SINGULARITIES IN GENERAL RELATIVITY

JONATHAN LUK

ABSTRACT

We survey some recent mathematical progress in understanding singularities arising in solutions to the Einstein equations. After some quick discussions of background material, we focus on the following three topics:

- constructions of singular solutions to the Einstein vacuum equations,
- the singularity structure in the interior of generic dynamical black holes and the relation to the strong cosmic censorship conjecture,
- the formation of trapped surfaces, instabilities for the Einstein vacuum equations, and the relation to singularities.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 83-02; Secondary 83C57, 83C75, 35Q75, 35Q76, 35B44

KEYWORDS

General relativity, singularities, cosmic censorship conjectures, black holes, gravitational collapse, weak null singularities

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 5, pp. 4120–4141
DOI 10.4171/ICM2022/9

Published by EMS Press

1. INTRODUCTION

We study the Cauchy problem for the celebrated Einstein equations for a $(3+1)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$ with appropriate matter fields:

$$\mathrm{Ric}(g) - \frac{1}{2}S(g)g + \Lambda g = 8\pi T, \tag{1.1}$$

where $\mathrm{Ric}(g)$ and $S(g)$ are, respectively, the Ricci- and scalar-curvature tensors of g, $\Lambda \in \mathbb{R}$ is the cosmological constant, and T is the stress–energy–momentum tensor describing the matter content in $\mathcal{M}$. Equation (1.1) is already highly nontrivial in vacuum, i.e., when $T \equiv 0$, and with vanishing cosmological constant $\Lambda = 0$, in which case (1.1) reduces to

$$\mathrm{Ric}(g) = 0. \tag{1.2}$$

A fascinating feature of solutions to (1.2), or more generally (1.1), is the presence of *singularities*, which can arise even from regular initial data. The most well-known singularities are those occurring at the big bang or in the interior of black holes, though more exotic singularities are known. Viewing (1.1) as a system of partial differential equations, it is desirable to give a complete description of all possible singularities, a goal which at present seems far out of reach.

In this article, we instead survey some recent mathematical progress in the following specific physically interesting settings:

(i) We first discuss some local constructions of different types of singular solutions to (1.2) (Section 2).

(ii) We then turn to the discussion of singularities in the interior of dynamical black holes. This is closely related to the strong cosmic censorship conjecture, stated as Conjecture 1.3 below (Section 3).

(iii) Finally, we discuss how trapped surfaces form dynamically in solutions to (1.1). As we will see below, the formation of trapped surfaces is closely related to black holes and singularities (Section 4).

Before we turn to these topics, we first give some further context regarding singularities in general relativity in the remainder of the introduction.

1.1. The Cauchy problem in general relativity

Any discussion of the Cauchy problem in general relativity begins with the following fundamental theorem (see also the earlier [30]):

Theorem 1.1 (Choquet-Bruhat–Geroch [12]). *Let $(\Sigma, \hat{g})$ be a Riemannian 3-manifold and $\hat{k}$ be a symmetric 2-tensor. Suppose $(\hat{g}, \hat{k})$ are sufficiently regular and satisfy the constraint equations. Then there exists a unique maximal Cauchy development $(\mathcal{M}, g)$ such that*

(1) *the metric g solves (1.2),*

(2) *$(\Sigma, \hat{g}) \hookrightarrow (\mathcal{M}, g)$ isometrically, and $\hat{k}$ is the induced second fundamental form,*

(3) *any other development* $(\mathcal{M}', g')$ *satisfying* (1) *and* (2) *embeds into* $(\mathcal{M}, g)$ *isometrically.*

In general, Theorem 1.1 does not guarantee the maximal Cauchy development to be geodesically complete. Thus, from the point of view of PDE theory, Theorem 1.1 should be viewed as a *local existence* result.

Under suitable *smallness* assumptions, the Choquet-Bruhat–Geroch theorem can be extended to a *global* result. More precisely, if the initial data are close to that of Minkowski spacetime, then the maximal Cauchy development is geodesically complete and converges to Minkowski for large times. This is the monumental *stability of Minkowski* theorem by Christodoulou–Klainerman [19].

In general, however, one must face the possibility of *singularities*. In particular, as we will see, singularities can arise from complete asymptotically flat initial data sets.

1.2. Singularities and black hole spacetimes

The simplest example of formation of singularity for (1.2) can be found in the Schwarzschild solution $(\mathcal{M}_{M,0}, g_{M,0})$, where $M > 0$ is the mass parameter, $\mathcal{M}_{M,0} = \mathbb{R}^2 \times \mathbb{S}^2$, and in a local coordinate system, $g_{M,0}$ is given by

$$g_{M,0} = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \gamma_{\mathbb{S}^2(1)},$$

where $\gamma_{\mathbb{S}^2(1)}$ denotes the round metric on $\mathbb{S}^2(1)$. The Schwarzschild solution is depicted by the Penrose diagram in Figure 1.

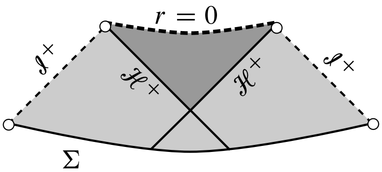

FIGURE 1
Schwarzschild as the maximal future Cauchy development of Σ.

Despite having smooth asymptotically flat initial data, the maximal future Cauchy development of Schwarzschild data has *singularity* inside the black hole region, depicted as the $\{r = 0\}$ surface. What is a singularity? There are a few <u>in</u>equivalent ways to capture the "singular nature" of $\{r = 0\}$ of Schwarzschild:

(i) (Geodesic incompleteness) Any causal geodesic entering the black hole must be incomplete and reach $\{r = 0\}$ in finite time.

(ii) (Blowup of curvature) The curvature invariant $R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} \to \infty$ as $r \to 0$.

(iii) (Infinitude of tidal deformation) Any observer heading towards the singularity will be infinitely torn apart.

1.3. Trapped surfaces and Penrose's incompleteness theorem

At first, one may hope that the Schwarzschild singularity only arises because Schwarzschild data are very special (e.g., because it is spherically symmetric). This was initially supported by the heuristics of Lifshitz–Khalatnikov [56]: they considered a class of asymptotically Kasner singularities (of which the Schwarzschild singularity is a particular example) and showed that they have one fewer functional degree of freedom compared to the Cauchy problem, which should mean that these singularities are highly nongeneric.

However, in a breakthrough work, Penrose [72] proved that singularities – at least in the sense of geodesic incompleteness – is a stable phenomenon. More precisely, he proved

Theorem 1.2 (Penrose). *If Σ is noncompact, and the maximal Cauchy development $(\mathcal{M}, g)$ contains a compact trapped surface, then $(\mathcal{M}, g)$ is future causally geodesically incomplete.*

Since trappedness is a stable condition, and the Schwarzschild solution contains many compact trapped surfaces, Theorem 1.2 implies that given any sufficiently small perturbations of Schwarzschild data, the corresponding maximal Cauchy development must be future causally geodesically incomplete.

It should be noted that Penrose's fundamental theorem (Theorem 1.2) only asserts the geodesic incompleteness of the spacetime; indeed, one important goal of the subject is to understand when the incompleteness is tied to stronger senses of singularities of curvature or tidal deformation. Already for small perturbations of Schwarzschild data, the geodesic incompleteness can look very different from Schwarzschild! To see this, one needs not look further than the explicit Kerr family of solutions $(\mathcal{M}_{M,a} = \mathbb{R}^2 \times \mathbb{S}^2, g_{M,a})$ for $|a| \leq M$, $M > 0$. When $a = 0$, this reduces to the Schwarzschild subfamily. However, when $0 < |a| < M$, the black hole region terminates with a smooth Cauchy horizon (see Figure 2); in particular, the solution remains completely smooth despite being geodesically incomplete!

