the case of infinite time blow-up. We also refer to Jendrej [45] for the construction of near ground state blow up solutions for the energy critical wave equation in dimension 5.

In a contrast to the Krieger–Schlag–Tataru solutions, the blow-up regimes exhibited in [95, 97] arise from C^∞ finite energy initial data and are characterized by some specific blow-up rates. Namely, the following was proved in [95]: for any $m \geq 1$, there exists a set of C^∞ m-equivariant initial data arbitrary close to ϕ_m in the energy space such that the corresponding solution blows up in finite time T and, as $t \to T$, has the form $u(t,x) \approx$

$\phi_m(\lambda(t)x)$ with

$$\lambda(t) = \begin{cases} (T-t)^{-1}e^{\sqrt{|\ln(T-t)|}+O(1)} & \text{if } m=1, \\ c_m(T-t)^{-1}|\ln(T-t)|^{\frac{1}{2m-2}}(1+o(1)) & \text{if } m \geq 2, \end{cases} \quad \text{as } t \to T.$$

Furthermore, it was shown that these blow-up regimes are stable under smooth equivariant perturbations of the initial data. Similar results were obtained for the focusing energy critical nonlinear wave equation in dimension $d = 4$ by Hillairet and Raphaël [39].

Compared to [95], the construction of [60] gives no information on the stability/instability of the corresponding solutions. Recently, Krieger and Miao [54] proved that for ν small, these solutions are stable under sufficiently smooth corotational initial perturbations. Furthermore, Krieger, Miao, and Schlag [55] showed that this stability persists under nonequivariant smooth perturbations that vanish near the light cone. See also Burzio and Krieger [9] for the related results for the 3D energy critical nonlinear wave equation.

While for m-equivariant Schrödinger maps with $m \geq 3$ the possibility of blow-up near ϕ_m is excluded by the stability results of Gustafson, Kang, Nakanishi, and Tsai, for $m = 1$ near ϕ_1 blow-up does occur. This was proved by Merle, Raphaël, and Rodnianski [77].

Theorem 5.1 (Merle, Raphael, Rodnianski [77]). *There exists a set of C^∞ 1-equivariant initial data with elements arbitrary close to ϕ_1 in the energy space such that the corresponding solution to the Schrödinger map equation* (5.1) *blows up in finite time T and, as $t \to T$, one has*

$$u(t) = e^{\alpha(t)R}\phi_1\big(\lambda(t)\cdot\big) + u^* + o_{\dot{H}^1}(1),$$
$$\lambda(t) = c\frac{|\ln(T-t)|^2}{T-t}\big(1+o(1)\big), \quad \alpha(t) = \alpha_0\big(1+o(1)\big),$$

with some $u^* \in \dot{H}^1 \cap \dot{H}^2$, $\alpha_0 \in \mathbb{R}$, and $c > 0$.

In contrast to the wave map result of [95], the initial data in Theorem 5.1 form a set of codimension one.

In [91] we complemented the result of [77] by showing that (5.1) admits Krieger–Schlag–Tataru-type blow-up solutions as well. Namely, we proved:

Theorem 5.2 ([91]). *For any $\nu > 1$, $\alpha_0 \in \mathbb{R}$, there exist 1-equivariant initial data arbitrary close to ϕ_1 in $\dot{H}^1 \cap \dot{H}^3$ such that the corresponding solution to the Schrödinger map equation* (5.1) *blows up in finite time T and, as $t \to T$, one has*

$$u(t) = e^{\alpha(t)R}\phi_1\big(\lambda(t)\cdot\big) + u^* + o_{\dot{H}^1\cap\dot{H}^2}(1), \tag{5.8}$$

where $\lambda(t) = (T-t)^{-1/2-\nu}$, $\alpha(t) = \alpha_0 \ln(T-t)$, and $u^* \in H^{1+2\nu-}$. Furthermore, $u^*(x) = e^{\theta R}v^*(r)$, $v^* = (v_1^*, v_2^*, v_3^*)$, is compactly supported, C^∞ away from $x = 0$ and, as $|x| \to 0$, behaves as

$$v_1^*(r) + iv_2^*(r) \sim c_{\alpha,\nu}r^{2i\alpha_0+2\nu}\ln r.$$

In fact, the solutions constructed in [91] belong to $\dot{H}^{1+2\nu-}$. Observe that the regularity of the limiting profile u^* in (5.8) is also related to the blow-up rate (as in the case of the mass critical NLS, see Section 3.2).

