HYDRODYNAMIC STABILITY AT HIGH REYNOLDS NUMBER

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ABSTRACT

The hydrodynamic stability theory is mainly concerned with how laminar flows become unstable and transit to turbulence at high Reynolds number. To shed some light on the transition mechanism, Trefethen et al. [Science 261(1993)] proposed the transition threshold problem: *how much disturbance will lead to the instability of the flow and the dependence of disturbance on the Reynolds number*. Many effects such as 3D lift-up, inviscid damping, enhanced dissipation, and boundary layer play a crucial role in determining the transition threshold. In this note, we will first survey some important progress on linear inviscid damping and enhanced dissipation for shear flows. Then we will outline key ingredients in our proof of transition threshold for the 3D Couette flow in a finite channel.

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1. INTRODUCTION

The hydrodynamic stability has been an active field in the fluid mechanics since Reynolds's experiment in 1883 [43]. This field focuses on how the laminar flows become unstable and transit to turbulence [20,46,57]. A fundamental model describing the motion of the incompressible fluid is the Navier–Stokes (NS) equations:

$$\begin{cases} \partial_t v - \nu\Delta v + v\cdot\nabla v + \nabla p = 0, \\ \nabla\cdot v = 0, \end{cases} \tag{1.1}$$

where $v = (v^1(t,x,y,z), v^2(t,x,y,z), v^3(t,x,y,z))$ is the velocity, $p(t,x,y,z)$ is the pressure, and $\nu = \mathrm{Re}^{-1} > 0$ (Re Reynolds number) is the viscosity coefficient. Let us recall some well-known laminar solutions of (1.1): the plane Couette flow $(y,0,0)$, the plane Poiseuille flow $(1-y^2,0,0)$, and the pipe Poiseuille flow $(0,0,1-r^2)$ with $r^2 = x^2+y^2$. Our aim is to study the stability of these laminar flows at high Reynolds number, i.e., $\mathrm{Re} \gg 1$.

The plane Couette flow is spectrally stable for any Reynolds number $\mathrm{Re} \geq 0$ [44]. It has been a folklore conjecture that the pipe Poiseuille flow is spectrally stable for any Reynolds number. Recently, we (jointly with Chen) [15] proved that the pipe Poiseuille flow is spectrally stable at high Reynolds number. On the other hand, the experiments and numerics observed that these flows could be unstable and transit to turbulence for small but finite perturbations when the Reynolds number exceeds some critical number [13,22,42]. In addition, some laminar flows such as plane Poiseuille flow become turbulent at a much lower Reynolds number than that predicted by the eigenvalue analysis. These are the so-called Sommerfeld paradoxes. The resolution of these paradoxes is a long-standing problem in fluid mechanics. For many works dedicated to resolving these paradoxes, see [13] and references therein.

Trefethen et al. [48] provided an explanation about the linear instability via the ε-pseudospectra of the linearized NS operator $\mathcal{L}$ defined by

$$\sigma_\varepsilon(\mathcal{L}) = \left\{\lambda \in \mathbb{C} : \left\|(\lambda-\mathcal{L})^{-1}\right\| \geq \varepsilon^{-1}\right\}.$$

For the plane Couette flow, the spectrum of the linearized operator lies in the stable lower half-plane, but the pseudospectrum extends significantly into the upper half-plane. The pseudomode may be excited to a substantial amplitude by a very small input. This phenomenon is due to the nonnormality of the linear operator. Now the psuedospectrum has become an important concept in the study of nonnormal operators [47]. Li and Lin [35] provided a resolution from the following point of view: there is a sequence of linearly unstable shears which approach the linear shear in the kinetic energy norm but not in the enstrophy norm, and such linear instabilities offer an initiator for the transition from the linear shear to turbulence.

To shed some light on the transition mechanism to turbulence, Trefethen et al. [48] proposed the *transition threshold problem*: *how much disturbance will lead to the instability of the flow and the dependence of disturbance on the Reynolds number*. This idea may be traced back to Kelvin [30]. The following mathematical version was formulated by Bedrossian, Germain, and Masmoudi [7]:

Given a norm $\|\cdot\|_X$*, find a* $\beta = \beta(X)$ *so that*

$$\|u_0\|_X \leq \mathrm{Re}^{-\beta} \Rightarrow \text{stability},$$
$$\|u_0\|_X \gg \mathrm{Re}^{-\beta} \Rightarrow \text{instability}.$$

The exponent β is referred to as the transition threshold. It was conjectured in [48] that

> *"Notwithstanding these qualifications, we conjecture that transition to turbulence of eigenvalue-stable shear flows proceeds analogously to our model in that the destabilizing mechanism is essentially linear in the sense described above and the amplitude threshold for transition is* $O(\mathrm{Re}^{\gamma})$ *for some* $\gamma < -1$*."*

Later on, a lot of works were devoted to estimating β (see [13] and the references therein). To the best of our knowledge, the community never reached a consensus on what the thresholds should be. Numerical results by Lundbladh, Henningson, and Reddy [38] indicated that for the plane Couette flow, $\beta = 1$ for streamwise perturbation and $\beta = \frac{5}{4}$ for oblique perturbation; for the plane Poiseuille flow, $\beta = \frac{7}{4}$ for both streamwise and oblique perturbations. Asymptotic analysis results by Chapman [13] showed that for the plane Couette flow, $\beta = 1$ for streamwise and oblique perturbation; for the plane Poiseuille flow, $\beta = \frac{3}{2}$ for streamwise perturbation and $\beta = \frac{5}{4}$ for oblique perturbation.

In the absence of a physical boundary, Bedrossian, Germain, and Masmoudi (BGM) made important progress on the transition threshold problem for the 3D Couette flow in a series of works [5, 6, 8]. It was shown that $\beta \leq 1$ for the perturbations in Gevrey class and $\beta \leq \frac{3}{2}$ for the perturbations in Sobolev space. In [52], we improved the result of [6] to $\beta \leq 1$ in Sobolev space, which means that the regularity of the initial data (at least above H^2-regularity) does not play an important role in determining the transition threshold. In the presence of a physical boundary, the boundary layer could affect the stability of the flow at the high Reynolds number regime. To understand the boundary layer effect, we (jointly with Chen and Li) [14] studied the transition threshold problem for the 2D Couette flow in a finite channel $\mathbb{T} \times [-1, 1]$. We established various space–time estimates for the linearized NS system by developing the robust resolvent estimate method. Based on this work and [52], we (jointly with Chen) [16] proved that the transition threshold $\beta \leq 1$ in the Sobolev space for the 3D Couette flow in a finite channel $\mathbb{T} \times [-1, 1] \times \mathbb{T}$. Therefore, the transition threshold for the 3D Couette flow is inconsistent with the value (some $\beta > 1$) conjectured in [48] even in the presence of the boundary layer effect. The main reason may be that the infinite-dimensional mixing effects and special null structures in the nonlinearity suppress most of the nonlinear interactions rather than giving what could be predicted by the toy model in [48].

Both BGM's and our works show that these linear effects, namely 3D lift-up, inviscid damping, enhanced dissipation, and boundary layer, play a crucial role in determining the transition from a laminar to turbulent flow at high Reynolds number. In this note, we will first survey some recent important progress about linear inviscid damping and enhanced dissipation for shear flows. Then we will outline some key ingredients in our proof of transition threshold for the 3D Couette flow in a finite channel.

2. LINEAR INVISCID DAMPING FOR SHEAR FLOWS

We consider the 2D linearized Euler equation around shear flow $(u(y), 0)$ in a finite channel $\Omega = \{(x, y) : x \in \mathbb{T}, y \in [-1, 1]\}$:

$$\partial_t \omega + \mathcal{L}\omega = 0, \quad \omega|_{t=0} = \omega_0(x, y), \tag{2.1}$$

where $\mathcal{L} = u(y)\partial_x + u''(y)\partial_x(-\Delta)^{-1}$ and ω is the vorticity. Taking the Fourier transform with respect to x, the linearized Euler equation (2.1) in terms of the stream function ψ (i.e., $\Delta\psi = \omega$) is reduced to

$$\partial_t \widehat{\psi} + i\alpha\mathcal{R}_\alpha \widehat{\psi} = 0, \tag{2.2}$$

where $\mathcal{R}_\alpha \widehat{\psi} = -(\partial_y^2 - \alpha^2)^{-1}(u''(y) - u(\partial_y^2 - \alpha^2))\widehat{\psi}$.

For the Couette flow (i.e., $u(y) = y$), Orr [41] observed an important phenomenon that the velocity will tend to 0 as $t \to \infty$, although the Euler equation is a conserved system. This phenomenon is the so-called *inviscid damping*, which is the analogue in hydrodynamics of Landau damping [32]; see [45] for similar phenomena in various systems. For general shear flows, the problem is challenging due to the presence of the nonlocal operator $u''(y)\partial_x(-\Delta)^{-1}$. In this case, the linear dynamics is associated with the singularities at the critical layer $u = c$ of the solution of the Rayleigh equation

$$(u - c)(\Phi'' - \alpha^2\Phi) - u''\Phi = f.$$

Based on the Laplace transform and singularity analysis of the solution ϕ at the critical layer, Case [12] gave the first prediction of linear damping for monotone shear flows. However, Case's argument does not work for nonmonotone flows. Bouchet and Morita [11] may be the first to study the linear damping for nonmonotone shear flows. Based on Laplace tools and numerical computations, they found a new dynamic mechanism, i.e., *vorticity depletion phenomena*. Assume that for large time

$$\widehat{\omega}(t, \alpha, y) \sim \omega_\infty(y)\exp\big(-i\alpha u(y)t\big) + O(t^{-\gamma}).$$

The vorticity depletion means that $\omega_\infty(y)$ vanishes at stationary points of $u(y)$. This is another important mechanism leading to the damping for nonmonotone shear flows. Based on this observation and using stationary phase expansion, they predicted similar decay rates of the velocity as in the monotone case.