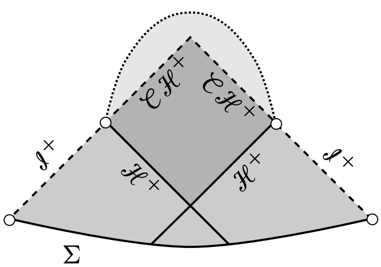

FIGURE 2
Kerr as the maximal future Cauchy development of Σ, with a non-unique extension.

1.4. The cosmic censorship conjectures

The further mathematical study of singularities is guided by two important conjectures of Penrose known as the cosmic censorship conjectures. In a sense, both conjectures assert that some desirable features of the Schwarzschild singularity should be *generic*.

As we discussed above, the interior of the Kerr black hole does not have any singularities. This poses a challenge to the deterministic nature of Einstein's theory as it reflects a breakdown of global uniqueness: the maximal Cauchy development of Kerr data (when $0 < |a| < M$) can be further extended (see Figure 2) – in infinitely many inequivalent ways – as a solution to the Einstein vacuum equations (1.2) beyond the smooth Cauchy horizon.

From this point of view, therefore, the Schwarzschild singularity is preferable to the smooth Kerr Cauchy horizon. Indeed, the first cosmic censorship conjecture asserts that the Schwarzschild case – as opposed to the Kerr case – should be generic.

Conjecture 1.3 (Strong cosmic censorship conjecture [17, 76]). *Maximal Cauchy developments of generic asymptotically flat initial data sets are inextendible as suitably regular Lorentzian manifolds.*

(See also [82, 83] for interesting works on an analogous conjecture for cosmological, i.e., compact, spacetimes. They will not be further discussed here.)

Conjecture 1.3, if true, would resolve the breakdown of determinism. In particular, the smooth Kerr Cauchy horizon would be nongeneric. From the point of view of PDE theory, Conjecture 1.3 can be viewed as a *global uniqueness* conjecture.

At this point, the formulation of Conjecture 1.3 is quite general: in the process of proving the conjecture, one must make precise the notions of "genericity" and "suitable regularity." The regularity class in which the solution is inextendible can be thought of as a convenient way to measure the strength of the singularity. We will refer to "the C^k formulation of Conjecture 1.3" when we mean to impose C^k-inextendibility of the metric. Note that C^2-inextendibility is related to curvature blowup, while C^0-inextendibility can be thought of as a more severe blowup, related to the infinitude of the tidal deformation seen in Schwarzschild. As we will see later (see Section 3), we must carefully distinguish the different formulations in order to capture the precise nature of the singularity in the interior of generic dynamical black holes.

Another preferable feature of the Schwarzschild singularity is that it is hidden behind an event horizon, and thus not visible to far-away observers. A mathematical reformulation of this fact without explicitly referring to the singularities is to say that null infinity of Schwarzschild is complete. In fact, the full Kerr family of black holes, not just the Schwarzschild subfamily, possess a complete null infinity. This is conjectured to be generic:

Conjecture 1.4 (Weak cosmic censorship conjecture [17, 73]). *Maximal Cauchy developments of generic asymptotically flat initial data sets possess a* complete *null infinity.*

Conjecture 1.4 can be viewed as a conjecture on *global existence* in the large; indeed, this is the best notion of global existence one can hope for in view of Theorem 1.2.

2. CONSTRUCTION OF SINGULARITIES

The first step towards understanding singularities in general relativity is to construct specific classes of singular solutions. Explicit singular solutions (including Schwarzschild and Kasner) have, of course, been known for a long time. There are also many results where singularities are constructed using simplifying assumptions of symmetry and analyticity. However, more general constructions of singularities have only been achieved quite recently.

2.1. Spacelike singularities

While perhaps Schwarzschild or Kasner singularities are the simplest to write down, Lifshitz–Khalatnikov (see Section 1.3) argued that such singularities depend only on three functional degrees of freedom (i.e., one fewer than that for the Cauchy problem) and are thus nongeneric. Nonetheless, one can construct the full class of such singular solutions:

Theorem 2.1 (Fournodavlos–Luk [31]). *There exists a class of asymptotically Kasner singular solutions to* (1.2) *parametrized by three functional degrees of freedom.*

See [44,51] and references therein for earlier works with symmetry and/or analyticity assumptions.

The key realization here is that the Einstein vacuum equations are, in fact, locally well-posed in a Gaussian coordinate system, i.e., in a gauge such that

$$g = -dt^2 + {}^{(3)}g_{ij}\,dx^i\,dx^j$$

for some Riemannian metric ${}^{(3)}g$, which is realized by considering the wave equation for the second fundamental form and appropriate renormalizations. In this gauge, we can carry out a Fuchsian-type analysis to construct an approximate solution, and then upgrade the construction to a bona fide solution by performing singular energy estimates.

The singularities constructed in Theorem 2.1 are not expected to be stable. Nonetheless, these singularities are stable after *restricting in suitable symmetry class*:

Theorem 2.2 (Alexakis–Fournodavlos [1], Founodavlos–Rodnianski–Speck [32]). *The singularities of Schwarzschild* [1] *and Kasner* [32] *are respectively stable under polarized axisymmetry and polarized* $\mathbb{U}(1)$ *symmetry.*

Note that in these symmetry classes, the Cauchy data depend only on two functional degrees of freedom. On the other hand, [32] treats a much more general case – the so-called subcritical regime – which includes a large class of Kasner singularities (a) in vacuum in high dimensions and (b) with matter fields in $(3 + 1)$ dimensions, without any symmetry assumptions.

It should be remarked that the influential paper [4] suggests that there should be a large class of spacelike singularities which are *oscillatory* (unlike the asymptotically Kasner singularities). Some progress has been made for a class of spatially homogeneous solutions [81]. However, its relevance in the spatially nonhomogeneous setting remains unclear.

2.2. Null singularities

It turns out that the Einstein vacuum equations admit a class of very different singular solutions which are much more stable! In contrast to Section 2.1, the singular hypersurfaces are null in these spacetimes. These solutions were first discovered in the context of the study of strong cosmic censorship conjecture for various matter models; see Section 3 below. Such singularities are often called "weak" null singularities, as the metric can be extended continuously beyond the singularities and the tidal deformation remains finite. However, they should also be thought of as "essential" singularities, since (at least conjecturally) they cannot be extended in $W^{1,p}$ for any $p > 1$.

Theorem 2.3 (Luk [57]). *A class of stable weak null singularities exist for the vacuum equation* (1.2) *without any symmetry assumptions.*

Analytic examples were previously constructed by Ori–Flanagan [71].

Like in the proof of Theorem 2.1 (and Theorem 2.2), the choice of coordinates lies at the heart of the proof. The proof uses a local coordinate system $(u, \underline{u}, \theta^1, \theta^2)$ adapted to a double null foliation, i.e., the metric takes the form

$$g = -4\Omega^2 du d\underline{u} + \gamma_{AB}(d\theta^A - b^A du)(d\theta^B - b^B du), \tag{2.1}$$

and the singular null hypersurface is a constant-u or constant-$\underline{u}$ hypersurface (or both in the case of a bifurcate null singularity). The Einstein equations in this gauge have remarkable – both linear and nonlinear – structure. First, by introducing appropriately "renormalized" curvature components, one can recast the Einstein equations in the gauge (2.1) as a coupled system of hyperbolic–elliptic–transport system which avoids the most singular (non-L^1) components of curvature. Moreover, the system of equations have important *nonlinear null structure* so that the potentially most dangerous singular terms do not appear.

