BOOTSTRAP APPROACH TO 1+1-DIMENSIONAL INTEGRABLE QUANTUM FIELD THEORIES: THE CASE OF THE SINH-GORDON MODEL

KAROL KAJETAN KOZLOWSKI

ABSTRACT

1+1-dimensional integrable quantum field theories correspond to a sparse subset of quantum field theories where the calculation of physically interesting observables can be brought to explicit, closed, and manageable expressions thanks to the factorizability of the **S** matrices which govern the scattering in these models. In particular, the correlation functions are expressed in terms of explicit series of multiple integrals, this nonperturbatively for all values of the coupling. However, the question of convergence of these series, and thus the mathematical well-definiteness of these correlators, is mostly open. This paper reviews the overall setting used to formulate such models and discusses the recent progress relative to solving the convergence issues in the case of the 1+1-dimensional massive integrable Sinh-Gordon quantum field theory.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 82B23; Secondary 81Q80,45E10,33E20,30E25

KEYWORDS

Integrable quantum field theories, convergence, exactly solvable models, Riemann–Hilbert problems, concentration of measure

INTERNATIONAL CONGRESS OF MATHEMATICIANS
2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 5, pp. 4096–4118
DOI 10.4171/ICM2022/96

Published by EMS Press

1. INTRODUCTION

1.1. Scattering matrices for quantum integrable field theories

It was discovered in the early 20th century that the description of matter at low-scales demands to wave-off some of the existing at the time paradigms governing the motion and very structure of particles in interactions. This led to the development of the theory of relativity on the one hand, and quantum mechanics on the other. In the latter setting, the state of a physical system is described by a vector, the wave function, belonging to some Hilbert space and supposed to encapsulate all the physical degrees of freedom of that system. On the classical level, the time evolution of particles' momenta and positions is governed by a set of generically nonlinear ordinary differential equations which can be written in the form of Hamilton's equations. In its turn, the time evolution of a wave function is governed by a first-order ordinary linear differential equation driven by the Hamiltonian operator. This operator is obtained through a quantization procedure: its symbol is given by the classical Hamiltonian of the system or, said differently, it is obtained from the classical Hamiltonian upon replacing the classical momenta and positions by operators. While the success of the approach was astonishing relatively to the amount of experiments which could have been explained, soon after the early development of the theory it became clear that in order to describe physics at even smaller scales or higher energies, one needs to develop a quantum theory of fields which would bring together the quantum and relativistic features in the setting of uncountably many degrees of freedom. In loose words, such a theory would be reached by producing operator valued generalized functions, viz. formal kernels of distributions, depending on the space-time coordinates which would satisfy analogues of nonlinear, relativistically invariant, evolution equations arising in classical field theory. While it was rather straightforward to construct the quantum theory of the free field (and nowadays such a construction is fully rigorous), the construction of interacting theories which are the sole relevant for physics appeared to be a tremendously hard task, this even on a formal level of rigor. The various approaches that were developed quickly met serious problems: the most prominent being the divergence of coefficients supposed to describe the formal perturbative expansions of physical observables around the free theories. Eventually, these problems could have been formally circumvented in certain cases by the use of the so-called renormalization procedure. The latter, while being able to produce numbers which were measured with great agreement in collider experiments, eluded for very long any attempts at making it rigorous. Some progress was eventually achieved for several instances of truly interacting, viz. nonfree, quantum field theories within the so-called constructive quantum field theory approach, see [35] for a review. While successful in rigorously showing the existence and certain overall properties of such theories, the approach did not lead yet to rigorous and manageable expressions for the correlation functions, which are the quantities measured in experiments and thus of prime interest to the theory.

Among the various alternatives to renormalization, one may single out the **S**-matrix program which aimed at describing a quantum field theory directly in terms of the quantities that are measured in experiments. This led to a formulation of the theory in terms of matrix-valued functions in n complex variables, with $n = 0, 1, 2, \ldots$, that correspond to the entries of the $\mathbf{S}$-matrix between asymptotic states. The $\mathbf{S}$-matrix program was actively investigated in the 1960s and 1970s and numerous attempts were made to characterize the $\mathbf{S}$-matrix which is the central object in this approach, see, e.g., [14,17]. However, these investigations led to rather unsatisfactory results in spacial dimensions higher than one, mainly due to the incapacity of constructing viable, explicit, $\mathbf{S}$-matrices for nontrivial models.

The interest in the $\mathbf{S}$-matrix approach was revived by the pioneering work of Gryanik and Vergeles [16]. These authors set forth the first features of an integrable structure based method for determining $\mathbf{S}$-matrices for the 1+1-dimensional quantum field theories whose classical analogues exhibit an infinite set of independent local integrals of motion. Indeed, the existence of analogous conservation laws on the quantum level heavily constrains the possible form of the scattering basically by reducing it to a concatenation of two-body processes and hence making the calculations of $\mathbf{S}$-matrices feasible. The work [16] focused on the case of models only exhibiting one type of asymptotic particles, the main example being given by the quantum Sinh-Gordon model. This 1+1-dimensional quantum field theory will be taken as a guiding example from now on. It corresponds to the appropriate quantization of the classical evolution equation of a scalar field $\varphi(x,t)$ under the partial differential equation

$$\big(\partial_t^2 - \partial_x^2\big)\varphi + \frac{m^2}{g}\sinh(g\varphi) = 0, \quad (x,t) \in \mathbb{R}^2. \tag{1.1}$$

For this model, the asymptotic "in" states of the theory are described by vectors $\boldsymbol{f} = (f^{(0)}, \ldots, f^{(n)}, \ldots)$ which belong to the Fock Hilbert space

$$\mathfrak{h}_{\mathrm{in}} = \bigoplus_{n=0}^{+\infty} L^2\big(\mathbb{R}^n_>\big) \quad \text{with } \mathbb{R}^n_> = \big\{\boldsymbol{\beta}_n = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n : \beta_1 > \cdots > \beta_n\big\}. \tag{1.2}$$

This means that $f^{(n)} \in L^2(\mathbb{R}^n_>)$ has the physical interpretation of an incoming n-particle wave-packet density in rapidity space. More precisely, on physical grounds, one interprets elements of the Hilbert space $\mathfrak{h}_{\mathrm{in}}$ as parameterized by n-particles states, $n \in \mathbb{N}$, arriving, in the remote past, with well-ordered rapidities $\beta_1 > \cdots > \beta_n$ prior to any scattering which would be enforced by the interacting nature of the model.

For the 1+1-dimensional quantum Sinh-Gordon model, the $\mathbf{S}$-matrix proposed in [16] is purely diagonal and thus fully described by one scalar function of the relative "in" rapidities of the two particles:

$$\mathbf{S}(\beta) = \frac{\tanh[\frac{1}{2}\beta - \mathrm{i}\pi\mathfrak{b}]}{\tanh[\frac{1}{2}\beta + \mathrm{i}\pi\mathfrak{b}]} \quad \text{with } \mathfrak{b} = \frac{1}{2}\frac{g^2}{8\pi + g^2}. \tag{1.3}$$

This $\mathbf{S}$-matrix satisfies the unitarity $\mathbf{S}(\beta)\mathbf{S}(-\beta) = 1$ and crossing $\mathbf{S}(\beta) = \mathbf{S}(\mathrm{i}\pi - \beta)$ symmetries. These are, in fact, fundamental symmetry features of an $\mathbf{S}$-matrix and arise in many other integrable quantum field theories. Within the physical picture, throughout the flow of time, the "in" particles approach each other, interact, scatter and finally travel again as asymptotically free outgoing, viz. "out," particles. Within such a scheme, an "out" n-particle state is then parameterized by n well-ordered rapidities $\beta_1 < \cdots < \beta_n$ and can be seen as a component of a vector belonging to the Hilbert space

$$\mathfrak{h}_{\text{out}} = \bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^n_<) \quad \text{with} \quad \mathbb{R}^n_< = \{\boldsymbol{\beta}_n = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n : \beta_1 < \cdots < \beta_n\}. \tag{1.4}$$

The $\mathbf{S}$-matrix will allow one to express the "out" state $\boldsymbol{g} = (g^{(0)}, \dots, g^{(n)}, \dots)$ which results from the scattering of an "in" state $\boldsymbol{f} = (f^{(0)}, \dots, f^{(n)}, \dots)$ as

$$g^{(n)}(\beta_1, \dots, \beta_n) = \prod_{a<b}^{n} \mathbf{S}(\beta_a - \beta_b) \cdot f^{(n)}(\beta_n, \dots, \beta_1). \tag{1.5}$$

Note that in this integrable setting, there is *no* particle production and that the scattering is a concatenation of two-body processes.

Over the years, it turned out to be possible to characterize thoroughly the $\mathbf{S}$-matrices for more involved quantum field theories underlying to other integrable classical field theories in 1+1 dimensions. Such models possess several types of asymptotic particles which can also form bound states. Then, the "in" Fock Hilbert space is more complicated and takes the form $\bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^n_>, \otimes^n \mathbb{C}^p)$ where the L^2-space refers to $\otimes^n \mathbb{C}^p$ valued functions on $\mathbb{R}^n_>$, with p corresponding to the number of different asymptotic particles in the given theory. The most celebrated example corresponds to the Sine-Gordon quantum field theory. Building on Faddeev–Korepin's [22] semiclassical quantization results of the solitons in the classical Sine-Gordon model, one concludes that the underlying quantum field theory possesses two distinct types of asymptotic particles of equal mass, the soliton and the antisoliton, as well as a certain number, which depends on the coupling constant, of bound states thereof. These all have distinct masses and are called breathers. Zamolodchikov argued the explicit form of the $\mathbf{S}$-matrix governing the soliton–antisoliton scattering [39] upon using the factorizability of the n-particle $\mathbf{S}$-matrix into two-particle processes, the independence of the order in which a three particle scattering process arises from a concatenation of two-particle processes as well as the fact that equal mass particles may *solely* exchange their momenta during scattering, this due to the existence of many conservation laws. This enforces that the $\mathbf{S}$ matrix satisfies the Yang–Baxter equation, which originally appeared in rather different contexts [5, 37], and strongly restricts its form. We do stress that the Yang–Baxter equation is the actual cornerstone of quantum integrability, so that it is not astonishing to recover it also in this setting. The missing pieces of the Sine-Gordon $\mathbf{S}$-matrix capturing the soliton–breather and breather–breather scattering were then proposed in [18]. Nowadays, $\mathbf{S}$-matrices of many other models have been proposed, see e.g., [1, 38].

1.2. The operator content and the bootstrap program

1.2.1. The basic operators

Having in mind the per se full construction of the quantum field theory, identifying the content in asymptotic particles, viz. the "in" particles' Hilbert space $\mathfrak{h}_{\text{in}}$, and the $\mathbf{S}$-matrix which describes their scattering only arises as the first step. Indeed, one should build, in a way that is compatible with the form of the scattering encapsulated in the $\mathbf{S}$-matrix of interest, a family $\mathtt{O}_\alpha$ of operator-valued distributions, α running through some set $\mathcal{S}$. More precisely, the $\mathtt{O}_\alpha$ should be distributions acting on smooth, compactly supported functions $d(\boldsymbol{x})$ of the Minkowskian space-time coordinate

$$\boldsymbol{x} = (x_0, x_1) \in \mathbb{R}^{1,1} \quad \text{with } \boldsymbol{x} \cdot \boldsymbol{y} = x_0 y_0 - x_1 y_1. \tag{1.6}$$

Then $\mathtt{O}_\alpha[d]$ is some densely defined operator on $\mathfrak{h}_{\text{in}}$ whose domain could, in principle, depend on d. It is useful from the point of view of connecting this picture to physics to express $\mathtt{O}_\alpha$ directly in terms of its generalized operator valued function

$$\mathtt{O}_\alpha[d] = \int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{x}\, d(\boldsymbol{x}) \mathtt{O}_\alpha(\boldsymbol{x}). \tag{1.7}$$

In fact, in physics' terminology, it is the $\mathtt{O}_\alpha(\boldsymbol{x})$s which correspond to the quantum fields of the theory. Moreover, as will be apparent in the following, it turns out that in most handlings $\mathtt{O}_\alpha(\boldsymbol{x})$ does actually make sense as a bona fide operator valued *function* on the Minkowski space having a well-defined dense domain. Hence, unless it is mandatory so as to make an appropriate sense out of the formula, we will make use of the generalized function notation $\mathtt{O}_\alpha(\boldsymbol{x})$.

