CLASSIFICATION OF GAPPED GROUND STATE PHASES IN QUANTUM SPIN SYSTEMS

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ABSTRACT

Recently, classification problems of gapped ground state phases attracted a lot of attention in quantum statistical mechanics. We explain our operator algebraic approach to these problems.

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1. INTRODUCTION

In quantum mechanics, physical models are determined in terms of some self-adjoint operators called Hamiltonians. Recently, Hamiltonians whose spectrum has a gap between the lowest eigenvalue (which coincides with the infimum of the spectrum) and the rest of the spectrum attracted a lot of attention. Physically, these models are considered to be in normal phases, where no critical phenomena occur. Despite that, it has turned out that the structure of these normal gapped phases is actually mathematically interesting when we introduce some equivalence relation to them. Roughly speaking, we say that two models are equivalent if we can connect them smoothly within those normal phases. In spacial dimensions higher than one, it is believed (and partially proven) that there are multiple phases with respect to such classifications. If we further introduce some symmetry to the game, we obtain interesting mathematical structures, even in one dimension. In this paper, we explain the operator-algebraic approach to those problems.

2. FINITE-DIMENSIONAL QUANTUM MECHANICS

In order to motivate us for the operator algebraic framework of quantum statistical mechanics, we first recall finite-dimensional quantum mechanics in this section. In finite-dimensional quantum mechanics, physical observables are represented by elements of M_n, the algebra of $n \times n$-matrices. Each positive matrix ρ with $\operatorname{Tr} \rho = 1$ (called a density matrix) defines a physical state by

$$\omega_\rho : \mathrm{M}_n \ni A \mapsto \operatorname{Tr}(\rho A) \in \mathbb{C}.$$

We call this map ω_ρ a state. Clearly, it is positive, i.e., $\omega_\rho(A^* A) \geq 0$ and normalized $\omega_\rho(\mathbb{I}) = 1$. This corresponds to the procedure of taking expectation values of each physical observables $A \in \mathrm{M}_n$, in the physical state ω_ρ. Note that the set of all states forms a convex compact set. Its extremal points are called pure states. A state ω_ρ is pure if and only if ρ is a rank-one projection.

Time evolution (Heisenberg dynamics) is given by a self-adjoint matrix H, called a Hamiltonian, via the formula

$$\mathrm{M}_n \ni A \mapsto \tau_t(A) := e^{itH} A e^{-itH}, \quad t \in \mathbb{R}. \tag{2.1}$$

Let p be the spectral projection of H corresponding to the lowest eigenvalue. A state $\omega_\rho(A) := \operatorname{Tr} \rho A$ on M_n is said to be a ground state of H if the support of ρ is under p. The ground state is unique if and only if p is a rank one projection, i.e., if the lowest eigenvalue of H is nondegenerate. In this case, the unique ground state is of the form $\omega_p(A) := \operatorname{Tr} pA$, and it is pure because p has rank one.

Sometimes we consider time-dependent Hamiltonians $H(t)$. Then the time evolution of an observable $A \in \mathrm{M}_n$ is given by a solution $\tau_t(A)$ of the differential equation

$$\frac{d}{dt}\tau_t(A) = i\left[H(t), \tau_t(A)\right], \quad \tau_0(A) = A, \quad A \in \mathrm{M}_n.$$

When the Hamiltonian is time-dependent $H(t) = H$, this reduces to the above Heisenberg dynamics $e^{itH} A e^{-itH}$.

Symmetry plays an important role in physics. Let G be a finite group and suppose that there is a group action $\beta : G \to \mathrm{Aut}(\mathrm{M}_n)$ given by unitaries V_g, $g \in G$,

$$\beta_g(A) := \mathrm{Ad}(V_g)(A), \quad A \in \mathrm{M}_n,\ g \in G.$$

Here and thereafter, $\mathrm{Aut}(\mathcal{A})$ for a $*$-algebra $\mathcal{A}$ denotes the automorphism group of $\mathcal{A}$. If a Hamiltonian H satisfies $\beta_g(H) = H$ for all $g \in G$, we say that H is β-invariant. If a β-invariant Hamiltonian H has a unique ground state $\omega_p(A) := \mathrm{Tr}\, pA$, then this unique ground state ω_p is β-invariant $\omega_p(\beta_g(A)) = \omega_p(A)$, $A \in \mathrm{M}_n$, because the spectral projection p is β-invariant, i.e., $\beta_g(p) = p$.

3. QUANTUM SPIN SYSTEMS

Operator-algebraic framework of quantum statistical mechanics allows us to extend the framework of finite-dimensional quantum mechanical systems to infinite dimensions. Let $2 \le d \in \mathbb{N}$ and $\nu \in \mathbb{N}$ be fixed. Physically, $\frac{d-1}{2}$ denotes the size of on-site spin (spin quantum number) and ν denotes the spacial dimension. We denote by $\mathfrak{S}_{\mathbb{Z}^\nu}$ the set of all finite subsets of $\mathbb{Z}^\nu$. To each finite subset $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$ we associate a finite-dimensional C^*-algebra

$$\mathcal{A}_\Lambda := \bigotimes_\Lambda \mathrm{M}_d.$$

Here, M_d is the algebra of $d \times d$-matrices. The ν-dimensional quantum spin system $\mathcal{A}_{\mathbb{Z}^\nu}$ is the C^*-inductive limit of this inductive net, given by the natural inclusion. For each infinite subset Γ, we may define $\mathcal{A}_\Gamma$ in exactly the same manner. The C^*-algebra $\mathcal{A}_\Gamma$ can be naturally regarded as a C^*-subalgebra of $\mathcal{A}_{\mathbb{Z}^\nu}$. We say that an element A has support in Γ if it belongs to $\mathcal{A}_\Gamma$. If an automorphism α acts trivially on $\mathcal{A}_{\Gamma^c}$ for some $\Gamma \subset \mathbb{Z}^\nu$, we say that α has support in Γ. The set of all elements in $\mathcal{A}_{\mathbb{Z}^\nu}$ with finite support is called a local algebra and denoted by $\mathcal{A}_{\mathrm{loc}}$.

A state ω on $\mathcal{A}_\Gamma$ is defined to be a linear functional on $\mathcal{A}_\Gamma$ with $\omega(\mathbb{I}) = 1$ which is positive in the sense that $\omega(A^*A) \ge 0$ for any $A \in \mathcal{A}_\Gamma$. The map $\mathcal{A}_\Gamma \ni A \mapsto \omega(A) \in \mathbb{C}$ corresponds to the procedure of taking the expectation value of a physical observable A in our physical state ω. The set of all states on $\mathcal{A}_\Gamma$ forms a convex weak$*$-compact set. Its extremal points are called pure states. By the Krein–Milman theorem, the set of states is the weak$*$-closure of the convex envelope of pure states. See [6] for more details.

For each state, we can associate a representation of $\mathcal{A}_\Gamma$ essentially uniquely.

Theorem 3.1 (GNS representation). *For each state ω on $\mathcal{A}_\Gamma$, there exist a representation π_ω of $\mathcal{A}_\Gamma$ on a Hilbert space $\mathcal{H}_\omega$ and a unit vector $\Omega_\omega \in \mathcal{H}_\omega$ such that*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle, \quad A \in \mathcal{A}_\Gamma, \quad \textit{and} \quad \mathcal{H}_\omega = \overline{\pi_\omega(\mathcal{A}_\Gamma)\Omega_\omega}. \tag{3.1}$$

Here, $\bar{\cdot}$ denotes the norm closure. It is unique up to unitary equivalence.

The triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is called the GNS triple of ω. We frequently consider the commutant or bicommutant of $\pi_\omega(\mathcal{A}_\Gamma)$. For a $*$-algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$, we denote by $\mathcal{M}'$ the set of all elements in $\mathcal{B}(\mathcal{H})$ (the set of all bounded operators on $\mathcal{H}$) commuting with every element in $\mathcal{M}$. The algebra $\mathcal{M}'$ is called a commutant of $\mathcal{M}$, and the commutant of $\mathcal{M}'$ is called bicommutant and denoted by $\mathcal{M}''$.