In a series of works [53–55], we (jointly with Zhao) confirmed Case's prediction on linear damping for monotone shear flows and Bouchet–Morita's prediction for nonmonotone shear flows, including Poiseuille and Kolmogorov flows. Let us review these results. The first result is the linear inviscid damping for monotone flows [53].

Theorem 2.1. *Let $u(y) \in C^4([0, 1])$ be a monotone function. Suppose that the linearized operator $\mathcal{L}$ has no embedding eigenvalues. Assume that $\int_{\mathbb{T}} \omega_0(x, y)dx = 0$ and $P_{\mathcal{L}}\omega_0 = 0$, where $P_{\mathcal{L}}$ is the spectral projection to $\sigma_d(\mathcal{L})$. Then it holds that*

1. *If $\omega_0(x, y) \in H_x^{-1}H_y^1$, then*

$$\big\|V(t)\big\|_{L^2} \le \frac{C}{\langle t\rangle}\|\omega_0\|_{H_x^{-1}H_y^1};$$

2. *If* $\omega_0(x,y) \in H_x^{-1}H_y^2$, *then*

$$\|V^2(t)\|_{L^2} \leq \frac{C}{\langle t\rangle^2}\|\omega_0\|_{H_x^{-1}H_y^2}.$$

Now we introduce a class of nonmonotone flows denoted by $\mathcal{K}$, which consists of the functions $u(y)$ satisfying $u(y) \in H^3(-1,1)$ and $u''(y) \neq 0$ for critical points (i.e., $u'(y) = 0$) and $u'(\pm 1) \neq 0$. For the flows in $\mathcal{K}$, we prove the following linear inviscid damping result and confirm the vorticity depletion phenomenon [54].

Theorem 2.2. *Assume that* $u(y) \in \mathcal{K}$ *and the linearized operator* $\mathcal{R}_\alpha$ *has no embedding eigenvalues. Assume that* $\widehat{\omega}_0(\alpha,y) \in H_y^1(-1,1)$ *and* $P_{\mathcal{R}_\alpha}\widehat{\psi}_0(\alpha,y) = 0$, *where* ψ_0 *is the stream function and* $P_{\mathcal{R}_\alpha}$ *is the spectral projection to* $\sigma_d(\mathcal{R}_\alpha)$. *Then it holds that*

$$\|\hat{V}(\cdot,\alpha,\cdot)\|_{L_t^2L_y^2} + \|\partial_t\hat{V}(\cdot,\alpha,\cdot)\|_{L_t^2L_y^2} \leq C_\alpha\|\widehat{\omega}_0(\alpha,\cdot)\|_{H_y^1}.$$

In particular, $\lim_{t\to+\infty}\|\hat{V}(t,\alpha,\cdot)\|_{L_y^2} = 0$. *If* $u'(y_0) = 0$, *then*

$$\lim_{t\to+\infty}\widehat{\omega}(t,\alpha,y_0) = 0.$$

Remark 2.1. For a class of symmetric shear flows, including the Poiseuille and Kolmogorov flows, we can obtain the explicit decay estimates as in the monotone case [54, 55]. A very interesting question is to prove the explicit decay estimates for general flows in $\mathcal{K}$.

The proof of Theorem 2.1 is based on the representation formula of the solution. Let Ω_ϵ be a simply connected domain including the spectrum $\sigma(\mathcal{R}_\alpha)$ of $\mathcal{R}_\alpha$. Then the solution $\widehat{\psi}(t,\alpha,y)$ is given by the following Dunford integral:

$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi i}\int_{\partial\Omega_\epsilon} e^{-i\alpha tc}(c-\mathcal{R}_\alpha)^{-1}\widehat{\psi}(0,\alpha,y)dc.$$

Let $\Phi(\alpha,y,c)$ be the solution of the inhomogeneous Rayleigh equation with $f(\alpha,y,c) = \frac{\widehat{\omega_0}(\alpha,y)}{i\alpha(u-c)}$ and $c \in \Omega_\epsilon$:

$$\Phi'' - \alpha^2\Phi - \frac{u''}{u-c}\Phi = f, \quad \Phi(-1) = \Phi(1) = 0. \tag{2.3}$$

Then we find that

$$(c-\mathcal{R}_\alpha)^{-1}\widehat{\psi}(0,\alpha,y) = i\alpha\Phi(\alpha,y,c).$$

Therefore, we have

$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi}\int_{\partial\Omega_\epsilon}\alpha\Phi(\alpha,y,c)e^{-i\alpha ct}dc. \tag{2.4}$$

Thus, the key ingredient of the proof is reduced to solving the inhomogeneous Rayleigh equation (2.3) and deriving uniform estimates of the solution Φ in ϵ. For this, we need to construct two independent solutions to the homogeneous Rayleigh equation for $c \in \Omega_\epsilon$:

$$\phi'' - \alpha^2\phi - \frac{u''}{u-c}\phi = 0.$$

Our idea is as follows. Let $\phi = (u(y) - c)\phi_1$. Then ϕ_1 satisfies

$$\big((u(y)-c)^2\phi_1'\big)' = \alpha^2\phi_1\big(u(y)-c\big)^2.$$

If $\phi_1(y_c, c) = 1$ and $\phi_1'(y_c, c) = 0$ at y_c, then we have

$$\begin{aligned}\phi_1(y,c) &= 1 + \int_{y_c}^{y} \frac{\alpha^2}{(u(y')-c)^2}\int_{y_c}^{y'}\phi_1(z,c)\big(u(z)-c\big)^2 dz dy' \\ &= 1 + \alpha^2 T\phi_1(y,c).\end{aligned}$$

Assume that u is monotone and let $y_c = u^{-1}(c_r)$ with $c_r = \mathrm{Re}\, c$. The following estimate is crucial: there exists a constant C independent of A so that

$$\left\| \frac{Tf(y,c)}{\cosh A(y-y_c)} \right\|_{L^\infty_{y,c}} \le \frac{C}{A^2}\left\| \frac{f(y,c)}{\cosh A(y-y_c)} \right\|_{L^\infty_{y,c}}.$$

Then $\phi_1(y,c) = \sum_{k=0}^{\infty}(\alpha^2 T)^k(1)$ by taking A large enough.

The proof of Theorem 2.2 is based on the limiting absorption principle. Consider the inhomogeneous Rayleigh equation:

$$(u-c)(\Phi'' - \alpha^2\Phi) - u''\Phi = \omega, \quad \Phi(-1) = \Phi(1) = 0,$$

where $c \in \Omega \setminus D_0$, $D_0 = \mathrm{Ran}\, u$. Using blow-up analysis and a compactness argument, we prove the limiting absorption principle for shear flows $u \in \mathcal{K}$.

Proposition 2.1. *If $\mathcal{R}_\alpha$ has no embedding eigenvalues, then there exists an ϵ_0 such that for $c \in \Omega_{\epsilon_0} \setminus D_0$, Φ has the the following uniform bound:*

$$\|\Phi\|_{H^1(-1,1)} \le C\|\omega\|_{H^1(-1,1)}.$$

Here C is a constant independent of ϵ_0. Moreover, there exists $\Phi_\pm(\alpha, y, c) \in H_0^1(-1,1)$ for $c \in \mathrm{Ran}\, u$, such that $\Phi(\alpha,\cdot,c \pm i\epsilon) \to \Phi_\pm(\alpha,\cdot,c)$ in $C([-1,1])$ as $\epsilon \to 0+$ and

$$\big\|\Phi_\pm(\alpha,\cdot,c)\big\|_{H^1(-1,1)} \le C\|\omega\|_{H^1(-1,1)}.$$

From (2.4) and Plancherel's formula, we infer that

$$\begin{aligned}\|\hat{V}(t,\alpha,y)\|^2_{H^1_t L^2_y} &= \int_{\mathbb{R}}\big(\|\hat{V}(t,\alpha,\cdot)\|^2_{L^2_y} + \|\partial_t\hat{V}(t,\alpha,\cdot)\|^2_{L^2_y}\big)dt \\ &\le C\int_{\mathrm{Ran}\, u}\|\widetilde{\Phi}(\alpha,\cdot,c)\|^2_{H^1_y}dc \le C\|\widehat{\omega}_0(\alpha,\cdot)\|^2_{H^1_y}.\end{aligned}$$

For monotone shear flows, we (jointly with Zhu) also developed the vector field method in the sprit of wave equation [56]. The idea is as follows. We first proved the space–time estimate of the velocity via the limiting absorption principle. Consider

$$\partial_t\omega + i\alpha\mathcal{R}_\alpha'\omega = f, \quad \mathcal{R}_\alpha'\omega = -\big(u''(\partial_y^2 - \alpha^2)^{-1} - u\big)\omega.$$

Using the limiting absorption principle, we can prove that

$$\begin{aligned}&\|\omega(T)\|^2_{L^2} + \alpha^2\int_0^T\big(\|\partial_y\psi(t)\|^2_{L^2} + \alpha^2\|\psi(t)\|^2_{L^2}\big)dt \\ &\quad \le C\|\omega(0)\|^2_{L^2} + C\alpha^{-2}\int_0^T\big(\|\partial_y f(t)\|^2_{L^2} + \alpha^2\|f(t)\|^2_{L^2}\big)dt = RHS. \qquad (2.5)\end{aligned}$$

Moreover, if $f(t,0) = f(t,1) = 0$, then we also have

$$\alpha \int_0^T \left(\left|\partial_y \psi(t,0)\right|^2 + \left|\partial_y \psi(t,1)\right|^2\right) dt \le RHS. \tag{2.6}$$

Then we introduce the vector field $X = (1/u')\partial_y + i\alpha t$, which commutes with $\partial_t + i\alpha u$. We denote

$$\omega_1 = X\omega, \quad \psi_2 = -\left(\partial_y^2 - \alpha^2\right)^{-1}\left(\partial_y \omega / u'\right), \quad \psi_3 = \psi_2 - \partial_y \psi / u'.$$

Then we have

$$\partial_t \omega_1 + i\alpha \mathcal{R}'_{\alpha,\beta}\omega_1 = -i\alpha (u'''/u')\psi + i\alpha u'' \psi_3.$$

Based on the space–time estimate (2.5) and (2.6), we can obtain a uniform estimate for $\|X\omega\|_{L^2}$, which implies that $\|V(t)\|_{L^2} \le C\langle t\rangle^{-1}$. More work is needed to prove $\|V^2(t)\|_{L^2} \le C\langle t\rangle^{-2}$. See Section 2 in [56] for the details.

