HYDRODYNAMIC LIMIT AND STOCHASTIC PDES RELATED TO INTERFACE MOTION

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ABSTRACT

The hydrodynamic limit gives a link between microscopic and macroscopic systems via a space–time scaling. Its notable feature is the averaging effect due to the local ergodicity under the local equilibria. In this article, as the microscopic system, we consider several types of interacting particle systems, in which particles perform random walks with interaction. We derive, under the hydrodynamic limit or its nonlinear fluctuation limit, three different objects: the motion by mean curvature, Stefan free boundary problem, and coupled KPZ equation. These are all related to the interface motion. The Boltzmann–Gibbs principle plays a fundamental role. We discuss the coupled KPZ equation from the aspect of singular SPDEs and renormalizations. Ginzburg–Landau $\nabla\phi$-interface model, stochastic motion by mean curvature, and stochastic eight-vertex model are also briefly discussed.

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1. INTRODUCTION

The hydrodynamic limit is a scaling limit in space and time for interacting systems at the microscopic level, and leads to macroscopic evolutional rules usually prescribed by nonlinear PDEs, via an averaging effect due to the local ergodicity in local equilibria, cf. [17,44,48]. It is formulated as a law of large numbers. Its fluctuation limit is also studied, and we obtain linear or nonlinear stochastic PDEs (SPDEs) in the limit.

In this review article, we discuss the derivation of three different objects from interacting particle systems: the motion by mean curvature (MMC, Section 2.1), Stefan free boundary problem (Section 3) and coupled Kardar–Parisi–Zhang (KPZ) equation (Section 4.2). We also discuss the coupled KPZ equation from the aspect of singular SPDEs (Section 4.1). This is an ill-posed equation in a classical sense and requires renormalizations.

We consider particle systems, in which each particle moves performing a random walk and interacting with other particles on the d-dimensional discrete torus $\mathbb{T}_N^d = \{1, 2, \dots, N\}^d$ (with periodic boundary) with large N. Specifically, we consider a zero-range process, in which several particles may occupy each site of $\mathbb{T}_N^d$ and interact only at the same site, or Kawasaki dynamics sometimes called exclusion process, in which particles obey the hard-core exclusion rule so that at most one particle can occupy each site. To derive Stefan problem or coupled KPZ equation, we consider multiple types of particles. In addition, we introduce the Glauber mechanism, which governs the creation and annihilation of particles. More precisely, we consider both creation and annihilation for MMC problem, annihilation only for Stefan problem, and neither creation nor annihilation for the coupled KPZ problem.

Our problems have a common feature that relates to the interface motion. The system leading to MMC exhibits a phase separation to sparse and dense regions of particles and, macroscopically, the interface is created to separate these two phases and evolves under the MMC, while, in that leading to the Stefan problem, we observe the segregation of different species. Scalar KPZ equation was originally introduced as an equation for a growing interface. Technically, the so-called Boltzmann–Gibbs principle plays a fundamental role.

2. MOTION BY MEAN CURVATURE

2.1. From particle systems

Here, to illustrate the idea and the results, we take Glauber–zero-range process as a microscopic model based on El Kettani et al. [7,8]. Instead of zero-range process, one can take simple Kawasaki dynamics (independent random walks with exclusion rule, [26,43]) or Kawasaki dynamics with speed change (see [21]).

Properly tuning the Glauber part, the system exhibits phase separation and one can derive the MMC as a macroscopic evolutional rule for the phase separation surface. Our method is a combination of the techniques of the hydrodynamic limit, based on the relative entropy method (Proposition 2.3) and Boltzmann–Gibbs principle (Theorem 2.6), and the

PDE technique called the sharp interface limit (Proposition 2.7) and the discrete Schauder estimate (Proposition 2.5).

2.1.1. Glauber–zero-range process and hydrodynamic limit with fixed K

Glauber–zero-range process on $\mathbb{T}_N^d$ is the Markov process $\eta^N(t) = \{\eta_x^N(t)\}_{x\in\mathbb{T}_N^d}$ on the configuration space $\mathcal{X}_N = \mathbb{Z}_+^{\mathbb{T}_N^d}$ ($\mathcal{X}_N = \{0,1\}^{\mathbb{T}_N^d}$ in Kawasaki case) with the generator given by $L_N = N^2 L_Z + K L_G$ with $K > 0$, where