On top of being compatible with the scattering date, the operators $\mathtt{O}_\alpha(\boldsymbol{x})$ should form an algebra, viz. the product $\mathtt{O}_\alpha(\boldsymbol{x})\mathtt{O}_{\alpha'}(\boldsymbol{y})$ should be a well-defined dense operator for almost all $\boldsymbol{x}$ and $\boldsymbol{y}$, and satisfy causality, viz. that for purely Bosonic theories as the Sinh-Gordon model

$$\big[\mathtt{O}_\alpha(\boldsymbol{x}), \mathtt{O}_{\alpha'}(\boldsymbol{y})\big] \equiv \mathtt{O}_\alpha(\boldsymbol{x})\mathtt{O}_{\alpha'}(\boldsymbol{y}) - \mathtt{O}_{\alpha'}(\boldsymbol{y})\mathtt{O}_\alpha(\boldsymbol{x}) = 0 \quad \text{if } (\boldsymbol{x} - \boldsymbol{y})^2 < 0, \tag{1.8}$$

namely when $\boldsymbol{x} - \boldsymbol{y}$ is space-like. The family $\mathtt{O}_\alpha(\boldsymbol{x})$ should in particular contain the per se quantized counterparts of the classical fields arising in the original evolution equation, for instance, $\boldsymbol{\Phi}(\boldsymbol{x})$ or $\mathrm{e}^{\gamma\boldsymbol{\Phi}}(\boldsymbol{x})$ in the Sinh-Gordon quantum field theory case. Moreover, these operators should comply with the various other symmetries imposed on a quantum field theory, such as invariance under Lorentz boosts of space-time coordinates or translational invariance. In the quantum Sinh-Gordon field theory on which we shall focus from now on, the latter means that the model is naturally endowed with a unitary operator $\mathtt{U}_{\mathtt{T}_{\boldsymbol{y}}}$ such that for any operator $\mathtt{O}(\boldsymbol{x})$

$$\mathtt{U}_{\mathtt{T}_{\boldsymbol{y}}} \cdot \mathtt{O}(\boldsymbol{x}) \cdot \mathtt{U}_{\mathtt{T}_{\boldsymbol{y}}}^{-1} = \mathtt{O}(\boldsymbol{x} + \boldsymbol{y}). \tag{1.9}$$

The operator $\mathtt{U}_{\mathtt{T}_{\boldsymbol{y}}}$ acts diagonally on $\mathfrak{h}_{\text{in}}$ given in (1.2):

$$\mathtt{U}_{\mathtt{T}_{\boldsymbol{y}}} \cdot \boldsymbol{f} = \big(\mathtt{U}_{\mathtt{T}_{\boldsymbol{y}}}^{(0)} \cdot f^{(0)}, \dots, \mathtt{U}_{\mathtt{T}_{\boldsymbol{y}}}^{(n)} \cdot f^{(n)}, \dots\big) \quad \text{with } \boldsymbol{f} = \big(f^{(0)}, \dots, f^{(n)}, \dots\big) \tag{1.10}$$

and where

$$\mathtt{U}_{\mathtt{T}_{\boldsymbol{y}}}^{(n)} \cdot f^{(n)}(\boldsymbol{\beta}_n) = \exp\Big\{\mathrm{i} \sum_{a=1}^{n} \boldsymbol{p}(\beta_a) \cdot \boldsymbol{y}\Big\} f^{(n)}(\boldsymbol{\beta}_n), \tag{1.11}$$

with $\boldsymbol{p}(\beta) = (m\cosh(\beta), m\sinh(\beta))$ and $\boldsymbol{\beta}_n = (\beta_1, \dots, \beta_n)$.

Should the construction of quantum fields fulfilling to the above be achieved, the ultimate goal would consist in computing in closed and explicit form the model's vacuum-to-vacuum n-point correlation functions:

$$\big\langle \mathtt{O}_{\alpha_1}(\boldsymbol{x}_1) \cdots \mathtt{O}_{\alpha_n}(\boldsymbol{x}_n) \big\rangle = \mathrm{Tr}_{\mathfrak{h}_{\text{in}}}\big[\mathtt{P}_0 \mathtt{O}_{\alpha_1}(\boldsymbol{x}_1) \cdots \mathtt{O}_{\alpha_n}(\boldsymbol{x}_n) \mathtt{P}_0\big], \tag{1.12}$$

with $\mathtt{P}_0$ being the orthogonal projection on the 0-particle Fock space. We do stress that the above objects are still generalized functions and, as such, should be considered in an appropriate distributional interpretation. That will be made precise below.

1.2.2. The bootstrap program for the zero particle sector

By virtue of the above, in the case of the $\mathfrak{h}_{\text{in}}$ Hilbert space, one may represent an operator $\mathtt{O}(\boldsymbol{x})$ as an integral operator acting on the L^2-based Fock space

$$\mathtt{O}(\boldsymbol{x}) \cdot \boldsymbol{f} = \big(\mathtt{O}^{(0)}(\boldsymbol{x}) \cdot \boldsymbol{f}, \dots, \mathtt{O}^{(n)}(\boldsymbol{x}) \cdot \boldsymbol{f}, \dots\big) \tag{1.13}$$

with $\mathtt{O}^{(n)}(\boldsymbol{x}) : \mathfrak{h}_{\text{in}} \to L^2(\mathbb{R}^n_>)$. Later on, we will discuss more precisely the structure of the operators $\mathtt{O}^{(n)}(\boldsymbol{x})$ that one needs to impose so as to end up with a consistent quantum field theory. However, first, we focus our attention on the 0^{th} space operators whose action may be represented, whenever it makes sense, as

$$\mathtt{O}^{(0)}(\boldsymbol{x}) \cdot \boldsymbol{f} = \sum_{m \geq 0} \int_{\mathbb{R}^m_>} \mathrm{d}^m \beta \, \mathcal{M}^{(\mathtt{O})}_{0;m}(\boldsymbol{\beta}_m) \prod_{a=1}^{m} \big\{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}(\beta_a)\cdot \boldsymbol{x}}\big\} f^{(m)}(\boldsymbol{\beta}_m). \tag{1.14}$$

The oscillatory $\boldsymbol{x}$-dependence is a simple consequence of the translation relation (1.9) along with the explicit form of the action of the translation operator (1.11).

In order for $\mathtt{O}^{(0)}(\boldsymbol{x})$ to comply with the scattering data encoded by $\mathtt{S}$, one needs to impose a certain amount of constraints on the integral kernels $\mathcal{M}^{(\mathtt{O})}_{0;m}(\boldsymbol{\beta}_m)$. First of all, general principles of quantum field theory impose that, in order for these to correspond to kernels of quantum fields, the $\mathcal{M}^{(\mathtt{O})}_{0;m}(\boldsymbol{\beta}_m)$ have to correspond to a $+$ boundary value $\mathcal{F}^{(\mathtt{O})}_{m;+}(\boldsymbol{\beta}_m)$ on $\mathbb{R}^m_>$ of a meromorphic function $\mathcal{F}^{(\mathtt{O})}_{m}(\boldsymbol{\beta}_m)$ of the variables β_a belonging to the strip

$$\mathscr{S} = \big\{z \in \mathbb{C} : 0 < \Im(z) < 2\pi\big\}. \tag{1.15}$$

Traditionally, in the physics literature, the functions $\mathcal{F}^{(\mathtt{O})}_{m}(\boldsymbol{\beta}_m)$ are called form factors.

Further, one imposes a set of equations on the $\mathcal{F}^{(\mathtt{O})}_{m}$s. These constitute the so-called form factor bootstrap program. On mathematical grounds, one should understand the form factor bootstrap program as a set of *axioms* that one imposes as a starting point of the theory given the data $(\mathfrak{h}_{\text{in}}, \mathtt{S})$. Upon solving them, one has to check a posteriori that their solutions do provide one, through (1.14) and (1.16), with a collection of operators satisfying all of the requirements of the theory discussed earlier on.

The bootstrap program axioms take the form of a Riemann–Hilbert problem for a collection of functions in many variables. In the case of the Sinh-Gordon model, since there are no bound states, these take the below form.

Form Factor Axioms 1.1. *Find functions $\mathcal{F}^{(\mathcal{O})}_{n}$, $n \in \mathbb{N}$, such that, for each $k \in [\![1; n]\!]$ and fixed $\beta_a \in \mathscr{S}$, $a \neq k$, the maps $\beta_k \mapsto \mathcal{F}^{(\mathcal{O})}_{n}(\boldsymbol{\beta}_n)$ are*

- *meromorphic on $\mathscr{S}$;*
- *admit $+$, resp. $-$, boundary values $\mathcal{F}^{(\mathcal{O})}_{n;+}$ on $\mathbb{R}$, resp. $\mathcal{F}^{(\mathcal{O})}_{n;-}$ on $\mathbb{R} + 2\mathrm{i}\pi$;*
- *are bounded at infinity by $C \cdot \cosh(\ell \Re(\beta_k))$ for some n and k independent ℓ.*

The $\mathcal{F}_n^{(\text{o})}$ satisfy the multivariate system of Riemann–Hilbert problems:

(i) *agreeing upon $\beta_{ab} = \beta_a - \beta_b$, one has $\mathcal{F}_n^{(\text{o})}(\beta_1, \dots, \beta_a, \beta_{a+1}, \dots, \beta_n) = \boldsymbol{S}(\beta_{aa+1}) \cdot \mathcal{F}_n^{(\text{o})}(\beta_1, \dots, \beta_{a+1}, \beta_a, \dots, \beta_n)$;*

(ii) *For $\beta_1 \in \mathbb{R}$, and given generic $\boldsymbol{\beta}_n' = (\beta_2, \dots, \beta_n) \in \mathscr{S}^{n-1}$, it holds $\mathcal{F}_{n;-}^{(\text{o})}(\beta_1 + 2\mathrm{i}\pi, \boldsymbol{\beta}_n') = \mathcal{F}_{n;+}^{(\text{o})}(\boldsymbol{\beta}_n', \beta_1) = \prod_{a=2}^{n} \boldsymbol{S}(\beta_{a1}) \cdot \mathcal{F}_{n;+}^{(\text{o})}(\boldsymbol{\beta}_n)$;*

(iii) *The only poles of $\mathcal{F}_n^{(\text{o})}$ are simple, located at $\mathrm{i}\pi$ shifted rapidities and*

$$-\mathrm{i}\mathrm{Res}\big(\mathcal{F}_{n+2}^{(\text{o})}(\alpha + \mathrm{i}\pi, \beta, \boldsymbol{\beta}_n) \cdot \mathrm{d}\alpha, \alpha = \beta\big) = \left\{1 - \prod_{a=1}^{n} \boldsymbol{S}(\beta - \beta_a)\right\} \cdot \mathcal{F}_n^{(\text{o})}(\boldsymbol{\beta}_n);$$

(iv) *$\mathcal{F}_n^{(\text{o})}(\boldsymbol{\beta}_n + \theta \overline{\boldsymbol{e}}_n) = \mathrm{e}^{\theta \boldsymbol{s}_{\text{o}}} \cdot \mathcal{F}_n^{(\text{o})}(\boldsymbol{\beta}_n)$ for some number $\boldsymbol{s}_{\text{o}}$ and with $\overline{\boldsymbol{e}}_n = (1, \dots, 1)$.*

Note that the reduction occurring at the residues of $\mathcal{F}_n^{(\text{o})}(\boldsymbol{\beta}_n)$ when $\beta_{ab} = \mathrm{i}\pi$ can be readily inferred from (i) and (iii).