Finally, let us mention some recent important results on linear inviscid damping [4,23,58,59] and nonlinear inviscid damping [10,19,27–29,37,39]. However, when the boundary effect is involved, nonlinear inviscid damping is still a challenging problem [58].

3. LINEAR ENHANCED DISSIPATION FOR KOLMOGOROV FLOW

Let us first consider the diffusion–convection equation in $\mathbb{T} \times \mathbb{R}$:

$$\partial_t \omega - \nu \Delta \omega + y \partial_x \omega = 0.$$

Introduce new variables $(\bar{x}, y) = (x - ty, y)$ and set $\widetilde{\omega}(t, \bar{x}, y) = \omega(t, x, y)$. Then the solution $\widehat{\widetilde{\omega}}(t,k,\xi) = \int_{\mathbb{T}\times\mathbb{R}} \widetilde{\omega}(t,x,y) e^{-2\pi i k x - i 2\pi \xi y}\, dx dy$ takes the form

$$\widehat{\widetilde{\omega}}_{\neq}(t,k,\xi) = e^{-\nu(2\pi)^2 \int_0^t (k^2 + (\xi - k\tau)^2) d\tau} \widehat{\omega}_{\neq}(0,k,\xi).$$

Due to $\int_0^t (k^2 + (\xi - k\tau)^2) d\tau \ge k^2 t^3/12$, we deduce that

$$\left\|\omega_{\neq}(t)\right\|_{L^2} \le e^{-c\nu t^3} \left\|\omega_{\neq}(0)\right\|_{L^2} \le C e^{-c\nu^{1/3} t} \left\|\omega_{\neq}(0)\right\|_{L^2}.$$

Here the exponent νt^3 gives a dissipation time scale $\nu^{-1/3}$, which is much shorter than the dissipation time scale ν^{-1}. We refer to this phenomenon as the *enhanced dissipation*, which is also due to the mixing mechanism.

We are concerned with the enhanced dissipation phenomenon for the linearized Navier–Stokes equations around shear flows. In this note, we will review some progress on the enhanced dissipation estimates for the linearized 2D NS equations in the torus $\mathbb{T}_{2\pi\delta} \times \mathbb{T}_{2\pi}$ around the Kolmogorov flow $(-e^{-\nu t} \cos y, 0)$, which is a solution of the 2D NS equations:

$$\partial_t \omega + \mathcal{L}_\nu(t)\omega = 0, \quad \omega|_{t=0} = \omega_0(x,y), \tag{3.1}$$

where $\mathcal{L}_\nu(t) = -\nu\Delta - e^{-\nu t} \cos y \partial_x (1 + \Delta^{-1})$. Beck and Wayne [2] considered the following model equation by removing the nonlocal part Δ^{-1} of $\mathcal{L}_\nu(t)$:

$$\partial_t \omega - \nu \Delta \omega - e^{-\nu t} \cos y \partial_x \omega = 0.$$

Using the hypocoercivity method in [49], they proved the enhanced dissipation rate of the solution in some Banach space X (see (3.7) in [2]): for any $t \in [0, \tau/\nu]$,

$$\|\omega(t)\|_X \le Ce^{-M\sqrt{\nu t}}\|\omega_0\|_X.$$

Based on numerical results, Beck and Wayne [2] conjectured that the same decay result should hold for $\mathcal{L}_\nu(t)$. In a series of works [34,51,55], we have developed three approaches to solve this conjecture: resolvent estimate method, wave operator method, and hypocoercivity method.

In [51], by developing the hypocoercivity method from [2], we proved the following enhanced dissipation results.

Theorem 3.1. *Given $\delta \in (0,1)$ and $\tau > 0$, there exist constants $c_1 > 0$, $C > 0$ such that if ω satisfies (3.1) with $\omega_0 \in L^2$ and $\int_{\mathbb{T}_{2\pi\delta}} \omega_0(x,y)dx = 0$, then it holds that, for $0 < t \le \tau/\nu$,*

$$\|\omega(t)\|_{L^2} \le Ce^{-c_1\sqrt{\nu t}}\|\omega_0\|_{L^2},$$
$$\|V(t)\|_{\dot{H}^1_x L^2_y} \le \frac{Ce^{-c_1\sqrt{\nu t}}}{\sqrt{1+\nu t^3}}\|\omega_0\|_{L^2}.$$

When $\delta = 1$, it holds that, for $0 < t \le \tau/\nu$,

$$\|(I-P_1)\omega(t)\|_{L^2} \le Ce^{-c_1\sqrt{\nu t}}\|(I-P_1)\omega_0\|_{L^2},$$
$$\|(I-P_1)V(t)\|_{\dot{H}^1_x L^2_y} \le \frac{Ce^{-c_1\sqrt{\nu t}}}{\sqrt{1+\nu t^3}}\|(I-P_1)\omega_0\|_{L^2}.$$

Here P_1 is the orthogonal projection to the space W_1 spanned by $\{\cos x, \sin x\}$.

Remark 3.1. Here the enhanced dissipation rate is smaller than that for the Couette flow. This leads to conjecture that, for stable monotone shear flows to the Euler equations, the enhanced dissipation rate should be $\nu^{\frac{1}{3}}$, and the rate should be $\nu^{\frac{1}{2}}$ for stable shear flows with nondegenerate critical points.

Remark 3.2. In addition to the important application to the transition threshold problem [34,36,55], the enhanced dissipation also plays an important role for the suppression of blow-up in the Keller–Segel system [9,24,31] and axisymmetrization of 2D viscous vortices [21]. Let us refer to [3,17,18,23] and the references therein for more relevant works.

Taking Fourier transform with respect to x to (3.1), we obtain

$$\partial_t\widehat{\omega} + \mathcal{L}_\nu(\alpha,t)\widehat{\omega} = 0, \quad \mathcal{L}_\nu = \nu(-\partial_y^2+\alpha^2) - i\alpha e^{-\nu t}\cos y\big(1+(\partial_y^2-\alpha^2)^{-1}\big).$$

We write

$$A = \sin y\big(1+(\partial_y^2-\alpha^2)^{-1}\big), \quad B = \cos y\big(1+(\partial_y^2-\alpha^2)^{-1}\big), \quad \gamma(t) = \alpha e^{-\nu t}.$$

Next we introduce an important inner product structure

$$\langle u, w\rangle_* = \langle u, w - (\alpha^2-\partial_y^2)^{-1}w\rangle.$$

An important observation is that under this inner product, the operators A and B are symmetric, i.e.,

$$\langle u, Aw\rangle_* = \langle Au, w\rangle_*, \quad \langle u, Bw\rangle_* = \langle Bu, w\rangle_*.$$

Moreover, for $|\alpha| > 1$, the norm $\|u\|_* = \langle u, u\rangle_*^{\frac{1}{2}}$ is equivalent to the usual L^2-norm:

$$(1-\alpha^{-2})\|u\|_{L^2}^2 \le \|u\|_*^2 \le \|u\|_{L^2}^2.$$

We introduce the energy functional:

$$\begin{aligned} &E_0(t) = \|\widehat{\omega}(t)\|_*^2, \quad E_1(t) = \|\partial_y\widehat{\omega}(t)\|_*^2, \quad E_2(t) = \|\partial_y^2\widehat{\omega}(t)\|_*^2, \\ &\mathcal{E}_1(t) = \mathrm{Re}\big\langle\partial_y\widehat{\omega}(t), iA\widehat{\omega}(t)\big\rangle_*, \quad \mathcal{E}_2(t) = \|\widehat{\omega}(t)\|_*^2 - \|B\widehat{\omega}(t)\|_*^2. \end{aligned}$$

Then we construct the total energy functional as follows:

$$\Phi(t) = E_0(t) + \alpha_0\nu tE_1(t) + \beta_0\nu t^2\mathcal{E}_1(t) + \gamma_0\nu t^3\mathcal{E}_2(t)$$

with the constants $\alpha_0, \beta_0, \gamma_0$ depending on $\gamma(0)$ so that

$$\Phi'(t) \le -c\big|\gamma(0)\big|^2\nu^2t^3E_0(t), \quad \Phi(t) \ge E_0(t).$$

Then the bound

$$E_0(t) \le \big(1 + c_2\big|\gamma(0)\big|^2\nu^2t^4\big)^{-1}E_0(0)$$

follows from the fact that $E_0(t)$ is decreasing in t. Once the polynomial decay is obtained, the exponential decay can be proved by iteration. Compared with [2], the key difference is that we introduce the new inner product and time dependent weights. This modification is also very effective in removing the logarithmic loss in [2] when achieving the dissipation in the usual L^2-norm.