$$(L_Z f)(\eta) = \sum_{x\in\mathbb{T}_N^d} \sum_{e\in\mathbb{Z}^d:|e|=1} g(\eta_x)\{f(\eta^{x,x+e}) - f(\eta)\},$$
$$(L_G f)(\eta) = \sum_{x\in\mathbb{T}_N^d} \sum_{\pm} c_x^{\pm}(\eta)\{f(\eta^{x,\pm}) - f(\eta)\},$$

for $\eta = \{\eta_x\}_{x\in\mathbb{T}_N^d} \in \mathcal{X}_N$ and functions f on $\mathcal{X}_N$. Here, $\eta_x \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$ denotes the number of particles at x, $\eta^{x,y}$ is η after one particle jumps from x to y, $\eta^{x,+}$ is η after one particle is created at x, and $\eta^{x,-}$ is η after one particle is annihilated at x.

The flip rates of the Glauber part are shift-invariant, that is, $c_x^{\pm}(\eta) = c^{\pm}(\tau_x\eta)$ with the creation and annihilation rates $c^{\pm}(\eta)(= c_0^{\pm}(\eta))$ of a particle at $x = 0$ and the spatial shift τ_x acting on $\mathcal{X}_N$. We assume $c^-(\eta) = 0$ if $\eta_0 = 0$. The jump rate $g(k)$, $k \in \mathbb{Z}_+$, of the zero-range part is bounded from above and below by linear functions of k. In particular, $g(0) = 0$.

The invariant measures or equilibrium states, being shift-invariant in space, of the zero-range process, that is, the leading part of our dynamics, are superpositions of the product measures $\bar\nu_\varphi$ on $\mathcal{X}_N$ (or on $\mathcal{X} = \mathbb{Z}_+^{\mathbb{Z}^d}$) with one-site marginal distribution given by

$$\bar\nu_\varphi(k) = \frac{1}{Z_\varphi}\frac{\varphi^k}{g(k)!}, \quad Z_\varphi = \sum_{k=0}^{\infty}\frac{\varphi^k}{g(k)!},$$

with parameter $\varphi \ge 0$ called fugacity, where $g(k)! = \prod_{i=1}^k g(i)$, $k \ge 1$, and $g(0)! = 1$. We denote $\nu_\rho := \bar\nu_{\varphi(\rho)}$ by changing the parameter with its mean $\rho \ge 0$. In fact, ρ and $\varphi = \varphi(\rho)$ are related by $\rho = \varphi(\log Z_\varphi)' = E^{\bar\nu_\varphi}[k] \equiv E^{\bar\nu_\varphi}[\eta_0]$, and $\varphi = E^{\bar\nu_\varphi}[g(k)]$ holds.

The macroscopic empirical measure (density field of particles) on $\mathbb{T}^d$ $(= [0,1)^d$ with periodic boundary), which is the macroscopic region corresponding to microscopic $\mathbb{T}_N^d$, associated with the configuration $\eta \in \mathcal{X}_N$, is defined by

$$\alpha^N(dv;\eta) = \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} \eta_x \delta_{\frac{x}{N}}(dv), \quad v \in \mathbb{T}^d, \tag{2.1}$$

or equivalently, for a test function $G \in C^\infty(\mathbb{T}^d)$,

$$\langle \alpha^N(\cdot;\eta), G\rangle = \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} \eta_x G\left(\frac{x}{N}\right). \tag{2.2}$$

Thus, the scaling from micro to macro is given by $\frac{1}{N}$ in space, $\frac{1}{N^d}$ in mass, as well as N^2 (for the zero-range part) and K (for the Glauber part) in time. Our problem is to study the limit as $N \to \infty$.

For a fixed K, one can expect that the hydrodynamic limit holds, that is,

$$\alpha^N\big(dv;\eta^N(t)\big) \to \rho(t,v)dv \quad \text{as } N\to\infty$$

holds in probability multiplying a test function G on $\mathbb{T}^d$, if this holds at $t=0$, where $\rho(t,v)$ is a unique weak solution of the reaction–diffusion equation with a nonlinear diffusion term

$$\partial_t\rho = \Delta\varphi(\rho) + Kf(\rho), \quad v\in\mathbb{T}^d, \tag{2.3}$$

with initial value $\rho(0)$ and

$$f(\rho) = E^{\nu_\rho}\big[c^+(\eta) - c^-(\eta)\big]. \tag{2.4}$$

Recall that $\varphi(\rho) = E^{\nu_\rho}[g]$ and Δ is the Laplacian on $\mathbb{T}^d$. This was shown for Glauber–Kawasaki dynamics in [5], and the result for the zero-range process without Glauber part (i.e., when $K=0$) is found in [44]. See Section 4.2.2 for some related heuristic arguments to derive (2.3).