For a pure state ω, it is known that π_ω is irreducible (i.e., there is no nontrivial closed subspace of $\mathcal{H}_\omega$ invariant under $\pi_\omega(\mathcal{A}_\Gamma)$) and $\pi_\omega(\mathcal{A}_\Gamma)$ is dense in $\mathcal{B}(\mathcal{H}_\omega)$ with respect to the strong operator topology. This property can be rephrased as $\pi_\omega(\mathcal{A}_\Gamma)'' = \mathcal{B}(\mathcal{H}_\omega)$.

Given GNS representations, we can introduce some equivalence relation between states. We say that two states ω, φ on $\mathcal{A}_\Gamma$ are equivalent (denoted $\omega \simeq \varphi$) if and only if the corresponding GNS representations are unitarily equivalent. For a state ω and an automorphism α on $\mathcal{A}_\Gamma$, if ω and $\omega \circ \alpha$ are equivalent, then there is a unitary u on the GNS Hilbert space $\mathcal{H}_\omega$ implementing α in the sense

$$\mathrm{Ad}(u) \circ \pi_\omega = \pi_\omega \circ \alpha. \tag{3.2}$$

This is because $\pi_\omega \circ \alpha$ is a GNS representation of $\omega \circ \alpha$. In our context of quantum spin systems, we can see that two states ω, φ are equivalent if they can be approximated by a local perturbation of each other. More precisely, ω can be approximated arbitrarily well in the norm topology of $\mathcal{A}_{\mathbb{Z}^2}^*$ by states of the form $\varphi(A^* \cdot A)$, with $A \in \mathcal{A}_{\mathrm{loc}}$, and vice versa. Physically, it means that ω and φ are macroscopically the same.

There is yet another equivalence relation between states, which is called quasiequivalence. Two states ω, φ are said to be quasiequivalent if there is a $*$-isomorphism $\iota : \pi_\omega(\mathcal{A}_\Gamma)'' \to \pi_\varphi(\mathcal{A}_\Gamma)''$ such that $\pi_\varphi(A) = \iota \circ \pi_\omega(A)$, for all $A \in \mathcal{A}_\Gamma$. Note that if two states are equivalent, they are quasiequivalent. The converse is not true in general, but if the states are pure, it is true.

In the operator-algebraic framework of quantum spin systems, physical models are specified with a map called interaction. An interaction Φ is a map $\Phi : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\mathrm{loc}}$ satisfying

$$\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$$

for all $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$. Physically, this $\Phi(X)$ indicates an interaction term between spins inside of X.

The easiest type of interaction is an on-site interaction, satisfying

$$\Phi(X) = 0 \quad \text{if } |X| \neq 1. \tag{3.3}$$

It means that the only possibly nonzero interaction terms are of the form $\Phi(\{\mathbf{x}\})$, with $\mathbf{x} \in \mathbb{Z}^\nu$. (Here and thereafter, $|X|$ indicates the number of elements in X.) Note that all interaction terms commute with each other for such interactions.

Physically, we are more interested in interactions that have nonzero interaction terms between different sites of $\mathbb{Z}^\nu$. For example, let $\{S_j\}_{j=1,2,3}$ be generators of the irreducible representation of $\mathfrak{su}(2)$ on $\mathbb{C}^d$. Then an interaction of $\mathcal{A}_\mathbb{Z}$ given by

$$\Phi\big(\{x, x+1\}\big) = \sum_{j=1}^{3} S_j^{(x)} S_j^{(x+1)}, \quad x \in \mathbb{Z}, \tag{3.4}$$

is called the antiferromagnetic Heisenberg chain, which has been extensively studied.

Now, given an interaction, we would like to define a dynamics on $\mathcal{A}_{\mathbb{Z}^\nu}$ out of it. For this, we need to assume that Φ is "suitably local." The simplest condition among such is the condition of the uniform boundedness and finite range. An interaction is of finite range if there exists an $m \in \mathbb{N}$ such that $\Phi(X) = 0$ for X with a diameter larger than m. It is uniformly bounded if it satisfies $\sup_{X \in \mathfrak{S}_{\mathbb{Z}^\nu}} \|\Phi(X)\| < \infty$. We can relax this restriction extensively. More generally, we define norms on interactions and consider interactions with finite norms; see **[40]**.

Given a suitably local interaction, we may define a C^*-dynamics, i.e., strongly continuous one-parameter group of automorphisms on $\mathcal{A}_{\mathbb{Z}^\nu}$. For an interaction Φ and a finite set $\Lambda \subset \mathbb{Z}^\nu$, we define the local Hamiltonian on Λ by

$$(H_\Phi)_\Lambda := \sum_{X \subset \Lambda} \Phi(X). \tag{3.5}$$

Then we consider the Heisenberg dynamics given by the local Hamiltonian $e^{it(H_\Phi)_\Lambda} A e^{-it(H_\Phi)_\Lambda}$ and take the thermodynamic limit. If our interaction Φ is suitably local, for example, if it is a uniformly bounded finite-range interaction, the limit

$$\tau_\Phi^t(A) = \lim_{\Lambda \to \mathbb{Z}^\nu} e^{it(H_\Phi)_\Lambda} A e^{-it(H_\Phi)_\Lambda}, \quad t \in \mathbb{R},\ A \in \mathcal{A}_{\mathbb{Z}^\nu} \tag{3.6}$$

exists and defines a dynamics τ_Φ on $\mathcal{A}_{\mathbb{Z}^\nu}$. The reason why we consider the dynamics τ_Φ instead of Hamiltonians is because there is no mathematically meaningful limit of local Hamiltonians $(H_\Phi)_\Lambda$ as $\Lambda \to \mathbb{Z}^\nu$, while the limit (3.6) makes sense. For this reason, in the operator-algebraic framework of quantum statistical mechanics, we talk about dynamics instead of Hamiltonians.

For the same reason, a ground state is defined in terms of the dynamics τ_Φ. Let δ_Φ be the generator of τ_Φ. A state ω on $\mathcal{A}_{\mathbb{Z}^\nu}$ is called a τ_Φ-ground state if the inequality

$$-i\omega\big(A^*\delta_\Phi(A)\big) \geq 0 \tag{3.7}$$

holds for any element A in the domain $\mathcal{D}(\delta_\Phi)$ of δ_Φ. We occasionally say a ground state of Φ instead of a τ_Φ-ground state. We denote by $\mathcal{G}_\Phi$ the set of all ground states of Φ. Clearly, $\mathcal{G}_\Phi$ is a weak$*$-compact convex set, and it is known that its extremal points $\operatorname{ex}\mathcal{G}_\Phi$ consists of pure states (see **[7, THEOREM 5.3.37]**).

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS triple of a τ_Φ-ground state ω. Then there exists a unique positive operator $H_{\omega,\Phi}$ on $\mathcal{H}_\omega$ such that $e^{itH_{\omega,\Phi}}\pi_\omega(A)\Omega_\omega = \pi_\omega(\tau_\Phi^t(A))\Omega_\omega$, for all $A \in \mathcal{A}_{\mathbb{Z}^\nu}$ and $t \in \mathbb{R}$. We call this $H_{\omega,\Phi}$ the bulk Hamiltonian associated with ω. Note that Ω_ω is an eigenvector of $H_{\omega,\Phi}$ with eigenvalue 0 (see **[7, PROPOSITION 5.3.19]**).