The proof of Theorem 2.3 was inspired by earlier works [61,62] by Luk–Rodnianski on the propagation and interaction of *impulsive gravitational waves* without symmetry assumptions. These solutions to (1.2), first discovered in symmetry classes (see [43,74]), contain null singularities which are weaker so that a local well-posedness theory still holds. (In this context, note also the more recent work [65,66] which considers the interaction of *three* impulsive gravitational waves, for which one needs geometric constructions beyond (2.1).)

2.3. κ-self-similar singularities and naked singularities

Self-similar singularities play an important role in many evolutionary PDEs. For the Einstein vacuum equations (1.2), Rodnianski–Shlapentokh-Rothman recently constructed a class of (what they called) κ-self-similar singularities. In fact, the singularities they constructed are *naked singularities*, i.e., spacetimes with incomplete future null infinities (cf. Conjecture 1.4).

Theorem 2.4 (Rodnianski–Shlapentokh-Rothman [84]). *The Einstein vacuum equations* (1.2) *admit solutions with naked singularities.*

If Conjecture 1.4 is true, then the naked singularities in Theorem 2.4 (and indeed any naked singularities) would be unstable. Nevertheless, Theorem 2.4 shows that in order to resolve Conjecture 1.4, one must come to terms with understanding "genericity."

A closely related construction was previously achieved by Christodoulou for the Einstein–scalar field system in spherical symmetry [15]. There are also numerical evidence of other regimes of (discretely) self-similar singularities [11,53].

3. BLACK HOLE INTERIORS AND THE STRONG COSMIC CENSORSHIP CONJECTURE

We now turn to the singularities that arise in black hole interiors. We have already seen the example of the singularity in a Schwarzschild black hole. We will soon also encounter black hole interiors with null singularities, just like those constructed in Section 2.2. However, unlike in Section 2, our main concern here is not only the *local* structure of the singularities (as in Sections 2.1, 2.2), but instead we are interested in what singularities are *formed* in dynamical evolution inside black holes.

In particular, we will be interested in the question of strong cosmic censorship (see Section 1.4), i.e., whether black hole interiors are indeed generically singular as in the Schwarzschild case.

3.1. Spherically symmetric model problems

The first results concerning the issues of black hole interiors and strong cosmic censorship were obtained under the assumption of spherical symmetry. The spherical symmetry assumption rules out the Kerr solution; nevertheless, if one couples the Einstein equations with a Maxwell field, the two-parameter family (parametrized by the mass and charge M, Q) of the Reissner–Nordström solution (when $0 < |Q| < M$) also has a Penrose diagram given by Figure 2. In particular, these solutions have a smooth global bifurcate Cauchy horizon which can be extended nonuniquely as solutions to the Einstein–Maxwell system.

The early breakthroughs [37,78,79] concerned the Einstein–Maxwell–null dust system in spherical symmetry. In these works of Hiscock, Poisson–Israel, it was already shown that both stability and instability aspects are present: the perturbed solution still has a Cauchy horizon, and the metric remains continuous up to the Cauchy horizon; it is only the higher derivatives, for instance, the Hawking mass, that blow up.

A more satisfactory spherically symmetry model, which involves a wave-type dynamical degree of freedom, is the Einstein–Maxwell–scalar field system:

$$\begin{aligned} \mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= 2\big(T^{(\mathrm{sf})}_{\mu\nu} + T^{(\mathrm{em})}_{\mu\nu}\big), \\ T^{(\mathrm{sf})}_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu}(g^{-1})^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \\ T^{(\mathrm{em})}_{\mu\nu} &= (g^{-1})^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4} g_{\mu\nu}(g^{-1})^{\alpha\beta}(g^{-1})^{\gamma\sigma}F_{\alpha\gamma}F_{\beta\sigma}, \end{aligned} \tag{3.1}$$

where ϕ is a real-valued scalar function and F is a 2-form satisfying

$$\Box_g\phi = 0, \quad dF = 0, \quad \nabla_\nu F^{\mu\nu} = 0. \tag{3.2}$$

It turns out that spherical symmetry breaks the supercriticality of the problem, and, in fact, one can study the structure of the black hole interior for large data (i.e., not only those which are small perturbations of Reissner–Nordström). For this model, it was proven that the Cauchy horizon is a generic feature!

Theorem 3.1 (Dafermos [20], Dafermos–Rodnianski [26]). *Given any asymptotically flat, spherically symmetric, admissible data on* $\Sigma = \mathbb{R} \times \mathbb{S}^2$*, if the initial charge is not identically* 0*, then the solution to* (3.1) *and* (3.2) *satisfies the following:*

(1) *each component of the black hole exterior converges to Reissner–Nordström,*

(2) *the black hole interior has a (null) Cauchy horizon as (at least) part of the boundary,*

(3) *the solution is extendible up to the Cauchy horizon with a continuous metric.*

In particular, the C^0*-formulation of the strong cosmic censorship conjecture is false.*

In fact, it can be shown [21,52] based on Theorem 3.1 that when the initial charge is nonvanishing, then the solution either has the Penrose diagram of Reissner–Nordström, or else the boundary of the black hole interior has both null and spacelike components, as indicated in Figure 3. Put differently, Theorem 3.1 shows that when the charge is nonvanishing, the black hole interior, at least near timelike infinity i^+, looks more like Reissner–Nordström than Schwarzschild. This phenomenon is due to a subtle interplay between the amplification effect in the black hole interior and the decay in the black hole exterior. On the one hand, the local *blue-shift* effect present at the Reissner–Nordström Cauchy horizon [86] causes an exponential growth of waves. On the other hand, [26] established that waves in the black hole exterior decay with at least an inverse polynomial rate (as predicted by the linear heuristics of Price [80]), by understanding the dispersion of waves in the far-away region and the red-shift effect near the black hole event horizon. This decay competes with the exponential growth induced by the blue-shift effect, resulting in a black hole interior which is still C^0-extendible.

The C^0-extendibility result in Theorem 3.1, however, is not the end of the story. While the solution is extendible for *all* data with nonvanishing charge, the result is consistent with spacetime metrics arising from *generic* data having derivatives that blow up at the

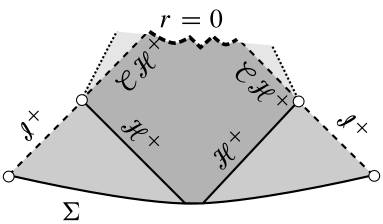

FIGURE 3
A possible Penrose diagram for Theorem 3.1.

Cauchy horizon. In fact, a conditional result was proven in [20], showing that the derivatives of the metric indeed blow up *assuming* some pointwise inverse polynomial lower bound.