2.1.2. Mesoscopic Glauber perturbation and derivation of MMC

We consider the Glauber–zero-range process $\eta^N(t)$, that is, the $\mathcal{X}_N$-valued process with generator $L_N = N^2L_Z + KL_G$, now with $K=K(N)\to\infty$. One can construct flip rates $c^\pm(\eta)$ of the Glauber part in such a way that the corresponding f determined by (2.4) is bistable, that is, f has exactly three zeros $0<\alpha_1<\alpha_*<\alpha_2<\infty$ and $f'(\alpha_1)<0$, $f'(\alpha_2)<0$ hold, and satisfies the φ-balance condition $\int_{\alpha_1}^{\alpha_2} f(\rho)\varphi'(\rho)d\rho = 0$. We actually take $c_x^+(\eta) = \frac{\hat{c}^+(\tau_x\eta)}{g(\eta_x+1)}$ and $c_x^-(\eta) = \hat{c}^-(\tau_x\eta)1_{\{\eta_x\ge 1\}}$, where $\hat{c}^\pm(\eta)$ are nonnegative local functions on $\mathcal{X} = \mathbb{Z}_+^{\mathbb{Z}^d}$ (regarded as those on $\mathcal{X}_N$), which do not depend on η_0. Microscopically, there are two phases: sparse phase (with density α_1 of particles) and dense phase (density α_2). Macroscopically, these two phases are separated by an interface Γ_t in $\mathbb{T}^d$. The creation and annihilation mechanism at the microscopic level forces the macroscopic density to one of those two stable phases.

For a function $u = \{u(\frac{x}{N})\}_{x\in\mathbb{T}_N^d}$, we define the local equilibrium state ν_u as the product measure on $\mathcal{X}_N$ defined by $\nu_u(d\eta) = \prod_{x\in\mathbb{T}_N^d}\nu_{u(\frac{x}{N})}(d\eta_x)$. For two probability measures μ and ν, the relative entropy of μ with respect to ν is defined by

$$H(\mu|\nu) := \int \frac{d\mu}{d\nu}\log\frac{d\mu}{d\nu}\cdot d\nu.$$

For the initial distribution μ_0^N of $\eta^N(0)$, we assume $H(\mu_0^N|\nu_0^N) = O(N^{d-\varepsilon_0})$ with some $\varepsilon_0>0$, where $\nu_0^N = \nu_{u^N(0)}$ for some $u^N(0) = \{u^N(0,\frac{x}{N})\}_{x\in\mathbb{T}_N^d}$ which satisfies

- $u^N(0,\frac{x}{N}) = u_0(\frac{x}{N})$, $x\in\mathbb{T}_N^d$, with some $u_0\in C^5(\mathbb{T}^d)$ such that $u_0>0$;
- $\Gamma_0 := \{v\in\mathbb{T}^d; u_0(v)=\alpha_*\}$ is a $(d-1)$-dimensional $C^{5+\theta}$-hypersurface, $\theta>0$, without boundary in $\mathbb{T}^d$ and ∇u_0 is nondegenerate in the normal direction to Γ_0.

Theorem 2.1 ([7]). *We assume $d\ge 2$, the above conditions, and that $K(N)$ diverges to ∞ satisfying $1\le K(N)\le \delta_0(\log N)^{\frac{\sigma}{2}}$ with small enough $\delta_0>0$ and the Hölder exponent $\sigma\in$*

$(0,1)$ *determined by Nash estimate, see Proposition 2.5. Let* $\alpha^N(t,dv) := \alpha^N(dv;\eta^N(t))$ *be the macroscopic empirical measure associated with* $\eta^N(t)$. *Then, we have for* $t \in (0,T]$,

$$\alpha^N(t) \to \chi_{\Gamma_t} := \begin{cases} \alpha_1, & \text{on one side of } \Gamma_t, \\ \alpha_2, & \text{on the other side of } \Gamma_t, \end{cases} \tag{2.5}$$

in probability, where the hypersurface Γ_t *in* $\mathbb{T}^d$ *moves according to the MMC,* $V = \lambda_0 \kappa$.

Here, κ *is the mean curvature of* Γ_t *multiplied by* $d-1$ *and* V *is the normal velocity of* Γ_t *from the* α_1*-side to* α_2*-side. The sides of* Γ_t *are determined continuously from* Γ_0. *We assume* Γ_t *is* $C^{5+\theta}$ *for* $t \le T$.

