ON THE ASYMPTOTICS FOR MINIMIZERS OF DONALDSON FUNCTIONAL IN TEICHMÜLLER THEORY

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ABSTRACT

We discuss the asymptotic behavior of minimizers for a Donaldson functional of interest in Teichmüller theory. For example, such minimizers allow one to parametrize the moduli space of constant mean curvature immersions of a closed surface S of genus $\mathfrak{g} \geq 2$ into a 3-manifold with sectional curvature -1, by elements of the tangent bundle of the Teichmüller space of S. The minimizers are governed by a system of PDEs which include a Gauss equation of Liouville type and a holomorphic κ-differential.

In our asymptotic analysis, we face the difficulty to describe the possible blow-up behavior of minimizers, especially when it occurs at a point where different zeroes of the holomorphic κ-differential coalesce. Therefore, we need to pursue accurate estimates of the blow-up profile of solutions for Liouville type equations, in the "collapsing" case.

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1. INTRODUCTION

In this note we discuss the asymptotics for minimizers of a Donaldson-type functional whose relevance in Teichmüller theory was pointed out in [29] and [33]. Such minimizers are governed by the system of equations (1.3) below, which includes a Liouville-type equation (as a Gauss consistency condition) and a holomorphic κ-differential, $\kappa \geq 2$, over a closed surface of genus $\mathfrak{g} \geq 2$.

Since such a holomorphic κ-differential admits $2\kappa(\mathfrak{g}-1)$ zeroes counted with multiplicity, to pursue the asymptotics of such minimizers, we must keep handy the detailed blow-up analysis and profile estimates developed for solutions of Liouville-type equations involving a weight function with a finite number of zeroes of integral multiplicity (see [3, 10, 11, 67]).

We recall that Liouville-type equations arise in many contexts of interest in mathematics and physics, and have attracted much attention after their first encounter by Liouville in his model of Field Theory. Since then, a rich literature is now available revealing the many facets of Liouville-type equations and the crucial role they played towards a successful development of Liouville Field Theory, see [54].

In [47], Liouville furnished a local formula for solutions of Liouville equations in terms of a meromorphic complex function, the so-called "developing map." In this way, Liouville equations were introduced into the realm of Complex Analysis and Algebraic Geometry. In fact, exploring the solvability of Liouville equations has led to tackle many fundamental issues about modular functions and forms, normal families, Fuchsian, Lamé, and Painlevé equations, and about various moduli spaces, see [7, 13–18, 28, 40, 42] and the references therein.

One may focus also on Liouville equations involving Dirac measures, whose poles replace the role of the zeroes. Indeed, the poles will correspond to the zeroes of the weight function which appears in the equation governing the "regular" part of the solution.

In (bidimensional) abelian Gauge Field Theory, at a self-dual regime, we have that vortex configurations are governed by the Bogomolny equations. They involve a (gauge invariant) Cauchy–Riemann equation for the (complex valued) Higgs field. Thus, the (gauge independent) zeroes of the Higgs field are isolated with integral multiplicity and identify the so-called "vortex-points." As a consequence, around a vortex-point we can confirm the "quantization" properties for the electric and magnetic fields, as already observed experimentally (e.g., in superconductivity).

Taubes showed how to express Bogomolny's self-dual equations in the form of Liouville-type equations with Dirac measures supported exactly at the vortex points, see [34]. By virtue of Taubes' approach, it has been possible to obtain a rigorous description of self-dual vortices for various models proposed in the context of Maxwell–Chern–Simons–Higgs theory, Electroweak theory, Comics strings, etc. We refer to the monographs [59, 68] for details.

We mention that, the analytical construction of physically meaningful vortices has motivated the accurate blow-up analysis and profile estimates for solutions of (singular) Liouville equations, contained in [8–11, 38].

From the geometrical side, such an analysis has helped also tackle the classical "uniformization" problem of surfaces with conical singularities prescribed along a given "divisor." In this direction, the most delicate situation occurs when the prescribed conical angle is bigger than 2π. For smaller angles, a complete description of conical metrics with constant Gauss curvature is contained in [48, 62, 63]. On the other hand, for the standard 2-sphere $S^2 = \mathbb{C} \cup \{\infty\}$, it is yet not clear when a spherical metric with prescribed conical singularities and relative angles (bigger than 2π) exists. Clearly, beside the constraint dictated by the Gauss–Bonnet theorem, there are other less obvious obstructions to prevent the existence of such (spherical) metrics. For example, in the case of two singularities, only the "American" football is possible, where both conical angles must coincide.

There is a rich literature concerning spherical metrics on S^2 (see, e.g., [12, 19, 22–27, 49–51, 57, 65, 69, 70] and the references therein), where different points of view have been adopted and yielded to interesting (partial) results. Only recently Mondello–Panov [52, 53] have identified (almost) sharp necessary and sufficient conditions on the conical angles so that a corresponding spherical metric exists. The sharp results in [52, 53] are established using strategies and techniques developed in algebraic geometry. At the moment, such results seem out of reach by mere analytical techniques. On the other hand, a blow-up approach to solutions of the singular Liouville equation over the flat torus has permitted to reveal surprising results, where nonexistence or (sharp) existence results may hold, according to the "geometry" of the periodic cell domain. Thus, for example, a flat torus with a *square* lattice and a single singularity with conical angle 4π cannot admit a metric with constant Gauss curvature, while this is possible for a *rhombus* lattice, see [42]. Many other surprising phenomena have been identified for the moduli space of tori and their metrics with conical singularities and constant Gauss curvature, see [15] and [42].

For those and other reasons, it has emerged the need to describe what happens when singularities (i.e., vortex or conical points) coalesce into a single point. Naturally, such an investigation furnishes a better grasp about the uniformization problem, see [51]. But also it helps in the understanding of *non-abelian* self-dual vortices which are described in terms of *systems* of Liouville equations (see [4, 36, 37, 43–46]). Indeed, it is difficult to have a firm grasp about the blow-up of solutions for systems, especially when various components blow-up at the same point, but with different blow-up rates. In such a situation, "concentration" phenomena introduce terms in the equations which behave as Dirac measures whose poles, however, may "collapse" together, see [35, 41].

We encounter an analogous "collapsing" issue in the asymptotic description of *minimizers* for the Donaldson functional, considered in [29] and [33]. Such a functional is inspired by [20, 21, 56], and relates to the representation of the fundamental group of a closed surface into various character varieties, or to the parametrization of the moduli space of minimal or constant mean curvature (CMC) immersions into a 3-manifold with constant sectional curvature -1, see [29, 33].

To be more precise, for a given oriented closed surface S with genus $\mathfrak{g} \geq 2$, we denote by $\mathcal{T}_{\mathfrak{g}}(S)$ the Teichmüller space of conformal structures on S, modulo biholomorphisms in the homotopy class of the identity.

For minimal immersions, Uhlenbeck in [64] proposed a parametrization of the corresponding moduli space in terms of elements of the cotangent bundle of $\mathcal{T}_{\mathfrak{g}}(S)$, described by pairs $(X, \alpha) \in \mathcal{T}_{\mathfrak{g}}(S) \times C_2(X)$, where $C_2(X)$ is the space of holomorphic quadratic differentials on X. In this way, minimal immersions are sought with assigned second fundamental form $\mathrm{II} = \mathrm{Re}(\alpha)$, simply by solving the Gauss equation of Liouville type for the conformal factor of the pullback metric on X from the minimal immersion. However, as discussed in [31,32], such an immersion may not exist, or when it exists, it may not be unique (see also [30]). So, by this approach, one does not obtain a one-to-one correspondence between a minimal immersion and the pair (X, α).