More recently, it was proven that in fact the following version of the C^2-formulation of the strong cosmic censorship conjecture holds (unconditionally):

Theorem 3.2 (Luk–Oh [59,60]). *There exists an open and dense subset of the set of initial data in Theorem 3.1 such that the maximal future Cauchy development is future inextendible as time-oriented Lorentzian manifold with a C^2-metric.*

Like Theorem 3.1, the blowup in the interior proven in Theorem 3.2 results from an interplay of the decay in the exterior and the growth in the interior. Indeed, the proof of Theorem 3.2 proceeds by first showing that *generically*, waves in the black hole exterior obey an inverse polynomial *lower bound* (slightly different from that in [20]), and then proving that the solution is C^2-future inextendible whenever such a lower bound holds. In the course of the proof, a condition at null infinity is identified: we define a functional $\mathfrak{L}$, which can be computed only in terms of the radiation field and the Bondi mass at null infinity, such that $\mathfrak{L} \neq 0$ generically, and $\mathfrak{L} \neq 0$ implies the desired inverse polynomial lower bound.

From the point of view of PDE theory, one may even hope that generic solutions are inextendible in $W^{1,2}_{\mathrm{loc}}$, so as to exclude the possibility of any extension as weak solutions to the Einstein equations [18]. The estimates in Theorem 3.2 indeed suggest that this may be true, though such a geometric statement is still unknown. Very recently, Sbierski [85] proved that generic solutions as in Theorem 3.2 are C^1-inextendible.

3.2. C^0-stability of the Kerr Cauchy horizon

While the above results completed the story for (3.1) in the spherically symmetric setting, it should be noted that the strong cosmic censorship conjecture concerns *generic* data. Spherically symmetric data are, of course, by definition far from generic!

In order to make progress towards the strong cosmic censorship conjecture *without any symmetry assumptions*, we investigate a perturbative regime near the Kerr solution. It has been shown that the presence of Cauchy horizons is a generic feature even outside of symmetry!

Theorem 3.3 (Dafermos–Luk [24]). *Consider general vacuum initial data corresponding to the expected induced geometry of a dynamical black hole settling down to Kerr (with parameters $0 < |a| < M$) on a suitable spacelike hypersurface Σ_0 in the black hole interior. Then the maximal future development spacetime $(\mathcal{M}, g)$ corresponding to Σ_0 is globally covered by a double null foliation and has a nontrivial Cauchy horizon* CH^+ *across which the metric is continuously extendible.*

If the Kerr exterior is stable – as is widely expected (see [23,50]) – then Theorem 3.3 in particular implies that any small perturbations of 2-ended Kerr initial data lead to a black hole interior with a Cauchy horizon across which the metric is continuously extendible. In fact, assuming stability of Kerr exterior, it can be proven that for small perturbations of two-ended Kerr data, the maximal future Cauchy development has a global bifurcate Cauchy horizon, as in given by Figure 2 [25].

However, the implications of Theorem 3.3 go beyond small perturbations of Kerr data. Indeed, it is sometimes conjectured – in the so-called final state conjecture [77] – that generic solutions settle down to finitely many Kerr exterior solutions moving away from each other. Moreover, one expects the asymptotically Schwarzschild (or asymptotically extremal; see Section 3.3.3) solutions to occur only for nongeneric data [23]. If this is true, then Theorem 3.3 would in fact apply to generic black hole interior near timelike infinity.

It should be noted that Theorem 3.3 does not indicate whether the Cauchy horizon is actually singular. In fact, Theorem 3.3 is proven as a *stability* theorem. The proof, however, relies only on very weak norms, consistent with the Cauchy horizon possibly being a weak null singularity.

One of the challenges of the proof of Theorem 3.3 is to control solutions to the Einstein vacuum equation using only weak norms consistent with the solution being only – at least when measured in the worst direction – $C^0 \cap W^{1,1}$. This is way below the threshold for well-posedness for the Einstein equations. Instead, the proof relies on the estimates developed in the construction of local weak null singularities without symmetry assumptions (cf. Theorem 2.3). At the same time, Theorem 3.3 requires an understanding of the decay towards timelike infinity in order to close the global problem. In particular, one needs to extend ideas in Theorem 3.1 to a setting without symmetry assumptions.

Even though Theorem 3.3 by itself does not show any blowup, on the basis of Theorem 3.2 and some model linear problems [3, 27, 35, 64], it seems reasonable to expect that with generic data, the Cauchy horizon is a weak null singularity as in Theorem 2.3:

Conjecture 3.4. Generic *small perturbations of two-ended Kerr data lead to a maximal Cauchy development where the global bifurcate Cauchy horizon is a weak null singularity.*

In a similar manner to Theorem 3.2, one expects that the key to Conjecture 3.4 is to understand the precise rates of convergence in the black hole exterior.

3.3. Further problems concerning black hole interiors

While Theorem 3.3 (and Conjecture 3.4) gives the structure of the interior of generic dynamic asymptotically flat black hole near timelike infinity, we survey some other situations here, where the black hole interior is expected to be different. At the moment these are only understood under spherical symmetry or even just in a linear setting.

3.3.1. Breakdown of weak null singularities

For astrophysical gravitational collapse, the initial hypersurface does not have two ends. Instead, a black hole is expected to form from initial data on $\Sigma = \mathbb{R}^3$ (cf. Section 4). These solutions are in particular not globally close to Kerr, though as discussed in Section 3.2, Kerr is still relevant as they may arise as the asymptotic state for the black hole exterior. In this case, Theorem 3.3 still applies to show that the metric is close to Kerr in C^0

near timelike infinity. However, there are regions in the black hole which are far away from Kerr and cannot be treated by perturbative arguments.

To gain some insight, one returns to spherical symmetry: a convenient model to simultaneously study gravitational collapse and the (in)stability of the Cauchy horizons in black hole interior in spherical symmetry is the Einstein–Maxwell–charged scalar field system, i.e., unlike (3.1), the scalar field is complex-valued and charged.

With the above model, Van de Moortel considered that problem where the initial data are posed on $\mathbb{R}^3$. He proved that if a black hole forms and converges to Reissner–Nordström in the exterior with appropriate rates, then the Cauchy horizon in the black hole interior is a weak null singularity as in Theorems 3.1 and 3.2 [89]. Even more interestingly, he also proved that the null boundary in the black hole interior must break down [90]! Conjecturally, this would mean that the singular boundary in the black hole has a null component and a spacelike component. See also the numerical works [7,10].

3.3.2. Other singularities in the presence of matter

Other singularities can occur in the interior of black holes; some of these singularities may be specific to the matter models involved. Some examples include highly oscillatory null singularities arising in the interior of black holes when a charged scalar field oscillates in the black hole exterior [42], and violent nonlinear spacelike singularities in the interior of hairy black holes [34,91].

3.3.3. Extremal black holes

When $|a| = M > 0$ for Kerr or $|Q| = M > 0$ for Reissner–Nordström, these black hole spacetimes are known to be *extremal*. Though their black hole interiors have a somewhat different global structure, they have a smooth Cauchy horizon as in the subextremal case. Unlike their subextremal counterparts, however, the local blue shift at the Cauchy horizon degenerates in the extremal case.

The degeneration of the local blue shift suggests that a dynamical black hole settling down to an extremal black hole may in fact have a Cauchy horizon so that the solution is not only C^0-extendible, but also extendible as a weak solution to the Einstein equations. For a spherically symmetric model, this was studied numerically in [69] and has been later proven by Gajic–Luk [33]. Amusingly, in order to go beyond symmetry, even though the nonlinear theory is expected to be simpler than Theorem 3.3 in view of the weaker singularity, the linear theory appears to be more complicated.

Notice that this phenomenon, by itself, does not pose a threat to strong cosmic censorship, since asymptotically extremal black holes are only expected to arise from a non-generic set of data!