The constant λ_0 *is determined by the homogenization effect from the nonlinear Laplacian and given by*

$$\lambda_0 = \frac{\int_{\alpha_1}^{\alpha_2} \varphi'(u)\sqrt{W(u)}du}{\int_{\alpha_1}^{\alpha_2} \sqrt{W(u)}du}$$

with the potential defined by $W(u) = \int_u^{\alpha_2} f(s)\varphi'(s)ds$, $u > 0$. *Note that* $\lambda_0 = 1$ *if* $g(k) = k$ *so that* φ *is linear,* $\varphi(u) = u$.

2.1.3. Proof of Theorem 2.1

(a) Probabilistic part. Let μ_t^N be the distribution of $\eta^N(t)$ on $\mathcal{X}_N$. Let $u^N(t) = \{u^N(t,\frac{x}{N})\}_{x\in\mathbb{T}_N^d}$ be the solution of the quasilinear discrete PDE (ODE):

$$\partial_t u^N\left(t,\frac{x}{N}\right) = \Delta^N \varphi\left(u^N\left(t,\frac{x}{N}\right)\right) + Kf\left(u^N\left(t,\frac{x}{N}\right)\right), \tag{2.6}$$

with initial value $u^N(0)$, where Δ^N is the discrete Laplacian defined by

$$\Delta^N \psi\left(\frac{x}{N}\right) = N^2 \sum_{y\in\mathbb{T}_N^d:|y-x|=1} \left(\psi\left(\frac{y}{N}\right) - \psi\left(\frac{x}{N}\right)\right), \tag{2.7}$$

for $\psi = \{\psi(\frac{x}{N})\}_{x\in\mathbb{T}_N^d}$. Note that (2.6) is a discretized version of (2.3). Let $\nu_t^N = \nu_{u^N(t)}$ be the local equilibrium state on $\mathcal{X}_N$ with mean density $\{u^N(t,\frac{x}{N})\}_{x\in\mathbb{T}_N^d}$.

The main estimate in the probabilistic part is the following:

Theorem 2.2. *Under the condition* $H(\mu_0^N|\nu_0^N) = O(N^{d-\varepsilon_0})$ *for some* $\varepsilon_0 > 0$, *if* $1 \le K(N) \le \delta_0(\log N)^{\frac{\sigma}{2}}$ *with small enough* $\delta_0 > 0$, *then we have* $H(\mu_t^N|\nu_t^N) = o(N^d)$ *as* $N \to \infty$.

Once this is shown, one can show that $\alpha^N(t)$ is close to $u^N(t)$ in the sense that

$$\lim_{N\to\infty} \mu_t^N(\mathcal{A}_{N,t}) = 0, \tag{2.8}$$

for the event $\mathcal{A}_{N,t} := \{\eta \in \mathcal{X}_N;\ |\langle \alpha^N, G\rangle - \langle u^N(t,\cdot), G\rangle| > \delta\}$ and every $\delta > 0$ and $G \in C^\infty(\mathbb{T})$. Indeed, we may combine the entropy inequality, $\mu(A) \le \frac{\log 2 + H(\mu|\nu)}{\log(1+1/\nu(A))}$, and the large deviation estimate for the product measure ν_t^N, $\nu_t^N(\mathcal{A}_{N,t}) \le e^{-CN^d}$ for some $C = C_{\delta,G} > 0$.

The proof of Theorem 2.2 is divided into five steps.

(1) The time derivative of the relative entropy.

Proposition 2.3 ([33, 40, 51]). *Let m be a reference measure on $\mathcal{X}_N$ with full support and set $\psi_t^N = \frac{d\nu_t^N}{dm}$. Then, we have*

$$\partial_t H(\mu_t^N | \nu_t^N) \le -N^2 \mathfrak{D}\left(\sqrt{\frac{d\mu_t^N}{d\nu_t^N}}; \nu_t^N\right) + \int_{\mathcal{X}_N} \{L_N^{*,\nu_t^N} 1 - \partial_t \log \psi_t^N\} d\mu_t^N,$$

where $L^{*,\nu}$ denotes the adjoint of L on $L^2(\nu)$ in general, and $\mathfrak{D}(f;\nu) \ge 0$ is the Dirichlet form associated with L_Z, which may be dropped since we actually do not use it.

(2) Computation of $L_N^{*,\nu} 1$ and $\partial_t \log \psi_t^N$. We write $\sum_x$ for $\sum_{x \in \mathbb{T}_N^d}$ for simplicity.