On the contrary, as we shall see below, we have a better chance when we choose to parametrize minimal or (CMC) immersions of S in terms of elements of the tangent bundle $\mathcal{T}_{\mathfrak{g}}(S)$.

To this purpose, for given $X \in \mathcal{T}_{\mathfrak{g}}(S)$, we let $T_X^{1,0}$ denote the holomorphic tangent bundle of X and define $E = \otimes^{\kappa-1} T_X^{1,0}$ with $\kappa \geq 2$. Moreover, letting $A^0(E)$ be the space of smooth sections of E and $A^{0,1}(X, E)$ the space of $(0,1)$-forms of X valued on E, we consider the $(0,1)$-Dolbeault cohomology group $\mathcal{H}^{0,1}(X, E) = A^{0,1}(X, E)/\overline{\partial}(A^0(E))$, where $\overline{\partial} : A^0(E) \to A^{0,1}(X, E)$ is the d-bar operator.

Using the Hodge star operator $*_E : A^{0,1}(X, E) \to A^{1,0}(X, E^*)$ and Serre duality theorem, we know that $C_\kappa(X)$, the space of holomophic κ-differential on X, satisfyies:

$$C_\kappa(X) \simeq (\mathcal{H}^{0,1}(X, E))^*,$$

see ([66]). Therefore, for $\kappa = 2$, we can use the pair $(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(S) \times \mathcal{H}^{0,1}(X, E)$ to parametrize the tangent bundle of $\mathcal{T}_{\mathfrak{g}}(S)$.

At this point, we consider on X the unique hyperbolic metric g_X with induced norm $|\cdot|$ and volume element dA. Also for $\beta \in A^{0,1}(X, E)$ the corresponding norm (in local coordinates) is given by $\|\beta\| = |\beta|(z)(g_X)^{\frac{\kappa-2}{2}}$.

Moreover, every $\beta \in A^{0,1}(X, E)$ admits the (unique) decomposition $\beta = \beta_0 + \overline{\partial}\eta$, with *harmonic* $\beta_0 \in A^{0,1}(X, E)$ and $\eta \in A^0(E)$. Therefore the class $[\beta] \in \mathcal{H}^{0,1}(X, E)$ is uniquely identified by its harmonic representative β_0 with respect to the metric g_X.

Thus, for any pair $(X, [\beta])$ and $t > 0$, we define the Donaldson functional:

$$D_t(u, \eta) = \int_X \left(\frac{1}{4}|\nabla u|^2 - u + te^u + 4e^{(\kappa-1)u}\|\beta_0 + \overline{\partial}\eta\|^2 \right) dA, \tag{1.1}$$

with the function u and the section η in the appropriate Sobolev spaces.

As observed in [29], it is possible to construct a (CMC) immersion with constant c, directly from a critical point of the Donaldson functional D_t with

$$t = 1 - c^2 > 0 \quad \text{and} \quad \kappa = 2. \tag{1.2}$$

Indeed, in this case, if (u, η) is a critical point of D_t, then it satisfies

$$\begin{cases} \Delta u + 2 - 2te^u - 8(\kappa - 1)e^{(\kappa-1)u}\|\beta_0 + \overline{\partial}\eta\|^2 = 0 \quad \text{in } X, \\ \overline{\partial}(e^{(\kappa-1)u} *_E (\beta_0 + \overline{\partial}\eta)) = 0. \end{cases} \tag{1.3}$$

Therefore, one may check that, if (1.2) holds, then $(X, e^u g_X)$ can be immersed as a (CMC) surface with constant $\pm c$ into a suitable hyperbolic 3-manifold $M^3 \simeq S \times \mathbb{R}$ with second fundamental form given by $\mathrm{II} = \mathrm{Re}(\alpha)$ and $\alpha = 8e^u *_E (\beta_0 + \overline{\partial}\eta) \in C_2(X)$, see [29,32,33] for details. Interestingly, as discussed in [33], system (1.3) can be recasted as Hitchin's selfduality equations for a suitable nilpotent $SL(2, \mathbb{C})$-Higgs bundle (of rank 2) and for this reason we refer to D_t as a Donaldson functional.

As also anticipated in [29], the following holds:

Theorem 1 ([33]). *For given $c \in (-1, 1)$, there is a one-to-one correspondence between the space of constant mean curvature c immersions into a 3-manifold of constant sectional curvature -1 and the tangent bundle of $\mathcal{T}_{\mathfrak{g}}(S)$, the latter parametrized by the pairs*

$$\big(X, [\beta]\big) \in \mathcal{T}_{\mathfrak{g}}(S) \times \mathcal{H}^{0,1}(X, E), \quad E = T_X^{1,0}.$$

Theorem 1 is a particular case of a more general result established in [33], showing that, for all $t > 0$ and $[\beta] \in \mathcal{H}^{0,1}(X, E)$, the Donaldson functional D_t admits a *unique* critical point (u_t, η_t), which is smooth and corresponds to the *global* minimum of D_t.

Such a uniqueness result yields also several interesting algebraic consequences. For example (for $\kappa = 2$ and $c = 0$), we derive a one-to-one correspondence between minimal immersions of S into a (germ of) hyperbolic 3-manifold and the irreducible representation of $\pi_1(S)$ into the group $\mathrm{PSL}(2, \mathbb{C})$ of the (orientation preserving) isometry group of $\mathbb{H}^3$.

On the grounds of Theorem 1, we can adventure to investigate the existence of (CMC) immersions with constant c reaching the limiting values $c = \pm 1$. Thus, for (u_c, η_c), the (unique) global minimum of D_t with $t = 1 - c^2$ and $\kappa = 2$, we can investigate if it survives the passage to the limit, as $|c| \to 1^-$. But we run immediately into trouble, since u_c could "blow up", as $|c| \to 1^-$. In fact, by using the blow-up analysis developed for solutions of Liouville equations, we find that actually blow-up can only occur around a finite number of (blow-up) points. We face a particularly delicate situation, when the blow-up point occurs at the "collapsing" of different zeroes of the holomorphic quadratic differential $\alpha_c = e^{u_c} *_E (\beta_0 + \overline{\partial}\eta_c) \in C_2(X)$. Recall that any holomorphic quadratic differential admits $4(\mathfrak{g} - 1) \geq 4$ zeroes in X (counted with multiplicity).

Thus, we devote the following sections to illustrate such a new scenario where, as pointed out in [35] and [41], we have to handle the new phenomenon of "blow-up without concentration." We present the recent results contained in [60,61]. Interestingly, when we deal with blow-up solutions carrying the least possible 'blow-up" mass 8π (see (3.19) and (3.20)), the pointwise estimates we obtain in the collapsing case are in striking analogy with the sharp "single bubble" estimates obtained in [8] and [38] for the nonvanishing (hence noncollapsing) case. Observe that no "bubble" is available in the "collapsing" situation.

By using the full power of the whole system (1.3), beyond the information encompassed by the mere Liouville equation, we are able to provide a useful description of (CMC) immersions with constant c “close” to ± 1 in some interesting cases, see Theorems 8 and 9.

In particular, we show that, for genus $\mathfrak{g} = 2$ and $[\beta] \neq 0$, the Donaldson functional at $t = 0$ is always bounded from below. This is a nontrivial information, since for $[\beta] = 0$ and $t = 0$, $D_{t=0}$ is always unbounded.