3.3.4. Nonvanishing cosmological constant

When the cosmological constant $\Lambda \neq 0$ (but still in vacuum), (1.1) admits the Kerr–(anti-)de Sitter black hole solutions, which (when $|a| \neq 0$) like Kerr, admit smooth Cauchy horizons in the interior of the black hole. However, the stability properties of these Cauchy horizons may turn out to be quite different from the $\Lambda = 0$ case in Theorem 3.3 and Conjecture 3.4!

When $\Lambda > 0$, at least when $|a|$ is sufficiently small, the nonlinear stability of Kerr–de Sitter has been established by Hintz–Vasy [36] and (unlike the Kerr case) is no longer a conjecture. Moreover, [36] shows that perturbations of Kerr–de Sitter data lead to solutions that converge *exponentially fast* back to Kerr–de Sitter. Thus the proof of Theorem 3.3 applies, mutatis mutandis, to show that the Kerr–de Sitter Cauchy horizon is C^0-stable.

However, the rapid exponential decay has the possibility to make Conjecture 3.4 false! To understand whether this happens seems to require determining the precise exponential rate of decay. This problem has attracted much heuristic and numerical works; see [6, 9, 28] and references therein.

When $\Lambda < 0$, the situation is difficult (and interesting) for a different reason: even linear waves on Kerr–anti-de Sitter spacetime decay only logarithmically [39]. Kehle [41] has made some interesting progress for the linear (in)stability of the Cauchy horizon, showing that the stability properties depend on the Diophantine properties of the black hole parameters in a subtle way.

4. GRAVITATIONAL COLLAPSE, FORMATION OF TRAPPED SURFACES, AND THE WEAK COSMIC CENSORSHIP CONJECTURE

As discussed in Section 1.3, Penrose's theorem (Theorem 1.2) shows that geodesic incompleteness is intimately related to the presence of trapped surfaces. In this final section, we discuss how trapped surfaces are formed dynamically from initial data without trapped surfaces. This is particularly relevant in gravitational collapse where black holes form. Finally, in Section 4.4, we will discuss how trapped surface formation relates to the weak cosmic censorship conjecture.

4.1. Formation of trapped surfaces by focussing of gravitational radiation

In the explicit Schwarzschild and Kerr solutions, either a (marginally) trapped surface or an antitrapped surface is present in any initial hypersurface. Physically, one expects that trapped surfaces may form dynamically in gravitational collapse, i.e., they may arise even if the initial hypersurface has trivial topology and is far from having a trapped surface.

Examples of formation of trapped surfaces in the presence of matter have been constructed very early on [70, 75]. The problem is much harder for the vacuum equations since any such construction is necessarily large data and (by Birkhoff's theorem) outside spherical symmetry. In a monumental breakthrough, Christodoulou constructed a large set of (stable) solutions in vacuum where trapped surfaces form dynamically from dispersed data via focussing of gravitational radiation.

Theorem 4.1 (Christodoulou [18]). *Consider the characteristic initial value problem with data on two intersecting null hypersurfaces H_0 and $\underline{H}_0$ such that*

(1) *the data on $\underline{H}_0$ is that of an incoming cone in Minkowski space;*

(2) *the data on* H_0 *is given in a region where* $0 \le \underline{u} \le \delta$ *and the initial shear* $\hat{\chi}$ *obeys the upper bound*

$$\sum_{i+j \le 10} \delta^{\frac{1}{2}} \left\| \nabla_4^i \nabla^j \hat{\chi} \right\|_{L^\infty} \le C,$$

(3) *the initial* $\hat{\chi}$ *on* H_0 *obeys the lower bound*

$$\inf_{\vartheta \in S^2} \int_0^\delta |\hat{\chi}|^2(\underline{u}', \vartheta)\, d\underline{u}' \ge c > 0.$$

Then, for $\delta > 0$ *sufficiently small, a trapped surface forms in the causal domain of the data.*

The significance of the breakthrough work of Christodoulou goes beyond the trapped surface formation problem, as it is also the first large data long time result regarding the dynamics of the Einstein vacuum equation without any symmetry assumptions. What allowed Christodoulou to handle a large data regime was a novel idea of "short pulse": the incoming radiation is concentrated in a region with a short length scale δ, so that despite the largeness, the nonlinear structure of the equations allowed Christodoulou to propagate a hierarchy of large and small estimates quantified by δ and to close all the estimates.

As was observed later in [63], as $\delta \to 0^+$, the spacetimes constructed by Christodoulou limit to a spacetime in which a null dust shell (i.e., the null dust is a delta measure on a null hypersurface) collapses and trapped surfaces form (thus the limit metric solves the Einstein equations with matter, even though for each $\delta > 0$ the spacetime is vacuum). In other words, after understanding that solutions to the Einstein–null dust system can, in fact, arise as limits of vacuum solutions [8, 40], the Christodoulou construction can be conceptually thought of as an approximation of the trapped surface formation examples with matter in [75, 87].

There are many subsequent simplifications and extensions of Theorem 4.1; see, for instance, [48, 54, 55]. We record two results that strengthen Theorem 4.1. The first improvement allows the focussing to occur only in some (as opposed to all) directions:

Theorem 4.2 (Klainerman–Luk–Rodnianski [46]). *Suppose* (1) *and* (2) *of Theorem* 4.1 *hold, and the* inf *in* (3) *is replaced by a* sup, *i.e.,*

$$\sup_{\vartheta \in S^2} \int_0^\delta |\hat{\chi}|^2(\underline{u}', \vartheta)\, d\underline{u}' \ge c > 0,$$

then a trapped surface forms in the causal domain of the data.

Theorem 4.2 is achieved by combining the existence theorem in [18] with a deformation argument, which identifies a trapped surface by solving an elliptic inequality.

The second improvement allows the incoming radiation to be much weaker: on the one hand, the incoming radiation is only required to be large in a scale-invariant norm; on the other hand, in some situations the required lower bound can be much smaller than the upper bound:

Theorem 4.3 (An–Luk [2]). *Assume* (1) *of Theorem* 4.1 *and replace* (2) *and* (3) *by*

(2) *the data on H_0 is given in a region where $0 \leq \underline{u} \leq \delta$ and the initial $\hat{\chi}$ obeys the upper bound*

$$\sum_{i+j\leq 10} \delta^{\frac{1}{2}} \|\nabla_4^i \nabla^j \hat{\chi}\|_{L^\infty} \leq a^{\frac{1}{2}},$$

(3) *the initial $\hat{\chi}$ on H_0 obeys the lower bound*

$$\inf_{\vartheta\in S^2} |\hat{\chi}|^2(\underline{u}', \vartheta)\, d\underline{u}' \geq 4ba^{\frac{1}{2}}\delta,$$

where $b \leq a$, $\delta a^{\frac{1}{2}} b < 1$, and $b \geq b_0$ for some universal large constant b_0. Then there exists a trapped surface in the causal domain of the data.

The scale-invariant results in Theorem 4.3 are proven using weighted estimates capturing the precise growth rate of the geometric quantities close to the vertex of $\underline{H}_0$.

4.2. Instability of anti-de Sitter spacetime

In Theorem 4.1, Christodoulou arranged gravitational waves to focus so that the nonlinear effect on the geometry causes a trapped surface to form dynamically. In spectacular recent works, Moschidis has demonstrated – albeit only in a spherically symmetric setting – a new trapped surface formation mechanism that goes beyond mere focussing of waves: he showed that nonlinear interaction of waves can enhance the focussing effect, which finally leads to trapped surface formation.