Let us consider the corresponding condition for a finite quantum system M_n with dynamics given by a Hamiltonian H (2.1). Let p be the spectral projection of H corresponding to the lowest eigenvalue E_0. Recall that a state ω on M_n is given by a density matrix ρ with the formula $\omega(A) = \operatorname{Tr}\rho A$. Let $s(\rho)$ be the support projection of this ρ. Then one can check that ω is a τ-ground state if and only if $s(\rho)$ satisfies $s(\rho) \leq p$. Recall that the last condition is the very definition of the ground state in finite-dimensional quantum mechanics. In fact, note that the generator δ of τ in (2.1) is $\delta(A) = i[H, A]$. If $s(\rho) \leq p$, then we have

$$-i\omega\big(A^*\delta(A)\big) = \omega\big(A^*(H - E_0)A\big) \geq 0, \quad A \in \mathrm{M}_n,$$

hence ω is a τ-ground state. Conversely, suppose that ω is a τ-ground state. For any unit eigenvectors ξ, η of H with $H\xi = E_0\xi$, $H\eta = E\eta$, for $E > E_0$, set $A \in \mathrm{M}_n$ to be a matrix satisfying $A\zeta = \langle \eta, \zeta\rangle \xi$ for any $\zeta \in \mathbb{C}^n$. Substituting this A, we get

$$0 \le -i\omega\big(A^*\delta(A)\big) = (E_0 - E)\langle \eta, \rho\eta\rangle.$$

Because $E_0 - E < 0$, this means that $\langle \eta, \rho\eta\rangle = 0$ for any such η. Hence we conclude that $p\rho p = \rho$, namely, $s(\rho) \le p$. It means that our definition in the operator-algebraic framework can be regarded as a natural generalization of the usual definition of a ground state to infinite systems.

Note, in general, that there can be many states satisfying condition (3.7). Namely, the ground state need not be unique. If the ground state is unique, it is automatically an extremal point of $\mathcal{G}_\Phi$. As a result, it is pure.

The systems we are interested in, in this paper, are those with gapped ground states.

Definition 3.1. We say that Φ has gapped ground states in the bulk if the following hold:

(i) The bulk Hamiltonian $H_{\omega,\Phi}$ of any pure τ_Φ-ground state ω has 0 as its nondegenerate eigenvalue.

(ii) There exists a constant $\gamma > 0$ such that

$$\sigma(H_{\omega,\Phi}) \setminus \{0\} \subset [\gamma, \infty), \tag{3.8}$$

for any pure τ_Φ-ground state ω. Here $\sigma(H_{\omega,\Phi})$ denotes the spectrum of $H_{\omega,\Phi}$.

We denote by $\mathcal{P}$ the set of all uniformly bounded finite-range interactions with gapped ground states in the bulk.

An interaction Φ is said to have a unique gapped ground state if its ground state is unique and gapped in the sense of Definition 3.1; see [1, 17, 18, 42–44] for examples of such models. If we consider the corresponding condition for a finite system M_n with dynamics (2.1). This condition corresponds to the situation that "the lowest eigenvalue of H is nondegenerate and the difference between the lowest eigenvalue and the second-lowest eigenvalue is at least γ." One remarkable property of the unique gapped ground state is the exponential decay of correlation functions.

Theorem 3.2 ([22, 37, 39]). *Let Φ be a uniformly bounded finite-range interaction with a unique gapped ground state ω_Φ. Then the correlation functions of ω_Φ decay exponentially fast: there exist constants $\mu > 0$ and $C > 0$ such that for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, with finite $X, Y \subset \mathbb{Z}^\nu$,*

$$\big|\omega_\Phi(AB) - \omega_\Phi(A)\omega_\Phi(B)\big| \le C\|A\|\|B\||X|e^{-\mu d(X,Y)}$$

holds. Here $d(X,Y)$ denotes the distance between X and Y.

This means ω_Φ is "almost like a product state."

4. PATHS OF AUTOMORPHISMS GENERATED BY TIME-DEPENDENT INTERACTIONS

In the previous section, we considered time-independent interactions, and derived a C^*-dynamics out of them. The same procedure can be carried out for time-dependent interactions to derive strongly continuous paths of automorphisms. (Recall that in finite-dimensional quantum mechanics, we also considered time-dependent Hamiltonians.) Let $\Phi : [0,1] \ni t \to \Phi_t = (\Phi(X;t))$ be a piecewise-continuous path of interactions. Namely, for each finite X, the matrix-valued function $[0,1] \ni t \to \Phi(X;t) \in \mathcal{A}_X$ is piecewise continuous. We then define the path of local Hamiltonians $(H_{\Phi_t})_\Lambda := \sum_{X \subset \Lambda} \Phi(X;t)$ for each finite subset Λ of $\mathbb{Z}^\nu$ and consider the solution $\alpha_{\Phi,t,\Lambda}(A)$ of the differential equation

$$\frac{d}{dt}\alpha_{\Phi,t,\Lambda}(A) = i\left[(H_{\Phi_t})_\Lambda, \alpha_{\Phi,t,\Lambda}(A)\right], \quad \alpha_{\Phi,0,\Lambda}(A) = A.$$

If the interactions along this path are suitably local, analogous to those considered in the previous section, then the thermodynamic limit

$$\alpha_{\Phi,t}(A) = \lim_{\Lambda \to \mathbb{Z}^\nu} \alpha_{\Phi,t,\Lambda}(A), \quad A \in \mathcal{A}_{\mathbb{Z}^\nu}$$

exists and defines a strongly continuous path of automorphisms $\alpha_{\Phi,t}$. We denote by $\mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ the set of all automorphisms $\alpha = \alpha_{\Phi,t}$ generated by some time-dependent interactions Φ in this manner. It forms a subgroup of the automorphism group $\mathrm{Aut}(\mathcal{A}_{\mathbb{Z}^\nu})$ on $\mathcal{A}_{\mathbb{Z}^\nu}$.

Due to the fact that $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ is given out of local interactions, it shows some nice locality properties. The most famous one is the Lieb–Robinson bound, which has been extensively studied and used [4,22,37,39,40]. It gives an estimate on $\|[\alpha(A), B]\|$ for $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, which decays as the distance between finite subsets X and Y goes to infinity.

The other property that is satisfied by $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ is the factorization property. It basically says that we can split α into two along any cut of the system modulo some error terms localized around the boundary. For example, in one-dimensional systems, if we cut the system into two parts at the origin, we have

$$\alpha = \mathrm{Ad}(v) \circ (\alpha_L \otimes \alpha_R), \tag{4.1}$$

where α_L is an automorphism on the left infinite chain $\mathcal{A}_L := \mathcal{A}_{(-\infty,-1]\cap\mathbb{Z}}$, while α_R is an automorphism on the right infinite chain $\mathcal{A}_R := \mathcal{A}_{[0,\infty)\cap\mathbb{Z}}$. The term $\mathrm{Ad}(v)$ is an inner automorphism given by some unitary v in $\mathcal{A}_\mathbb{Z}$, which corresponds to the "error around the boundary." In a two-dimensional system, for example, we have the following when we cut the system into two by the y-axis. For $0 < \theta < \frac{\pi}{2}$, we define a double cone C_θ by

$$C_\theta := \left\{(x,y) \in \mathbb{Z}^2 \mid |y| \le \tan\theta \cdot |x|\right\}. \tag{4.2}$$

Furthermore, H_L, H_R, H_U, H_D denotes left/right and upper/lower half-planes, and $C_{\theta,L} := C_\theta \cap H_L$, $C_{\theta,R} := C_\theta \cap H_R$. For any $0 < \theta < \frac{\pi}{2}$, there is $\alpha_L \in \mathrm{Aut}\,\mathcal{A}_{H_L}$, $\alpha_R \in \mathrm{Aut}\,\mathcal{A}_{H_R}$, and $\Theta \in \mathrm{Aut}\,\mathcal{A}_{(C_\theta)^c}$ such that

$$\alpha = \mathrm{Ad}(v)(\alpha_L \otimes \alpha_R) \circ \Theta. \tag{4.3}$$

Actually, α can be cut in many directions simultaneously. The factorization property is a simple but strong analytical property, which turns out to be useful in the analysis of gapped ground state phases [36,45–47,49].

Another property we note about $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ is that it does not create a long-range entanglement. For example, it satisfies the following property. If A and B are observables localized in finite regions far away from each other, then α almost preserves the tensor product form of $A \otimes B$, namely, there are operators $\tilde{A}$, $\tilde{B}$ strictly localized in some finite disjoint areas such that $\tilde{A} \otimes \tilde{B}$ approximates $\alpha(A \otimes B)$ in the norm topology. In fact, our α can be regarded as a version of a quantum circuit with finite depth, which is regarded as a quantum circuit which does not create long-range entanglement [3]. From this point of view, we say a state has a short-range entanglement if it is of the form

$$\left(\bigotimes_{\boldsymbol{x}\in\mathbb{Z}^\nu} \rho_{\boldsymbol{x}}\right) \circ \alpha, \tag{4.4}$$

with infinite tensor product state $\bigotimes_{\boldsymbol{x}\in\mathbb{Z}^\nu} \rho_{\boldsymbol{x}}$ and an automorphism $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$. Otherwise, we say it has a long-range entanglement.