The seminal contribution contained in this note awaits improvements and some geometrical interpretation. We hope that our discussion will stimulate further investigation and new ideas in the pursuit of more complete results.

2. BLOW-UP AT COLLAPSING ZEROES: LOCAL ANALYSIS

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, and regular set, and consider the sequence:

$$\eta_k \in C^2(\Omega) \cap C^0(\overline{\Omega}),$$

satisfying the following Liouville-type problem:

$$\begin{cases} -\Delta \eta_k = W_k e^{\eta_k} \quad \text{in } \Omega, & (2.1) \\ \max_{\partial\Omega} \eta_k - \min_{\partial\Omega} \eta_k \leq C, & (2.2) \\ \int_\Omega W_k e^{\eta_k} \leq C, & (2.3) \end{cases}$$

with a weight function $W_k \geq 0$.

After the pioneering work of Brezis–Merle [6], a vast literature is now available, concerning the asymptotic behavior of η_k (possibly along a subsequence), as $k \to +\infty$, according to various assumptions on W_k and its vanishing behavior, see [2, 9, 11, 39, 55, 59]. Motivated by our applications, here we shall take W_k to satisfy

$$W_k \geq 0 \quad \text{and} \quad \|W_k\|_{L^\infty(\Omega)} + \int_\Omega \frac{1}{(W_k)^{\varepsilon_0}} \leq C, \quad \text{for some } \varepsilon_0 > 0. \tag{2.4}$$

As in [6], we say that η_k admits a *blow-up* point at $z_0 \in \Omega$, if

$$\exists z_k \to z_0 \quad \text{with } \eta_k(z_k) \to +\infty, \quad \text{as } k \to \infty \tag{2.5}$$

(possibly along a subsequence), and the value

$$\sigma(z_0) = \lim_{r\to 0} \liminf_{k\to+\infty} \int_{B_r(z_0)} W_k e^{\eta_k} \tag{2.6}$$

is called the “*blow-up mass*” of η_k at z_0.

The following result was pointed out in [61], as a general version of previous results contained in [2, 6, 55, 59]. We hope it can be useful in other contexts as well.

Proposition 2.1. *Let η_k satisfy (2.1)–(2.3) with $W_k \to W$ uniformly in $C^0_{\mathrm{loc}}(\Omega)$, and assume that (2.4) holds. Then (along a subsequence) η_k satisfies one of the following alternatives, as $k \to +\infty$:*

(i) $\eta_k \to -\infty$ *uniformly on compact sets of* Ω;

(ii) $\eta_k \to \eta_0$ *in* $C^0_{\mathrm{loc}}(\Omega)$, *with* η_0 *satisfying*

$$\begin{cases} -\Delta \eta_0 = W e^{\eta_0} & \text{in } \Omega, \\ \int_\Omega W e^{\eta_0} \le C; \end{cases}$$

(iii) *(blow-up) there exists a finite set* $\mathcal{S}$ *of blow-up points of* η_k *in* Ω. *Moreover, either*

("concentration") $\quad W_k e^{\eta_k} \rightharpoonup \sum_{q \in \mathcal{S}} \sigma(q) \delta_q$ *weakly in the sense of measures,*

and in particular $\eta_k \to -\infty$, *uniformly on compact sets of* $\Omega \setminus \mathcal{S}$;

or

("no concentration") $\quad \eta_k \to \eta_0$ *in* $C^0_{\mathrm{loc}}(\Omega \setminus \mathcal{S})$,

$$W_k e^{\eta_k} \rightharpoonup \sum_{q \in \mathcal{S}} \sigma(q) \delta_q + W e^{\eta_0}$$

weakly in the sense of measures, and η_0 *satisfies*

$$\begin{cases} -\Delta \eta_0 = W e^{\eta_0} + \sum_{q \in \mathcal{S}} \sigma(q) \delta_q & \text{in } \Omega, \\ \int_\Omega W e^{\eta_0} \le C. \end{cases} \tag{2.7}$$

Moreover, the blow-up mass satisfies $\sigma(q) \ge 4\pi$, $\forall q \in \mathcal{S}$.

Clearly, when alternative (iii) holds, in order to better understand the behavior of η_k around a blow-up point $q \in \mathcal{S}$, it is crucial to identify the specific value of the blow-up mass $\sigma(q)$ in (2.6).

In this respect, we recall the result of Li–Shafrir [39] and Bartolucci–Tarantello [2] in case,

$$W_k(x) = |x - p_k|^{2\alpha_k} h_k(x) \quad \text{in } B_r(q), \tag{2.8}$$

for $r > 0$ sufficiently small, with $p_k \in B_r(q)$ and

$$\begin{aligned} & h_k \to h \quad \text{uniformly with } 0 < a \le h \le b \text{ and } |\nabla h_k| \le A; \\ & 0 \le \alpha_k \to \alpha, \quad p_k \to q, \text{ as } k \to +\infty. \end{aligned} \tag{2.9}$$

Theorem 2 ([2, 39]). *If* η_k *in Proposition* 2.1 *satisfies alternative* (iii) *and for some* $q \in \mathcal{S}$ *the weight function* W_k *satisfies* (2.8)–(2.9), *then* (iii)(a) *holds, in the sense that blow-up occurs with a "concentration" property. Furthermore,*

(i) *if* $W(q) > 0$ *(i.e.,* $\alpha_k \equiv 0$ *and* (2.9)*) then* $\sigma(q) = 8\pi$,

(ii) *if* $\alpha > 0$ *in* (2.9) *then* $\sigma(q) = 8\pi(1 + \alpha)$.

Therefore, we focus on a blow-up point $q \in \mathcal{S}$ with $W(q) = 0$ and q being the accumulation point of different zeroes of W_k (collapsing zeroes). In view of the applications we have in mind, we assume that the zeroes of W_k have integral multiplicity.

In [35] this situation was handled in case only two zeroes of W_k coalesce at q, while the general case was treated in [61], see also [36], [37]. The following "quantization" property for the "blow-up" mass holds:

Theorem 3 ([35, 61]). *Suppose that η_k in Proposition 2.1 satisfy alternative* (iii). *Let $q \in \mathcal{S}$ and assume that, for $r > 0$ sufficiently small, we have*

$$W_k(x) = \left(\prod_{j=1}^{s} |x - p_{j,k}|^{2\alpha_j}\right) h_k(x), \quad \text{for } x \in B_r(q) \text{ and } s \geq 2, \tag{2.10}$$

and h_k satisfies (2.9) *in $B_r(q)$, $\alpha_j \in \mathbb{N}$ and $p_{j,k} \to q$, as $k \to +\infty$, $\forall j = 1, \ldots, s$. Then $\sigma(q) \in 8\pi\mathbb{N}$.*

The "local" results above can be used to describe the asymptotic behavior of solutions for Liouville-type equations on a compact Riemann surface (X, g). Denote by $d_g(\cdot,\cdot)$ the distance in (X, g). We consider a sequence $\xi_k \in C^{2,\alpha}(X)$ satisfying

$$-\Delta\xi_k = R_k e^{\xi_k} - f_k \quad \text{in } X, \tag{2.11}$$

where

$$R_k(z) = \left(\prod_{j=1}^{N} \left(d_g(z, z_{j,k})\right)^{2\alpha_j}\right) g_k(z), \quad z \in X; \tag{2.12}$$

$$g_k \in C^1(X) : a \leq g_k \leq b, \quad |\nabla g_k| \leq A \text{ and } g_k \to g_0 \text{ in } C^0(X); \tag{2.13}$$

$$z_{j,k} \in X : z_{j,k} \neq z_{l,k}, \quad j \neq l \in \{1, \ldots, N\} \text{ and } z_{j,k} \to z_j, \ j = 1, \ldots, N; \tag{2.14}$$

$$f_k \in C^{0,\alpha}(X), \quad f_k \to f_0 \text{ in } L^p(X), p > 1, \int_X f_0 \, dA \neq 0. \tag{2.15}$$