One may already comment on the origin of the axioms. The first one illustrates the scattering properties of the model on the level of the operator's kernel. The second and third axioms may be interpreted heuristically as a consequence of the LSZ reduction [25], and locality of the operator, see, e.g., [2, 34] for heuristics on that matter. Finally, the last axiom is a manifestation of the Lorentz invariance of the theory. The number $\boldsymbol{s}_{\text{o}}$ arising in (iv) is called the spin of the operator. Moreover, the number ℓ depends on the type of operator being considered. Finally, for more complex models, one would also need to add an additional axiom which would encapsulate the way how the presence of bound states in the model governs certain additional poles in the form factors, cf. [34].

1.2.3. The bootstrap program for the multiparticle sector

It is convenient to represent the action of the operators $\text{O}^{(n)}(\boldsymbol{x})$ in the form

$$\big(\text{O}^{(n)}(\boldsymbol{x}) \cdot \boldsymbol{f}\big)(\boldsymbol{\gamma}_n) = \sum_{m \geq 0} \prod_{a=1}^{n} \{\mathrm{e}^{\mathrm{i}\boldsymbol{p}(\gamma_a)\cdot \boldsymbol{x}}\} \cdot \mathbf{M}_{\text{O}}^{(m)}(\boldsymbol{x} \mid \boldsymbol{\gamma}_n)\big[f^{(m)}\big]. \tag{1.16}$$

There $\mathbf{M}_{\text{O}}^{(m)}(\boldsymbol{x} \mid \boldsymbol{\gamma}_n)$ are distribution-valued functions which act on appropriate spaces of sufficiently regular functions in m variables. The regularity assumptions will clear out later on, once that we provide the explicit expressions (1.18) for these distributions. In fact, it is convenient, in order to avoid heavy notations, to represent their action as generalized integral operators

$$\mathbf{M}_{\text{O}}^{(m)}(\boldsymbol{x} \mid \boldsymbol{\gamma}_n)\big[f^{(m)}\big] = \int_{\mathbb{R}^m_>} \mathrm{d}^m\beta \, \mathcal{M}_{n;m}^{(\text{o})}(\boldsymbol{\gamma}_n; \boldsymbol{\beta}_m) \prod_{a=1}^{m} \{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}(\beta_a)\cdot \boldsymbol{x}}\} f^{(m)}(\boldsymbol{\beta}_m), \tag{1.17}$$

in which one understands of the kernels $\mathcal{M}_{n;m}^{(\text{o})}(\boldsymbol{\gamma}_n; \boldsymbol{\beta}_m)$ as generalized functions.

The last axiom of the bootstrap program provides one with a way to compute these kernels. Heuristically, it can be seen as a consequence of the LSZ reduction [25]:

(v)

$$\mathcal{M}^{(\mathrm{O})}_{n;m}(\boldsymbol{\alpha}_n;\boldsymbol{\beta}_m) = \mathcal{M}^{(\mathrm{O})}_{n-1;m+1}\big(\boldsymbol{\alpha}'_n;(\alpha_1+\mathrm{i}\pi,\boldsymbol{\beta}_m)\big) + 2\pi\sum_{a=1}^{m}\delta_{\alpha_1;\beta_a}\prod_{k=1}^{a-1}\mathsf{S}(\beta_k-\alpha_1)\cdot\mathcal{M}^{(\mathrm{O})}_{n-1;m-1}\big(\boldsymbol{\alpha}'_n;(\beta_1,\dots,\widehat{\beta}_a,\dots,\beta_m)\big).$$

In the above expression, $\widehat{\beta}_a$ means that the variable β_a should be omitted and $\delta_{x;y}$ refers to the Dirac mass distribution centered at x and acting on functions of y. Finally, the evaluation at $\alpha_1+\mathrm{i}\pi$ is understood in the sense of a boundary value of the meromorphic continuation in the strip $0\le\Im(z)\le\pi$ from $\mathbb{R}$ up to $\mathbb{R}+\mathrm{i}\pi$. This axiom is to be complemented with the initialization condition $\mathcal{M}^{(\mathrm{O})}_{0;n}(\emptyset;\boldsymbol{\beta}_n)=\mathcal{F}^{(\mathrm{O})}_{n;+}(\boldsymbol{\beta}_n)$ when $\boldsymbol{\beta}_n\in\mathbb{R}^n_>$. It is direct to establish that the recursion may be solved in closed form allowing one to determine the distributional kernel $\mathcal{M}^{(\mathrm{O})}_{n;m}(\boldsymbol{\alpha}_n;\boldsymbol{\beta}_m)$ in terms of $\mathcal{F}^{(\mathrm{O})}_n(\boldsymbol{\beta}_n)$:

$$\mathcal{M}^{(\mathrm{O})}_{n;m}(\boldsymbol{\alpha}_n;\boldsymbol{\beta}_m) = \sum_{p=0}^{\min(n,m)}\sum_{\substack{k_1<\dots<k_p\\1\le k_a\le n}}\sum_{\substack{i_1\neq\dots\neq i_p\\1\le i_a\le m}}\prod_{a=1}^{p}\{2\pi\delta_{\alpha_{k_a};\beta_{i_a}}\}\mathsf{S}\big(\overleftarrow{\boldsymbol{\alpha}}_n\mid\overleftarrow{\boldsymbol{\alpha}}^{(1)}_n\big) \times\mathsf{S}\big(\boldsymbol{\beta}^{(1)}_n\mid\boldsymbol{\beta}_n\big)\cdot\mathcal{F}_{n+m-2p;-}\big(\overleftarrow{\boldsymbol{\alpha}}^{(2)}_n+\mathrm{i}\pi\overline{\boldsymbol{e}}_{n-p},\boldsymbol{\beta}^{(2)}_m\big). \tag{1.18}$$

There, we have used the shorthand notations $\boldsymbol{\alpha}^{(1)}_n=(\alpha_{k_1},\dots,\alpha_{k_p})$ and $\boldsymbol{\alpha}^{(2)}_n=(\alpha_{\ell_1},\dots,\alpha_{\ell_{n-p}})$ where $\{\ell_1,\dots,\ell_{n-p}\}=[\![1;n]\!]\setminus\{k_a\}_1^p$, $\ell_1<\dots<\ell_{n-p}$, and analogously $\boldsymbol{\beta}^{(1)}_m=(\beta_{i_1},\dots,\beta_{i_p})$ and $\boldsymbol{\beta}^{(2)}_m=(\beta_{j_1},\dots,\beta_{j_{m-p}})$ where $\{j_1,\dots,j_{m-p}\}=[\![1;m]\!]\setminus\{i_a\}_1^p$, $j_1<\dots<j_{m-p}$. Moreover, we have introduced

$$\mathsf{S}\big(\overleftarrow{\boldsymbol{\alpha}}_n\mid\overleftarrow{\boldsymbol{\alpha}}^{(1)}_n\big)=\prod_{a=1}^{p}\prod_{\substack{b=1\\k_a>\ell_b}}^{n-p}\mathsf{S}(\alpha_{k_a}-\alpha_{\ell_b}),$$

$$\mathsf{S}\big(\boldsymbol{\beta}^{(1)}_n\mid\boldsymbol{\beta}_n\big)=\prod_{a=1}^{p}\prod_{\substack{b=1\\b<i_a}}^{m}\mathsf{S}(\beta_b-\beta_{i_a})\cdot\prod_{\substack{a>b\\i_a>i_b}}\mathsf{S}(\beta_{i_a}-\beta_{i_b}).$$

Finally, we agree upon $\overleftarrow{\boldsymbol{\gamma}}_N=(\gamma_N,\dots,\gamma_1)$ for any $\boldsymbol{\gamma}_N=(\gamma_1,\dots,\gamma_N)$.

It is clear on the level of the explicit expression (1.18) that this generalized function is well defined, even though it involves a multiplication of distributions.

1.2.4. The road towards the bootstrap program

The first calculation of certain of the operators' kernels was initiated by Weisz [36] who built on the full characterization of the S matrix of the Sine-Gordon model to argue with the help of general principles of quantum field theory an expression for the kernel $\mathcal{M}^{(\mathrm{O})}_{1;1}(\alpha;\beta)$ of the electromagnetic current operator only involving one-dimensional variables α,β. The setting up of a systematic approach allowing one to calculate all the collection of kernels characterizing an operator starting from a given model's S-matrix has been initiated by Karowski and Weisz [19] who proposed a set of equation satisfied by that model's equivalent of $\mathcal{F}^{(O)}_n(\boldsymbol{\beta}_n)$. These allowed them to provide closed-form expressions for two-particle form factors in several models. However, these equations were still far from forming the full bootstrap program as described above.

After long investigations [30, 31, 33] which revealed a deeper structure of the form factors of the Sine-Gordon model, Smirnov [32] formulated the equivalent of axioms (i)–(ii) in that model. Subsequently, Kirillov and Smirnov [20] proposed the full set of the bootstrap program axioms, exemplified in the case of the Massive Thirring model; see also [34].

2. SOLVING THE BOOTSTRAP PROGRAM

The resolution of the bootstrap program was systematized over the years and these efforts led to explicit expressions for the form factors of local operators in numerous 1+1-dimensional massive quantum field theories, see, e.g., [34]. The first expressions for the form factors were rather combinatorial in nature. Later, a substantial progress was achieved in simplifying the latter, in particular by exhibiting a deeper structure at their root. Notably, one can mention the free field based approach, also called angular quantization, to the calculation of form factors. It was introduced by Lukyanov [26] and allowed obtaining convenient representations for certain form factors solving the bootstrap program. In particular, the construction lead to closed-form and manageable expressions [10, 27] for the form factors of the exponential of the field operators in the Sinh-Gordon and the Bullough–Dodd models. Later, Babujian, Fring, Karowski, Zapetal [2] and Babujian, Karowski [3, 4] developed the more powerful $\mathcal{K}$-transform approach which will be described below on the example of the Sinh-Gordon model. The construction of [3, 4] was improved in [15, 24] so as to encompass more complicated operators, the so-called descendants of the Sinh-Gordon exponential of the field operator.

2.1. The 2-particle sector solution

The constructions of solutions to the bootstrap program starts from obtaining a specific solution to the equations (i)–(iv) when $n = 2$, i.e., for two variables. This was first achieved in [19].

Lemma 2.1 ([19]). *Let $\mathcal{F}_2^{(O)}(\beta_1, \beta_2)$ solve (i)–(iv) at $n = 2$. Then, there exist $k \in \{1, \dots, \ell/2\}$, $\varkappa_a \in \mathbb{C}$, $a = 1, \dots, k$, such that, with $\beta_{12} = \beta_1 - \beta_2$*

$$\mathcal{F}_2^{(O)}(\beta_1, \beta_2) = \mathcal{N}_O \prod_{a=1}^{k} \left\{ \sinh\left[\frac{\beta_{12} - \varkappa_a}{2}\right] \cdot \sinh\left[\frac{\beta_{12} + \varkappa_a}{2}\right] \right\} e^{\frac{\mathfrak{s}_O}{2}(\beta_1+\beta_2)} \mathbf{F}(\beta_{12}) \quad (2.1)$$

for some $\mathcal{N}_O \in \mathbb{C}$ and where $\mathbf{F}$ is given by the integral representation valid for $0 < \Im(\beta) < 2\pi$:

$$\mathbf{F}(\beta) = \exp\left\{ -4 \int_0^{+\infty} \mathrm{d}x \, \frac{\sinh(x\mathfrak{b}) \cdot \sinh(x\hat{\mathfrak{b}}) \cdot \sinh(\frac{1}{2}x)}{x \sinh^2(x)} \cos\left(\frac{x}{\pi}(\mathrm{i}\pi - \beta) \right) \right\}, \quad (2.2)$$

with $\hat{\mathfrak{b}} = \frac{1}{2} - \mathfrak{b}$.