As in [60,61], the proof of Theorem 5.2 relies on obtaining an approximate solution to an arbitrary high order $O((T-t)^N)$, which we construct using matching asymptotic expansions. We also refer to [31] for some closely related constructions in the parabolic setting. To convert the approximate solution into an exact solution, one then solves the problem for the small remainder backward in time with zero initial data at $t = T$. Once one can solve the equation up to any order, some very rough energy estimates are enough to control the remainder, in contrast to the approach of [77] that requires more advanced mixed energy/Morawetz estimates. Of course, a drawback of this procedure is that it gives no information on the stability of the constructed solutions.

Although we do not discuss the parabolic problems in this note, let us stress that as far as slow blow-up is concerned, there are a lot of direct connections between Schrödinger-type equations and their parabolic counterpart.

5.2. Energy critical NLS

In this subsection, we consider the energy critical focusing nonlinear Schrödinger equation

$$iu_t = -\Delta u - |u|^{\frac{4}{d-2}} u, \quad x \in \mathbb{R}^d, \quad d \geq 3, \tag{5.9}$$

restricting ourselves to the case of radial solutions

$$u|_{t=0} = u_0 \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d). \tag{5.10}$$

Recall that this equation admits a family of stationary states $W_{\alpha,\lambda}(x) = e^{i\alpha}\lambda^{\frac{d-2}{2}} W(\lambda x)$, $\alpha \in \mathbb{R}$, $\lambda > 0$, with W given by (2.6). We denote by $\mathcal{S}$ the two-dimensional manifold of these solutions, $\mathcal{S} = \{W_{\alpha,\lambda}, \alpha \in \mathbb{R}, \lambda > 0\}$.

The dynamics for the energies below the ground state energy was classified by Kenig and Merle [48] for radial data in dimensions 3, 4, 5, and by Killip–Visan [52] ($d \geq 5$) and Dodson [18] ($d = 4$) for general initial data in dimension $d \geq 4$. The results of [48, 52] ensure that for $u_0 \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^d)$ with $E(u_0) < E(W)$ one has global existence and scattering if $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, and finite time blow-up both forward and backward in time if $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$ and $u_0 \in L^2$.

A classification of radial solutions with critical energy $E(u_0) = E(W)$ was obtained by Duyckaerts and Merle [27] in dimensions 3, 4, 5, and by Li and Zhang [65] in dimension $d \geq 6$. In this case, in addition to scattering in both directions if $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$ and finite time blow-up in both directions if $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$ and $u_0 \in L^2$, there exist solutions that converge to W in one direction and scatter or blow-up in the opposite direction. More precisely, there exist unique, up to the symmetries, solutions W^-, W^+ that converge to W in $\dot{H}^1$ as $t \to +\infty$ satisfying $\|W^-\|_{L^2} < \|\nabla W\|_{L^2}$, $\|W^+\|_{L^2} > \|\nabla W\|_{L^2}$; W^- is global and scatters as $t \to -\infty$, and W^+ blows up in finite negative time, at least if $d \geq 5$.