As before, we assume that

$$\alpha_j \in \mathbb{N}, \quad j = 1, \ldots, N. \tag{2.16}$$

In particular, we have that $R_k \to R_0$ uniformly in X, as $k \to +\infty$, with

$$R_0(z) = \left(\prod_{j=1}^{N} \left(d_g(z, z_j)\right)^{2\alpha_j}\right) g_0(z).$$

We denote by

$$Z = \left\{z \in X : R_0(z) = 0\right\} \tag{2.17}$$

the zero set of R_0. Clearly, $Z = \{z_1, \ldots, z_N\}$ with the point z_j given in (2.14) for $j = 1, \ldots, N$. We must keep in mind that such points are *not* necessarily distinct, as different zeroes of R_k could coalesce to the same zero of R_0. Therefore, we let Z_0 be the set (possibly empty) of such "collapsing" zeroes, namely

$$\begin{aligned} Z_0 = \big\{&z \in Z : \exists s \geq 2, 1 \leq j_1 < \cdots < j_s \leq N \text{ such that} \\ &z = z_{j_1} = \cdots = z_{j_s} \text{ and } z \notin Z \setminus \{z_{j_1}, \ldots, z_{j_s}\}\big\}. \end{aligned} \tag{2.18}$$

By combining the "local" results stated above, we can establish the following:

Theorem 4 (**[61]**). *Let ξ_k satisfy* (2.11) *and assume* (2.12)–(2.16). *Then, along a subsequence, one of the following alternatives holds:*

(i) *(compactness) $\xi_k \to \xi_0$ in $C^2(X)$ with*

$$-\Delta\xi_0 = R_0 e^{\xi_0} - f_0 \quad \text{in } X, \tag{2.19}$$

(ii) *(blow-up) there exists a finite blow-up set*

$$\mathcal{S} = \{q \in X : \exists q_k \to q \text{ and } \xi_k(q_k) \to +\infty, \text{ as } k \to +\infty\}$$

such that ξ_k is uniformly bounded in $C^2_{\text{loc}}(X \setminus \mathcal{S})$ and, as $k \to +\infty$,

(a) *either (blow-up with concentration)*

$$\begin{aligned} &\xi_k \to -\infty \quad \text{uniformly on compact sets of } X \setminus \mathcal{S}; \\ &R_k e^{\xi_k} \rightharpoonup \sum_{q\in\mathcal{S}} \sigma(q)\delta_q \quad \text{weakly in the sense of measures}, \sigma(q) \in 8\pi\mathbb{N}. \end{aligned} \tag{2.20}$$

In particular, $\int_X f_0 \, dA \in 8\pi\mathbb{N}$ in this case.

(b) *or (blow-up without concentration)*

$$\begin{aligned} &\xi_k \to \xi_0 \quad \text{in } C^2_{\text{loc}}(X \setminus \mathcal{S}); \\ &R_k e^{\xi_k} \rightharpoonup R_0 e^{\xi_0} + \sum_{q\in\mathcal{S}} \sigma(q)\delta_q \quad \text{weakly in the sense of measures;} \\ &-\Delta\xi_0 = R_0 e^{\xi_0} + \sum_{q\in\mathcal{S}} \sigma(q)\delta_q - f_0 \quad \text{in } X, \ \sigma(q) \in 8\pi\mathbb{N}. \end{aligned}$$

Furthermore, in case alternative (ii)(b) *holds, $\mathcal{S} \subset Z_0$ and so any blow-up point occurs at a collapsing of zeroes of R_k.*

See **[61]** for details. As discussed in **[35]** and **[41]**, all the alternatives of Theorem 4 can actually occur.

Remark 2.1. If in (ii) we have $\mathcal{S} \setminus Z_0 \neq \emptyset$, then blow-up always occurs with the “concentration” property. So (2.20) holds and, by Theorem 2, for $q \in \mathcal{S} \setminus Z_0$, we have:

(1) $\sigma(q) = 8\pi$, if $q \notin Z$;

(2) $\sigma(q) = 8\pi(1 + \alpha_j)$, if $q = z_j \in Z \setminus Z_0$.

As a direct consequence of Theorem 4, we may extend to the “collapsing” case the “compactness” result, well known to hold in the “non-collapsing” situation:

Corollary 2.1. *Under the assumption of Theorem 4, if*

$$\limsup_{k\to+\infty} \int_X R_k e^{\xi_k} \, dA < 8\pi,$$

then alternative (i) *holds.*

Next, we wish to provide more precise information around $q \in \mathcal{S} \cap Z_0$, a blow-up point of "collapsing" zeroes of R_k. To this purpose, we "localize" our analysis by introducing in X local holomorphic coordinates around q centered at the origin. Thus, with the obvious manipulations (see, e.g., [1, 2, 38]), and with abuse of notation, for $r > 0$ small, in $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$ we may consider a sequence $\xi_k \in C^{2,\alpha}(B_r) \cap C(\overline{B}_r)$ satisfying

$$\begin{cases} -\Delta \xi_k = W_k e^{\xi_k} \quad \text{in } B_r, & (2.21) \\ \max_{\partial B_r} \xi_k - \min_{\partial B_r} \xi_k \le C, \quad \int_{B_r} W_k e^{\xi_k} \le C, & (2.22) \\ \max_{\overline{B}_r} \xi_k = \xi_k(0) \to +\infty, \quad \text{as } k \to +\infty, & (2.23) \end{cases}$$

with

$$\begin{aligned} &W_k(x) = \left(\prod_{j=1}^{s} |x - p_{j,k}|^{2\alpha_j} \right) h_k(x), \quad \text{where } h_k \text{ satisfies (2.9) in } B_r; \\ &s \ge 2, \quad \alpha_j \in \mathbb{N}, p_{j,k} \ne p_{l,k} \text{ for } j \ne l; \\ &p_{j,k} \to 0, \quad \text{as } k \to +\infty, \ \forall j = 1, \dots, s. \end{aligned} \tag{2.24}$$

Let us just recall that the bounded oscillation property stated in (2.22) follows from the global problem (2.11) by means of the Green representation formula. We have

$$|\nabla W_k| \le A \quad \text{and} \quad W_k \to W \text{ in } C^0_{\text{loc}}(B_r), \text{ as } k \to +\infty, \tag{2.25}$$

with

$$W(x) = |x|^{2\alpha} h(x) \quad \text{and} \quad \alpha = \sum_{j=1}^{s} \alpha_j \in \mathbb{N}. \tag{2.26}$$

Furthermore, by taking $r > 0$ smaller if necessary, we may assume that zero is the *only* blow-up point of ξ_k in B_r, that is,

$$\forall 0 < \delta < r \ \exists C_\delta > 0 : \max_{\overline{B}_r \setminus B_\delta} \xi_k \le C_\delta. \tag{2.27}$$

Clearly, under the assumptions above, Theorem 3 applies to ξ_k and implies the following for the "blow-up" mass:

$$\sigma := \lim_{\delta \to 0^+} \left(\lim_{k \to +\infty} \int_{B_\delta(0)} W_k e^{\xi_k} \right) \in 8\pi \mathbb{N}. \tag{2.28}$$

Here, we focus on the case of the *least* "blow-up" mass, namely when (2.28) holds with

$$\sigma = 8\pi. \tag{2.29}$$

Interestingly, in this case we are able to provide sharp pointwise estimates for ξ_k in B_r. This should be considered a first relevant step. Indeed, the analysis of multiple "blow-up," where $\sigma = 8\pi m$ with $m \in \mathbb{N}$ and $m \ge 2$, typically reduces to the case $\sigma = 8\pi$ after multiple rescaling, unless one ends up with a blow-up point at a "noncollapsing" zero of W, described in (ii) of Theorem 2. But in the latter case one can take advantage of the recent estimates in [3] and [67] to complete the analysis. Also we mention [35], where blow-up was analyzed when "collapsing" occurs between two zeroes, i.e., when $s = 2$ in (2.24).