Moschidis' work is in the context of the AdS stability problem. The anti-de Sitter (AdS) spacetime ($M_{\mathrm{AdS}} = \mathbb{R}^{3+1}, g_{\mathrm{AdS}}$), with g_{AdS} given by

$$g_{\mathrm{AdS}} = -\left(1 - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2\gamma_{\mathbb{S}^2(1)},$$

is a solution to the Einstein vacuum equations with $\Lambda < 0$. Since it is not globally hyperbolic, one needs to impose boundary conditions to study its stability properties. AdS is conjectured [22] to be unstable under reflective boundary conditions, and this has been studied heuristically and numerically [5]. (Nevertheless, it is expected to be stable under maximally dissipative conditions; see [38].)

In a series of remarkable recent works, Moschidis resolved the AdS instability conjecture for various matter models in spherical symmetry, showing that the AdS spacetime is unstable against *trapped surface formation*.

Theorem 4.4 (Moschidis). *There exist arbitrarily small spherically symmetric perturbations of AdS data for*

(1) *the Einstein–null dust system with an inner mirror* [67]

(2) *the Einstein–massless Vlasov system* [68]

(3) *the Einstein–scalar field system* [Moschidis, in preparation]

with reflective boundary conditions such that a trapped surface forms dynamically.

In particular, viewed as a solution to the above Einstein–matter systems with reflective boundary conditions, the AdS solution is unstable.

The proof of Theorem 4.4 constructs small perturbations of AdS consisting of many spherically symmetric matter beams with judiciously chosen widths and amplitudes. The basic underlying instability mechanism is as follows: whenever two such matter beams interact, the energy of the incoming beam is concentrated. Due to the reflective boundary condition, these beams interact many times, so much so that the nonlinear interaction eventually causes a trapped surface to form.

4.3. The bounded L^2-curvature theorem and beyond

Returning to the Einstein vacuum equations, we now know that the Moschidis instability – first established in spherical symmetry for various matter models in the AdS instability problem – can be adapted in the vacuum case *without symmetry assumptions*.

To provide some context for this instability in vacuum, we recall the celebrated bounded L^2-curvature theorem (first conjectured in [45]):

Theorem 4.5 (Klainerman–Rodnianski–Szeftel [49]). *There exists $\epsilon_0 > 0$ such that if the initial data have H^2-norm $\leq \epsilon_0$, then the solution has H^2-norm $O(\epsilon_0)$ up to time $O(1)$.*

As pointed out in [47, 49], H^2 is sharp for estimating the null conjugacy radius, which is an important step in the construction of the parametrix. It turns out that not only the techniques cannot be extended below H^2, but the result itself also cannot be improved:

Theorem 4.6 (Luk–Moschidis, in progress). *There exists $\delta > 0$ such that the following holds for every $s \in [2-\delta, 2)$: For any $\epsilon > 0$, there exist initial data such that the initial data have H^s-norms of size ϵ, but the H^s-norms at time $O(1)$ are $\gtrsim \epsilon^{-1}$.*

Notice that this is not an ill-posedness result in a *fixed gauge*. For instance, it has been previously proven in [29] that the Einstein vacuum equations *in wave coordinates* are ill-posed in H^2. In contrast, in Theorem 4.6 we proved that the H^s-norm becomes large in *any* coordinate system so that the metric remains C^0-close to the Minkowski metric.

Theorem 4.6 is an instability result in a Sobolev space *above scaling*. (The scaling-invariant norm would be $H^{3/2}$.) In particular, a corresponding instability result is false for model problems such as wave maps [88] or for the Einstein–scalar field system in spherical symmetry [14]. The underlying instability mechanism, which is based on quasilinear interaction of gravitational "wave packets," is quasilinear and anisotropic, and is inspired by the Moschidis mechanism used in Theorem 4.4.

Of course, Theorem 4.6 is only an instability result, and it does not say anything about the formation of trapped surfaces per se. We remark, however, that it is not so difficult to show that if smallness is imposed on the H^s norm for $s > \frac{3}{2}$, then the initial hypersurface does not contain any trapped surfaces [58]. Extending the ideas in Theorem 4.6, one may imagine a scenario where the data are small in H^s (for $s \in (\frac{3}{2}, 2)$), but then there is an evolutionary formation of trapped surfaces associated with the growth in the H^s-norm:

Problem 4.7. Is it possible to dynamically form a trapped surface with localized initial data that are small in H^s for some $s \in [\frac{3}{2}, 2)$?

If the answer to Problem 4.7 is positive, then there would be a trapped surface formation mechanism for data even weaker than in Theorems 4.2 and 4.3.

4.4. Weak cosmic censorship conjecture

In [17], Christodoulou proposed a program to tackle the weak cosmic censorship conjecture (Conjecture 1.4). This program in particular relates weak cosmic censorship to the formation of trapped surfaces.

The strategy suggested in [17] is inspired by the spectacular work [16] which resolved the weak cosmic censorship conjecture for the Einstein–scalar field system in spherical symmetry. This latter work is, in fact, so far the only mathematical work which gives us some insights as to why naked singularities should be nongeneric. The strategy in [16] combines two ingredients: (i) a sharp trapped surface formation criterion [13], and (ii) a scale-invariant breakdown criterion [14]. Using (ii), Christodoulou further showed that small perturbations of naked singularities must be blue-shifted so that, using (i), he showed that trapped surfaces must form arbitrarily close by in the perturbed spacetime. As a result, for generic data, first singularities are preceded by trapped surfaces arbitrarily close by. From this, Christodoulou deduced that weak cosmic censorship holds for this model.

To tackle the weak cosmic censorship conjecture in vacuum without symmetry assumptions, Christodoulou introduced the following conjecture:

Conjecture 4.8 (Trapped surface conjecture, Christodoulou [17]). *For generic asymptotically flat vacuum data on Σ, the maximal globally hyperbolic development has the following property: Given a terminal indecomposable past set $\mathcal{P}$, if $\mathcal{P} \cap \Sigma$ has compact closure, then for every $\Sigma \supset U \supset \overline{\mathcal{P} \cap \Sigma}$, the domain of dependence of U contains a closed trapped surface.*

As pointed out in [17], Conjecture 4.8 has the advantage of being formulated locally, without referring to future null infinity (as in Conjecture 1.4).

At present, Conjecture 4.8 is far out of reach. In spherical symmetry, a sharp trapped surface formation result [13] has turned out to play a fundamental role for the analogue of Conjecture 4.8. The analysis outside symmetry is of course fundamentally more difficult, but one may hope that understanding the mechanisms for trapped surface formation (Sections 4.1–4.3) will likewise be relevant for Conjecture 4.8.

ACKNOWLEDGMENTS

I thank my many collaborators for their patience, inspiration, and friendship. I am particularly grateful to Igor Rodnianski, who, in addition, introduced me to this beautiful subject. I also thank Mihalis Dafermos for the figures in this article.

FUNDING

This work was partially supported by a Terman fellowship and the NSF grant DMS-2005435.

REFERENCES

[1] S. Alexakis and G. Fournodavlos, Stable space-like singularity formation for axi-symmetric and polarized near-Schwarzschild black hole interiors. 2020, arXiv:2004.00692.

[2] X. An and J. Luk, Trapped surfaces in vacuum arising dynamically from mild incoming radiation. *Adv. Theor. Math. Phys.* **21** (2017), no. 1, 1–120.

[3] Y. Angelopoulos, S. Aretakis, and D. Gajic, Late-time tails and mode coupling of linear waves on Kerr spacetimes. 2021, arXiv:2102.11884.