Proof. Axiom (iv) ensures that $\mathcal{F}_2^{(O)}(\beta_1, \beta_2) = e^{\frac{\mathfrak{s}_O}{2}(\beta_1+\beta_2)} \tilde{\mathbf{F}}(\beta_1 - \beta_2)$ for some function $\tilde{\mathbf{F}}(\beta)$ that is holomorphic on the strip $0 < \Im(\beta) < 2\pi$, bounded at infinity by $C \cosh(\ell\beta)$, and such that $\tilde{\mathbf{F}}_-(\beta + 2i\pi) = \mathbf{S}(\beta)\tilde{\mathbf{F}}_+(-\beta) = \tilde{\mathbf{F}}_+(\beta)$, $\beta \in \mathbb{R}$. One first looks for a particular solution to this scalar Riemann–Hilbert problem, namely a holomorphic function $\mathbf{F}$ in the strip $0 < \Im(\beta) < 2\pi$ which behaves as $\mathbf{F}(\beta) = 1 + \mathrm{O}(\beta^{-2})$ as $\Re(\beta) \to \pm\infty$ uniformly in $0 \le \Im(\beta) \le 2\pi$ and satisfies $\mathbf{F}_-(\beta + 2i\pi) = \mathbf{S}(-\beta)\mathbf{F}_+(\beta) = \mathbf{F}_+(-\beta)$, $\beta \in \mathbb{R}$.

Starting from the below integral representation

$$\mathbf{S}(\beta) = \exp\left\{ 8 \int_0^{+\infty} \mathrm{d}x \, \frac{\sinh(x\mathfrak{b}) \cdot \sinh(x\hat{\mathfrak{b}}) \cdot \sinh(\frac{1}{2}x)}{x \sinh(x)} \sinh\left(\frac{x\beta}{i\pi}\right) \right\}, \tag{2.3}$$

one readily checks that the solution is provided by the below $2i\pi$-periodic Cauchy transform

$$\mathbf{F}(\beta) = \exp\left\{ \int_{\mathbb{R}} \frac{\mathrm{d}s}{4i\pi} \coth\left[\frac{1}{2}(s - \beta)\right] \ln \mathbf{S}(s) \right\}. \tag{2.4}$$

The s integral can then be taken by means of the integral representation (2.3) for $\ln \mathbf{S}(s)$ and leads to (2.2). Now it is easy to check that the holomorphic function $\mathbf{G}(\beta) = \tilde{\mathbf{F}}(\beta)/\mathbf{F}(\beta)$ on the strip $0 < \Im(\beta) < 2\pi$ admits $\pm$ boundary values and satisfies $\mathbf{G}_-(\beta + 2i\pi) = \mathbf{G}_+(\beta)$ for $\beta \in \mathbb{R}$ and is bounded by $C\cosh(\ell\Re(\beta))$ as $\Re(\beta) \to \infty$ in this strip. As a consequence, it admits a unique extension into a $2i\pi$-periodic entire function bounded by $C\cosh(\ell\beta)$ and hence is of the form $P_\ell(e^\beta)$, where P_ℓ is a Laurent polynomial of maximal positive and negative degree ℓ. Since it is $2i\pi$-periodic and even, $P_\ell(e^\beta)$ necessarily takes the form

$$P_\ell(e^\beta) = \prod_{a=1}^{k} \left\{ \sinh\left[\frac{\beta - \varkappa_a}{2}\right] \cdot \sinh\left[\frac{\beta + \varkappa_a}{2}\right] \right\} \quad \text{for some } 2k \le \ell. \tag{2.5}$$

■

2.2. The *n*-particle sector solution

Proposition 2.2. *Consider the change of unknown functions*

$$\mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n) = \prod_{a<b}^{n} \mathbf{F}(\beta_{ab}) \cdot \mathcal{K}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n) \quad \textit{with } \beta_{ab} = \beta_a - \beta_b, \tag{2.6}$$

with $\mathbf{F}$ as defined through (2.2). Then $\mathcal{F}_n^{(\mathcal{O})}$ solves the bootstrap axioms (i)–(iv) *if any only if*

(I) *$\mathcal{K}_n^{(\mathcal{O})}$ is a symmetric function of $\boldsymbol{\beta}_n$;*

(II) *$\mathcal{K}_n^{(\mathcal{O})}$ is a $2i\pi$ periodic and meromorphic function of each variable taken singly;*

(III) *the only poles of $\mathcal{K}_n^{(\mathcal{O})}$ are simple and located at $\beta_a - \beta_b \in i\pi(1 + 2\mathbb{Z})$. The associated residues are given by*

$$\operatorname{Res}\left(\mathcal{K}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n) \cdot \mathrm{d}\beta_1, \beta_{12} = i\pi\right) = \frac{i}{\mathbf{F}(i\pi)} \cdot \frac{1 - \prod_{a=3}^{n} \mathbf{S}(\beta_{2a})}{\prod_{a=3}^{n}\{\mathbf{F}(\beta_{2a} + i\pi)\mathbf{F}(\beta_{2a})\}} \cdot \mathcal{K}_{n-2}^{(\mathcal{O})}(\boldsymbol{\beta}_n'') \tag{2.7}$$

where $\boldsymbol{\beta}_n'' = (\beta_3, \ldots, \beta_n)$;

(IV) $\mathcal{K}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n + \theta\overline{\boldsymbol{e}}_n) = e^{\theta s_{\mathcal{O}}} \cdot \mathcal{K}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n)$.

This first transformation simplifies the symmetry properties of the problem. However, the inductive reductions provided by the computation of the residues are still quite intricate. The idea is then to proceed to yet another change of unknown function, this time by means of a more involved transform. The latter will then lead to structurally much simpler, and thus easier to solve, equations satisfied by the new unknown function. As already mentioned, the dawn of this approach goes back to [10, 27] and it was put in the present form in [2–4]. In particular, we refer to [3] for the proof.

Proposition 2.3 ([3]). *Let* $\boldsymbol{\ell}_n \in \{0,1\}^n$ *and* $p_n^{(\mathcal{O})}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n)$ *be a solution to the below constraints:*

(a) $\boldsymbol{\beta}_n \mapsto p_n^{(\mathcal{O})}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n)$ *is a collection of* $2\mathrm{i}\pi$*-periodic holomorphic functions on* $\mathbb{C}$ *that are symmetric in the two sets of variables jointly, viz. for any* $\sigma \in \mathfrak{S}_n$ *it holds* $p_n^{(\mathcal{O})}(\boldsymbol{\beta}_n^{\sigma} \mid \boldsymbol{\ell}_n^{\sigma}) = p_n^{(\mathcal{O})}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n)$ *with* $\boldsymbol{\beta}_n^{\sigma} = (\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)})$;

(b) $p_n^{(\mathcal{O})}(\beta_2 + \mathrm{i}\pi, \boldsymbol{\beta}_n' \mid \boldsymbol{\ell}_n) = g(\ell_1,\ell_2)\, p_{n-2}^{(\mathcal{O})}(\boldsymbol{\beta}_n'' \mid \boldsymbol{\ell}_n'') + h(\ell_1,\ell_2 \mid \boldsymbol{\beta}_n')$ *where* h *does not depend on the remaining set of variables* $\boldsymbol{\ell}_n''$ *and*

$$g(0,1) = g(1,0) = \frac{-1}{\sin(2\pi \mathfrak{b})\boldsymbol{F}(\mathrm{i}\pi)}; \tag{2.8}$$

(c) $p_n^{(\mathcal{O})}(\boldsymbol{\beta}_n + \theta \overline{\boldsymbol{e}}_n \mid \boldsymbol{\ell}_n) = \mathrm{e}^{\theta s_{\mathcal{O}}} \cdot p_n^{(\mathcal{O})}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n)$.

Then, its $\mathcal{K}$*-transform*

$$\mathcal{K}_n\big[p_n^{(\mathcal{O})}\big](\boldsymbol{\beta}_n) = \sum_{\boldsymbol{\ell}_n \in \{0,1\}^n} (-1)^{\overline{\boldsymbol{\ell}}_n} \prod_{a<b}^{n} \Big\{ 1 - \mathrm{i} \frac{\ell_{ab} \cdot \sin[2\pi \mathfrak{b}]}{\sinh(\beta_{ab})} \Big\} \cdot p_n^{(\mathcal{O})}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n), \tag{2.9}$$

in which $\overline{\boldsymbol{\ell}}_n = \sum_{a=1}^{n} \ell_k$, *solves* (I)–(IV).

Note that arguments were given in [15] in favor of some form of bijection between certain classes of solutions to (a)–(c) and (I)–(IV). However, we do stress that, so far, the question whether there does exist a clear cut correspondence between all solutions to (a)–(c) and (I)–(IV) is still open.

3. TOWARDS PHYSICAL OBSERVABLES AND THE CONVERGENCE PROBLEM

The resolution of the bootstrap program provides one with the expressions for the integral kernels of certain operators which are candidates for the quantum fields of the 1+1-dimensional Sinh-Gordon quantum field theory. However, for this construction to really provide one with the quantum field theory of interest, one should establish several facts. First of all, the operators so constructed should form an algebra in the sense discussed in Section 1.2.1. By virtue of the translational invariance (1.9), this means that, for any $n, m \in \mathbb{N}$, the series of multiple integrals arising in the operator product $\mathtt{U}_{\mathtt{T}_x}^{-1} \mathtt{P}_n \mathtt{O}_1(\boldsymbol{x}) \mathtt{O}_2(\mathbf{0}) \mathtt{P}_m$, where $\mathtt{P}_k$ is the orthogonal projector on the k-particle Fock space, should converge in the weak sense. Namely, for any sufficiently regular functions $\boldsymbol{\alpha}_n \mapsto f^{(n)}(\boldsymbol{\alpha}_n)$ and $\boldsymbol{\beta}_m \mapsto g^{(m)}(\boldsymbol{\beta}_m)$ belonging respectively to a dense subset of $L^2(\mathbb{R}^n_>)$ and $L^2(\mathbb{R}^m_>)$, and for any $d \in \mathcal{C}^\infty_c(\mathbb{R}^2)$,

$$\sum_{\ell\geq 0}\int_{\mathbb{R}^\ell_>}\frac{\mathrm{d}^\ell\gamma}{(2\pi)^\ell}\left\{\int_{\mathbb{R}^n_>}\frac{\mathrm{d}^n\alpha}{(2\pi)^n}f^{(n)}(\boldsymbol{\alpha}_n)\mathcal{M}^{(O_1)}_{n;\ell}(\boldsymbol{\alpha}_n;\boldsymbol{\gamma}_\ell)\right\}$$
$$\times\left\{\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{x}\,d(\boldsymbol{x})\prod_{a=1}^{\ell}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}(\gamma_a)\cdot\boldsymbol{x}}\right\}\cdot\left\{\int_{\mathbb{R}^m_>}\frac{\mathrm{d}^m\beta}{(2\pi)^m}\mathcal{M}^{(O_2)}_{\ell;m}(\boldsymbol{\gamma}_\ell;\boldsymbol{\beta}_m)g^{(m)}(\boldsymbol{\beta}_m)\right\} \tag{3.1}$$

should converge. The simplest case corresponds to establishing the convergence of the series of multiple integrals subordinate to operator products $\mathtt{P}_0\mathtt{O}_1(\boldsymbol{x})\mathtt{O}'_2(\mathbf{0})\mathtt{P}_0$, viz. for $n=m=0$. Since the zeroth Fock space is one-dimensional, this exactly amounts to the convergence of the series of multiple integrals which represents the two-point generalized function $\langle\mathtt{O}_1(\boldsymbol{x})\mathtt{O}'_2(\mathbf{0})\rangle$. However, even for this specific instance, proving this property on rigorous grounds remained an open problem for a very long time. It has only recently been solved by the author [23] in the case of space-like separation between the operators, viz. $\boldsymbol{x}^2<0$. The scheme of proof of this result will be discussed in Section 4. From the proof's structure, it is rather clear that one can build on minor modifications of this method so as to establish convergence in the time-like regime, i.e. when $\boldsymbol{x}^2>0$, although this has not been done yet. Moreover, the combinatorial expressions for the kernels $\mathcal{M}^{(O)}_{n;m}(\boldsymbol{\alpha}_n;\boldsymbol{\beta}_m)$ in terms of the base form factors $\mathcal{F}^{(O)}_p$, $0\leq p\leq m+n$ indicates that the method outlined below would also allow one to tackle the convergence problem for general multipoint correlation functions.