The following estimates were derived in [61].

Theorem 5 ([61]). *Let ξ_k satisfy the assumptions above. If (2.29) holds, then*

(i) $\xi_k(0) = -(\min_{\partial B_r} \xi_k + 2\sum_{j=1}^{s} 2\alpha_j \log|p_{j,k}|) + O(1)$;

(ii) $\xi_k(x) = \log \frac{e^{\xi_k(0)}}{(1+\frac{1}{8}W_k(0)e^{\xi_k(0)}|x|^2)^2} + O(1)$;

(iii) $\int_{B_r} |\nabla \xi_k|^2 = -16\pi(\min_{\partial B_r} \xi_k + \sum_{j=1}^{s} 2\alpha_j \log|p_{j,k}|) + O(1)$.

It is interesting to compare the above estimates with those available in [8] and [38] (see, e.g., Theorem 0.3 in [38]) for solutions of (2.21)–(2.23), when (2.25) holds with $W(0) > 0$ (instead of (2.26) as considered here). In this case, (2.29) is automatically satisfied (see (i) of Theorem 2) and the estimate (ii) of Theorem 5 is the striking exact analogue of the pointwise estimate provided in Theorem 0.3 of [38]. Furthermore, by considering the sequence

$$u_k(x) = \xi_k(x) + \sum_{j=1}^{s} 2\alpha_j \log|x - p_{j,k}|,$$

$$\text{satisfying } -\Delta u_k = h_k e^{u_k} - 4\pi \sum_{j=1}^{s} \alpha_j \delta_{p_j,k} \text{ in } B_r,$$

we realize that the estimate (i) stated for ξ_k in Theorem 5 reduces just to the following "sup + inf" estimate of Harnack type [5] for u_k:

$$u_k(0) + \min_{\partial B_r} u_k = O(1), \tag{2.30}$$

which was established in this form in [58] when the origin is a "noncollapsing" zero of W. Therefore, we expect that the estimate (2.30) should remain valid in the "collapsing" case as well, without the assumption (2.29).

We shall use those estimates to describe the asymptotic behavior of minimizers of the Donaldson functional, considered in [29,33].

3. ASYMPTOTICS FOR MINIMIZERS OF THE DONALDSON FUNCTIONAL

Let S be a smooth, closed, oriented surface of genus $\mathfrak{g} \geq 2$, and denote by $\mathcal{T}_{\mathfrak{g}}(S)$ the Teichmüller space of conformal structures on S, modulo biholomorphisms in the homotopy class of the identity.

We fix a conformal structure $X \in \mathcal{T}_{\mathfrak{g}}(S)$ and denote by g_X the corresponding hyperbolic metric on X, which will be used as the background metric, with norm $|\cdot|$ and volume element dA.

On (X, g_X) we consider a Donaldson functional assigned in terms of a pair of (conformal) data $(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(S) \times \mathcal{H}^{0,1}(X, E)$, where $E = \otimes^{k-1} T_X^{1,0}$ with $\kappa \geq 2$ and $T_X^{1,0}$ the holomorphic tangent bundle of X and $\mathcal{H}^{0,1}(X, E) = A^{0,1}(X, E)/\bar{\partial}(A^0(E))$ is the $(0,1)$-Dolbeault cohomology group. We recall that $A^{0,1}(X, E)$ is the space of $(0,1)$-forms in X

valued in E, $A^0(E)$ is the space of smooth sections of E and $\overline{\partial} : A^0(E) \to A^{0,1}(X, E)$ is the d-bar operator. For $\beta \in A^{0,1}(X, E)$, we have the decomposition $\beta = \beta_0 + \overline{\partial}\eta$, with β_0 a unique *harmonic* $(0, 1)$-form valued on E and $\eta \in A^0(E)$. So the class $[\beta] \in \mathcal{H}^{0,1}(X, E)$ is uniquely identified by its harmonic representative β_0. We also recall that, by means of the Hodge star operator $*_E : A^{0,1}(E) \to A^{1,0}(E^*)$ and by Serre's duality theorem (see [66]), for any class $[\beta] = [\beta_0 + \overline{\partial}\eta] \in \mathcal{H}^{0,1}(X, E)$ with β_0 harmonic, we can uniquely identify $*_E \beta_0$ with a holomorphic κ-differential on X. In other words, denoting by $C_\kappa(X)$ the space of κ-holomorphic differentials, we have that $C_\kappa(X) \simeq (\mathcal{H}^{0,1}(X, E))^*$. Moreover, the linear complex space $C_\kappa(X)$ is finite dimensional and $\dim_{\mathbb{C}} C_\kappa(X) = (2\kappa - 1)(\mathfrak{g} - 1)$. Since $\mathcal{T}_{\mathfrak{g}}(S)$ is a complex cell of dimension $3(\mathfrak{g} - 1)$, we find that, for $\kappa = 2$, the pair $(X, [\beta]) \in \mathcal{T}_{\mathfrak{g}}(S) \times \mathcal{H}^{0,1}(X, E)$ can be used to parametrize the tangent bundle of $\mathcal{T}_{\mathfrak{g}}(S)$.

In addition, recall that in local holomorphic coordinates $\{z\}$, any $\alpha \in C_\kappa(X)$ takes the expression $\alpha = h(dz)^k$, with h holomorphic. In this way, a zero for α is well understood, and actually, it is known that α admits $2\kappa(\mathfrak{g} - 1)$ zeroes in X, counted with multiplicity.

At this point, for a given pair $(X, [\beta])$ and $t > 0$, we define the *Donaldson functional*

$$D_t(u, \eta) = \int_X \left(\frac{|\nabla u|^2}{4} - u + te^u + 4e^{(\kappa-1)u} \|\beta_0 + \overline{\partial}\eta\|^2 \right) dA \tag{3.1}$$

with "natural" (convex) domain

$$\Lambda = \left\{ (u, \eta) \in H^1(X) \times W^{1,2}(X, E) : \int_X e^{(\kappa-1)u} \|\beta_0 + \overline{\partial}\eta\|^2 \, dA < +\infty \right\}.$$

Here, $H^1(X)$ and $W^{1,2}(X, E)$ are the usual Sobolev spaces. Clearly, the functional D_t is bounded from below in Λ.

In [33], the authors have shown that, for any $[\beta] \in \mathcal{H}^{0,1}(X, E)$ and $t > 0$, the functional D_t attains its infimum on Λ at a *smooth* pair (u_t, η_t) satisfying

$$\begin{cases} \Delta u + 2 - 2te^u - 8(\kappa - 1)e^{(\kappa-1)u} \|\beta_0 + \overline{\partial}\eta\|^2 = 0 & \text{in } X, \\ \overline{\partial}\big(e^{(\kappa-1)u} *_E (\beta_0 + \overline{\partial}\eta)\big) = 0. \end{cases} \tag{3.2}$$

More importantly, it is possible to show the *unique* solvability of (3.2).