[4] V. A. Belinski, E. M. Lifshitz, and I. Khalatnikov, Oscillatory approach to the singular point in relativistic cosmology. *Sov. Phys., Usp.* **13** (1971), no. 6, 745.

[5] P. Bizoń and A. Rostworowski, Weakly turbulent instability of anti-de Sitter spacetime. *Phys. Rev. Lett.* **107** (2011), no. 3, 031102.

[6] P. R. Brady, I. G. Moss, and R. C. Myers, Cosmic censorship: as strong as ever. *Phys. Rev. Lett.* **80** (1998), no. 16, 3432.

[7] P. R. Brady and J. D. Smith, Black hole singularities: a numerical approach. *Phys. Rev. Lett.* **75** (1995), no. 7, 1256–1259.

[8] G. A. Burnett, The high-frequency limit in general relativity. *J. Math. Phys.* **30** (1989), no. 1, 90–96.

[9] V. Cardoso, J. a. L. Costa, K. Destounis, P. Hintz, and A. Jansen, Quasinormal modes and strong cosmic censorship. *Phys. Rev. Lett.* **120** (2018), 031103.

[10] P. M. Chesler, R. Narayan, and E. Curiel, Singularities in Reissner–Nordström black holes. *Classical Quantum Gravity* **37** (2020), no. 2, 025009.

[11] M. W. Choptuik, Universality and scaling in gravitational collapse of a massless scalar field. *Phys. Rev. Lett.* **70** (1993), 9–12.

[12] Y. Choquet-Bruhat and R. P. Geroch, Global aspects of the Cauchy problem in general relativity. *Comm. Math. Phys.* **14** (1969), 329–335.

[13] D. Christodoulou, The formation of black holes and singularities in spherically symmetric gravitational collapse. *Comm. Pure Appl. Math.* **44** (1991), no. 3, 339–373.

[14] D. Christodoulou, Bounded variation solutions of the spherically symmetric Einstein–scalar field equations. *Comm. Pure Appl. Math.* **46** (1993), no. 8, 1131–1220.

[15] D. Christodoulou, Examples of naked singularity formation in the gravitational collapse of a scalar field. *Ann. of Math. (2)* **140** (1994), no. 3, 607–653.

[16] D. Christodoulou, The instability of naked singularities in the gravitational collapse of a scalar field. *Ann. of Math.* **149** (1999), 183–217.

[17] D. Christodoulou, On the global initial value problem and the issue of singularities. *Classical Quantum Gravity* **16** (1999), A23–A35.

[18] D. Christodoulou, *The formation of black holes in general relativity*. EMS Monogr. Math., European Mathematical Society (EMS), Zürich, 2009.

[19] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*. Princeton Math. Ser. 41, Princeton University Press, Princeton, 1993.

[20] M. Dafermos, The interior of charged black holes and the problem of uniqueness in general relativity. *Comm. Pure Appl. Math.* **58** (2005), no. 4, 445–504.

[21] M. Dafermos, Black holes without spacelike singularities. *Comm. Math. Phys.* **332** (2014), no. 2, 729–757.

[22] M. Dafermos and G. Holzegel, Dynamic instability of solitons in 4 + 1-dimensional gravity with negative cosmological constant. 2006, https://www.dpmms.cam.ac.uk/~md384/ADSinstability.pdf.

[23] M. Dafermos, G. Holzegel, I. Rodnianski, and M. Taylor, The non-linear stability of the Schwarzschild family of black holes. 2021, arXiv:2104.08222.

[24] M. Dafermos and J. Luk, The interior of dynamical vacuum black holes I: the C^0-stability of the Kerr Cauchy horizon. 2017, arXiv:1710.01722.

[25] M. Dafermos and J. Luk, The interior of dynamical vacuum black holes III: the C^0-stability of the bifurcation sphere of the Kerr Cauchy horizon. In preparation.

[26] M. Dafermos and I. Rodnianski, A proof of Price's law for the collapse of self-gravitating scalar field. *Invent. Math.* **162** (2005), 381–457.

[27] M. Dafermos and Y. Shlapentokh-Rothman, Time-translation invariance of scattering maps and blue-shift instabilities on Kerr black hole spacetimes. *Comm. Math. Phys.* **350** (2017), no. 3, 985–1016.

[28] O. J. C. Dias, F. C. Eperon, H. S. Reall, and J. E. Santos, Strong cosmic censorship in de Sitter space. *Phys. Rev. D* **97** (2018), no. 10, 104060.

[29] B. Ettinger and H. Lindblad, A sharp counterexample to local existence of low regularity solutions to Einstein equations in wave coordinates. *Ann. of Math. (2)* **185** (2017), no. 1, 311–330.

[30] Y. Foures-Bruhat, Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Math.* **88** (1952), no. 1, 141–225.

[31] G. Fournodavlos and J. Luk, Asymptotically Kasner-like singularities. 2020, arXiv:2003.13591.

[32] G. Fournodavlos, I. Rodnianski, and J. Speck, Asymptotically Kasner-like singularities. 2020, arXiv:2012.05888.

[33] D. Gajic and J. Luk, The interior of dynamical extremal black holes in spherical symmetry. *Pure Appl. Anal.* **1** (2019), no. 2, 263–326.

[34] S. A. Hartnoll, G. T. Horowitz, J. Kruthoff, and J. E. Santos, Gravitational duals to the grand canonical ensemble abhor Cauchy horizons. *J. High Energy Phys.* **(10):102, 23** (2020).

[35] P. Hintz, A sharp version of Price's law for wave decay on asymptotically flat spacetimes. 2020, arXiv:2004.01664.

[36] P. Hintz and A. Vasy, The global non-linear stability of the Kerr–de Sitter family of black holes. *Acta Math.* **220** (2018), no. 1, 1–206.

[37] W. A. Hiscock, Evolution of the interior of a charged black hole. *Phys. Rev. Lett. A* **83** (1981), 110–112.

[38] G. Holzegel, J. Luk, J. Smulevici, and C. Warnick, Asymptotic properties of linear field equations in anti-de Sitter space. *Comm. Math. Phys.* **374** (2020), no. 2, 1125–1178.

[39] G. Holzegel and J. Smulevici, Quasimodes and a lower bound on the uniform energy decay rate for Kerr–AdS spacetimes. *Anal. PDE* **7** (2014), no. 5, 1057–1090.

[40] C. Huneau and J. Luk, High-frequency backreaction for the Einstein equations under polarized $U(1)$-symmetry. *Duke Math. J.* **167** (2018), no. 18, 3315–3402.

[41] C. Kehle, Diophantine approximation as Cosmic Censor for Kerr–AdS black holes. 2020, arXiv:2007.12614.

[42] C. Kehle and M. Van de Moortel, Strong Cosmic Censorship in the presence of matter: the decisive effect of horizon oscillations on the black hole interior geometry. 2021, arXiv:2105.04604.

[43] K. A. Khan and R. Penrose, Scattering of two impulsive gravitational plane waves. *Nature* **229** (1971), 185–186.

[44] S. Kichenassamy and A. D. Rendall, Analytic description of singularities in Gowdy spacetimes. *Classical Quantum Gravity* **15** (1998), no. 5, 1339–1355.

[45] S. Klainerman, PDE as a unified subject. In *GAFA 2000 number special volume. Part I*, pp. 279–315. 2000 (Tel Aviv, 1999).

[46] S. Klainerman, J. Luk, and I. Rodnianski, A fully anisotropic mechanism for formation of trapped surfaces in vacuum. *Invent. Math.* **198** (2014), no. 1, 1–26.