In [55], we (jointly with Zhao) used the wave operator method. This idea was first introduced in [33] to study the pseudospectral bound of the Oseen vortices operator. The aim is to construct a wave operator $\mathbb{D}$ so that

$$\mathbb{D}\cos y\big(1 + \big(\partial_y^2 - \alpha^2\big)^{-1}\big)\omega = \cos y\mathbb{D}\omega.$$

Then $w = \mathbb{D}\omega$ satisfies

$$\partial_t w - \nu\big(\partial_y^2 - \alpha^2\big)w - i\alpha e^{-\nu t}\cos yw = -\nu\big[\partial_y^2, \mathbb{D}\big]\omega.$$

Moreover, the wave operator $\mathbb{D}$ we constructed has the following important properties:

- $\|\mathbb{D}(\omega)\|_{L^2}^2 = \langle\omega, \omega + (\partial_y^2 - \alpha^2)^{-1}\omega\rangle$;
- There exists a constant C independent of α so that

$$\begin{aligned} &\big\|\sin y\mathbb{D}(\omega)\big\|_{L^2}^2 \ge \|\partial_y\psi\|_{L^2}^2 + (\alpha^2-1)\|\psi\|_{L^2}^2, \\ &\big\|\partial_y\mathbb{D}(\omega)\big\|_{L^2} \le C|\alpha|^{\frac{1}{2}}\|\omega\|_{H^1}, \\ &\big\|\partial_y^2\mathbb{D}(\omega)\big\|_{L^2} \le C|\alpha|^{\frac{3}{2}}\|\omega\|_{H^2}, \end{aligned}$$

where $-(\partial_y^2 - \alpha^2)\psi = \omega$;

- Commutator estimate holds:

$$\left\|\sin y\left[\partial_y^2, \mathbb{D}\right]\omega\right\|_{L^2} \le C\left(|\alpha|\|\omega\|_{L^2} + \|\partial_y \omega\|_{L^2}\right).$$

The construction of the wave operator was motivated by our study of linear inviscid damping. More precisely, we may write the solution of (2.2) in the form

$$e^{-i\alpha t\mathcal{R}_\alpha}\psi_0 = \frac{1}{2\pi i}\int_{\operatorname{Ran} u} e^{-i\alpha t c}\Gamma(y,c)\tilde{\mathbb{D}}[\omega_0](c)\,dc.$$

An important observation is that

$$-\tilde{\mathbb{D}}\left[\left(\partial_y^2 - \alpha^2\right)e^{-i\alpha t\mathcal{R}_\alpha}\psi_0\right](c) = e^{-i\alpha t c}\tilde{\mathbb{D}}[\omega_0](c).$$

Taking the time derivative at $t = 0$, we get

$$-\tilde{\mathbb{D}}\left[\left(\partial_y^2 - \alpha^2\right)\mathcal{R}_\alpha\psi_0\right](c) = c\tilde{\mathbb{D}}[\omega_0](c),$$

which implies, by taking $c = u(y)$, that

$$\mathbb{D}\left[u\omega_0 + u''\psi_0\right] = u(y)\mathbb{D}[\omega_0].$$

Here $u(y) = -\cos y$, $\mathbb{D}[\omega_0](y) = \Lambda_o(y)\tilde{\mathbb{D}}[\omega_0](u(y))$ if ω_0 is odd and $\mathbb{D}[\omega_0](y) = \Lambda_e(y)\tilde{\mathbb{D}}[\omega_0](u(y))$ if ω_0 is even. See Section 2.2 in **[55]** for the details.

In **[34]**, we (jointly with Li) used the resolvent estimate method developed in **[33]** to prove the enhanced dissipation estimates for the linearized NS equations with time-independent coefficient:

$$\partial_t\omega + \mathcal{L}_\nu\omega = 0, \quad \mathcal{L}_\nu = -\nu\Delta - \sin y\,\partial_x\left(1 + \Delta^{-1}\right).$$

The key ingredient is to establish the following resolvent estimate: given $0 < \nu \le 1$ and $|\beta| > 1$, there exists a constant $C > 0$, independent of ν, λ, β, such that

$$\left\|(L_\nu - i\lambda)w\right\|_{L^2} \ge C\nu^{\frac{1}{2}}|\beta|^{\frac{1}{2}}\left(1 - \beta^{-2}\right)\|w\|_{L^2}, \tag{3.2}$$

where $L_\nu w = -\nu\partial_y^2 w + i\beta(\sin y w + \sin y\varphi)$ with $(\partial_y^2 - \beta^2)\varphi = w$.

In order to deduce the semigroup bound from (3.2), we use Gearhart–Prüss-type lemma for an m-accretive operator proved by the first author **[50]**.

Lemma 3.2. *Let H be an m-accretive operator in a Hilbert space X. Then it holds that, for any $t \ge 0$,*

$$\left\|e^{-tH}\right\| \le e^{-t\Psi + \pi/2},$$

where $\Psi(H) = \inf\{\|(H - i\lambda)u\|; u \in D(H),\ \lambda \in \mathbb{R},\ \|u\| = 1\}$.

Now the operator L_ν is m-accretive with respect to the new inner product $\langle\cdot,\cdot\rangle_*$. From (3.2), we infer that $\Psi(L_\nu) \ge c\nu^{\frac{1}{2}}|\beta|^{\frac{1}{2}}(1 - \beta^{-2})$. Then it follows from Lemma 3.2 that $\|e^{-tL_\nu}\| \le Ce^{-t\nu^{\frac{1}{2}}}$. In **[26]**, the authors also derive the semigroup bound via establishing the pseudospectral bound of the linearized operator.

Next we give a simple sketch of the proof of (3.2). Notice that

$$(L_\nu - i\beta\lambda)w = -\nu\partial_y^2 w + i\beta\left(\sin y(w + \varphi) - \lambda w\right).$$

We introduce $u = w + \varphi$. Then it suffices to show that

$$\|\mathcal{L}_\lambda u\|_{L^2} \geq C|\nu\beta|^{\frac{1}{2}}\|u\|_{L^2},$$

where $(\partial_y^2 - \tilde{\beta}^2)\varphi = u$ with $\beta^2 - 1 = \tilde{\beta}^2$ and

$$\mathcal{L}_\lambda u = i\beta\big[(\sin y - \lambda)u + \lambda\varphi\big] - \nu\partial_y^2 u.$$

Consider the case of $\lambda > 1$. Integration by parts gives

$$\big|\mathrm{Im}\langle\mathcal{L}_\lambda u, u\rangle\big| = \beta\left(\int_0^{2\pi}(\lambda - \sin y)|u|^2 dy + \lambda\big\|\varphi'\big\|_{L^2}^2 + \lambda\tilde{\beta}^2\|\varphi\|_{L^2}^2\right),$$

which implies

$$\int_0^{2\pi}(\lambda - \sin y)|u|^2 dy + \lambda\big\|\varphi'\big\|_{L^2}^2 + \lambda\tilde{\beta}^2\|\varphi\|_{L^2}^2 \leq \beta^{-1}\|\mathcal{L}_\lambda u\|_{L^2}\|u\|_{L^2}.$$

Let $\delta \in (0, 1]$. Then we have

$$\begin{aligned}\|u\|_{L^2}^2 \leq \|u\|_{L^2(\frac{\pi}{2}+\delta, \frac{5\pi}{2}-\delta)}^2 + 2\delta\|u\|_{L^\infty}^2 &\lesssim \delta^{-2}\int_0^{2\pi}(\lambda - \sin y)|u|^2 dy + \delta\|u\|_{L^\infty}^2 \\ &\lesssim \beta^{-1}\delta^{-2}\|\mathcal{L}_\lambda u\|_{L^2}\|u\|_{L^2} \\ &\quad + \nu^{-\frac{1}{2}}\delta\|\mathcal{L}_\lambda u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{3}{2}} + \delta\|u\|_{L^2}^2.\end{aligned}$$

Here we used the fact that

$$\|u\|_{L^\infty} \leq \big\|u'\big\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2} \leq \nu^{-\frac{1}{4}}\|\mathcal{L}_\lambda u\|_{L^2}^{\frac{1}{4}}\|u\|_{L^2}^{\frac{3}{4}} + \|u\|_{L^2},$$

due to $\nu\|u'\|_{L^2}^2 = |\mathrm{Re}\langle\mathcal{L}_\lambda u, u\rangle|$. Taking $\delta = \beta^{-\frac{1}{4}}\nu^{\frac{1}{4}} \ll 1$, we infer

$$\|u\|_{L^2}^2 \lesssim (\beta\nu)^{-\frac{1}{2}}\|\mathcal{L}_\lambda u\|_{L^2}\|u\|_{L^2} + (\beta\nu)^{-\frac{1}{4}}\|\mathcal{L}_\lambda u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{3}{2}},$$

which implies that

$$\|\mathcal{L}_\lambda u\|_{L^2} \gtrsim |\beta\nu|^{\frac{1}{2}}\|u\|_{L^2}.$$

The case of $|\lambda| < 1$ is much more difficult. Let $0 \leq y_1 \leq \frac{\pi}{2} \leq y_2 \leq \pi$ so that $\lambda = \sin y_1 = \sin y_2$. Let $\delta = \beta^{-\frac{1}{4}}\nu^{\frac{1}{4}} \ll 1$. Then we need to consider the following four types of energy estimates:

$$\begin{aligned}&\mathrm{Im}\langle\mathcal{L}_\lambda u, \chi_{(y_1,y_2)}u\rangle, \quad \mathrm{Im}\left\langle\mathcal{L}_\lambda u, \chi_{(y_1+\delta,y_2-\delta)}\frac{u}{\sin y - \lambda}\right\rangle, \\ &\mathrm{Im}\langle\mathcal{L}_\lambda u, \chi_{(y_2,y_1+2\pi)}u\rangle, \quad \mathrm{Im}\left\langle\mathcal{L}_\lambda u, \chi_{(y_2+\delta,y_1+2\pi-\delta)}\frac{u}{\sin y - \lambda}\right\rangle.\end{aligned}$$

See Section 3 in [34] for the details.