Theorem 6 ([33]). *For given $t > 0$ and $[\beta] \in \mathcal{H}^{0,1}(X, E)$, the functional D_t admits a unique critical point (u_t, η_t), which corresponds to its global minimum in Λ. Furthermore, (u_t, η_t) is smooth and it is the only solution of* (3.2).

Such a uniqueness result implies relevant information about the moduli space of minimal, constant mean curvature, and Lagrangean immersions into hyperbolic 3-manifolds, and also about the irreducible representation of the fundamental group $\pi_1(S)$ in various character varieties. We refer to [33] and the references therein for more details. Here, we only mention the following consequence of Theorem 6 about the immersion of constant mean curvature (CMC) surfaces:

Corollary 3.1 ([29, 33]). *For a given $c \in (-1, 1)$, there is a one-to-one correspondence between the space of constant mean curvature c immersions of S into 3-manifolds of constant sectional curvature -1 and the tangent bundle of $\mathcal{T}_{\mathfrak{g}}(S)$, the latter parametrized by the* *pair*

$$\big(X,[\beta]\big)\in\mathcal{T}_{\mathfrak{g}}(S)\times\mathcal{H}^{0,1}(X,E).$$

Clearly, Corollary 3.1 is a direct consequence of Theorem 6, once we take $\kappa=2$ and $t=1-c^2>0$. We refer the reader to [33] for details.

For $X\in\mathcal{T}_{\mathfrak{g}}(S)$ fixed, by this approach, one may be tempted to look for (CMC) immersions of X with constant $c=\pm1$, simply by taking $t=1-c^2$ and by following the solution (u_t,η_t) to the limit, as $t\to0^+$. However, this requires a rather delicate analysis. Indeed, it is not even clear for which data $(X,[\beta])$, the functional $D_0=D_{t=0}$, given by

$$D_0(u,\eta)=\int_X\left(\frac{|\nabla u|^2}{4}-u+4e^{(\kappa-1)u}\|\beta_0+\overline{\partial}\eta\|^2\right)dA, \tag{3.3}$$

is bounded from below in Λ. Notice, for example, that for $t>0$ and $[\beta]=0$ (i.e., $\beta_0=0$),

$$u_t=\log\frac{1}{t},\quad \eta_t=0\text{ and }D_t(u_t,\eta_t)=\log t\to-\infty,\quad \text{as } t\to0^+.$$

Obviously, for $\beta_0=0$ and $t=0$, the system of equations (3.2) admits no solutions.

On the other hand, for $[\beta]\neq0$, the following has been established in [60].

Theorem 7 ([60]). *Let $[\beta]\neq0$ and assume that D_0 admits a critical point (u_0,η_0) (or equivalently, the system (3.2) for $t=0$ is solvable). Then D_0 is bounded from below in Λ and (u_0,η_0) is unique, smooth, and it corresponds to the global minimum of D_0 in Λ.*

Therefore, as anticipated, to find out if D_0 admits a critical point, we need to analyze the convergence of (u_t,η_t) (the global minimum of D_t), as $t\to0^+$. We shall see that the failure of convergence of (u_t,η_t) (along a subsequence) is due to "blow-up" phenomena.

For $\kappa=2$, such an asymptotic analysis allows us to obtain information about (CMC)-immersions, when the constant c approaches the limiting values ±1. On the other hand, when $\kappa\geq2$, such an asymptotic analysis permits to follow the behavior of the global minimizer (u_λ,η_λ) of the Donaldson functional

$$D(u,\eta)=\int_X\left(\frac{|\nabla u|^2}{4}-u+e^u+4e^{(\kappa-1)u}\|\lambda\beta_0+\overline{\partial}\eta\|^2\right)dA \tag{3.4}$$

along the (0, 1)-Dolbeault cohomology classes $[\lambda\beta]$, as λ varies in $(0,+\infty)$ and $[\beta]\neq0$ is fixed in $\mathcal{H}^{0,1}(X,E)$. Indeed, via the transformations

$$\begin{gathered} t=\lambda^{-\frac{2}{\kappa-1}},\quad u_t=u_\lambda+\frac{2}{\kappa-1}\log\lambda,\quad \eta_t=\frac{1}{\lambda}\eta_\lambda,\quad\text{and}\\ D_t(u_t,\eta_t)=D(u_\lambda,\eta_\lambda)-4\pi(\mathfrak{g}-1)\log\lambda^{\frac{2}{\kappa-1}}, \end{gathered}\tag{3.5}$$

we can recast the analysis of (u_λ,η_λ) (the global minimum of D in (3.4)), as $\lambda\to+\infty$, to the analysis of (u_t,η_t) (the global minimum of D_t in (3.1)), as $t\to0^+$.

To start, we notice that, by the strict positivity of the Hessian D_t'' at (u_t,η_t) (see [29,33]) and the Implicit Function Theorem, we can show the C^2-dependence of (u_t,η_t) with respect to $t\in(0,+\infty)$. We refer for details to [60], where it is also shown that the expression

$t\int_X e^{u_t}\,dA$ is increasing, as a function of $t \in (0,+\infty)$. Since, after integration over X of the first equation in (3.2), we have

$$t\int_X e^{u_t}\,dA + 4(\kappa-1)\int_X e^{(k-1)u_t}\|\beta_0+\bar\partial\eta_t\|^2\,dA = 4\pi(\mathfrak{g}-1), \tag{3.6}$$

we may conclude that

$$\rho_t\big([\beta]\big) := 4(\kappa-1)\int_X e^{(\kappa-1)u_t}\|\beta_0+\bar\partial\eta_t\|^2\,dA \in \big(0,4\pi(\mathfrak{g}-1)\big) \tag{3.7}$$

is *decreasing* in $(0,+\infty)$. These facts lead us to ask the following question:

Question 1. Can we identify the value

$$\rho\big([\beta]\big) = \rho(\beta_0) = \lim_{t\to 0^+} 4(\kappa-1)\int_X e^{(\kappa-1)u_t}\|\beta_0+\bar\partial\eta_t\|^2\,dA \tag{3.8}$$

in terms of the given cohomology class $[\beta] = [\beta_0+\bar\partial\eta] \in \mathcal{H}^{0,1}(X,E)$?

To emphasize the relevance of the value $\rho([\beta])$ in (3.8), we observe that, for $[\beta] = [\beta_0+\bar\partial\eta] \neq 0$, the interval $(0,\rho([\beta]))$ provides the range of the (decreasing) function

$$\rho_t\big([\beta]\big) = 4(\kappa-1)\int_X e^{(\kappa-1)u_t}\|\beta_0+\bar\partial\eta_t\|^2\,dA, \quad \text{as } t \text{ varies in } (0,+\infty).$$

We summarize the following consequences of the above discussion:

Proposition 3.1. *Given* $[\beta] \in \mathcal{H}^{0,1}(X,E)$*, there hold:*

(i) $\rho([\beta]) \in [0,4\pi(\mathfrak{g}-1)]$ *and* $\rho([\beta]) = 0 \iff [\beta] = 0$*;*

(ii) *If* $[\beta] \neq 0$*, then for every* $0 < \rho < \rho([\beta])$*, there exists a unique* $\lambda \in (0,+\infty)$ *such that* $\rho = 4(\kappa-1)\int_X e^{(\kappa-1)u_\lambda}\|\lambda\beta_0+\bar\partial\eta_\lambda\|^2\,dA$*, where* (u_λ,η_λ) *is the global minimum (and unique critical point) for* D *in* (3.4)*.*

Letting $c_t = D_t(u_t,\eta_t) = \min_\Lambda D_t$, we see that it is increasing for $t \in (0,+\infty)$ and therefore,

$$D_0 \quad \text{is bounded from below on } \Lambda \iff \inf_{t>0} c_t = \lim_{t\to 0^+} c_t = c_0 > -\infty \tag{3.9}$$

and $\inf_\Lambda D_0 = c_0$. More importantly, it has been proved in [60] that the following holds:

Proposition 3.2 ([60]). *If* D_0 *is bounded from below on* Λ *then* $\rho([\beta]) = 4\pi(\mathfrak{g}-1)$.