The dynamics of the radial solutions with the energies slightly above $E(W)$ in the 3D case was studied by Nakanishi and Roy [86]. In continuation of the previous results of Nakanishi and Schlag [87,88] for the energy subcritical Klein–Gordon and Schrödinger equations and Krieger, Nakanishi, and Schlag [56] for the energy critical wave equation, Nakanishi and Roy proved that any radial $\dot{H}^1$ solution to (5.9) with $E(u) \leq E(W) + \varepsilon^2$, $\varepsilon \ll 1$, can stay $\dot{H}^1$-close to the ground state family $\mathcal{S}$ only on an interval of time, although it can be the entire lifespan. Once the solution leaves a neighborhood of $\mathcal{S}$, it either scatters or blows up (in the latter case, one has to assume in addition that $u_0 \in L^2$). Furthermore, all four combinations of scattering and blow-up forward/ backward in time occur for large sets of initial data. One might expect a similar result to hold in higher dimensions. The solutions that stay $\dot{H}^1$-close to $\mathcal{S}$ forward (backward) in time are expected to form a codimension one center-stable (center-unstable) manifold that divides a neighborhood of $\mathcal{S}$ into two regions exhibiting blow-up and scattering, respectively, forward (backward) in time (see Krieger, Nakanishi, and Schlag [57] for the corresponding result for the energy critical nonlinear wave equation). In low dimensions, the near ground state solutions can exhibit nontrivial dynamical behavior, including, along with scattering to the ground states, finite and infinite time type II blow-up. In dimension 3, the examples of infinite time near ground state blow-up at prescribed power law rate were constructed in [89]. Combining the linear analysis around W that we developed in [89] with the construction of approximate solutions of [91], one gets also $H^1_{\mathrm{rad}}(\mathbb{R}^3) \cap \dot{H}^{1+\nu-}(\mathbb{R}^3)$ finite time blow-up solutions of the form

$$u(t) = W_{\alpha(t),\lambda(t)} + u^* + o_{\dot{H}^1}(1), \quad \text{as } t \to 0,$$
$$\lambda(t) = t^{-1/2-\nu}, \quad \alpha(t) = \alpha_0 \ln t, \quad \|u^*\|_{\dot{H}^1 \cap \dot{H}^{1+\nu-}} \ll 1, \quad \nu > 0, \quad \alpha_0 \in \mathbb{R}.$$

Similar results can be proved for $d = 4$. As in the case of Schrödinger maps (5.1), these blow-up dynamics are closely related to the slow decay of the ground state in low dimensions and are expected to disappear starting from $d = 7$ (for $d = 5, 6$, finite or infinite near ground state concentration is still expected). Some partial results in this direction were obtained in [92], where we showed that for any $d \geq 7$, radial solutions staying in a neighborhood of $\mathcal{S}$ are global and scatter to a fixed ground state as soon as the linearization around W satisfies some suitable spectral assumptions that we were able to prove for d sufficiently large. In the parabolic setting, much more complete results are available. Namely, for the energy critical nonlinear heat equation in dimension $d \geq 7$, Collot, Merle, and Raphaël [15] obtained a complete classification of the dynamics for initial data $\dot{H}^1$ close to W (without radial symmetry assumption), showing that both the set of initial data leading to blow-up in the ODE type I regime and the set of initial data leading to global solutions that dissipate as $t \to +\infty$ are open in $\dot{H}^1$ and are separated by a codimension one set of global solutions that converge as $t \to +\infty$ to a fixed ground state.

5.3. Radial multibubble dynamics

In the breakthrough paper [25], Duyckaerts, Kenig, and Merle obtained a complete classification of radial, energy bounded solutions of the 3D focusing energy critical nonlinear wave equation:

$$\begin{aligned} u_{tt} &= \Delta u + u^5, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ (u, \partial_t u)|_{t=0} &= (u_0, u_1) \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^3) \times L^2_{\mathrm{rad}}(\mathbb{R}^3), \end{aligned} \tag{5.11}$$

showing that they asymptotically decompose into a finite sum of scale separated ground states and a radiation term which solves the linear wave equation. More precisely, one has

Theorem 5.3 (Soliton resolution for (5.11), Duyckaerts, Kenig, Merle [25]). *Let $(u_0, u_1) \in \dot{H}^1_{\mathrm{rad}} \times L^2_{\mathrm{rad}}$ and $(u, \partial_t u) \in C(]T_-, T_+[, \dot{H}^1 \times L^2)$ be the corresponding maximal solution to (5.11). Then one of the following holds:*

(i) *Type I blow-up: $T_+ < +\infty$ and $\|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} \overset{t \to T_+}{\longrightarrow} \infty$.*