In the physics literature, the classification of states with respect to local unitaries is considered [14]. Two states are equivalent if there is a local unitary connecting them. In our framework, these local unitaries can be understood as automorphisms in $\mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$, and the classification in [14] can be reformulated as follows. For two states ω_1, ω_0 on $\mathcal{A}_{\mathbb{Z}^\nu}$, we write $\omega_1 \sim_{\mathrm{l.u.}} \omega_0$ if there is an automorphism $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ such that $\omega_1 = \omega_0 \circ \alpha$. This gives some equivalence relation. From the fact that automorphisms in $\mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ do not create long-range entanglement, this is one physically natural criterion of classification of states.

5. THE CLASSIFICATION OF GAPPED GROUND STATE PHASES

The automorphisms in $\mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ are of fundamental importance in the classification problem of gapped ground state phases. In a word, ground state spaces of two interactions $\Phi_0, \Phi_1 \in \mathcal{P}$ (Definition 3.1) are connected to each other via such automorphisms if they are equivalent in the classification of gapped ground state phases. In this section, we introduce such a theorem, called the automorphic equivalence. The automorphic equivalence started as Hasting's adiabatic lemma [23] in finite-dimensional quantum mechanical system. There have been seminal mathematical improvements and generalizations after that [4,40] in the context of the thermodynamic limit of quantum spin systems. Here we introduce a version from [33], where we require the spectral gap only in the infinite systems (i.e., the setting in Section 3).

The classification problem of gapped ground states in infinite systems can be roughly described as follows.

We say that two interactions $\Phi_0, \Phi_1 \in \mathcal{P}$ are equivalent if there is a path of interactions $\Phi : [0,1] \to \mathcal{P}$ satisfying the following conditions:

(1) $\Phi(0) = \Phi_0$ and $\Phi(1) = \Phi_1$;

(2) $[0,1] \ni s \mapsto \Phi(X;s) \in \mathcal{A}_X$ is continuous and piecewise C^1. The interaction $\Phi(s)$ and its derivative are of finite range, bounded with respect to some norm uniformly in $s \in [0,1]$ (see (ii)–(iv) of Assumption 1.2 in [33]);

(3) For each pure τ_{Φ_0}-ground state φ_0, there is a unique smooth path of states φ_s where each φ_s is a pure $\tau_{\Phi(s)}$-ground state. (Here, smooth means the expectation value of some class of elements in $\mathcal{A}_{\mathbb{Z}^\nu}$ with respect to φ_s is differentiable, and its derivative is not too large compared to some norm; see [33, Assumption 1.2(vii)].) For each $s \in [0,1]$, the map $\mathrm{ex}\,\mathcal{G}_{\Phi_0} \ni \varphi_0 \mapsto \varphi_s \in \mathrm{ex}\,\mathcal{G}_{\Phi_s}$ gives a bijection;

(4) The gap is uniformly bounded from below by some $\gamma > 0$ along the path, i.e., $\sigma(H_{\psi_s,\Phi(s)}) \setminus \{0\} \subset [\gamma,\infty)$ for all $s \in [0,1]$ and a pure τ_{Φ_s}-ground state ψ_s.

We write $\Phi_0 \sim \Phi_1$ if $\Phi_0, \Phi_1 \in \mathcal{P}$ are equivalent in this sense.

The automorphic equivalence in this setting is given as follows.

Theorem 5.1 ([33]). *If $\Phi_0 \sim \Phi_1$, then there is an $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ such that*

$$\mathcal{G}_{\Phi_1} = \mathcal{G}_{\Phi_0} \circ \alpha. \tag{5.1}$$

Proof. We use the notation above for $\Phi_0 \sim \Phi_1$. From Remark 1.4 of [33], there is a path of automorphisms $\alpha_s \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ satisfying $\varphi_s = \varphi_0 \circ \alpha_s$ for each state φ_0, φ_s in (3). This α_s is independent of the choice of φ_0. Because $\mathcal{G}_{\Phi(s)}$ is a convex weak$*$-compact set, it coincides with the weak$*$-closure of the convex hull of extremal points of $\mathcal{G}_{\Phi(s)}$. Hence we see that this α_s maps $\mathcal{G}_{\Phi(0)}$ to $\mathcal{G}_{\Phi(s)}$ bijectively. ■

Hence automorphisms in $\mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ connect ground state spaces of Φ_0 and Φ_1. For this reason, this class of automorphisms is of fundamental importance. The point here is that it is not only that there is some automorphism connecting the ground state spaces, but also that we know the details of the automorphisms.

Note that for interactions $\Phi_1, \Phi_0 \in \mathcal{P}$ with unique ground states $\omega_{\Phi_1}, \omega_{\Phi_0}$, $\Phi_1 \sim \Phi_0$ implies $\omega_{\Phi_1} \sim_{\mathrm{l.u.}} \omega_{\Phi_0}$ by Theorem 5.1. At the moment of writing, it is not clear to us if the converse is true.

We call an on-site interaction (defined in (3.3)) with a unique gapped ground state a trivial interaction. The unique ground state ω_{Φ_0} of a trivial interaction Φ_0 is of infinite tensor product form. One can easily see that any two trivial interactions are equivalent. The equivalence class $\mathcal{P}_0$ of interactions including these trivial interactions is called a trivial phase. Any interaction Φ in the trivial phase has a unique ground state, and, by Theorem 5.1, it has a short-range entanglement (4.4).

6. SYMMETRY PROTECTED TOPOLOGICAL (SPT) PHASES

The trivial phase $\mathcal{P}_0$ consists of interactions that are connected to trivial interactions, and as a result, its ground state has a short-range entanglement which is basically the same as product states. From this point of view, the trivial phase itself may not be that interesting. However, if we introduce some symmetry to the game, we can extract some interesting mathematical structure out of it. This is so-called symmetry protected topological (SPT) phases, which were introduced by Gu and Wen [12,13,21]. Throughout this section, ω_Φ for $\Phi \in \mathcal{P}_0$ indicates the unique ground state of Φ.

In this talk, as a symmetry, we consider an on-site finite group symmetry, which is defined as follows. (A study on the global reflection symmetry in one-dimensional systems can be found in [46].) We fix a finite group G and a (projective) unitary representation U of G on $\mathbb{C}^d$. Then there is a unique automorphism β_g satisfying

$$\beta_g(A) = \left(\bigotimes_{x\in\Lambda} U(g)\right) A \left(\bigotimes_{x\in\Lambda} U(g)^*\right), \quad g \in G,\ A \in \mathcal{A}_\Lambda,\ \Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}.$$

Clearly, this gives an action of G on $\mathcal{A}_{\mathbb{Z}^\nu}$, i.e., $\beta_g\beta_h = \beta_{gh}$ for $g, h \in G$. We call this action of G, an on-site symmetry given by G and U. We say an interaction Φ is β-invariant if $\beta_g(\Phi(X)) = \Phi(X)$ for all $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$ and $g \in G$. For a ground state φ of a β-invariant interaction Φ, one can check that $\varphi \circ \beta_g$ is also a ground state of Φ. Therefore, if a β-invariant interaction Φ has a unique ground state ω_Φ, the ground state is β-invariant, $\omega_\Phi \circ \beta_g = \omega_\Phi$.

What we are interested in, in this section, is the set of all β-invariant interactions in the trivial phase $\mathcal{P}_0$. We denote the set of all such interactions by $\mathcal{P}_{0,\beta}$. We would like to classify them with respect to the following criterion. Two interactions Φ_0, Φ_1 are β-equivalent if there is a smooth path of interactions in $\mathcal{P}_{0,\beta}$ satisfying the conditions (1)–(4) we saw in Section 5. We write $\Phi_0 \sim_\beta \Phi_1$ in this case. The difference between $\sim$ and $\sim_\beta$ is that we require the symmetry to be preserved along the path. Because of this additional condition, there can be interactions $\Phi_0, \Phi_1 \in \mathcal{P}_{0,\beta}$, which satisfy $\Phi_0 \sim \Phi_1$ (by definition) but not $\Phi_0 \sim_\beta \Phi_1$. In other words, $\mathcal{P}_{0,\beta}$ may split into possibly multiple equivalence classes. The resulting equivalence classes are the symmetry-protected topological (SPT) phases.