Now the delicate questions are the following:

Question 2. (i) If $\rho([\beta]) = 4\pi(\mathfrak{g}-1)$, is it true that D_0 is bounded from below in Λ?

(ii) If D_0 is bounded from below in Λ, for which $[\beta] \neq 0$ is the infimum attained?

In order to investigate the questions raised above, we set

$$\beta_t = \beta_0 + \bar\partial\eta_t \in A^{0,1}(X,E) \quad \text{and} \quad \alpha_t = e^{u_t} *_E \beta_t. \tag{3.10}$$

By virtue of the second equation in (3.2), we know that

$$\alpha_t \in C_\kappa(X) = \{\alpha \in A^{1,0}(X, E^*) : \bar{\partial}\alpha = 0\}, \quad \alpha_t \neq 0,$$

namely $\alpha_t \neq 0$ is a holomorphic κ-differential in X, and so it admits $2\kappa(\mathfrak{g}-1)$ zeroes in X, counted with multiplicity. Moreover, since $C_\kappa(X)$ is finite dimensional, all norms of α_t are equivalent. Let

$$s_t \in \mathbb{R} : e^{(\kappa-1)s_t} = \|\alpha_t\|^2_{L^\infty} \quad \text{and} \quad \hat{\alpha}_t = \frac{\alpha_t}{\|\alpha_t\|_{L^\infty}} = e^{-\frac{(k-1)s_t}{2}}\alpha_t. \tag{3.11}$$

Then, as $t \to 0^+$ (along a subsequence), we have $\hat{\alpha}_t \to \hat{\alpha}_0$ with

$$\hat{\alpha}_0 \in C_\kappa(X) \quad \text{and} \quad \|\hat{\alpha}_0\|_{L^\infty} = 1.$$

So, also $\hat{\alpha}_0$ must vanish at $2\kappa(\mathfrak{g}-1)$ points (counted with multiplicity), which correspond to the limits of the zeroes of $\hat{\alpha}_t$ (along a subsequence). Obviously, different zeroes of $\hat{\alpha}_t$ could coalesce into the same zero of $\hat{\alpha}_0$. It is shown in **[60]** that, in order to describe the asymptotic behavior of (u_t, η_t) satisfying (3.2), it is possible to use the blow-up analysis discussed in Section 2 for (a subsequence of)

$$\xi_t = -u_t + s_t. \tag{3.12}$$

With this information, it is possible to obtain the following (nontrivial) lower bound:

Proposition 3.3 (**[60]**). *For $[\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}$, there holds:*

$$\rho([\beta]) \geq \frac{4\pi}{\kappa - 1} \quad \textit{with } \rho([\beta]) \textit{ in } (3.8).$$

For details, we refer the interested reader to **[60]**.

Next, along a suitable sequence $t_k \to 0^+$, we are going to analyze more closely the sequence $\xi_k = \xi_{t_k}$ in (3.12). To reduce technicalities, from now on we focus on the case

$$\kappa = 2. \tag{3.13}$$

We let $u_k = u_{t_k}$, $\eta_k = \eta_{t_k}$, $s_k = s_{t_k}$, and $\alpha_k = \alpha_{t_k}$, so that the function

$$\xi_k = -(u_k - s_k) \tag{3.14}$$

satisfies

$$-\Delta\xi_k = 8\|\hat{\alpha}_k\|^2 e^{\xi_k} - f_k \quad \text{in } X, \tag{3.15}$$

with $f_k = 2(1 - t_k e^{u_k})$ and $\hat{\alpha}_k = e^{-\frac{s_k}{2}}\alpha_k$ satisfying $\|\hat{\alpha}_k\|_{L^\infty} = 1$. By the maximum principle, we also know that $\|f_k\|_{L^\infty(X)} \leq 2$. So, along the given sequence, we can further assume that

$$f_k \to f_0 \quad \text{in } L^p(X), p > 1, \quad \text{and} \quad \hat{\alpha}_k \to \hat{\alpha}_0 \in C_2(X), \|\hat{\alpha}_0\|_{L^\infty} = 1.$$

So, for $N = 4(\mathfrak{g}-1)$ (recall (3.13)), we let $Z = \{z_1, \ldots, z_N\}$ be the set of zeroes of $\hat{\alpha}_0$, repeated according to their multiplicity. Clearly, the set Z is formed by the limit points of the zeroes of $\hat{\alpha}_k$, which may coalesce into the same zero of $\hat{\alpha}_0$. Thus, we let Z_0 the set (possibly empty) of such "collapsing" zeroes of $\hat{\alpha}_k$, as defined in (2.18).

Theorem 4 applies to ξ_k and (possibly along a subsequence) implies that:

(i) either (compactness) $\xi_k \to \xi_0$ in $C^2(X)$, as $k \to +\infty$, and D_0 is bounded from below and attains its infimum in Λ;

(ii) or (blow-up) ξ_k admits a finite blow-up set

$$\mathcal{S} = \{q_1, \dots, q_n : 1 \le n \le \mathfrak{g} - 1\},$$

and we may have "blow-up with concentration," or "blow-up without concentration," as described respectively in parts (ii)(a) and (ii)(b) of Theorem 4 (with $R_k = 8\|\hat{\alpha}_k\|^2$ and $R_0 = \|\hat{\alpha}_0\|^2$).

At this point, by exploiting the full power of the whole system (3.2), it is possible to provide a careful description of the minimizer (u_k, η_k) of D_{t_k} in case of blow-up.

We start to discuss the case where we assume that $\mathcal{S} \cap Z = \emptyset$, namely no blow-up point coincides with a zero of $\hat{\alpha}_0$. In this situation, by Remark 2.1, we know that only "blow-up with concentration" occurs **[6, 38]**. Therefore,

$$8e^{u_k}\|\alpha_k\|^2 = 8\|\hat{\alpha}_k\|^2 e^{\xi_k} \rightharpoonup 8\pi \sum_{l=1}^{n} \delta_{q_l}, \quad \text{and we obtain } \rho\big([\beta]\big) = 4\pi n.$$

To proceed further, we follow **[60]**, and for a given set $P = \{x_1, \dots, x_\nu\} \subset X$ with $1 \le \nu \le (\mathfrak{g} - 1)$, we introduce the following subspace of $C_2(X)$:

$$Q_2[P] = Q_2\big[\{x_1, \dots, x_\nu\}\big] = \big\{\alpha \in C_2(X) : \alpha \text{ vanishes exactly at the set } P\big\}.$$

By the Riemann–Roch theorem, we have

$$\dim_{\mathbb{C}} Q_2\big[\{x_1, \dots, x_\nu\}\big] = 3(\mathfrak{g} - 1) - \nu.$$