(ii) *Type II blow-up: $T_+ < +\infty$ and there exist $(v_0, v_1) \in \dot{H}^1 \times L^2$, an integer $J \in \mathbb{N} \setminus \{0\}$, and for each $1 \le j \le J$, a sign $\varepsilon_j \in \{-1, 1\}$ and a positive function $\lambda_j(t)$ defined for t close to T_+, verifying*

$$\frac{\lambda_j}{\lambda_{j+1}}(t) \overset{t \to T_+}{\longrightarrow} \infty, \quad \forall 1 \le j \le J, \quad \lambda_{J+1}(t) = (T_+ - t)^{-1},$$

such that

$$u(t) = \sum_{j=1}^{J} \varepsilon_j \lambda_j^{1/2}(t) W\big(\lambda_j(t)\cdot\big) + v_0 + o_{\dot{H}^1}(1), \quad \partial_t u(t) = v_1 + o_{L^2}(1),$$

as $t \to T_+$.

(iii) *Global solution: $T_+ = +\infty$ and there exist a solution v_L of the linear wave equation, an integer $J \in \mathbb{N}$, and for each $1 \le j \le J$, a sign $\varepsilon_j \in \{-1, 1\}$ and a positive function $\lambda_j(t)$ defined for t sufficiently large, verifying*

$$\frac{\lambda_j}{\lambda_{j+1}}(t) \overset{t \to +\infty}{\longrightarrow} \infty, \quad \forall 1 \le j \le J, \quad \lambda_{J+1}(t) = t^{-1},$$

such that

$$\begin{aligned} u(t) &= \sum_{j=1}^{J} \varepsilon_j \lambda_j^{1/2}(t) W\big(\lambda_j(t)\cdot\big) + v_L(t) + o_{\dot{H}^1}(1), \\ \partial_t u(t) &= \partial_t v_L(t) + o_{L^2}(1), \quad \text{as } t \to +\infty. \end{aligned}$$

Later similar results were proved for the radial energy critical NLW equation in all odd dimensions and in dimension 4, and for critical equivariant wave maps, see Duyckaerts, Kenig, and Merle [26], Duyckaerts, Kenig, Martel, and Merle [24], Jendrej and Lawrie [47], as well as the references therein.

In view of the above results, a natural question is to determine which type of configurations of solitons and radiation can really occur. A similar question can be asked the NLS equation. In dimensions $d = 3, 4, 5$, for both the radial energy critical wave and radial energy critical Schrödinger equations, no examples with $J \ge 2$ are known. For the energy critical wave equation in dimension $d \ge 6$ and for the energy critical Schrödinger equation in dimension $d \geq 7$, global ($T_+ = +\infty$) radial pure ($v_L = 0$) two-bubble solutions, with one bubble developing at scale 1 and the other concentrating at infinite time, were constructed by Jendrej [44, 46].

5.4. Further generalizations

Blow-up by concentration of stationary states can also occur in the energy supercritical models. Among the known examples are the focusing energy supercritical NLS and NLW equations. The focusing energy supercritical NLS,

$$iu_t = -\Delta u - |u|^{2p} u, \quad x \in \mathbb{R}^d, \quad d \geq 3, \quad p > \frac{2}{d-2}, \tag{5.12}$$

has a two-parameter family of smooth radial stationary solutions $\varphi_{\alpha,\lambda}(x) = e^{i\alpha}\lambda^{\frac{1}{p}}\varphi(\lambda x)$, $\alpha \in \mathbb{R}$, $\lambda > 0$, where φ solves

$$\Delta\varphi + \varphi^{2p+1} = 0, \quad \varphi > 0, \quad \varphi(0) = 1,$$

and has the following behavior at infinity:

$$\varphi(x) \sim \frac{c_{p,d}}{|x|^{\frac{1}{p}}}, \quad c_{p,d}^{2p} = \frac{(d-2)p - 1}{p^2}, \quad \text{as } |x| \to \infty.$$