Once that the convergence problem is solved in full generality, hence guaranteeing that the operators $\mathtt{O}_\alpha(\boldsymbol{x})$ do form an algebra, one still needs to establish the local commutativity property of the quantum fields which ensures causality of the theory. The method for doing so is now well established. Indeed, under the hypothesis of convergence of the handled series of multiple integrals issuing from the operators products, Kirillov and Smirnov showed this property in the Sine-Gordon case in [20, 21]. Their method readily applies to the Sinh-Gordon case. Hence, convergence is the only remaining problem so as to set this construction of quantum field theories on rigorous grounds.

3.1. The well-poised series expansion for two-point functions

First of all, by translation invariance, it is enough to focus on $\langle\mathtt{O}_1(\boldsymbol{x})\mathtt{O}_2(\mathbf{0})\rangle$. Recall that, at least in principle, this quantity is a generalised function and should thus be understood, in the first place, as the formal integral kernel of the distribution $\langle\mathtt{O}_1\mathtt{O}_2\rangle$. For $d\in\mathcal{C}^\infty_c(\mathbb{R}^2)$, provided convergence holds, one has

$$\langle\mathtt{O}_1\mathtt{O}_2\rangle[d] = \int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{x}\,d(\boldsymbol{x})\big\langle\mathtt{O}_1(\boldsymbol{x})\mathtt{O}_2(\mathbf{0})\big\rangle = \sum_{n\geq 0}\frac{1}{n!}\mathcal{J}^{(\mathtt{O}_1,\mathtt{O}_2)}_n[d] \tag{3.2}$$

with

$$\mathcal{J}^{(\mathtt{O}_1,\mathtt{O}_2)}_n[d]=\int_{\mathbb{R}^n}\frac{\mathrm{d}^n\beta}{(2\pi)^n}\mathcal{F}^{(\mathtt{O}_1)}_n(\boldsymbol{\beta}_n)\mathcal{M}^{(\mathtt{O}_2)}_{n;0}(\boldsymbol{\beta}_n;\emptyset)\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{x}\,d(\boldsymbol{x})\prod_{a=1}^{n}\big\{\mathrm{e}^{-\mathrm{i}m[t\cosh(\beta_a)-x\sinh(\beta_a)]}\big\}.$$

It is a direct consequence of the kernel reduction axiom (v) and of Lorentz invariance (iv) that

$$\mathcal{M}_{n;0}^{(\mathsf{O}_2)}(\boldsymbol{\beta}_n;\emptyset) = \mathcal{F}_n^{(\mathsf{O}_2)}(\overleftarrow{\boldsymbol{\beta}}_n + \mathrm{i}\pi\overline{\boldsymbol{e}}_n) = \mathrm{e}^{\mathrm{i}\pi \mathbf{s}_{\mathsf{O}_2}}\mathcal{F}_n^{(\mathsf{O}_2)}(\overleftarrow{\boldsymbol{\beta}}_n). \tag{3.3}$$

This identity, along with the growth bounds in each β_a of the form factors $\mathcal{F}_n^{(\mathsf{O})}(\boldsymbol{\beta}_n)$, ensures the well definiteness of the n-fold integrals since the space-time integral over $\boldsymbol{x}$ produces a decay in each β_a that is faster than any exponential $\mathrm{e}^{\pm k\beta_a}$, $\Re(\beta_a) \to \pm\infty$. By virtue of the Morera theorem, this rapid decay at infinity along with the holomorphy properties of the integrands allow one to deform, *simultaneously* for each integration variable β_a, $a = 1, \dots, n$, the integration curves to $\mathbb{R} + \mathrm{i}\frac{\pi}{2}\mathrm{sgn}(x)$ when $\boldsymbol{x}$ is space-like and, when $\boldsymbol{x}$ is time-like, to $\gamma(\mathbb{R})$ where $\gamma(u) = u + \mathrm{i}\vartheta(u)$, where ϑ is smooth, $|\vartheta| < \pi/4$, and such that there exists $M > 0$ large enough and $0 < \varepsilon < \pi/2$ so that $\vartheta(u) = -\mathrm{sgn}(t)\mathrm{sgn}(u)\varepsilon$ when $|u| \geq M$. This operation turns the $\boldsymbol{\beta}_n$ integrals into absolutely convergent ones irrespectively of the presence of $d(\boldsymbol{x})$. In particular, for the space-like regime, one gets that

$$\mathcal{J}_n^{(\mathsf{O}_1,\mathsf{O}_2)} = \mathrm{e}^{\eta(\boldsymbol{x})}\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{x}\, d(\boldsymbol{x})\int_{\mathbb{R}^n}\frac{\mathrm{d}^n\beta}{(2\pi)^n}\mathcal{F}_n^{(\mathsf{O}_1)}(\boldsymbol{\beta}_n)\mathcal{F}_n^{(\mathsf{O}_2)}(\overleftarrow{\boldsymbol{\beta}}_n)\prod_{a=1}^{n}\mathrm{e}^{-mr\cosh(\beta_a)},$$

in which $r = \sqrt{x^2 - t^2}$, $\tanh(\vartheta) = t/x$ while

$$\eta(\boldsymbol{x}) = \mathrm{i}\pi\mathbf{s}_{\mathsf{O}_2} + (\mathrm{i}\tfrac{\pi}{2}\mathrm{sgn}(x) + \vartheta)(\mathbf{s}_{\mathsf{O}_1} + \mathbf{s}_{\mathsf{O}_2}).$$

Hence, provided convergence holds, one has the well-defined in the usual sense of numbers representation for the two-point function

$$\langle \mathsf{O}_1(\boldsymbol{x})\mathsf{O}_2(\boldsymbol{0})\rangle = \mathrm{e}^{\eta(\boldsymbol{x})}\sum_{n\geq 0}\frac{1}{n!}\int_{\mathbb{R}^n}\frac{\mathrm{d}^n\beta}{(2\pi)^n}\mathcal{F}_n^{(\mathsf{O}_1)}(\boldsymbol{\beta}_n)\mathcal{F}_n^{(\mathsf{O}_2)}(\overleftarrow{\boldsymbol{\beta}}_n)\prod_{a=1}^{n}\mathrm{e}^{-mr\cosh(\beta_a)}. \tag{3.4}$$

3.2. Convergence of series representation for two-point functions

Thus, the well-definiteness of the two-point functions boils down to providing an appropriate upper bound for the below class of N-fold integrals for $\varkappa > 0$,

$$\mathcal{Z}_N(\varkappa) = \int_{\mathbb{R}^N}\mathrm{d}^N\beta\prod_{a\neq b}^{N}\mathrm{e}^{\frac{1}{2}\mathfrak{w}(\beta_{ab})}\cdot\prod_{a=1}^{N}\{\mathrm{e}^{-2\varkappa\cosh(\beta_a)}\}\mathcal{K}_N[p_N^{(\mathsf{O}_1)}](\boldsymbol{\beta}_N)\mathcal{K}_N[p_N^{(\mathsf{O}_2)}](\overleftarrow{\boldsymbol{\beta}}_N). \tag{3.5}$$

The two-body potential $\mathfrak{w}$ is defined through the relation $\mathbf{F}(\lambda)\mathbf{F}(-\lambda) = \mathrm{e}^{\mathfrak{w}(\lambda)}$.

Theorem 3.1 ([23]). *Assume that there exist C_1, C_2, and $k \in \mathbb{N}$ such that given $s \in \{1, 2\}$,*

$$\left|p_N^{(\mathsf{O}_s)}(\boldsymbol{\beta}_N \mid \boldsymbol{\ell}_N)\right| \leq C_1^N\cdot\prod_{a=1}^{N}\mathrm{e}^{C_2\beta_a^k}\quad \textit{for any } \boldsymbol{\ell}_N \in \{0,1\}^N, \tag{3.6}$$

uniformly in N. Then, it holds

$$\left|\mathcal{Z}_N(\varkappa)\right| \leq \exp\left[-\frac{3\pi^2\mathfrak{b}\hat{\mathfrak{b}}\cdot N^2}{4\cdot(\ln N)^3}\left\{1 + \mathrm{O}\left(\frac{1}{\ln N}\right)\right\}\right]. \tag{3.7}$$

The proof of this theorem was the goal of the author's work [23]. The proof relies on Riemann–Hilbert problem techniques for inverting singular integral operators of truncated-Wiener–Hopf type along with the Deift–Zhou nonlinear steepest descent method [12, 13], concentration of measure, and large deviation techniques which were developed for dealing with certain β-ensembles multiple integrals [7,9,28], and some generalizations thereof to the case of N-dependent integrands in N-dimensional integrals as it was developed in [8].

4. THE PROOF OF THE CONVERGENCE OF THE FORM FACTOR SERIES

In this section we shall describe the main steps of the proof. The details can be found in Proposition 3.1 of [23].

4.1. An simpler upper bound

The starting point consists in obtaining a structurally simpler upper bound on $\mathcal{Z}_N(\varkappa)$ when $\varkappa > 0$.

Proposition 4.1. *There exists $C > 0$ such that*

$$\left|\mathcal{Z}_N(\varkappa)\right| \leq (C \cdot \ln N)^N \cdot \max_{p \in [\![0;N]\!]} |\mathscr{Z}_{N,p}(\varkappa)|, \tag{4.1}$$

where $\mathscr{Z}_{N,p}(\varkappa) = \int_{\mathbb{R}^{N-p}} \mathrm{d}^{N-p}\lambda \int_{\mathbb{R}^p} \mathrm{d}^p \nu \overline{\varrho}_{N,p}(\boldsymbol{\lambda}_{N-p}, \boldsymbol{\nu}_p)$ whose integrand is expressed as

$$\overline{\varrho}_{N,p}(\boldsymbol{\lambda}_{N-p}, \boldsymbol{\nu}_p) = \prod_{a=1}^{p} \left\{\mathrm{e}^{-V_N(\nu_a)}\right\} \cdot \prod_{a=1}^{N-p} \left\{\mathrm{e}^{-V_N(\lambda_a)}\right\}$$
$$\times \prod_{a<b}^{p} \left\{\mathrm{e}^{\mathfrak{w}_N(\nu_{ab})}\right\} \cdot \prod_{a<b}^{N-p} \left\{\mathrm{e}^{\mathfrak{w}_N(\lambda_{ab})}\right\} \cdot \prod_{a=1}^{p} \prod_{b=1}^{N-p} \left\{\mathrm{e}^{\mathfrak{w}_{\mathrm{tot};N}(\nu_a - \lambda_b)}\right\}. \tag{4.2}$$