4. TRANSITION THRESHOLD PROBLEM FOR THE 3D COUETTE FLOW

We consider the transition threshold problem for the 3D Couette flow $U_*(y) = (y, 0, 0)$ in a finite channel $\Omega = \mathbb{T} \times [-1, 1] \times \mathbb{T}$. We introduce the perturbation $u(t, x, y, z) = v(t, x, y, z) - U_*(y)$, which solves

$$\begin{cases} \partial_t u - \nu\Delta u + y\partial_x u + (u^2, 0, 0) + \nabla p^L + u \cdot \nabla u + \nabla p^{NL} = 0, \\ \nabla \cdot u = 0, \\ u(t, x, \pm 1, z) = 0, \quad u(0, x, y, z) = u_0(x, y, z). \end{cases} \tag{4.1}$$

Here the pressure p^L and p^{NL} are determined by

$$\begin{cases} \Delta p^L = -2\partial_x u^2, \\ \Delta p^{NL} = -\mathrm{div}(u \cdot \nabla u) = -\partial_i u^j \partial_j u^i, \\ \big(\partial_y p^L - \nu\Delta u^2\big)|_{y=\pm 1} = 0, \quad \partial_y p^{NL}|_{y=\pm 1} = 0. \end{cases} \tag{4.2}$$

We define

$$P_0 f = \bar{f} = \frac{1}{2\pi}\int_{\mathbb{T}} f(x, y, z)dx, \quad P_{\neq} f = f_{\neq} = f - P_0 f.$$

In [16], we prove the following stability result, which implies that the transition threshold $\beta \le 1$ for the 3D Couette flow in a finite channel.

Theorem 4.1. *Assume that $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ with $\mathrm{div}\, u_0 = 0$. There exist constants $\nu_0, c_0, \epsilon, C > 0$, independent of ν, so that if $\|u_0\|_{H^2} \le c_0\nu$, $0 < \nu \le \nu_0$, then the solution u of the system (4.1) is global in time and satisfies the following stability estimates:*

- *(Uniform bounds and decay of the background streak)*

$$\begin{aligned} &\big\|\bar{u}^1(t)\big\|_{H^2} + \big\|\bar{u}^1(t)\big\|_{L^\infty} \le C\nu^{-1}\min(\nu t + \nu^{2/3}, e^{-\nu t})\|u_0\|_{H^2}, \\ &\big\|\bar{u}^2(t)\big\|_{H^2} + \big\|\bar{u}^3(t)\big\|_{H^1} + \big\|(\bar{u}^2, \bar{u}^3)(t)\big\|_{L^\infty} \le Ce^{-\nu t}\|u_0\|_{H^2}; \end{aligned}$$

- *(Rapid convergence to a streak)*

$$\begin{aligned} &\big\|(\partial_x, \partial_z)\partial_x u_{\neq}(t)\big\|_{L^2} + \big\|(\partial_x, \partial_z)\nabla u^2_{\neq}(t)\big\|_{L^2} + \big\|\big(\partial_x^2 + \partial_z^2\big)u^3_{\neq}(t)\big\|_{L^2} \\ &\quad + \nu^{1/4}\big\|u^2_{\neq}(t)\big\|_{H^2} + \nu^{1/3}\big\|\big(u^1_{\neq}, u^3_{\neq}\big)(t)\big\|_{H^1} + \big\|u^2_{\neq}(t)\big\|_{L^\infty} \\ &\quad + \nu^{1/6}\big\|\big(u^1_{\neq}, u^3_{\neq}\big)(t)\big\|_{L^\infty} \le Ce^{-2\epsilon\nu^{1/3}t}\|u_0\|_{H^2}, \\ &\|u_{\neq}\|_{L^\infty L^2} + \sqrt{\nu}\big\|t\big(u^1_{\neq}, u^3_{\neq}\big)\big\|_{L^2L^2} + \big\|\nabla u^2_{\neq}\big\|_{L^\infty L^2} \\ &\quad + \big\|\nabla u^2_{\neq}\big\|_{L^2L^2} \le C\|u_0\|_{H^2}. \end{aligned}$$

Let us give some remarks on our result.

1. Global stability estimates in particular imply that

$$\big\|u(t)\big\|_{L^\infty} \le Cc_0e^{-\nu t} \to 0 \quad \text{as} \quad t \to +\infty.$$

   This means that the 3D Couette flow is nonlinearly stable in the L^∞-sense when the perturbation is $o(\nu)$ in H^2.

2. Our rigorous analysis shows that various linear effects (including 3D lift-up effect, boundary layer effect, inviscid damping, and enhanced dissipation) play a crucial role in determining the transition threshold. Surprisingly, the transition threshold obtained in this paper is consistent with that for the case of $\Omega = \mathbb{T} \times \mathbb{R} \times \mathbb{T}$ obtained in [52]. This shows that the 3D lift-up may be the main mechanism leading to the instability of the flow even in the presence of the boundary layer effect. Our explanation of this surprise result is that weak nonlinear interaction (or null structure of nonlinear terms) and good linear mechanisms (inviscid damping and enhanced dissipation) counteract the bad effect of the boundary layer.

3. The transition threshold problem is very interesting in an infinite channel $\Omega = \mathbb{R} \times [-1, 1] \times \mathbb{T}$. In this case, we need to understand the long wave effect in the x variable. In fact, we conjecture that the threshold may be strictly less than 1 in this case.

4. The asymptotic analysis conducted in [13] indicates that the profile of shear flows may affect the transition threshold. From the results in [13], it seems reasonable to conjecture that the threshold $\beta \le \frac{3}{2}$ for the plane Poiseuille flow. In [34], Li, Wei, and Zhang proved that the threshold $\beta \le \frac{7}{4}$ for the 3D Kolmogorov flow. It is unclear whether one can improve it to $\beta \le \frac{3}{2}$.

5. The transition threshold problem for the pipe Poiseuille flow is completely open. This flow is probably the most interesting and important because it is close to the setting of the experiment conducted by Reynolds in 1883. The experimental result carried out by Hof, Juel, and Mullin [25] conclude that the minimum amplitude of a perturbation required to cause transition scales as the inverse of the Reynolds number, i.e., $O(\mathrm{Re}^{-1})$. The subsequent numerical result in [40] agrees with the experiment result in [25] for $\mathrm{Re} \gtrsim 4000$.

Now we give a sketch of some key ingredients of the proof.

First of all, we decompose the solution u into the zero mode $\bar{u}$ and nonzero mode $u_{\neq}$ due to their different behaviors. The zero mode $\bar{u}$ satisfies

$$(\partial_t - \nu\Delta)\bar{u}^1 + \bar{u}^2 + \overline{u \cdot \nabla u^1} = 0, \tag{4.3}$$

$$(\partial_t - \nu\Delta)\bar{u}^j + \partial_j \bar{p} + \big(\bar{u}^2\partial_y + \bar{u}^3\partial_z\big)\bar{u}^j + \overline{u_{\neq} \cdot \nabla u_{\neq}^j} = 0, \quad j = 2, 3. \tag{4.4}$$

To estimate nonzero modes, we will use a formulation in terms of the shearwise velocity u^2 and vorticity $\omega^2 = \partial_z u^1 - \partial_x u^3$:

$$\begin{cases} \partial_t(\Delta u^2) - \nu\Delta^2 u^2 + y\partial_x \Delta u^2 + \big(\partial_x^2 + \partial_z^2\big)(u \cdot \nabla u^2) \\ \quad - \partial_y\big[\partial_x(u \cdot \nabla u^1) + \partial_z(u \cdot \nabla u^3)\big] = 0, \\ \partial_t \omega^2 - \nu\Delta\omega^2 + y\partial_x\omega^2 + \partial_z u^2 + \partial_z(u \cdot \nabla u^1) - \partial_x(u \cdot \nabla u^3) = 0, \\ \partial_y u^2(t, x, \pm 1, z) = u^2(t, x, \pm 1, z) = 0, \quad \omega^2(x, \pm 1, z) = 0. \end{cases} \tag{4.5}$$

The idea of using Δu^2 may go back to Kelvin's original paper [30]. The main advantage of using Δu^2 is that the equation of Δu^2 does not destroy the linear structure. This important point has played an important role in the works [6,52].

The linearized system of zero mode $\bar{u}$ becomes

$$(\partial_t - \nu\Delta)\bar{u}^1 + \bar{u}^2 = 0, \quad (\partial_t - \nu\Delta)\bar{u}^j + \partial_j \bar{p} = 0, \quad j = 2, 3.$$

Then it is easy to see that $\|\bar{u}^1(t)\|_{L^2} \le C(1+t)e^{-\nu t}\|u_0\|_{L^2}$. When $t \lesssim \nu^{-1}$, $\bar{u}^1$ grows linearly in time. This phenomenon is referred to as the *3D lift-up*. To keep $\bar{u}^1$ small, the perturbation u_0 should be as small as $o(\nu)$. From this point of view, our result seems optimal. It also turns out that the 3D lift-up is the worst mechanism leading to the instability.