For this SPT classification problem, physicists and algebraic topologists have a conjecture [26,56]. They say that SPT-phases should be understood in terms of the invertible quantum field theory. As a result, for a finite group G, SPT-phases should be classified by the Pontryagin dual of bordism group on the classifying space BG of G. In one and two dimensions, these Pontryagin duals are $H^2(G, \mathrm{U}(1))$, $H^3(G, \mathrm{U}(1))$. In fact, we can derive these group-cohomology-valued invariants out of our general microscopic models of in those dimensions.

Theorem 6.1 ([45,47]). *There is an $H^2(G, \mathrm{U}(1))$-valued invariant for one-dimensional SPT-phases. There is an $H^3(G, \mathrm{U}(1))$-valued invariant for two-dimensional SPT-phases.*

For the rest of this section, we explain how to find such invariants out of general models. In the analysis of gapped ground state phases, there is a general guiding principle to find an invariant. That is, cut the system into two and look at the edge. This principle is sometimes called the bulk-edge correspondence. In order to derive the invariant in Theorem 6.1, we follow this principle and restrict our group action β to the half of the system.

Namely, we consider the group actions

$$\beta_g^R := \mathrm{id}_{\mathcal{A}_L} \otimes \bigotimes_{x\geq 0} \mathrm{Ad}\big(U(g)\big), \quad \beta_g^U := \mathrm{id}_{\mathcal{A}_{H_D}} \otimes \bigotimes_{(x,y)\in H_U} \mathrm{Ad}\big(U(g)\big), \tag{6.1}$$

in one and two dimensions, respectively. We investigate the effect of these actions on our unique ground state ω_Φ for $\Phi \in \mathcal{P}_{0,\beta}$.

Let us start with one-dimensional systems. Recall that ω_Φ has a short-range entanglement, and is β-invariant. From these facts, we expect that the effect of β^R is not much recognizable on the left infinite chain, far away from the origin. On the other hand, on the right infinite chain, far away from the origin, the differences between β and β^R are not much recognizable. Combining this and the fact that ω_Φ is β-invariant, we conclude that the effect of β^R is not much recognizable on the right infinite chain, far away from the origin. As a result, we expect that the effect of β^R on ω_Φ should be localized around the origin. In other words, ω_Φ and $\omega_\Phi \circ \beta_g^R$ are macroscopically the same. It turns out to be true, mathematically, in the following sense.

Proposition 6.1. *The states ω_Φ and $\omega_\Phi \circ \beta_g^R$ are equivalent.*

This can be seen very easily. Recall from the definition that $\Phi \in \mathcal{P}_0$ means $\Phi \sim \Phi_0$ with some trivial interaction Φ_0. By Theorem 5.1, we have $\omega_\Phi = \omega_{\Phi_0} \circ \alpha$ with some $\alpha \in \mathrm{QAut}(\mathcal{A}_\mathbb{Z})$. Recall that, as a trivial interaction, Φ_0 has a unique ground state of infinite tensor product form. In particular, we can write ω_{Φ_0} as $\omega_{\Phi_0} = \omega_L \otimes \omega_R$ with pure states ω_L, ω_R on the left and right infinite chains $\mathcal{A}_L$, $\mathcal{A}_R$, respectively. Recall also that our α satisfies the factorization property (4.1). Combining these facts, we conclude that

$$\omega_\Phi \simeq (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R), \tag{6.2}$$

with some automorphisms α_L, α_R on $\mathcal{A}_L, \mathcal{A}_R$. From this and the invariance of ω_Φ under β_g, we see that $\omega_L\alpha_L\beta_g^L \otimes \omega_R\alpha_R\beta_g^R \simeq \omega_L\alpha_L \otimes \omega_R\alpha_R$, where β^L, β^R are the restrictions of β to the left and right infinite chains, respectively. This implies $\omega_R\alpha_R\beta_g^R \simeq \omega_R\alpha_R$, hence we get

$$\omega_\Phi\beta_g^R \simeq \omega_L\alpha_L \otimes \omega_R\alpha_R\beta_g^R \simeq \omega_L\alpha_L \otimes \omega_R\alpha_R \simeq \omega_\Phi, \tag{6.3}$$

proving the claim.

Note from Section 3 that Proposition 6.1 means β_g^R is implementable by a unitary u_g in the GNS representation $(\mathcal{H}_{\omega_\Phi}, \pi_{\omega_\Phi})$ of ω_Φ, i.e.,

$$\mathrm{Ad}(u_g) \circ \pi_{\omega_\Phi} = \pi_{\omega_\Phi} \circ \beta_g^R. \tag{6.4}$$

Because β^R is a group action, we have

$$\mathrm{Ad}(u_g u_h) \circ \pi_{\omega_\Phi} = \pi_{\omega_\Phi} \circ \beta_g^R \beta_h^R = \pi_{\omega_\Phi} \circ \beta_{gh}^R = \mathrm{Ad}(u_{gh}) \circ \pi_{\omega_\Phi}, \quad g, h \in G. \tag{6.5}$$

Recall that ω_Φ is a unique ground state of Φ, hence it is pure. As a result, $\pi_{\omega_\Phi}(\mathcal{A}_\mathbb{Z})$ is dense in $\mathcal{B}(\mathcal{H}_{\omega_\Phi})$ with respect to the strong operator topology. From this, (6.5) implies that there is some $\sigma(g,h) \in \mathrm{U}(1)$ such that

$$u_g u_h = \sigma(g,h) u_{gh}, \quad g, h \in G. \tag{6.6}$$

In other words, (u_g) forms a projective representation. As a result, we obtain $H^2(G, \mathrm{U}(1))$-valued index out of it.

Using the automorphic equivalence Theorem 5.1 and the factorization property of the automorphism therein, one can show that it is in fact an invariant of our classification $\sim_\beta$ [45]. The point of the proof is, when $\Phi_0 \sim_\beta \Phi_1$, that the time-dependent interactions giving $\alpha \in \mathrm{QAut}(\mathcal{A}_\mathbb{Z})$ in Theorem 5.1 can be taken to be β-invariant. Proposition 6.1 itself holds for general β-invariant unique gapped ground state. This is thanks to the theorem by Matsui [31] showing the split property for unique gapped ground states. Projective representations associated to split states have been known since the year 2000 [30] among operator algebraists. What is new here is that the associated cohomology class is an invariant of our classification. In fact, this $H^2(G, \mathrm{U}(1))$-valued index is a complete invariant of pure β-invariant split states with respect to some classification [48]. This index can be used to show Lieb–Schultz–Mattis-type theorems [2,29,30,38] (no-go theorems for the existence of unique gapped ground state under some symmetry), for finite groups symmetries [50,51].

For two dimensions, $\omega_\Phi \circ \beta_g^U$ is not equivalent to ω_Φ in general. However, an analogous argument as in the one-dimensional case lets us expect that the effect of β_g^U should be localized around the x-axis. In fact, it turns out to be true mathematically.

Proposition 6.2. *For any $0 < \theta < \frac{\pi}{2}$, there are $\eta_{g,L} \in \mathrm{Aut}(\mathcal{A}_{C_{\theta,L}})$ and $\eta_{g,R} \in \mathrm{Aut}(\mathcal{A}_{C_{\theta,R}})$ such that*

$$\omega_\Phi \circ \beta_g^U \simeq \omega_\Phi(\eta_{g,L} \otimes \eta_{g,R}).$$

It means macroscopically that the effect of β_g^U on ω_Φ is localized around $C_{\theta,L}$ and $C_{\theta,R}$ for any $0 < \theta < \frac{\pi}{2}$. This $\eta_{g,R}$ is our source of the $H^3(G, \mathrm{U}(1))$-valued index.