Above, we have used the N-dependent functions

$$V_N(\lambda) = \varkappa \cosh(\tau_N \lambda), \quad \mathfrak{w}_N(\lambda) = \mathfrak{w}(\tau_N \lambda), \quad \mathfrak{w}_{\mathrm{tot};N}(\lambda) = \mathfrak{w}_{\mathrm{tot}}(\tau_N \lambda), \tag{4.3}$$

with $\tau_N = \ln N$ and

$$\mathfrak{w}_{\mathrm{tot}}(\lambda) = \mathfrak{w}(\lambda) + \mathfrak{v}_{2\pi\mathfrak{b},0^+}(\lambda) \quad \textit{with } \mathfrak{v}_{\alpha,\eta}(\lambda) = \ln\left(\frac{\sinh(\lambda + \mathrm{i}\alpha)\sinh(\lambda - \mathrm{i}\alpha)}{\sinh(\lambda + \mathrm{i}\eta)\sinh(\lambda - \mathrm{i}\eta)}\right). \tag{4.4}$$

4.2. Energetic bounds

Proposition 4.2. *The partition function $\mathscr{Z}_{N,p}(\varkappa)$ admits the upper bound*

$$\mathscr{Z}_{N,p}(\varkappa) \leq \exp\left\{-N^2 \inf\left\{\mathcal{E}_{N,\frac{p}{N}}[\mu,\nu] : (\mu,\nu) \in \mathcal{M}^1(\mathbb{R}) \times \mathcal{M}^1(\mathbb{R})\right\} + \mathrm{O}\left(N\tau_N^2\right)\right\}, \tag{4.5}$$

in which the control is uniform in $p \in [\![0;N]\!]$, and where

$$\begin{aligned}\mathcal{E}_{N,t}[\mu,\nu] = {} & \frac{1}{N}\left\{t \int V_N(s)\mathrm{d}\nu(s) + (1-t) \int V_N(s)\mathrm{d}\mu(s)\right\} \\ & - \frac{t^2}{2} \int \mathfrak{w}_N(s-u)\mathrm{d}\nu(s)\mathrm{d}\nu(u) - \frac{(1-t)^2}{2} \int \mathfrak{w}_N(s-u)\mathrm{d}\mu(s)\mathrm{d}\mu(u) \\ & - t(1-t) \int \mathfrak{w}_{\mathrm{tot};N}(s-u)\mathrm{d}\mu(s)\mathrm{d}\nu(u).\end{aligned}$$

One may obtain such an upper bound within the standard approach to establishing large deviation bounds for N-fold integrals as pioneered in [7], adjoined to the local regularization of the empirical distribution of the integration variables proposed in [28], and some fine bounds due to the N-dependence of the integrand which were also considered in [8]. The details can be found in Lemmata 4.2–4.3 of [23].

4.3. Characterization of the minimizer and a lower-bound minimizer

The upper bound established in Proposition 4.2 does not allow one to conclude directly on the convergence of the series. Indeed, even if one could prove that the infimum in (4.5) gives a strictly positive number, the N-dependence of the energy functional could make the infimum N-dependent and, in principle, the latter could give rise to a behaviour in N which, when multiplied by the N^2 prefactor, could turn out to be subdominant with respect to the corrections $\mathrm{O}(N\tau_N^2)$. Hence, the longest part of the proof is devoted to obtaining some sharp and explicit lower bound for the infimum which can then be computed in closed form so that one may explicitly check that the above scenario does hold.

For that purpose, one starts by showing

Proposition 4.3. *For $0 < t < 1$ $\mathcal{E}_{N,t}$ admits a unique minimizer $(\mu_{\mathrm{eq}}^{(N,t)}, \nu_{\mathrm{eq}}^{(N,t)})$ on $\mathcal{M}^1(\mathbb{R}) \times \mathcal{M}^1(\mathbb{R})$. Similarly, $\mathcal{E}_{N,0}$ and $\mathcal{E}_{N,1}$ admit unique minimizers on $\mathcal{M}^1(\mathbb{R})$.*

This is established by showing that, for $0 < t < 1$, $\mathcal{E}_{N,t}$ is lower semicontinuous and strictly convex on $\mathcal{M}^1(\mathbb{R}) \times \mathcal{M}^1(\mathbb{R})$, has compact level sets, is not identically $+\infty$, and is bounded from below. In principle, this result could be already enough to obtain sharp in N estimates for $\mathcal{E}_{N,t}[\mu_{\mathrm{eq}}^{(N,t)}, \nu_{\mathrm{eq}}^{(N,t)}]$. Indeed, by relying on the analogous to the case of β-ensembles variational characterization of the minimizers and showing that these are actually Lebesgue continuous with compact connected supports, one may establish a system of two-coupled singular linear integral equations of truncated Wiener–Hopf type depending on the large-parameter N. These may be analyzed within the method developed by Krein's school after generalizing the work [29] and solving the 4×4 associated Riemann–Hilbert problem in the large-N regime by the Deift–Zhou nonlinear steepest descent method [12,13]. However, these steps would definitely lead to an extremely cumbersome and long clamber, especially taken the minimal amount of information one needs, in the end, from such handlings. Therefore, it is more convenient to reduce the numbers of minimizers which ought to be thoroughly determined by providing a lower bound for $\mathcal{E}_{N,t}[\mu_{\mathrm{eq}}^{(N,t)}, \nu_{\mathrm{eq}}^{(N,t)}]$ whose estimation would demand less effort while still leading to the desired result.

A direct calculation shows that one has a simpler representation for $\mathcal{E}_{N,t}$ in terms of functionals only acting on one copy of a space of bounded measures:

$$\mathcal{E}_{N,t}[\mu,\nu] = \sum_{\upsilon=\pm} \mathcal{E}_N^{(\upsilon)}\big[\sigma_t^{(\upsilon)}\big] \quad \text{with } \sigma_t^{(\pm)} = t\nu \pm (1-t)\mu, \tag{4.6}$$

in which $\mathcal{E}_N^{(+)}$ is a functional on $\mathcal{M}^1(\mathbb{R})$ while $\mathcal{E}_N^{(-)}$ is a functional on $\mathcal{M}_{\mathfrak{s}}^{(2t-1)}(\mathbb{R})$, the space of signed, bounded, measures on $\mathbb{R}$ of total mass $2t-1$. These take the form

$$\mathcal{E}_N^{(+)}[\sigma] = \frac{1}{N}\int V_N(s)\,\mathrm{d}\sigma(s) - \frac{1}{2}\int \mathfrak{w}_N^{(+)}(s-u)\cdot \mathrm{d}\sigma(s)\,\mathrm{d}\sigma(u), \tag{4.7}$$

$$\mathcal{E}_N^{(-)}[\sigma] = -\frac{1}{2}\int \mathfrak{w}_N^{(-)}(s-t)\cdot \mathrm{d}\sigma(s)\,\mathrm{d}\sigma(u). \tag{4.8}$$

The two-body interactions appearing above involve $\mathfrak{w}$ and $\mathfrak{v}_{\alpha,\eta}$ introduced in (3.5) and (4.4)

$$\mathfrak{w}_N^{(\pm)}(u) = \mathfrak{w}^{(\pm)}(\tau_N u) \quad \text{with} \quad \begin{cases} \mathfrak{w}^{(+)}(u) = \mathfrak{w}(u) + \frac{1}{2}\mathfrak{v}_{2\pi\mathfrak{b},0^+}(u), \\ \mathfrak{w}^{(-)}(u) = -\frac{1}{2}\mathfrak{v}_{2\pi\mathfrak{b},0^+}(u). \end{cases} \tag{4.9}$$

By going to Fourier space, one observes that

$$\mathcal{E}_N^{(-)}[\sigma] = \frac{1}{2}\int \mathrm{d}\lambda \big|\mathcal{F}[\sigma](\lambda)\big|^2 \frac{\sinh(\pi \mathfrak{b}\lambda)\cdot \sinh(\pi \hat{\mathfrak{b}}\lambda)}{\lambda \sinh(\frac{\pi}{2}\lambda)} \geq 0, \tag{4.10}$$

where $\mathcal{F}[\sigma](\lambda)$ stands for the Fourier transform of the signed measure σ. Thus,

$$\mathcal{E}_{N,t}\big[\mu_{\mathrm{eq}}^{(N,t)}, \nu_{\mathrm{eq}}^{(N,t)}\big] \geq \mathcal{E}_N^{(+)}\big[t\nu_{\mathrm{eq}}^{(N,t)} + (1-t)\mu_{\mathrm{eq}}^{(N,t)}\big] \geq \mathcal{E}_N^{(+)}\big[\sigma_{\mathrm{eq}}^{(N)}\big]. \tag{4.11}$$

In the last line, we have used that $\mathcal{E}_N^{(+)}$ is lower-continuous, has compact level sets, is strictly convex on $\mathcal{M}^1(\mathbb{R})$, bounded from below, and not identically $+\infty$, so as to ensure the existence of a unique minimizer thereof: $\mathcal{E}_N^{(+)}[\sigma_{\mathrm{eq}}^{(N)}] = \inf\{\mathcal{E}_N^{(+)}[\sigma] : \sigma \in \mathcal{M}^1(\mathbb{R})\}$.

4.4. Singular integral equation characterization of the minimizer $\sigma_{\mathrm{eq}}^{(N)}$

By using the variational characterization of the minimizer, see e.g., [11] for an exposition in the β-ensemble case, one reduces the construction of $\sigma_{\mathrm{eq}}^{(N)}$ to finding a solution to a singular integral equation on the Sobolev space $H_s([a_N;b_N])$ driven by the operator

$$\mathcal{S}_N[\phi](\xi) = \fint_{a_N}^{b_N} \big(\mathfrak{w}^{(+)}\big)'\big[\tau_N(\xi-\eta)\big]\cdot \phi(\eta)\mathrm{d}\eta. \tag{4.12}$$

Indeed, upon introducing the effective potential subordinate to a function $\phi \in H_s([a_N;b_N])$,

$$V_{N;\mathrm{eff}}[\phi](\xi) = \frac{1}{N}V_N(\xi) - \fint_{a_N}^{b_N} \mathfrak{w}^{(+)}\big[\tau_N(\xi-\eta)\big]\cdot \phi(\eta)\mathrm{d}\eta, \tag{4.13}$$

one may formulate

Proposition 4.4. *Let $a_N < b_N$ and $\varrho_{\mathrm{eq}}^{(N)} \in H_s([a_N;b_N])$, $1/2 < s < 1$, solve*

$$\frac{1}{N\tau_N}V_N'(x) = \mathcal{S}_N\big[\varrho_{\mathrm{eq}}^{(N)}\big](x) \quad \text{on }]a_N;b_N[, \tag{4.14}$$

be subject to the conditions

$$\varrho_{\mathrm{eq}}^{(N)}(\xi) \geq 0 \quad \text{for } \xi \in [a_N;b_N], \qquad \int_{a_N}^{b_N} \varrho_{\mathrm{eq}}^{(N)}(\xi)\mathrm{d}\xi = 1, \tag{4.15}$$

and

$$V_{N;\mathrm{eff}}\big[\varrho_{\mathrm{eq}}^{(N)}\big](\xi) > \inf\big\{V_{N;\mathrm{eff}}\big[\varrho_{\mathrm{eq}}^{(N)}\big](\eta) : \eta \in \mathbb{R}\big\} \quad \text{for any } \xi \in \mathbb{R}\setminus[a_N;b_N]. \tag{4.16}$$

Then, the equilibrium measure $\sigma_{\mathrm{eq}}^{(N)}$ is supported on the segment $[a_N;b_N]$ and continuous in respect to Lebesgue's measure with density $\varrho_{\mathrm{eq}}^{(N)}$. Moreover, the density takes the form

$$\varrho_{\mathrm{eq}}^{(N)}(\xi) = \sqrt{(b_N-\xi)(\xi-a_N)}\cdot h_N(\xi) \quad \text{with } h_N \in \mathcal{C}^{\infty}\big([a_N;b_N]\big). \tag{4.17}$$

The above proposition thus provides one with the following strategy for determining the equilibrium measure. One starts by solving the singular integral equation (4.14) for *any* endpoints a_N and b_N. The inversion should be carried out in an appropriate functional space which is dictated by the local structure (4.17) of the equilibrium measure's density, as can be inferred from an analysis of the systems of loop equations associated with the probability measure on $\mathbb{R}^N$ naturally subordinate to the energy functional $\mathcal{E}_N^{(+)}$. The fact that $\mathcal{S}_N$ should be inverted on $H_s([a_N; b_N])$, $0 < s < 1$, imposes a constraint on a_N and b_N. A second constraint is obtained from the fact that the equilibrium measure has unit mass (4.15). This is still not enough so as to be sure that the solution constructed in this way provides one with the equilibrium measure. For that to happen, one still needs to verify that the two positivity constraints (4.15)–(4.16) are fulfilled. The realization of such a program demands to have a thorough control on the inversion of $\mathcal{S}_N$. The latter may be reached within the scheme developed in [29], by solving an auxiliary 2×2 Riemann–Hilbert problem.