To estimate Δu^2 and ω^2, we need to establish the space–time estimates for the following linearized system:

$$\begin{cases} \partial_t \omega - \nu\big(\partial_y^2 - \eta^2\big)\omega + iky\omega = -ikf_1 - \partial_y f_2 - i\ell f_3 - f_4, \\ \omega|_{y=\pm 1} = 0, \quad \omega|_{t=0} = \omega_{in}, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \partial_t \omega - \nu\big(\partial_y^2 - \eta^2\big)\omega + iky\omega = F, \\ \big(\partial_y^2 - \eta^2\big)\varphi = \omega, \ \partial_y\varphi|_{y=\pm 1} = \varphi|_{y=\pm 1} = 0, \\ \omega|_{t=0} = \omega_{in}. \end{cases} \tag{4.7}$$

Here $\eta^2 = k^2 + \ell^2$. In [16], we establish the following space–time estimates.

Theorem 4.2. *Let ω be a solution of (4.6) with $f_4(t, \pm 1) = 0$ and $\omega_{in}(\pm 1) = 0$. Then there exists $\epsilon_1 > 0$ so that, for any $a \in [0, \epsilon_1]$,*

$$\begin{aligned} &\big\|e^{a\nu^{1/3}t}\omega\big\|_{L^\infty L^2}^2 + \nu\big\|e^{a\nu^{1/3}t}\omega'\big\|_{L^2L^2}^2 + \big(\nu\eta^2 + (\nu k^2)^{1/3}\big)\big\|e^{a\nu^{1/3}t}\omega\big\|_{L^2L^2}^2 \\ &\quad \le C\big(\|\omega_{in}\|_{L^2}^2 + \nu^{-1}\big\|e^{a\nu^{1/3}t} f_2\big\|_{L^2L^2}^2 + \big(\eta|k|\big)^{-1}\big\|e^{a\nu^{1/3}t}\partial_y f_4\big\|_{L^2L^2}^2 \\ &\qquad + \eta|k|^{-1}\big\|e^{a\nu^{1/3}t} f_4\big\|_{L^2L^2}^2 + \min\big((\nu\eta^2)^{-1}, (\nu k^2)^{-1/3}\big)\big\|e^{a\nu^{1/3}t}(kf_1 + \ell f_3)\big\|_{L^2L^2}^2\big). \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\big\|e^{a\nu^{1/3}t}\omega'\big\|_{L^\infty L^2}^2 + \nu\big\|e^{a\nu^{1/3}t}\omega''\big\|_{L^2L^2}^2 + \nu\eta^2\big\|e^{a\nu^{1/3}t}\omega'\big\|_{L^2L^2}^2 \\ &\quad \le C\big\|\omega_{in}'\big\|_{L^2}^2 + C\nu^{-\frac{2}{3}}|k|^{\frac{2}{3}}\big(\|\omega_{in}\|_{L^2}^2 + \big(\eta|k|\big)^{-1}\big\|e^{a\nu^{1/3}t}\partial_y f_4\big\|_{L^2L^2}^2 \\ &\qquad + \eta|k|^{-1}\big\|e^{a\nu^{1/3}t} f_4\big\|_{L^2L^2}^2\big) + C\nu^{-1}\big(\big\|e^{a\nu^{1/3}t}(kf_1 + \ell f_3)\big\|_{L^2L^2}^2 \\ &\qquad + \nu^{-\frac{2}{3}}|k|^{\frac{2}{3}}\big\|e^{a\nu^{1/3}t} f_2\big\|_{L^2L^2}^2 + \big\|e^{a\nu^{1/3}t}\partial_y f_2\big\|_{L^2L^2}^2\big). \end{aligned}$$

Here $\omega' = \partial_y\omega$ and $\omega'' = \partial_y^2\omega$.

Theorem 4.3. *Let ω solve (4.7) with $\partial_y\varphi_{in}|_{y=\pm 1} = 0$ and $F = ikf_1 + \partial_y f_2 + i\ell f_3$. Then there exist $\epsilon_1 > 0$, $\nu_0 > 0$ so that, for any $a \in [0, \epsilon_1]$, $\nu \in (0, \nu_0)$,*

$$\begin{aligned} &|k\eta|^{\frac{1}{2}}\big\|e^{a\nu^{\frac{1}{3}}t}(\partial_y, \eta)\varphi\big\|_{L^2L^2} + \nu^{\frac{3}{4}}\big\|e^{a\nu^{\frac{1}{3}}t}\partial_y\omega\big\|_{L^2L^2} + \nu^{\frac{1}{2}}\eta\big\|e^{a\nu^{\frac{1}{3}}t}\omega\big\|_{L^2L^2} \\ &\quad + \eta\big\|e^{a\nu^{\frac{1}{3}}t}(\partial_y, \eta)\varphi\big\|_{L^\infty L^2} + \nu^{\frac{1}{4}}\big\|e^{a\nu^{\frac{1}{3}}t}\omega\big\|_{L^\infty L^2} \\ &\quad \le C\nu^{-\frac{1}{2}}\big\|e^{a\nu^{\frac{1}{3}}t}(f_1, f_2, f_3)\big\|_{L^2L^2} + C\big(\eta^{-1}\|\partial_y\omega_{in}\|_{L^2} + \|\omega_{in}\|_{L^2}\big). \end{aligned}$$

The proofs of Theorems 4.2 and 4.3 used the resolvent estimate method developed in [14]. The main idea is to separate the resolvent problem into two subproblems:

1. *The inhomogeneous problem with favorable boundary conditions*
   The good boundary conditions avoid the boundary terms caused by the integration by parts argument so that we can establish various resolvent estimates via the direct energy method by choosing suitable multipliers.

2. *The homogenous problem with nonvanishing boundary conditions*
   This step is to match the boundary conditions. We can first use the Airy function or the solution of a simple elliptic problem to construct an approximate solution. Then we can construct the solution to the homogenous problem via solving a perturbation problem with favorable boundary conditions.

The space–time estimates established in Theorems 4.2 and 4.3 encompass four kinds of important linear effects: heat diffusion, enhanced dissipation, inviscid damping, and boundary layer. These estimates should be enough to prove a transition threshold $\beta \leq \frac{5}{3}$. To achieve the sharp threshold, we have to handle the problem in a quasilinear way. That is, we need to consider the full linearized 3D Navier–Stokes system around the flow $(V(y,z),0,0)$, which is a small perturbation of the Couette flow, i.e.,

$$\|V-y\|_{H^4} \leq \varepsilon_0, \quad V(y,z)-y|_{y=\pm 1}=0,$$

with ε_0 small enough but independent of ν. We denote

$$\mathbb{A}_{\nu,V}u = \mathbb{P}\big(\nu\Delta u - V\partial_x u - \big(\partial_y V(u^2+\kappa u^3),0,0\big)\big),$$

here $\mathbb{P}$ is the Leray projection and $\kappa = \partial_z V/\partial_y V$. Then we study the following linearized system:

$$\partial_t u_{\neq} - \mathbb{A}_{\nu,V}u_{\neq} + \vec{g} = 0. \tag{4.8}$$

The key point is to exclude the unstable eigenvalues of the operator $\mathbb{A}_{\nu,V}$. This problem is highly nontrivial. Even for the following linearized equation:

$$\begin{cases} \partial_t w - \nu\Delta w + V\partial_x w = f, \\ \Delta\varphi = w, \quad \varphi|_{y=\pm 1} = \partial_y\varphi|_{y=\pm 1} = 0, \end{cases}$$

the linear stability when $V = V(y)$ is close to y was just proved by Almog and Helffer [1]. After applying the Fourier transform with respect to (t,x) and introducing $W = u^2 + \kappa u^3$ and $U = u^3$, the problem is reduced to the following linearized system in terms of (W,U):

$$\begin{cases} -\nu\Delta W + ik\big(V(y,z)-\lambda\big)W - a(\nu k^2)^{1/3}W + (\partial_y + \kappa\partial_z)p^{L1} \\ \quad + G_1 + \nu(\Delta\kappa)U + 2\nu\nabla\kappa\cdot\nabla U = 0, \\ -\nu\Delta U + ik\big(V(y,z)-\lambda\big)U - a(\nu k^2)^{1/3}U + G_2 + \partial_z p^{L1} = 0, \\ W|_{y=\pm 1} = \partial_y W|_{y=\pm 1} = U|_{y=\pm 1} = 0, \end{cases} \tag{4.9}$$

where $\lambda \in \mathbb{R}$ and

$$\Delta p^{L1} = -2ik\partial_y V W, \quad \partial_x W = ikW,\ \partial_x U = ikU, \quad \partial_x p^{L1} = ikp^{L1}.$$

Theorem 4.4. *Let $W \in H^4(\Omega)$, $U \in H^2(\Omega)$ be a solution of* (4.9). *Then there exist $\epsilon_1 > 0$, $\nu_0 > 0$ so that, for any $a \in [0, \epsilon_1]$, $\nu \in (0, \nu_0)$,*

$$\begin{aligned}
&\nu^{\frac{1}{3}}\big(\big\|\partial_x^2 U\big\|_{L^2}^2 + \big\|\partial_x(\partial_z - \kappa\partial_y)U\big\|_{L^2}^2\big) + \nu\big(\big\|\nabla\partial_x^2 U\big\|_{L^2}^2 + \big\|\nabla\partial_x(\partial_z - \kappa\partial_y)U\big\|_{L^2}^2\big) \\
&\quad + \nu^{\frac{1}{3}}\|\partial_x\nabla W\|_{L^2}^2 + \nu\|\partial_x\Delta W\|_{L^2}^2 + \nu^{\frac{5}{3}}\|\partial_x\Delta U\|_{L^2}^2 \\
&\quad \le C\nu^{-1}\big(\|\nabla G_1\|_{L^2}^2 + \|\partial_x G_2\|_{L^2}^2\big).
\end{aligned}$$

In particular, this result shows that *the 3D linearized Navier–Stokes system* (4.8) *around the Couette flow is linearly stable*. This theorem is the key and most difficult part in the proof of nonlinear stability. The proof was motivated by our work [52]. The key point is to introduce a good unknown $W_g = W - W_s$, where W_s is the singular part of W. Then $w_g = \Delta W_g$ satisfies

$$-\nu\Delta w_g + ik\big(V(y,z) - \lambda\big)w_g - a(\nu k^2)^{1/3}w_g = \text{good terms}.$$

For nonlinear stability, we introduce the following energy functionals, which are suitable adaptations of those introduced in [52].