Now we fix some $0 < \theta < \frac{\pi}{2}$, and set $\gamma_g^R := \beta_g^{UR} \circ \eta_{g,R}^{-1}$, $\gamma_g^L := \beta_g^{UL} \circ \eta_{g,L}^{-1}$ with $\eta_{g,R}$, $\eta_{g,L}$ for this θ. Here, β_g^{UR}, β_g^{UL} are group actions of G given by

$$\beta_g^{UR} := \mathrm{id}_{(H_U \cap H_R)^c} \otimes \bigotimes_{(x,y) \in H_U \cap H_R} \mathrm{Ad}\big(U(g)\big),$$
$$\beta_g^{UL} := \mathrm{id}_{(H_U \cap H_L)^c} \otimes \bigotimes_{(x,y) \in H_U \cap H_L} \mathrm{Ad}\big(U(g)\big).$$

From Proposition 6.2, we have

$$\omega_\Phi \circ \left(\gamma_g^L \otimes \gamma_g^R\right) \simeq \omega_\Phi, \quad g \in G. \tag{6.7}$$

On the other hand, recall from the definition that $\Phi \in \mathcal{P}_0$ means $\Phi \sim \Phi_0$ with some trivial interaction Φ_0. By Theorem 5.1, we have $\omega_\Phi = \omega_{\Phi_0} \circ \alpha$, with $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^\nu})$ satisfying the factorization property, i.e.,

$$\alpha = \mathrm{Ad}(v) \circ (\alpha_L \otimes \alpha_R) \circ \Theta, \quad \alpha_L \in \mathrm{Aut}\, \mathcal{A}_{H_L},\ \alpha_R \in \mathrm{Aut}\, \mathcal{A}_{H_R},\ \Theta \in \mathrm{Aut}\, \mathcal{A}_{C_\theta^c}, \tag{6.8}$$

for our fixed θ. Recall that as a trivial interaction, Φ_0 has a unique ground state ω_{Φ_0} of infinite tensor product form. In particular, we can write ω_{Φ_0} as $\omega_{\Phi_0} = \omega_L \otimes \omega_R$ with pure states ω_L, ω_R on $\mathcal{A}_{H_L}$, $\mathcal{A}_{H_R}$, respectively. Combining these, we conclude that

$$\omega_\Phi \simeq (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta. \tag{6.9}$$

Repeated use of (6.7) gives

$$\omega_\Phi \circ \left(\gamma_g^L \gamma_h^L \left(\gamma_{gh}^L\right)^{-1} \otimes \gamma_g^R \gamma_h^R \left(\gamma_{gh}^R\right)^{-1}\right) \simeq \omega_\Phi. \tag{6.10}$$

Applying (6.9) to this, we obtain

$$\begin{aligned}(\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta \circ \left(\gamma_g^L \gamma_h^L \left(\gamma_{gh}^L\right)^{-1} \otimes \gamma_g^R \gamma_h^R \left(\gamma_{gh}^R\right)^{-1}\right) \\ \simeq (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta.\end{aligned} \tag{6.11}$$

Note that

$$\gamma_g^R \gamma_h^R \left(\gamma_{gh}^R\right)^{-1} = \left(\beta_g^{UR} \eta_{g,R}^{-1} \left(\beta_g^{UR}\right)^{-1}\right)\left(\beta_{gh}^{UR} \eta_{h,R}^{-1} \eta_{gh,R} \left(\beta_{gh}^{UR}\right)^{-1}\right) \in \mathrm{Aut}(\mathcal{A}_{C_{\theta,R}}). \tag{6.12}$$

Similarly, we have $\gamma_g^L \gamma_h^L (\gamma_{gh}^L)^{-1} \in \mathrm{Aut}(\mathcal{A}_{C_{\theta,L}})$. Therefore, they commute with $\Theta \in \mathrm{Aut}(\mathcal{A}_{C_\theta^c})$. From this and (6.11), we obtain

$$(\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \left(\gamma_g^L \gamma_h^L \left(\gamma_{gh}^L\right)^{-1} \otimes \gamma_g^R \gamma_h^R \left(\gamma_{gh}^R\right)^{-1}\right) \simeq (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R),$$

which implies

$$\omega_R \alpha_R \gamma_g^R \gamma_h^R \left(\gamma_{gh}^R\right)^{-1} \simeq \omega_R \alpha_R. \tag{6.13}$$

Recall from Section 3 that this means the automorphism $\gamma_g^R \gamma_h^R (\gamma_{gh}^R)^{-1}$ is implementable by a unitary $u(g,h)$ in the GNS representation $(\mathcal{H}_R, \pi_R)$ of $\omega_R \alpha_R$, i.e.,

$$\mathrm{Ad}\big(u(g,h)\big)\pi_R = \pi_R \gamma_g^R \gamma_h^R \left(\gamma_{gh}^R\right)^{-1}. \tag{6.14}$$

Note also that (6.9) and (6.7) imply

$$(\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta \circ \left(\gamma_g^L \otimes \gamma_g^R\right) \simeq (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta. \tag{6.15}$$

Therefore, with $(\mathcal{H}_L, \pi_L)$ a GNS representation of $\omega_L \alpha_L$, there is a unitary W_g on $\mathcal{H}_L \otimes \mathcal{H}_R$ implementing $\Theta \circ (\gamma_g^L \otimes \gamma_g^R) \circ \Theta^{-1}$ in the GNS representation $(\mathcal{H}_L \otimes \mathcal{H}_R, \pi_L \otimes \pi_R)$ of $\omega_L \alpha_L \otimes \omega_R \alpha_R$, i.e.,

$$\mathrm{Ad}(W_g)(\pi_L \otimes \pi_R) = (\pi_L \otimes \pi_R) \circ \Theta \circ \left(\gamma_g^L \otimes \gamma_g^R\right) \circ \Theta^{-1}. \tag{6.16}$$

For these $u(g,h)$ (6.14) and W_g (6.16), we claim that there are $c(g,h,k) \in \mathrm{U}(1)$ such that

$$\begin{aligned}&\mathrm{Ad}(W_g)\big(\mathbb{I}_L \otimes u(h,k)\big) \cdot \big(\mathbb{I}_L \otimes u(g,hk)\big) \\ &\quad = c(g,h,k)\big(\mathbb{I}_L \otimes u(g,h)u(gh,k)\big), \quad g,h,k \in G.\end{aligned} \tag{6.17}$$

To see this, consider $\pi_L \otimes \pi_R \gamma_g^R \gamma_h^R \gamma_k^R$. On the one hand, with the repeated use of (6.14), we have

$$\begin{aligned}\pi_L \otimes \pi_R \gamma_g^R \gamma_h^R \gamma_k^R &= \mathrm{Ad}\big(\mathbb{I}_L \otimes u(g,h)\big)\big(\pi_L \otimes \pi_R \gamma_{gh}^R \gamma_k^R\big) \\ &= \mathrm{Ad}\big(\mathbb{I}_L \otimes u(g,h)u(gh,k)\big)\big(\pi_L \otimes \pi_R \circ \gamma_{ghk}^R\big).\end{aligned} \tag{6.18}$$

On the other hand, note that both of $\gamma_h^R\gamma_k^R(\gamma_{hk}^R)^{-1}$ and $\gamma_g^R(\gamma_h^R\gamma_k^R(\gamma_{hk}^R)^{-1})(\gamma_g^R)^{-1}$ commute with Θ as before. Hence we have

$$\begin{aligned}&\mathrm{id}_L\otimes\gamma_g^R\big(\gamma_h^R\gamma_k^R\big(\gamma_{hk}^R\big)^{-1}\big)\big(\gamma_g^R\big)^{-1}\\&\quad=\Theta\big(\gamma_g^L\otimes\gamma_g^R\big)\Theta^{-1}\big(\mathrm{id}_L\otimes\gamma_h^R\gamma_k^R\big(\gamma_{hk}^R\big)^{-1}\big)\Theta\big(\gamma_g^L\otimes\gamma_g^R\big)^{-1}\Theta^{-1}.\end{aligned}\tag{6.19}$$