Merle, Raphael, and Rodnianski [78] proved that in dimension $d \geq 11$, for generic integers p satisfying $p > p^*(d) = \frac{2}{d-4-2\sqrt{d-1}}$, equation (5.12) admits radial blow-up solutions of the form

$$u(t,x) \approx \lambda^{\frac{1}{p}}(t)\varphi\big(\lambda(t)x\big), \quad \lambda(t) \sim (T-t)^{-\kappa(p,d)l}, \quad l \in \mathbb{N} \setminus \{0\},$$

arising from C^∞ compactly supported initial data. Here $\kappa(p,d) > 0$ is an explicit constant. The H^s norms of these solutions remain bounded if $0 \leq s < s_c$, while the critical norm blows up logarithmically, $\|u\|_{H^{s_c}} \sim \sqrt{|\ln(T-t)|}$, as $t \to T$. The corresponding result for the energy supercritical nonlinear heat equation (in the same range of parameters and with the same sequence of blow-up rates) goes back to the work of Herrero and Velázquez [38], see also Mizoguchi [85]. The numerology $d \geq 11$, $p > p^*(d)$ is related to the stability properties of φ; for the radial energy supercritical heat equation, it is known (under some mild additional assumptions) that no type II blow-up occurs outside this range, see Matano– Merle [69] and the references therein. The analysis of [78] was extended to the energy supercritical wave equation by Collot [14]. One can also use the approach of [60, 91] to construct near φ blow-up solutions of finite Sobolev regularity with a continuum of power-type blow-up rates for both energy supercritical NLS and NLW equations.

Another example that we would like to mention is the hyperbolic vanishing mean curvature flow in the Minkowski space $\mathbb{R}^{2n,1}$ that we considered in [1] in the case of $n = 4$. The minimal hypersurfaces in $\mathbb{R}^{2n}$ are stationary solutions of the corresponding Cauchy problem. It is known that $\mathbb{R}^8$ is foliated by a scaling-invariant family of smooth birotational invariant minimal hypersurfaces asymptotic at infinity to the Simons cone:

$$C_4 = \big\{(x_1, \ldots, x_8) \in \mathbb{R}^8, x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\big\}.$$

In [1], we showed that this family of minimal hypersurfaces generates finite time blow-up for the hyperbolic vanishing mean curvature flow, again with a continuum of prescribed power-type blow-up rates. In the parabolic setting, that is, for the mean curvature flow, a similar result (but with a sequence of specific blow-up rates) was proved much earlier by Velázquez [110].

6. FINITE TIME BLOW-UP FOR THE ENERGY SUPERCRITICAL DEFOCUSING NLS

Consider the energy supercritical defocusing nonlinear Schrödinger equation

$$iu_t = -\Delta u + |u|^{2p}u, \quad x \in \mathbb{R}^d, \quad p > \frac{2}{d-2}, \quad d \geq 3. \tag{6.1}$$

The question whether finite time blow-up occurs for (6.1) remained completely open up to very recently. On the one hand, numerical simulations, global well-posedness results for the log-supercritical equations (see, e.g., Tao [107]), the nonexistence of soliton-like solutions, and the expected nonexistence of the self-similar blow-up supported the hypothesis of global well-posedness. On the other hand, in [108] Tao exhibited examples of the energy supercritical defocusing NLS systems for which finite time blow-up does occur.

A decisive breakthrough has been achieved recently by Merle, Raphaël, Rodnianski, and Szeftel [79] who showed that in dimension $5 \leq d \leq 9$ the energy supercritical NLS (6.1), at least for certain choices of p, admits finite time blow-up solutions arising from C^∞ well-localized initial data. The construction of [79] employs the hydrodynamic formulation of the NLS equation that relates (6.1) to a compressible Euler equation via the Madelung transform, $u = \sqrt{\rho}e^{i\varphi}$. In a companion paper [80], Merle, Raphaël, Rodnianski, and Szeftel proved that the underlying compressible Euler equation has a family of self-similar blow-up solutions with C^∞ profiles that well approximate the NLS dynamics and thus can be used to construct finite time blow-up solutions for (6.1). The smoothness of the Eulerian self-similar solutions plays an important role in the analysis of [79]. In contrast to the focusing energy supercritical blow-up regime discussed in Section 5.4, where all subcritical H^s norms remain bounded, the solutions constructed in [79] satisfy

$$\|u(t)\|_{H^s} \overset{t \to T}{\longrightarrow} +\infty, \quad \forall s > s^*, \tag{6.2}$$

for some $1 < s^* < s_c$, the growth in (6.2) being polynomial. The recent results of Bulut [8] indicate (6.2) as a general feature of the energy supercritical defocusing blow-up.

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