4.5. The Riemann–Hilbert based inversion of the operator

In the following, we adopt the shorthand notations

$$\overline{a}_N = \tau_N a_N, \quad \overline{b}_N = \tau_N b_N, \quad \overline{x}_N = \tau_N (b_N - a_N). \tag{4.18}$$

Consider the Riemann–Hilbert problem for a 2×2 matrix function $\chi \in \mathcal{M}_2(\mathcal{O}(\mathbb{C} \setminus \mathbb{R}))$:

- χ has continuous $\pm$-boundary values on $\mathbb{R}$;
- there exist constant matrices $\chi^{(a)}$ with $\chi_{12}^{(1)} \neq 0$ such that when $\lambda \to \infty$,

$$\chi(\lambda) = \begin{cases} \mathcal{P}_{L;\uparrow}(\lambda) \cdot \begin{pmatrix} -\mathfrak{s}_\lambda \cdot \mathrm{e}^{\mathrm{i}\lambda \overline{x}_N} & 1 \\ -1 & 0 \end{pmatrix} \cdot \frac{(-\mathrm{i}\lambda)^{\frac{3}{2}\sigma_3}}{\mathrm{e}^{-\mathrm{i}\frac{3\pi}{2}\sigma_3}} \\ \quad \times \big(I_2 + \frac{\chi^{(1)}}{\lambda} + \frac{\chi^{(2)}}{\lambda^2} + \mathrm{O}(\lambda^{-3})\big) \cdot \mathsf{Q}(\lambda), \quad \lambda \in \mathbb{H}^+, \\ \mathcal{P}_{L;\downarrow}(\lambda) \cdot \begin{pmatrix} -1 & \mathfrak{s}_\lambda \cdot \mathrm{e}^{-\mathrm{i}\lambda \overline{x}_N} \\ 0 & 1 \end{pmatrix} \cdot (\mathrm{i}\lambda)^{\frac{3}{2}\sigma_3} \\ \quad \times \big(I_2 + \frac{\chi^{(1)}}{\lambda} + \frac{\chi^{(2)}}{\lambda^2} + \mathrm{O}(\lambda^{-3})\big) \cdot \mathsf{Q}(\lambda), \quad \lambda \in \mathbb{H}^-, \end{cases}$$

  in which the matrix Q takes the form

$$\mathsf{Q}(\lambda) = \begin{pmatrix} 0 & -\chi_{12}^{(1)} \\ \{\chi_{12}^{(1)}\}^{-1} & \mathfrak{q}_1 + \lambda \end{pmatrix} \quad \text{with } \mathfrak{q}_1 = (\chi_{11}^{(1)}\chi_{12}^{(1)} - \chi_{12}^{(2)}) \cdot \{\chi_{12}^{(1)}\}^{-1};$$

- $\chi_+(\lambda) = G_\chi(\lambda) \cdot \chi_-(\lambda)$ for $\lambda \in \mathbb{R}$ where

$$G_\chi(\lambda) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\lambda \overline{x}_N} & 0 \\ \frac{1}{\mathrm{i}\pi} \cdot R(\lambda) & -\mathrm{e}^{-\mathrm{i}\lambda \overline{x}_N} \end{pmatrix}$$

$$\text{with } R(\lambda) = 2\frac{\sinh(\pi \mathfrak{b}\lambda) \cdot \sinh(\pi \hat{\mathfrak{b}}\lambda) \cdot \sinh(\frac{\pi}{2}\lambda)}{\cosh^2(\frac{\pi}{2}\lambda)}.$$

Here $\mathfrak{s}_\lambda = \mathrm{sgn}(\Re\lambda)$, $\mathcal{O}(A)$ stands for the ring of holomorphic functions on A, while the O remainder appearing in matrix equalities should be understood entrywise. Moreover, we point out that the matrix Q appearing in the asymptotic expansion for χ is chosen such that χ has the large-λ behavior

$$\chi(\lambda) = \chi_{\uparrow/\downarrow}^{(\infty)}(\lambda) \cdot (\mp \mathrm{i}\lambda)^{\frac{1}{2}\sigma_3}, \quad \lambda \in \mathbb{H}^\pm, \tag{4.19}$$

with $\chi_{\uparrow/\downarrow}^{(\infty)}(\lambda)$ bounded at ∞.

The Deift–Zhou nonlinear steepest descent method [12, 13] allows one to reduce the above Riemann–Hilbert problem into one that is uniquely solvable by the singular integral equation method of [6], provided that N is large enough and $b_N - a_N > c > 0$ uniformly in N.

The solution χ then provides one with a full description of the inverse of $\mathcal{S}_N$.

Proposition 4.5. *Let $0 < s < 1$. The operator $\mathcal{S}_N : H_s([a_N; b_N]) \to H_s(\mathbb{R})$ is continuous and invertible on its image:*

$$\mathfrak{X}_s(\mathbb{R}) = \left\{ H \in H_s(\mathbb{R}) : \int_{\mathbb{R}+\mathrm{i}\varepsilon'} \chi_{12}(\mu) \mathcal{F}[H](\tau_N \mu) \mathrm{e}^{-\mathrm{i}\mu \overline{b}_N} \cdot \frac{\mathrm{d}\mu}{(2\mathrm{i}\pi)^2} = 0 \right\}. \tag{4.20}$$

More specifically, one has the left and right inverse relations

$$\mathcal{W}_N \circ \mathcal{S}_N = \mathrm{id} \quad \text{on } H_s\big([a_N; b_N]\big) \quad \text{and} \quad \mathcal{S}_N \circ \mathcal{W}_N[H](\xi) = H(\xi) \quad \text{a.e. on } [a_N; b_N]$$

for any $H \in \mathfrak{X}_s(\mathbb{R})$. The operator $\mathcal{W}_N : \mathfrak{X}_s(\mathbb{R}) \to H_s([a_N; b_N])$ is given, whenever it makes sense, as an encased oscillatorily convergent Riemann integral transform

$$\mathcal{W}_N[H](\xi) = \frac{\tau_N^2}{\pi} \int_{\mathbb{R}+2\mathrm{i}\varepsilon'} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{\mathbb{R}+\mathrm{i}\varepsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \mathrm{e}^{-\mathrm{i}\tau_N \lambda(\xi - a_N)} W(\lambda, \mu) \mathrm{e}^{-\mathrm{i}\mu \overline{b}_N} \mathcal{F}[H](\tau_N \mu), \tag{4.21}$$

where $\varepsilon' > 0$ is small enough. The integral kernel

$$W(\lambda, \mu) = \frac{1}{\mu - \lambda} \left\{ \frac{\mu}{\lambda} \cdot \chi_{11}(\lambda)\chi_{12}(\mu) - \chi_{11}(\mu)\chi_{12}(\lambda) \right\} \tag{4.22}$$

is expressed in terms of the entries of the matrix χ.

These pieces of information, along with the explicit, uniform on $\mathbb{C}$, large-N expansion of the solution χ to the above Riemann–Hilbert problem and several technical estimates which allow one to check that (4.15)–(4.16) hold, allow one to formulate

Theorem 4.6. *Let $N \geq N_0$ with N_0 large enough. Then the unique minimizer $\sigma_{\mathrm{eq}}^{(N)}$ of the functional $\mathcal{E}_N^{(+)}$ introduced in (4.7) is absolutely continuous in respect to the Lebesgue measure with density $\varrho_{\mathrm{eq}}^{(N)}$ and is supported on the segment $[a_N; b_N]$. The endpoints are the unique solutions to the equations*

$$a_N + b_N = 0 \quad \text{and} \quad \vartheta \cdot \frac{(\overline{b}_N)^2 \mathrm{e}^{\overline{b}_N}}{N} \cdot \mathfrak{t}(2\overline{b}_N) \cdot \left\{ 1 + \mathrm{O}\big((\overline{b}_N)^5 \mathrm{e}^{-2\overline{b}_N(1-\varepsilon)} \big) \right\} = 1,$$

for any $1 > \varepsilon > 0$, and the remainder is smooth and differentiable in $\overline{b}_N$. Above, one has

$$\vartheta = \frac{2\varkappa}{3(2\pi)^{\frac{5}{2}}} \cdot \frac{\Gamma(\mathfrak{b}, \hat{\mathfrak{b}})}{\mathfrak{b}^{\mathfrak{b}} \hat{\mathfrak{b}}^{\hat{\mathfrak{b}}}},$$

while, upon using the constants w_k introduced below in (4.24),

$$\mathfrak{t}(\overline{x}_N) = \frac{6}{(\overline{x}_N)^2} \left\{ 2 + w_2 - w_1 - \frac{w_1 w_3}{w_2} \right\} \underset{\overline{x}_N \to +\infty}{\sim} 1 + \mathrm{O}\left(\frac{1}{\overline{x}_N} \right). \tag{4.23}$$

In particular, $\overline{b}_N$ is uniformly away from zero and admits the large-N expansion

$$\overline{b}_N = \ln N - 2 \ln \ln N - \ln \vartheta + \mathrm{O}\left(\frac{\ln \ln N}{\ln N} \right).$$

Finally, the density $\varrho_{\mathrm{eq}}^{(N)}$ of the equilibrium measure is expressed in terms of the integral transform of the potential $\varrho_{\mathrm{eq}}^{(N)} = \mathcal{W}_N[V_N']/(N\tau_N)$.