From this and repeated use of (6.14), (6.16), we have

$$\begin{aligned}&\pi_L\otimes\pi_R\gamma_g^R\gamma_h^R\gamma_k^R\\&\quad=(\pi_L\otimes\pi_R)\Theta\big(\gamma_g^L\otimes\gamma_g^R\big)\Theta^{-1}\big(\mathrm{id}_L\otimes\gamma_h^R\gamma_k^R\big(\gamma_{hk}^R\big)^{-1}\big)\Theta\big(\gamma_g^L\otimes\gamma_g^R\big)^{-1}\\&\qquad\times\Theta^{-1}\big(\mathrm{id}_L\otimes\gamma_g^R\gamma_{hk}^R\big)\\&\quad=\mathrm{Ad}\big(W_g\big(\mathbb{I}_L\otimes u(h,k)\big)W_g^*\big(\mathbb{I}_L\otimes u(gh,k)\big)\big)\big(\pi_L\otimes\pi_R\gamma_{ghk}^R\big).\end{aligned}\tag{6.20}$$

Comparing this and (6.18), we have

$$\begin{aligned}&\mathrm{Ad}\big(\mathbb{I}_L\otimes u(g,h)u(gh,k)\big)(\pi_L\otimes\pi_R)\\&\quad=\mathrm{Ad}\big(W_g\big(\mathbb{I}_L\otimes u(h,k)\big)W_g^*\big(\mathbb{I}_L\otimes u(gh,k)\big)\big)(\pi_L\otimes\pi_R).\end{aligned}\tag{6.21}$$

Note that, because $(\mathcal{H}_L\otimes\mathcal{H}_R,\pi_L\otimes\pi_R)$ is a GNS representation of a pure state $\omega_L\alpha_L\otimes\omega_R\alpha_R$, $(\pi_L\otimes\pi_R)(\mathcal{A}_{\mathbb{Z}^2})$ is dense in $\mathcal{B}(\mathcal{H}_L\otimes\mathcal{H}_R)$ with the strong operator topology. As a result, (6.21) implies our claim (6.17).

The situation in (6.14), (6.17) is pretty much similar to that of cocycle actions [15,24]. In fact, following the argument in [24], we can show that $c(g,h,k)$ satisfies the 3-cocycle relation. Hence, out of it, we obtain an $H^3(G,\mathrm{U}(1))$-valued index. Using the automorphic equivalence Theorem 5.1 and the factorization property of the automorphism therein, one can show that it is in fact an invariant of our classification $\sim_\beta$.

A derivation of indices for SPT-phases was initially carried out in tensor network models, matrix product states MPS [52–54] in one dimension, and projected entangled pair states [32]. Our indices coincide with theirs in those models. In other words, thanks to those works, there are many examples. Our approach introduced in this section is operator algebraic. Recently, some quantum information based approaches were reported [25,55].

7. ANYONS IN TOPOLOGICAL PHASES

In this section, we consider the classification $\sim_{\mathrm{l.u.}}$ in two dimensions. Recall that states which are equivalent to an infinite tensor product state with respect to $\sim_{\mathrm{l.u.}}$ are said to have a short-range entanglement, and otherwise they are said to have a long-range entanglement. It is frequently said that in the two-dimensional systems, the existence of an "anyon" means the long-range entanglement of the state [28]. In this section, we formulate this statement in our operator-algebraic setting.

An anyon is a string-like excitation with a braiding structure. How to formulate an anyon mathematically is a nontrivial question of mathematical physics. Our answer, motivated by AQFT [27] and studies of Kitaev models [10,19,34,35] is that it is a superselection sector. It is defined in terms of cones. By a cone we mean a subset of $\mathbb{Z}^2$ of the form

$$\Lambda_{\boldsymbol{a},\theta,\varphi} := \left\{ \boldsymbol{x} \in \mathbb{Z}^2 \mid (\boldsymbol{x} - \boldsymbol{a}) \cdot \boldsymbol{e}_\theta > \cos\varphi \cdot \|\boldsymbol{x} - \boldsymbol{a}\| \right\},$$

with some $\boldsymbol{a} \in \mathbb{R}$, $\theta \in \mathbb{R}$, and $\varphi \in (0, \pi)$. Here we set $\boldsymbol{e}_\theta := (\cos\theta, \sin\theta)$. For a cone $\Lambda := \Lambda_{\boldsymbol{a},\theta,\varphi}$ and $\boldsymbol{b} \in \mathbb{R}^2$, $\varepsilon > 0$, we set $\Lambda_\varepsilon + \boldsymbol{b} := \Lambda_{\boldsymbol{a}+\boldsymbol{b},\theta,\varphi+\varepsilon}$, $|\arg\Lambda| := 2\varphi$, and $\boldsymbol{e}_\Lambda := \boldsymbol{e}_\theta$.

Definition 7.1. Let $(\mathcal{H}, \pi_0)$ be an irreducible representation of $\mathcal{A}_{\mathbb{Z}^2}$. We say that a representation π of $\mathcal{A}_{\mathbb{Z}^2}$ on $\mathcal{H}$ satisfies the superselection criterion for π_0 if

$$\pi|_{\mathcal{A}_{\Lambda^c}} \simeq_{u.e.} \pi_0|_{\mathcal{A}_{\Lambda^c}},$$

for any cone Λ in $\mathbb{Z}^2$. (Here, $\simeq_{u.e.}$ means that the two representations are unitarily equivalent.) Such representations are called superselection sectors for π_0.

Superselection sectors are objects studied extensively in AQFT. In the context of quantum spin systems, P. Naaijkens and his coauthors carried out studies on Kitaev's quantum double model from the point of view of superselection sectors [10, 19, 34, 35], where they drove a braiding structure.

We can see the importance of the sector theory for us from the fact that it is an invariant of $\sim_{\mathrm{l}.u.}$.

Theorem 7.1 ([36]). *Let $(\mathcal{H}, \pi_0)$ be an irreducible representation and let $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^2})$. Suppose that a representation π satisfies the superselection criterion for π_0. Then $\pi \circ \alpha$ satisfies the superselection criterion for $\pi_0 \circ \alpha$.*

Let ω_1, ω_0 be pure states such that $\omega_1 \sim_{\mathrm{l}.u.} \omega_0$ with $\omega_1 = \omega_0 \circ \alpha$, $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^2})$. Then, by Theorem 7.1, α gives a bijection between the set of all superselection sectors of π_{ω_0} and the set of all superselection sectors of π_{ω_1}.

The proof of Theorem 7.1 is a simple argument using the factorization property. For $\varepsilon > 0$, analogous to (4.3), we have a decomposition

$$\alpha = \mathrm{Ad}(v) \circ \Xi \circ (\alpha_\Lambda \otimes \alpha_{\Lambda^c}), \tag{7.1}$$

where $\alpha_\Lambda, \alpha_{\Lambda^c}, \Xi$ are automorphisms on $\mathcal{A}_\Lambda, \mathcal{A}_{\Lambda^c}, \mathcal{A}_{\Lambda_\varepsilon}$, respectively. (We choose $\varepsilon > 0$ small enough so that Λ_ε is still a cone.) Then for a superselection sector π for π_0, we have

$$\begin{aligned} \pi \circ \alpha|_{\mathcal{A}_\Lambda} &\sim_{\mathrm{u}.e.} \pi \circ \Xi \circ \alpha_\Lambda|_{\mathcal{A}_\Lambda} = \pi|_{\mathcal{A}_{\Lambda_\varepsilon}} \circ \Xi \circ \alpha_\Lambda|_{\mathcal{A}_\Lambda} \\ &\sim_{\mathrm{u}.e.} \pi_0|_{\mathcal{A}_{\Lambda_\varepsilon}} \circ \Xi \circ \alpha_\Lambda|_{\mathcal{A}_\Lambda} \sim_{\mathrm{u}.e.} \pi_0 \circ \alpha|_{\mathcal{A}_\Lambda}, \end{aligned} \tag{7.2}$$

proving the claim.