(1) *Energy functional of zero mode*. We first decompose $\bar{u}^1 = \bar{u}^{1,0} + \bar{u}^{1,\neq}$ with

$$\begin{aligned}
&(\partial_t - \nu\Delta)\bar{u}^{1,0} + \bar{u}^2 + \bar{u}^2\partial_y\bar{u}^{1,0} + \bar{u}^3\partial_z\bar{u}^{1,0} = 0, \\
&(\partial_t - \nu\Delta)\bar{u}^{1,\neq} + \bar{u}^2\partial_y\bar{u}^{1,\neq} + \bar{u}^3\partial_z\bar{u}^{1,\neq} + \overline{u_{\neq}\cdot\nabla u^1_{\neq}} = 0, \\
&\bar{u}^{1,0}|_{t=0} = 0, \quad \bar{u}^{1,\neq}|_{t=0} = \bar{u}^1(0), \quad \bar{u}^{1,0}|_{y=\pm1} = 0, \quad \bar{u}^{1,\neq}|_{y=\pm1} = 0.
\end{aligned}$$

The main reason for making this decomposition is that $\bar{u}^{1,\neq}$ has better decay in ν, and thus $\bar{u}^{1,\neq}\partial_x$ could be viewed as a perturbation. In this way, we avoid estimating the higher-order derivatives of nonzero modes. Then we introduce the following energy functional to control the zero mode:

$$E_1 = E_{1,0} + \nu^{-2/3}E_{1,\neq},$$

where

$$\begin{aligned}
E_{1,0} &= \big\|\bar{u}^{1,0}\big\|_{L^\infty H^4} + \nu^{-1}\big\|\partial_t\bar{u}^{1,0}\big\|_{L^\infty H^2} + \nu^{-\frac{1}{2}}\big\|\partial_t\bar{u}^{1,0}\big\|_{L^2 H^3}, \\
E_{1,\neq} &= \big\|\bar{u}^{1,\neq}\big\|_{L^\infty H^2} + \nu^{\frac{1}{2}}\big\|\nabla\bar{u}^{1,\neq}\big\|_{L^2 H^2},
\end{aligned}$$

and the energy E_2 is defined by

$$\begin{aligned}
E_2 = {}& \big\|\Delta\bar{u}^2\big\|_{L^\infty L^2} + \nu^{\frac{1}{2}}\big\|\nabla\Delta\bar{u}^2\big\|_{L^2L^2} + \nu^{\frac{1}{2}}\big\|\Delta\bar{u}^2\big\|_{L^2L^2} + \nu^{-\frac{1}{2}}\big\|\partial_t\nabla\bar{u}^2\big\|_{L^2L^2} \\
&+ \big\|\nabla\bar{u}^3\big\|_{L^\infty L^2} + \nu^{\frac{1}{2}}\big\|\Delta\bar{u}^3\big\|_{L^2L^2} + \nu^{\frac{1}{2}}\big\|\nabla\bar{u}^3\big\|_{L^2L^2} + \nu^{-\frac{1}{2}}\big\|\partial_t\bar{u}^3\big\|_{L^2L^2} \\
&+ \big\|\min\big((\nu^{\frac{2}{3}} + \nu t)^{\frac{1}{2}}, 1 - y^2\big)\Delta\bar{u}^3\big\|_{L^\infty L^2} \\
&+ \nu^{-\frac{1}{2}}\big\|\min\big((\nu^{\frac{2}{3}} + \nu t)^{\frac{1}{2}}, 1 - y^2\big)\nabla\partial_t\bar{u}^3\big\|_{L^\infty L^2} \\
&+ \nu^{\frac{1}{2}}\big\|\min\big((\nu^{\frac{2}{3}} + \nu t)^{\frac{1}{2}}, 1 - y^2\big)\nabla\Delta\bar{u}^3\big\|_{L^2L^2}.
\end{aligned}$$

The estimates of E_1 and E_2 are based on direct energy estimates for the system (4.3) and (4.4).

(2) *Energy functional of nonzero mode (semilinear part)*. We consider

$$E_3 = E_{3,0} + E_{3,1},$$

where $E_{3,0}$ and $E_{3,1}$ are defined by

$$\begin{aligned} E_{3,0} &= \nu^{\frac{1}{2}} \| e^{2\epsilon\nu^{\frac{1}{3}}t}(\partial_x, \partial_z)\Delta u^2_{\neq}\|_{L^2L^2} + \nu^{\frac{3}{4}} \| e^{2\epsilon\nu^{\frac{1}{3}}t}\nabla\Delta u^2_{\neq}\|_{L^2L^2} \\ &\quad + \| e^{2\epsilon\nu^{\frac{1}{3}}t}(\partial_x, \partial_z)\nabla u^2_{\neq}\|_{L^\infty L^2} + \| e^{2\epsilon\nu^{\frac{1}{3}}t}\partial_x\nabla u^2_{\neq}\|_{L^2L^2} \\ &\quad + \| e^{2\epsilon\nu^{\frac{1}{3}}t}(\partial_x^2 + \partial_z^2)u^3_{\neq}\|_{L^\infty L^2} + \nu^{\frac{1}{2}} \| e^{2\epsilon\nu^{\frac{1}{3}}t}(\partial_x^2 + \partial_z^2)\nabla u^3_{\neq}\|_{L^2L^2}, \\ E_{3,1} &= \nu^{\frac{1}{3}}\big(\| e^{2\epsilon\nu^{\frac{1}{3}}t}\nabla\omega^2_{\neq}\|_{L^\infty L^2} + \nu^{\frac{1}{2}}\| e^{2\epsilon\nu^{\frac{1}{3}}t}\Delta\omega^2_{\neq}\|_{L^2L^2}\big). \end{aligned}$$

The estimate of E_3 is based on the space–time estimates for the coupled system (4.5) of $(\Delta u^2, \omega^2)$ via Theorems 4.2 and 4.3.

(3) *Energy functional of nonzero mode (quasilinear part)*. We now treat

$$E_5 = \nu^{1/6}\| e^{3\epsilon\nu^{1/3}t}\partial_x^2 u^2_{\neq}\|_{L^2L^2} + \nu^{1/6}\| e^{3\epsilon\nu^{1/3}t}\partial_x^2 u^3_{\neq}\|_{L^2L^2},$$

which is vital to control some nonlinear interaction terms with the lift-up effect such as $\bar{u}^1\partial_x u_{\neq}$ and $u^j_{\neq}\partial_j\bar{u}^1 (j = 2, 3)$. The estimate of E_5 relies on Theorem 4.4.

Based on the linear space–time estimates, combined with a nonlinear interaction estimate, we can derive the following uniform energy estimates:

$$\begin{aligned} E_{1,0} &\le C\nu^{-1}\big(\|u_0\|_{H^2} + E_2 + E_2E_{1,0}\big), \\ E_{1,\neq} &\le C\big(\|u_0\|_{H^2} + \nu^{-1}E_2E_{1,\neq} + \nu^{-\frac{4}{3}}E_3^2\big), \\ E_2 &\le C(1 + \nu^{-1}E_2)^2\big(\|u(0)\|_{H^2} + \nu^{-1}E_3^2\big), \end{aligned}$$

and

$$\begin{aligned} E_{3,0}^2 &\le C\|u_0\|^2_{H^2} + C\big(E_3^4/\nu^2 + E_2^2E_3^2/\nu^2 + E_1^2E_3E_5 + E_1^2E_3^{\frac{3}{2}}E_5^{\frac{1}{2}}\big), \\ E_{3,1}^2 &\le C\big(\|u_0\|^2_{H^2} + \nu^{-2}E_3^4 + \nu^{-\frac{4}{3}}E_2^2E_3^2 + E_1^2E_3E_5 + E_1^2E_3^{\frac{7}{4}}E_5^{\frac{1}{4}} + E_1^2E_3^{\frac{3}{2}}E_5^{\frac{1}{2}}\big), \end{aligned}$$

as well as

$$E_5^2 \le CE_6^2 \le C\|u_0\|^2_{H^2} + C\big(E_1^2 + \nu^{-2}E_2^2\big)E_6^2 + C\nu^{-2}E_3^4,$$

where E_6 is an auxiliary energy functional (see Section 14 in [16] for the definition of E_6). When the perturbation $\|u_0\|_{H^2} \le c_0\nu$, E_1 is small due to the lift-up effect, while E_2, E_3, and E_5 are as small as $o(\nu)$.

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REFERENCES

[1] Y. Almog and B. Helffer, On the stability of laminar flows between plates. *Arch. Ration. Mech. Anal.* **241** (2021), 1281–1401.

[2] M. Beck and C. E. Wayne, Metastability and rapid convergence to quasi-stationary bar states for the two-dimensional Navier–Stokes equations. *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013), 905–927.

[3] J. Bedrossian and M. Coti Zelati, Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows. *Arch. Ration. Mech. Anal.* **224** (2017), 1161–1204.

[4] J. Bedrossian, M. Coti Zelati, and V. Vicol, Vortex axisymmetrization, inviscid damping, and vorticity depletion in the linearized 2D Euler equations. *Ann. PDE* **5** (2019), Art. 4, 192 pp.