In the statement of the theorem, we made use of the coefficients w_k arising in the $\lambda \to 0$ expansion below

$$2\mathrm{i}\frac{\mathfrak{b}^{2\mathrm{i}\mathfrak{b}\lambda}\hat{\mathfrak{b}}^{2\mathrm{i}\hat{\mathfrak{b}}\lambda}2^{\mathrm{i}\lambda}}{\lambda^3\mathfrak{b}\hat{\mathfrak{b}}\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}}\Gamma^2\left(\begin{array}{c}\frac{1}{2}+\mathrm{i}\frac{\lambda}{2}\\ \frac{1}{2}-\mathrm{i}\frac{\lambda}{2}\end{array}\right)\Gamma\left(\begin{array}{c}1-\mathrm{i}\mathfrak{b}\lambda,1-\mathrm{i}\hat{\mathfrak{b}}\lambda,1-\mathrm{i}\frac{\lambda}{2}\\ \mathrm{i}\mathfrak{b}\lambda,\mathrm{i}\hat{\mathfrak{b}}\lambda,\mathrm{i}\frac{\lambda}{2}\end{array}\right)=\sum_{\ell=0}^{3}\frac{(-\mathrm{i})^{\ell}w_{\ell}}{\lambda^{3-\ell}}+\mathrm{O}(\lambda). \tag{4.24}$$

4.6. Estimation of the minimum

The closed-form expression for $\sigma_{\mathrm{eq}}^{(N)}$ in terms of the solution χ to the above Riemann–Hilbert problem and the close relation between the two-body interaction in the potential and the $\mathcal{S}_N$ operator's kernel allow one to exploit the system of jumps for χ so as to recast $\mathcal{E}_N^{(+)}[\sigma_{\mathrm{eq}}^{(N)}]$ only in terms of N, $\overline{b}_N$, and χ evaluated at special points:

$$\mathcal{E}_N^{(+)}\big[\sigma_{\mathrm{eq}}^{(N)}\big]=\frac{\varkappa}{2N}\cosh(\overline{b}_N)+\frac{\varkappa^2\mathrm{e}^{2\overline{b}_N}}{8\pi N^2}\big\{\chi_{12}^2(\mathrm{i})+2\big[\chi_{12}(\mathrm{i})\chi_{11}'(\mathrm{i})-\chi_{11}(\mathrm{i})\chi_{12}'(\mathrm{i})\big]\big\}$$
$$-\frac{\varkappa\mathrm{e}^{\overline{b}_N}}{4N}\big\{1+\mathrm{e}^{-\overline{x}_N}+\chi_{22;-}(0)\big[2\chi_{11}(\mathrm{i})+\mathrm{i}\chi_{12}(\mathrm{i})\big]-2\chi_{21;-}(0)\chi_{12}(\mathrm{i})\big\}.$$

Once that one arrives to the above closed expression, it is a matter of direct calculations which build on the uniform on $\mathbb{C}$ large-N asymptotic expansion for χ provided by the nonlinear steepest descent so as to infer the large-N asymptotics

Proposition 4.7. *One has the large-N asymptotic behavior*

$$\mathcal{E}_N^{(+)}\big[\sigma_{\mathrm{eq}}^{(N)}\big]=\frac{3\pi^4\mathfrak{b}\hat{\mathfrak{b}}\tilde{w}_1}{4(\overline{b}_N)^3\tilde{w}_2\mathfrak{t}(2\overline{b}_N)}+\frac{9\pi^4\mathfrak{b}\hat{\mathfrak{b}}}{8(\overline{b}_N)^4\mathfrak{t}^2(2\overline{b}_N)}\left\{1-\frac{2\tilde{w}_1}{\overline{b}_N\tilde{w}_2}\right\}+\mathrm{O}\big(\mathrm{e}^{-2\overline{b}_N(1-\varepsilon)}\big), \tag{4.25}$$

where $\mathfrak{t}$ *is as introduced in Theorem 4.6 and we have rescaled the* w_k *variables:*

$$w_1=2\overline{b}_N\tilde{w}_1,\quad w_2=2(\overline{b}_N)^2\tilde{w}_2,\quad \textit{with } \tilde{w}_k=1+\mathrm{O}\left(\frac{1}{\overline{b}_N}\right)\quad \textit{as } N\to+\infty. \tag{4.26}$$

Together with Propositions 4.2–4.3 and the lower bound in (4.11), the above theorem yields Theorem 3.1.

5. CONCLUSION

In this paper we reviewed the bootstrap program approach to the rigorous construction of 1+1-dimensional integrable quantum field theories arising as appropriate quantizations of integrable classical evolution equations of 1+1-dimensional field theory. This was done on the example of the Sinh-Gordon quantum field theory which is the simplest and nontrivial instance of such model. The approach starts by proposing an appropriate Hilbert space on which such a model is realized. Then, it produces the form of the **S**-matrix which governs the scattering in such a case. This **S**-matrix arises as a solution of certain symmetry constrains on the scattering in a relativistically-invariant theory along with the requirement of the factorizability of scattering into a concatenation of two-particle processes. Then, the quantum fields, which are operator-valued distributions on functions of the space-time variables, are constructed as integral operators whose integral kernels satisfy a set of equations, the bootstrap program axioms (i)–(v), which should be taken as the basic axioms of the theory. These axiomatic equations strongly depend on the form of the **S**-matrix for the given theory. It turns out that the bootstrap program equations can be solved explicitly with the help of the algebraic setting provided by the quantum integrability of the model and, in particular, the Yang–Baxter equation satisfied by the **S**-matrix. Once one ends up with the set of explicit solutions to (i)–(v), it remains to check the consistency of the whole construction, in particular, that the so-constructed quantum fields do form an algebra and that they commute at space-like separations. The latter requirement is crucial for guaranteeing the causality of the so-constructed theory and thus it being viable as a per se quantum field theory. To check these last steps of the construction, one must show that the series of multiple integrals resulting from the integral operator's multiplications do converge. This was a long standing open question in this field and its solution [23], in the simplest case scenario, was discussed by the author in the last section of this paper.

There are still numerous open questions related to these topic: first of all, to implement the method of [23], for establishing the convergence of form factor expansions for the time-like separated two-point functions as well as the multipoint correlation functions in all possible regimes of separation between the operators. These questions definitely seem to be manageable within a finite time. Further, one would like to extend the methods of proving the convergence to more challenging but also more physically relevant models such as the 1+1-dimensional integrable Sine-Gordon quantum field theory. There, the multitude of asymptotic particles, along with the presence of bound states and equal mass asymptotic particles, will definitely be a challenging, but hopefully surmountable task.

Last but not least, one should provide a thorough description of the correlation functions in the infrared limit, viz. when the Minkowski separation between the operator approaches zero. In the case of the two-point function given in (3.4) that would correspond to extracting the $r \to 0^+$ limit.

ACKNOWLEDGMENTS

We thank M. Karowski, M. Lashkevich, and F. Smirnov for stimulating discussions on various aspects of integrable quantum field theories.

FUNDING

This work was supported by CNRS. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while the author participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2021 semester.

REFERENCES

[1] A. E. Arinshtein, V. A. Fateev, and A. B. Zamolodchikov, Quantum S-matrix of the (1+1) dimensional Todd chain. *Phys. Lett. B* **87** (1979), 389–392.

[2] H. Babujian, A. Fring, M. Karowski, and A. Zapletal, Exact form factors in integrable quantum field theories: the sine-Gordon model. *Nuclear Phys. B* **538** (1999), 535–586.

[3] H. Babujian and M. Karowski, Exact form factors in integrable quantum field theories: the sine-Gordon model (II). *Nuclear Phys. B* **620** (2002), 407–455.

[4] H. Babujian and M. Karowski, Sine-Gordon breather form factors and quantum field equations. *J. Phys. A* **35** (2002), 9081–9104.

[5] R. J. Baxter, Partition function of the eight vertex lattice model. *Ann. Phys.* **70** (1972), 193–228.

[6] R. Beals and R. R. Coifman, Scattering and inverse scattering for first order systems. *Comm. Pure Appl. Math.* **37** (1984), 39–90.

[7] G. Ben Arous and A. Guionnet, Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields* **108** (1997), 517–542.

[8] G. Borot, A. Guionnet, and K. K. Kozlowski, *Asymptotic expansion of a partition function related to the sinh-model*. Math. Phys. Stud., Springer, 2016.

[9] A. Boutet de Monvel, L. Pastur, and M. Shcherbina, On the statistical mechanics approach in the random matrix theory: Integrated density of states. *J. Stat. Phys.* **79** (1995), no. 3–4, 585–611.

[10] V. Brazhnikov and S. Lukyanov, Angular quantization and form factors in massive integrable models. *Nuclear Phys. B* **512** (1998), 616–636.

[11] P. A. Deift, *Orthogonal polynomials and random matrices: a Riemann–Hilbert approach*. Courant Lect. Notes 3, New York University, 1999.

[12] P. A. Deift and X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems. *Bull. Amer. Math. Soc.* **26** (1992), no. 1, 119–123.

[13] P. A. Deift and X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics of the mKdV equation. *Ann. of Math.* **137** (1993), 297–370.

[14] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The analytic S matrix*. Cambridge University Press, 1966.

[15] B. Feigin and M. Lashkevich, Form factors of descendant operators: free field construction and reflection relations. *J. Phys. A: Math. Theor.* **42** (2009), 304014.

[16] V. M. Gryanik and S. N. Vergeles, Two-dimensional quantum field theories having exact solutions. *J. Nucl. Phys.* **23** (1976), 1324–1334.

[17] D. Iagolnitzer, *The S matrix*. North Holland Publishing Company, Amsterdam, New York, Oxford, 1978.

[18] M. Karowski and H. J. Thun, Complete S-matrix of the massive Thirring model. *Nuclear Phys. B* **130** (1978), 295–308.

[19] M. Karowski and P. Weisz, Exact form factors in (1+1)-dimensional field theoretic models with soliton behaviour. *Nuclear Phys. B* **139** (1978), 455–476.

[20] A. N. Kirillov and F. A. Smirnov, A representation of the current algebra connected with the SU(2)-invariant Thirring model. *Phys. Rev. B* **198** (1987), 506.

[21] A. N. Kirillov and F. A. Smirnov, Form-factors in the SU(2)-invariant Thirring model. *J. Sov. Math.* **47** (1989), 2423–2450.

[22] V. E. Korepin and L. D. Faddeev, Quantisation of solitons. *Theoret. Math. Phys.* **25** (1975), 1039–1049.

[23] K. K. Kozlowski, On convergence of form factor expansions in the infinite volume quantum Sinh-Gordon model in 1+1 dimensions. 2020, arXiv:2007.01740.

[24] M. Lashkevich and Y. Pugai, Form factors of descendant operators: resonance identities in the sinh-Gordon model. *J. High Energy Phys.* **2014** (2014), 112.

[25] K. S. H. Lehmann and W. Zimmerman, Zür Formulierung quantisierter Feldtheorien. *Nuovo Cimento* **1** (1955), 205–225.

[26] S. Lukyanov, Free field representation for massive integrable models. *Comm. Math. Phys.* **167** (1995), 183–226.

[27] S. Lukyanov, Form-factors of exponential fields in the sine-Gordon model. *Modern Phys. Lett. A* **12** (1997), 2543–2550.

[28] M. Maïda and E. Maurel-Segala, Free transport-entropy inequalities for nonconvex potentials and application to concentration for random matrices. *Probab. Theory Related Fields* **159** (2014), no. 1–2, 329–356.

[29] V. Yu. Novokshenov, Convolution equations on a finite segment and factorization of elliptic matrices. *Mat. Zametki* **27** (1980), 449–455.

[30] F. A. Smirnov, Quantum Gelfand–Levitan–Marchenko equations and form factors in the sine-Gordon model. *J. Phys. A: Math. Gen.* **17** (1984), L873–L878.

[31] F. A. Smirnov, Quantum Gelfand–Levitan–Marchenko equations for the sine-Gordon model. *Theoret. Math. Phys.* **60** (1984), 871–880.

[32] F. A. Smirnov, The general formula for solitons form factors in sine-Gordon model. *J. Phys. A* **19** (1986), L575–578.

[33] F. A. Smirnov, Solution of quantum Gel'fand–Levitan–Marchenko equations for the sine-Gordon model in the soliton sector for $\gamma = \pi/\nu$. *Theoret. Math. Phys.* **67** (1986), 344–351.

[34] F. A. Smirnov, *Form factors in completely integrable models of quantum field theory*. Adv. Ser. Math. Phys. 14, World Scientific, 1992.

[35] S. J. Summers, A perspective on constructive quantum field theory. 2016, arXiv:1203.3991.

[36] P. H. Weisz, Exact quantum sine-Gordon soliton form factors. *Phys. Rev. B* **67** (1977), 179.

[37] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.* **19** (1967), 1312–1315.

[38] A. B. Zamolodchikov and Al. B. Zamolodchikov, Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models. *Ann. Phys.* **120** (1979), 253–291.

[39] Al. B. Zamolodchikov, Exact two-particle S-matrix of quantum sine-Gordon solitons. *Comm. Math. Phys.* **55** (1977), 183–186.

KAROL KAJETAN KOZLOWSKI

Univ Lyon, ENS de Lyon, Univ Claude Bernard Lyon 1, CNRS, Laboratoire de Physique, F-69342 Lyon, France, karol.kozlowski@ens-lyon.fr