In **[60]** it has been shown that the following holds:

Theorem 8 (**[60]**). *Assume that ξ_k blows up (in the sense of (ii) above). If (3.13) holds and*

$$\mathcal{S} \cap Z = \emptyset, \tag{3.16}$$

then (along a subsequence), as $k \to +\infty$,

$$\alpha_k \to \alpha_0 \in C_2(X) \quad \textit{with } \alpha_0 \neq 0 \textit{ vanishing exactly at } Z,$$

$$e^{-u_k} \rightharpoonup 4\pi \sum_{q \in \mathcal{S}} \frac{1}{\|\alpha_0\|^2(q)} \delta_q \quad \textit{weakly in the sense of measures}, \tag{3.17}$$

$$c_k = D_{t_k}(u_k, \eta_k) = -4\pi(\mathfrak{g} - 1 - n)d_k + O(1), \quad \textit{with } d_k = \fint_X u_k \, dA \to +\infty,$$

$$\int_X \beta_0 \wedge \alpha \, dA = 0, \quad \forall \alpha \in Q_2[\mathcal{S}]. \tag{3.18}$$

Furthermore, $\rho([\beta]) = \int_X \beta_0 \wedge \alpha_0 \, dA = 4\pi n$.

Remark 3.1. Since $\dim_{\mathbb{C}} Q_2[\mathcal{S}] = 3(\mathfrak{g} - 1) - n$, the orthogonality condition (3.18), together with the estimate (3.17) for the global minimizer of D_{t_k}, seems to indicate that ξ_k should admit only *one* blow-up point ($n = 1$), where the holomorphic quadratic differential $*_E \beta_0$ does not vanish.

When (3.16) holds, the estimate (3.17) allows us to answer Question 2 posed above. Indeed, if $\rho([\beta]) = 4\pi(\mathfrak{g} - 1)$ then $n = \mathfrak{g} - 1$, and therefore, by using (3.17), we find that

D_0 is bounded from below in Λ. However, the analysis above seems to suggests that D_0 may not attain its infimum in Λ.

Next we wish to acquire some useful information about the blow-up behavior of (u_k, η_k) when we no longer assume (3.16). By taking advantage of the blow-up analysis developed in Section 2, we focus to the case where blow-up occurs with the "least" blow-up mass. More precisely, for the blow-up mass

$$\sigma(q) = \lim_{r\to 0^+}\left(\lim_{k\to+\infty}\int_{B_r(q)} e^{u_k}\|\beta_0 + \bar{\partial}\eta_k\|^2 dA\right) \in 8\pi\mathbb{N}, \quad \forall q \in \mathcal{S}, \tag{3.19}$$

we assume that

$$\sigma(q) = 8\pi, \quad \forall q \in \mathcal{S}. \tag{3.20}$$

Remark 3.2. When (3.20) holds, it is shown in **[60]** that every blow-up point $q \in \mathcal{S} \cap Z$ must correspond to a collapsing of zeroes, that is,

$$\mathcal{S} \cap Z = \mathcal{S} \cap Z_0. \tag{3.21}$$

For $q_l \in \mathcal{S}$ and $r > 0$ sufficiently small, let

$$x_{k,l} \in B_r(q_l) : \xi_k(x_{k,l}) = \max_{B_r(q_l)} \xi_k \to +\infty \quad \text{and} \quad x_{k,l} \to q_l, \text{ as } k \to +\infty, \tag{3.22}$$

and set

$$\mu_{k,l} = \|\alpha_k\|^2(x_{k,l}). \tag{3.23}$$

In **[60]**, it is shown that the following holds:

Theorem 9 (**[60]**). *Assume (3.20) and suppose that $\mathcal{S} \cap Z \neq \emptyset$. Then (along a subsequence)*

$$s_k \to +\infty, \quad \textit{as } k \to +\infty.$$

Moreover, there exists a set of indices $J \subseteq \{1, \dots, n\}$ such that, as $k \to +\infty$,

(i) *$\forall l \in J$ we have $q_l \in \mathcal{S} \cap Z = \mathcal{S} \cap Z_0$ and $\mu_{k,l} \to \mu_l > 0$,*

$$e^{-u_k} \rightharpoonup 4\pi \sum_{l\in J} \frac{1}{\mu_l}\delta_{q_l} \quad \textit{weakly in the sense of measures;}$$

(ii) *$\int_X \beta_0 \wedge \alpha \, dA = 0$, $\forall \alpha \in C_2(X)$ vanishing at $\mathcal{S}_0 = \{q_l \in \mathcal{S} : l \in J\} \subset Z_0$. In particular, $\int_X \beta_0 \wedge \hat{\alpha}_0 \, dA = 0$;*

(iii) *$\mu_{k,l} \to +\infty$, as $k \to +\infty$, $\forall l \in \{1, \dots, n\} \setminus J$ (if not empty),*

$$c_k = D_{t_k}(u_k, \eta_k) = -4\pi(\mathrm{g} - 1 - n)d_k - \sum_{l\in\{1,\dots,n\}\setminus J} \log(\mu_{k,l}) + O(1),$$

$$\textit{with } d_k = \fint_X u_k \, dA \to +\infty. \tag{3.24}$$

We can reveal a clearer relation between Theorems 8 and 9, when J covers the full set of possible indices, namely when $J = \{1, \dots, n\}$, which is reasonable, as we expect that $n = 1$. With the above notations, the following holds:

Corollary 3.2. *Under the assumptions of Theorem 9, if in part 2 we have*

$$J = \{1, \dots, n\},$$

then $\mathcal{S} \subset Z_0$*. Moreover, as* $k \to +\infty$*,*

(i) $e^{-u_k} \to 4\pi \sum_{l=1}^{n} \frac{1}{\mu_l} \delta_q$ *weakly in the sense of measures;*

(ii) $c_k = -4\pi(\mathfrak{g} - 1 - n)d_k + O(1)$ *with* $d_k = \fint_X u_k \, dA \to +\infty$*,*

and

(iii) $\int_X \beta_0 \wedge \alpha \, dA = 0$*,* $\forall \alpha \in Q_2[\mathcal{S}]$*.*

When $\mathfrak{g} = 2$, $\mathcal{S}$ contains at most one single point ($n = 1$), and by virtue of Propositions 3.1 and 3.3, we know that

$$\rho([\beta]) = 4\pi, \quad \text{for every } [\beta] \in \mathcal{H}^{0,1}(X, E) \setminus \{0\}.$$

Thus, as a consequence of Corollary 2.1 or Theorem 8 and Corollary 3.2, we obtain

Corollary 3.3. *For the genus* $\mathfrak{g} = 2$*, the functional* D_0 *in* (3.3) *is bounded from below, whenever* $[\beta] \neq 0$*.*

As a final observation, we add that in Theorem 9 it should be possible to remove the assumption (3.20). However, when (3.20) is no longer valid then also (3.21) cannot be expected to hold (recall Remark 3.2) and so we could end up with a blow-up point $q \in Z \setminus Z_0$. Namely, blow-up can occur at a zero of $\hat{\alpha}_0$ which *does not* coincide with a "collapsing" of zeroes of $\hat{\alpha}_k$. As well known, in this case one needs to deal with a "multiple bubble" situation where, after rescaling, the "bubbles" are symmetrically placed (see **[3]**). This fact causes some "cancelation" phenomena that prevents to obtain, as in **[60]**, a nice control on the sequence s_k. It is likely that the new sharper estimates obtained by Wei–Zhang in **[67]** may help resolve such difficulties.

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