We say that π_0 has a trivial sector theory if any representation satisfying the superselection criterion for π_0 is quasiequivalent to π_0. Otherwise, we say π_0 has a nontrivial sector theory. One can show that for a pure state of infinite tensor product form, its GNS representation has a trivial sector theory [36]. Combing this and Theorem 7.1, we obtain the following.

Corollary 7.1. *If a pure state has a short-range entanglement, then its GNS representation has a trivial sector theory.*

In other words, the existence of nontrivial superselection sectors implies the long-range entanglement. If we regard superselection sectors as anyons, it is a mathematical realization of the folklore saying that the existence of anyons implies long-range entanglement of the state.

The reason why we expect superselection sectors to be related to anyons comes from AQFT. Using the tools from AQFT, in [11] Cha–Naaijkens–Nachtergaele derived a braiding structure in a general setting of semigroup of almost localized endomorphisms in quantum spin systems. It is well known that anyons show up in AQFT surprisingly naturally [5, 8, 9, 16, 20, 27]. More precisely, under some condition called Haag duality, a braided C^*-tensor category can be associated to the irreducible representation with nontrivial sector theory. The Haag duality is the property $\pi_0(\mathcal{A}_{\Lambda^c})' = \pi_0(\mathcal{A}_\Lambda)''$, for all cones Λ in $\mathbb{Z}^2$.

The problem for us about introducing this condition in quantum spin systems is that it does not look to be plausible that this condition is stable under automorphisms in $\mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^2})$. Recalling that automorphisms in $\mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^2})$ are the fundamental operations in the classification problem of gapped ground state phases, this situation is not convenient for us. For this reason, we introduce a weaker version of Haag duality.

Definition 7.2 (Approximate Haag duality [49]). Let $(\mathcal{H}, \pi_0)$ be an irreducible representation of $\mathcal{A}_{\mathbb{Z}^2}$. We say that $(\mathcal{H}, \pi_0)$ satisfies the approximate Haag duality if the following conditions hold: For any $\varphi \in (0, 2\pi)$ and $\varepsilon > 0$ with $\varphi + 4\varepsilon < 2\pi$, there is some $R_{\varphi,\varepsilon} > 0$ and decreasing functions $f_{\varphi,\varepsilon,\delta}(t)$, $\delta > 0$ on $\mathbb{R}_{\geq 0}$ with $\lim_{t\to\infty} f_{\varphi,\varepsilon,\delta}(t) = 0$ such that

(i) for any cone Λ with $|\arg \Lambda| = \varphi$, there is a unitary $U_{\Lambda,\varepsilon} \in \mathcal{U}(\mathcal{H})$ satisfying
$$\pi_0(\mathcal{A}_{\Lambda^c})' \subset \mathrm{Ad}(U_{\Lambda,\varepsilon})\big(\pi_0(\mathcal{A}_{(\Lambda - R_{\varphi,\varepsilon}\boldsymbol{e}_\Lambda)_\varepsilon})''\big), \tag{7.3}$$
and

(ii) for any $\delta > 0$ and $t \geq 0$, there is a unitary $\tilde{U}_{\Lambda,\varepsilon,\delta,t} \in \pi_0(\mathcal{A}_{\Lambda_{\varepsilon+\delta} - t\boldsymbol{e}_\Lambda})''$ satisfying
$$\|U_{\Lambda,\varepsilon} - \tilde{U}_{\Lambda,\varepsilon,\delta,t}\| \leq f_{\varphi,\varepsilon,\delta}(t). \tag{7.4}$$

The good point about this weaker version is that we know it is stable under automorphisms in $\mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^2})$.

Proposition 7.1. *Let $(\mathcal{H}, \pi_0)$ be an irreducible representation of $\mathcal{A}_{\mathbb{Z}^2}$ satisfying the approximate Haag duality. Then for any automorphism $\alpha \in \mathrm{QAut}(\mathcal{A}_{\mathbb{Z}^2})$, $(\mathcal{H}, \pi_0 \circ \alpha)$ also satisfies the approximate Haag duality.*

It turns out that even with this weaker version of Haag duality and the setting of gapped ground state phases (which is different from that of AQFT), we can still derive a braided C^*-tensor category (see [41] for the definition) out of superselection sectors where, unlike endomorphisms, the multiplication rule is not a priori given [49]. The proof is a modification of the argument in AQFT and some additional argument using the gap condition Definition 3.1. More precisely, let Φ be a uniformly bounded finite range interaction on $\mathcal{A}_{\mathbb{Z}^2}$ with gapped ground states. Let ω be a pure τ_Φ-ground state with a GNS representation

$(\mathcal{H}, \pi_0, \Omega)$. We assume that π_0 has a nontrivial sector theory, and π_0 satisfies the approximate Haag duality. Fix some $\theta \in \mathbb{R}$ and $\varphi \in (0, \pi)$, and denote by $\mathcal{C}_{(\theta,\varphi)}$ the set of all cones whose angle does not intersects with $[\theta - \varphi, \theta + \varphi]$. We set

$$\mathcal{B}_{(\theta,\varphi)} := \overline{\bigcup_{\Lambda \in \mathcal{C}_{(\theta,\varphi)}} \pi_0(\mathcal{A}_{\Lambda^c})'}. \tag{7.5}$$

Here $\bar{\cdot}$ denotes the norm closure. Using the approximate Haag duality, using the argument in [9], each superselection sector $\rho : \mathcal{A}_{\mathbb{Z}^2} \to \mathcal{B}(\mathcal{H}_\omega)$ for π_0 extends to an endomorphism on $\mathcal{B}_{(\theta,\varphi)}$. We denote the extension by the same symbol ρ. Via these extensions, we can introduce compositions between superselection sectors. With this composition as a tensor, the superselection sectors of π_0 are the objects of our braided C^*-tensor category. Our morphisms are given by the intertwiners. Namely, for objects ρ, σ, the morphisms from ρ to σ are bounded operators R on $\mathcal{H}$ such that $R\rho(A) = \sigma(A)R$, for any $A \in \mathcal{A}_{\mathbb{Z}^2}$. The set of all morphisms from ρ to σ is denoted by (ρ, σ). Note that (ρ, σ) is a Banach space and (ρ, ρ) is a C^*-algebra. Following AQFT, the tensor of morphisms $R_1 \in (\rho_1, \sigma_1)$, $R_2 \in (\rho_2, \sigma_2)$ are defined by

$$R_1 \otimes R_2 := R_1 \rho_1(R_2) \in (\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2). \tag{7.6}$$

In fact, each intertwiner belongs to $\mathcal{B}_{(\theta,\varphi)}$ such that $\rho_1(R_2)$ is well-defined. Using the gap inequality and the nontriviality of the sector theory, we can show for any cone Λ that $\pi_0(\mathcal{A}_\Lambda)''$ is either type II_∞ or type III factor. It means that there are isometries $u_\Lambda, v_\Lambda \in \pi_0(\mathcal{A}_\Lambda)''$ such that $u_\Lambda u_\Lambda^* + v_\Lambda v_\Lambda^* = \mathbb{I}$. Using this, for any superselection sectors ρ, σ, we can define their direct sum $\rho \bigoplus \sigma : \mathcal{A}_{\mathbb{Z}^2} \to \mathcal{B}(\mathcal{H}_0)$ by

$$\left(\rho \bigoplus \sigma\right)(A) := u_\Lambda \rho(A) u_\Lambda^* + v_\Lambda \sigma(A) v_\Lambda^*, \quad A \in \mathcal{A}_{\mathbb{Z}^2}. \tag{7.7}$$

From the same fact, we can also define subobjects. Namely, if $p \in (\rho, \rho)$ is a nonzero projection, we can find some superselection sector σ and an isometry v such that $vv^* = p$ and $\rho(A)v = v\sigma(A)$ for all $A \in \mathcal{A}_{\mathbb{Z}^2}$. Hence we obtain the following theorem.

Theorem 7.2 ([49]). *In the above setting, superselection sectors of π_0 form a braided C^*-tensor category. If two of such states $\omega_{\Phi_1}, \omega_{\Phi_2}$ satisfy $\omega_{\Phi_1} \sim_{\mathrm{l.u.}} \omega_{\Phi_2}$, then corresponding braided C^*-tensor categories are monoidally equivalent.*

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