[5] J. Bedrossian, P. Germain, and N. Masmoudi, Dynamics near the subcritical transition of the 3D Couette flow II: Above threshold case. 2015, arXiv:1506.03721.

[6] J. Bedrossian, P. Germain, and N. Masmoudi, On the stability threshold for the 3D Couette flow in Sobolev regularity. *Ann. of Math.* **185** (2017), 541–608.

[7] J. Bedrossian, P. Germain, and N. Masmoudi, Stability of the Couette flow at high Reynolds number in 2D and 3D. *Bull. Amer. Math. Soc. (N.S.)* **56** (2019), 373–414.

[8] J. Bedrossian, P. Germain, and N. Masmoudi, Dynamics near the subcritical transition of the 3D Couette flow I: Below threshold case. *Mem. Amer. Math. Soc.* **266** (2020), no. 1294.

[9] J. Bedrossian and S. He, Suppression of blow-up in Patlak–Keller–Segel via shear flows. *SIAM J. Math. Anal.* **49** (2017), 4722–4766.

[10] J. Bedrossian and N. Masmoudi, Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. *Publ. Math. Inst. Hautes Études Sci.* **122** (2015), 195–300.

[11] F. Bouchet and H. Morita, Large time behavior and asymptotic stability of the 2D Euler and linearized Euler equations. *Phys. D* **239** (2010), 948–966.

[12] K. Case, Stability of inviscid plane Couette flow. *Phys. Fluids* **3** (1960), 143–148.

[13] S. J. Chapman, Subcritical transition in channel flows. *J. Fluid Mech.* **451** (2002), 35–97.

[14] Q. Chen, T. Li, D. Wei, and Z. Zhang, Transition threshold for the 2D Couette flow in a finite channel. *Arch. Ration. Mech. Anal.* **238** (2020), 125–183.

[15] Q. Chen, D. Wei, and Z. Zhang, Linear stability of pipe Poiseuille flow at high Reynolds number. 2019, arXiv:1910.14245. *Comm. Pure Appl. Math.*, in press.

[16] Q. Chen, D. Wei, and Z. Zhang, Transition threshold for the 3D Couette flow in a finite channel. 2020, arXiv:2006.00721. *Mem. Amer. Math. Soc.*, in press.

[17] P. Constantin, A. Kiselev, L. Ryzhik, and A. Zlatos, Diffusion and mixing in fluid flow. *Ann. of Math.* **168** (2008), 643–674.

[18] M. Coti Zelati, M. G. Delgadino, and T. M. Elgindi, On the relation between enhanced dissipation timescales and mixing rates. *Comm. Pure Appl. Math.* **73** (2020), 1205–1244.

[19] Y. Deng and N. Masmoudi, Long time instability of the Couette flow in low Gevrey spaces. 2018, arXiv:1803.01246.

[20] P. Drazin and W. Reid, *Hydrodynamic stability*. Cambridge Monogr. Mech. Appl. Math., Cambridge Univ. Press, New York, 1981.

[21] T. Gallay, Enhanced dissipation and axisymmetrization of two-dimensional viscous vortices. *Arch. Ration. Mech. Anal.* **230** (2018), 939–975.

[22] T. Gebhardt and S. Grossmann, Chaos transition despite linear stability. *Phys. Rev. E* **50** (1994), 3705–3711.

[23] E. Grenier, T. Nguyen, F. Rousset, and A. Soffer, Linear inviscid damping and enhanced viscous dissipation of shear flows by using the conjugate operator method. *J. Funct. Anal.* **278** (2020), 108339, 27 pp.

[24] S. He, Suppression of blow-up in parabolic-parabolic Patlak–Keller–Segel via strictly monotone shear flows. *Nonlinearity* **31** (2018), 3651–3688.

[25] B. Hof, A. Juel, and T. Mullin, Scaling of the turbulence transition threshold in a pipe. *Phys. Rev. Lett.* **91** (2004), 244502.

[26] S. Ibrahim, Y. Maekawa, and N. Masmoudi, On pseudospectral bound for non-selfadjoint operators and its application to stability of Kolmogorov flows. *Ann. PDE* **5** (2019), Paper No. 14, 84 pp.

[27] A. Ionescu and H. Jia, Nonlinear inviscid damping near monotonic shear flows. 2018, arXiv:2001.03087.

[28] A. Ionescu and H. Jia, Axi-symmetrization near point vortex solutions for the 2D Euler equation. 2019, arXiv:1904.09170.

[29] A. Ionescu and H. Jia, Inviscid damping near the Couette flow in a channel. *Comm. Math. Phys.* **374** (2020), 2015–2096.

[30] L. Kelvin, Stability of fluid motion-rectilinear motion of viscous fluid between two parallel plates. *Philos. Mag.* **24** (1887), 188–196.

[31] A. Kiselev and X. Xu, Suppression of chemotactic explosion by mixing. *Arch. Ration. Mech. Anal.* **222** (2016), 1077–1112.

[32] L. Landau, On the vibration of the electronic plasma. *J. Phys. USSR* **10** (1946), 25.

[33] T. Li, D. Wei, and Z. Zhang, Pseudospectral and spectral bounds for the Oseen vortices operator. *Ann. Sci. Éc. Norm. Supér.* **53** (2020), 993–1035.

[34] T. Li, D. Wei, and Z. Zhang, Pseudospectral bound and transition threshold for the 3D Kolmogorov flow. *Comm. Pure Appl. Math.* **73** (2020), 465–557.

[35] Y. C. Li and Z. Lin, A resolution of the Sommerfeld paradox. *SIAM J. Math. Anal.* **43** (2011), 1923–1954.

[36] Z. Lin and M. Xu, Metastability of Kolmogorov flows and inviscid damping of shear flows. *Arch. Ration. Mech. Anal.* **231** (2019), 1811–1852.

[37] Z. Lin and C. Zeng, Inviscid dynamical structures near Couette flow. *Arch. Ration. Mech. Anal.* **200** (2011), 1075–1097.

[38] A. Lundbladh, D. Henningson, and S. Reddy, Threshold amplitudes for transition in channel flows. In *Transition, turbulence and combustion*, pp. 309–318, Springer, New York, 1994.

[39] N. Masmoudi and Z. Zhao, Nonlinear inviscid damping for a class of monotone shear flows in finite channel. 2020, arXiv:2001.08564.

[40] F. Mellibovskya and A. Meseguer, Pipe flow transition threshold following localized impulsive perturbations. *Phys. Fluids* **19** (2007), 044102.

[41] W. Orr, The stability or instability of steady motions of a perfect liquid and of a viscous liquid. Part I: A perfect liquid. *Proc. R. Irish Acad. Sec. A: Math. Phys. Sci.* **27** (1907), 9–68.

[42] S. Orszag and L. Kells, Transition to turbulence in plane Poiseuille and plane Couette flow. *J. Fluid Mech.* **96** (1980), 159–205.

[43] O. Reynolds, An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. *Proc. R. Soc. Lond.* **35** (1883), 84.

[44] V. A. Romanov, Stability of plane-parallel Couette flow. *Funktsional. Anal. i Prilozhen.* **7** (1973), 62–73.

[45] D. Ryutov, Landau damping: half a century with the great discovery. *Plasma Phys. Control Fusion* **41** (1999), A1–A12.

[46] P. Schmid and D. Henningson, *Stability and transition in shear flows*. Appl. Math. Sci. 142, Springer, New York, 2001.

[47] L. N. Trefethen, Pseudospectra of linear operators. *SIAM Rev.* **39** (1997), 383–406.

[48] L. Trefethen, A. Trefethen, S. Reddy, and T. Driscoll, Hydrodynamic stability without eigenvalues. *Science* **261** (1993), 578–584.

[49] C. Villani, Hypocoercivity. *Mem. Amer. Math. Soc.* **202** (2009), no. 950, iv+141.

[50] D. Wei, Diffusion and mixing in fluid flow via the resolvent estimate. *Sci. China Math.* **64** (2021), 507–518.

[51] D. Wei and Z. Zhang, Enhanced dissipation for the Kolmogorov flow via the hypocoercivity method. *Sci. China Math.* **62** (2019), 1219–1232.

[52] D. Wei and Z. Zhang, Transition threshold for the 3D Couette flow in Sobolev space. *Comm. Pure Appl. Math.* DOI 10.1002/cpa.21948.

[53] D. Wei, Z. Zhang, and W. Zhao, Linear inviscid damping for a class of monotone shear flow in Sobolev spaces. *Comm. Pure Appl. Math.* **71** (2018), 617–687.

[54] D. Wei, Z. Zhang, and W. Zhao, Linear inviscid damping and vorticity depletion for shear flows. *Ann. PDE* **5** (2019), Paper No. 3, 101 pp.

[55] D. Wei, Z. Zhang, and W. Zhao, Linear inviscid damping and enhanced dissipation for the Kolmogorov flow. *Adv. Math.* **362** (2020), 106963.

[56] D. Wei, Z. Zhang, and H. Zhu, Linear inviscid damping for the β-plane equation. *Comm. Math. Phys.* **375** (2020), 127–174.

[57] A. Yaglom, *Hydrodynamic instability and transition to turbulence*. Fluid Mech. Appl. 100, Springer, New York, 2012.

[58] C. Zillinger, Linear inviscid damping for monotone shear flows in a finite periodic channel, boundary effects, blow-up and critical Sobolev regularity. *Arch. Ration. Mech. Anal.* **221** (2016), 1449–1509.

[59] C. Zillinger, Linear inviscid damping for monotone shear flows. *Trans. Amer. Math. Soc.* **369** (2017), 8799–8855.

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