[47] S. Klainerman and I. Rodnianski, Causal geometry of Einstein–vacuum spacetimes with finite curvature flux. *Invent. Math.* **159** (2005), no. 3, 437–529.

[48] S. Klainerman and I. Rodnianski, On the formation of trapped surfaces. *Acta Math.* **208** (2012), no. 2, 211–333.

[49] S. Klainerman, I. Rodnianski, and J. Szeftel, The bounded L^2 curvature conjecture. *Invent. Math.* **202** (2015), no. 1, 91–216.

[50] S. Klainerman and J. Szeftel, *Global nonlinear stability of Schwarzschild spacetime under polarized perturbations*. Ann. of Math. Stud. 210, Princeton University Press, Princeton, 2020.

[51] P. Klinger, A new class of asymptotically non-chaotic vacuum singularities. *Ann. Physics* **363** (2015), 1–35.

[52] J. Kommemi, The global structure of spherically symmetric charged scalar field spacetimes. *Comm. Math. Phys.* **323** (2013), no. 1, 35–106.

[53] L. Lehner and F. Pretorius, Black strings, low viscosity fluids, and violation of cosmic censorship. *Phys. Rev. Lett.* **105** (2010), no. 10, 101102.

[54] J. Li and J. Liu, Instability of spherical naked singularities of a scalar field under gravitational perturbations. 2017, arXiv:1710.02422.

[55] J. Li and P. Yu, Construction of Cauchy data of vacuum Einstein field equations evolving to black holes. *Ann. of Math. (2)* **181** (2015), no. 2, 699–768.
[56] E. M. Lifshitz and I. M. Khalatnikov, Investigations in relativistic cosmology. *Adv. Phys.* **12** (1963), 185–249.
[57] J. Luk, Weak null singularities in general relativity. *J. Amer. Math. Soc.* **31** (2018), no. 1, 1–63.
[58] J. Luk and G. Moschidis. In preparation.
[59] J. Luk and S.-J. Oh, Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data I. The interior of the black hole region. *Ann. of Math. (2)* **190** (2019), no. 1, 1–111.
[60] J. Luk and S.-J. Oh, Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data II: the exterior of the black hole region. *Ann. PDE* **5** (2019), no. 1, Paper No. 6, 194.
[61] J. Luk and I. Rodnianski, Local propagation of impulsive gravitational waves. *Comm. Pure Appl. Math.* **68** (2015), no. 4, 511–624.
[62] J. Luk and I. Rodnianski, Nonlinear interaction of impulsive gravitational waves for the vacuum Einstein equations. *Camb. J. Math.* **5** (2017), no. 4, 435–570.
[63] J. Luk and I. Rodnianski, High-frequency limits and null dust shell solutions in general relativity. 2020, arXiv:2009.08968.
[64] J. Luk and J. Sbierski, Instability results for the wave equation in the interior of Kerr black holes. *J. Funct. Anal.* **271** (2016), no. 7, 1948–1995.
[65] J. Luk and M. Van de Moortel, Nonlinear interaction of three impulsive gravitational waves I: Main result and the geometric estimates. 2020, arXiv:2101.08353.
[66] J. Luk and M. Van de Moortel, Nonlinear interaction of three impulsive gravitational waves II: The wave estimates. 2020, arXiv:2106.05479.
[67] G. Moschidis, A proof of the instability of AdS for the Einstein–null dust system with an inner mirror. 2017, arXiv:1704.08681.
[68] G. Moschidis, A proof of the instability of AdS for the Einstein–massless Vlasov system. 2018, arXiv:1812.04268.
[69] K. Murata, H. S. Reall, and N. Tanahashi, What happens at the horizon(s) of an extreme black hole? *Classical Quantum Gravity* **30** (2013), no. 23, 235007.
[70] J. R. Oppenheimer and H. Snyder, On continued gravitational contraction. *Phys. Rev.* **56** (1939), no. 5, 455.
[71] A. Ori and E. E. Flanagan, How generic are null spacetime singularities? *Phys. Rev. D* **53** (1996), 1754–1758.
[72] R. Penrose, Gravitational collapse and space-time singularities. *Phys. Rev. Lett.* **14** (1965), 57–59.
[73] R. Penrose, Gravitational collapse: The role of general relativity. *Riv. Nuovo Cim.* **1** (1969), 252–276.
[74] R. Penrose, The geometry of impulsive gravitational waves. In *General relativity (papers in honour of J. L. Synge)*, pp. 101–115, Clarendon Press, Oxford, 1972.
[75] R. Penrose, Naked singularities. *Ann. N.Y. Acad. Sci.* **224** (1973), no. 1, 125–134.

[76] R. Penrose, Gravitational collapse. In *Gravitational radiation and gravitational collapse*, edited by C. Dewitt-Morette, pp. 82–91, Symp., Int. Astron. Union 64, Springer, Berlin, 1974.

[77] R. Penrose, Some unsolved problems in classical general relativity. In *Seminar on differential geometry*, edited by S.-T. Yau, pp. 631–668, Ann. of Math. Stud. 102, Princeton University Press, Princeton, 1982.

[78] E. Poisson and W. Israel, Inner-horizon instability and mass inflation in black holes. *Phys. Rev. Lett.* **63** (1989), 1663–1666.

[79] E. Poisson and W. Israel, Internal structure of black holes. *Phys. Rev. D* **41** (1990), 1796–1809.

[80] R. H. Price, Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations. *Phys. Rev. D* **3** (1972), no. 5, 2419–2438.

[81] H. Ringström, The Bianchi IX attractor. *Ann. Henri Poincaré* **2** (2001), no. 3, 405–500.

[82] H. Ringström, *The Cauchy problem in general relativity*. ESI Lect. Math. Phys. European Mathematical Society (EMS), Zürich, 2009.

[83] H. Ringström, Strong cosmic censorship in T^3-Gowdy spacetimes. *Ann. of Math. (2)* **170** (2009), no. 3, 1181–1240.

[84] I. Rodnianski and Y. Shlapentokh-Rothman, Naked singularities for the Einstein vacuum equations: The exterior solution. 2019, arXiv:1912.08478.

[85] J. Sbierski, On holonomy singularities in general relativity and the $C^{0,1}_{\mathrm{loc}}$-inextendibility of spacetimes. 2020, arXiv:2007.12049.

[86] M. Simpson and R. Penrose, Internal instability in a Reissner–Nordström black hole. *Internat. J. Theoret. Phys.* **7** (1973), 183–197.

[87] J. L. Synge, A model in general relativity for the instantaneous transformation of a massive particle into radiation. *Proc. Roy. Irish Acad. Sect. A* **59** (1957), 1–13.

[88] T. Tao, Global regularity of wave maps. II. Small energy in two dimensions. *Comm. Math. Phys.* **224** (2001), no. 2, 443–544.

[89] M. Van de Moortel, Stability and instability of the sub-extremal Reissner–Nordström black hole interior for the Einstein–Maxwell–Klein–Gordon equations in spherical symmetry. *Comm. Math. Phys.* **360** (2018), no. 1, 103–168.

[90] M. Van de Moortel, The breakdown of weak null singularities inside black holes. 2019, arXiv:1912.10890.

[91] M. Van de Moortel, Violent nonlinear collapse in the interior of charged hairy black holes. 2021, arXiv:2109.10932.

JONATHAN LUK

Department of Mathematics, Stanford University, 450 Jane Stanford Way, Stanford, CA 94305-2125, USA, jluk@stanford.edu