Lemma 2.4. *Let $\nu = \nu_u$ and $\varphi(\frac{x}{N}) = \varphi(u(\frac{x}{N}))$ in the following first two equalities. Then,*

$$N^2 L_Z^{*,\nu} 1 = \sum_x \frac{(\Delta^N \varphi)(\frac{x}{N})}{\varphi(\frac{x}{N})} \left\{ g(\eta_x) - \varphi\left(\frac{x}{N}\right) \right\},$$

$$L_G^{*,\nu} 1 = \sum_x \left\{ \hat{c}^+(\tau_x \eta) \left(\frac{1(\eta_x \ge 1)}{\varphi(\frac{x}{N})} - \frac{1}{g(\eta_x + 1)} \right) + \hat{c}^-(\tau_x \eta) \left(\frac{\varphi(\frac{x}{N})}{g(\eta_x + 1)} - 1(\eta_x \ge 1) \right) \right\},$$

$$\partial_t \log \psi_t^N = \sum_x \frac{\partial_t \varphi(u^N(t, \frac{x}{N}))}{\varphi(u^N(t, \frac{x}{N}))} \left(\eta_x - u^N\left(t, \frac{x}{N}\right) \right).$$

(3) Schauder estimate. To bound the prefactor appearing in $N^2 L_Z^{*,\nu} 1$, we need

Proposition 2.5 (Schauder estimate for quasilinear discrete PDEs, [24]). *If $\sup_N \|u^N(0)\|_{C_N^4} < \infty$ (which holds under our assumption), the solution of* (2.6) *has the bound*

$$\|u^N(t)\|_{C_N^2} \le C K^{\frac{2}{\sigma}},$$

where $\|u\|_{C_N^k} = \sum_{i=0}^k \max_{x; e_1, \dots, e_i} |\nabla_{e_1}^N \cdots \nabla_{e_i}^N u(\frac{x}{N})|$, $\sigma \in (0,1)$ is the Hölder exponent obtained in Nash estimate and $\nabla_e^N u(\frac{x}{N}) = N(u(\frac{x+e}{N}) - u(\frac{x}{N}))$ is the discrete derivative of u in the direction $e \in \mathbb{Z}^d$, $|e| = 1$.

(4) First-order Boltzmann–Gibbs principle. For a local function $h = h(\eta)$ on $\mathcal{X}$ (i.e., h depends only on finitely many $\{\eta_x\}$) growing at most linearly in η, we set $\tilde{h}(\rho) = E^{\nu_\rho}[h]$, $\rho \ge 0$, and

$$f_x(\eta) = h(\tau_x \eta) - \tilde{h}(u_x) - \tilde{h}'(u_x)(\eta_x - u_x),$$

where we write $u_x = u^N(t, \frac{x}{N})$ for simplicity. Roughly saying, one can replace h by the first-order Taylor expansion of its equilibrium average.

Theorem 2.6 (Boltzmann–Gibbs principle). *Let $\{a_{t,x}\}_{t \ge 0, x \in \mathbb{T}_N^d}$ be nonrandom coefficients satisfying $|a_{t,x}| \le M$. Then, there exist $\varepsilon_1, C > 0$ such that*

$$E\left| \int_0^T \sum_x a_{t,x} f_x(\eta^N(t)) dt \right| \le CMKN^{d-\varepsilon_1} + CM \int_0^T H(\mu_t^N | \nu_t^N) dt.$$

For the proof, we apply truncation, entropy inequality, estimate on the exponential moment under ν_t^N, Feynman–Kac formula, Raleigh estimate, and equivalence of ensembles.

First, take $h = g(\eta_0) - \varphi(u_x)$ and $a_{t,x} = \frac{\Delta^N \varphi(u_x)}{\varphi(u_x)}$. Then, $\tilde{h}(\rho) = \varphi(\rho) - \varphi(u_x)$ and, noting $\tilde{h}(u_x) = 0$, by Theorem 2.6, $N^2 L_Z^{*,\nu_t^N} 1$ is replaced by

$$\sum_x \frac{\Delta^N \varphi(u_x)}{\varphi(u_x)} \varphi'(u_x)(\eta_x - u_x).$$

We use Proposition 2.5 to bound $|a_{t,x}|$ by $M = CK^{\frac{2}{\sigma}}$. Note that $\varphi(u_x) \geq c$ holds for some $c > 0$ by the maximum principle, the property of f, and the assumption on $u^N(0)$.

Next, take the function inside curly braces in $L_G^{*,\nu} 1$ in Lemma 2.4 replacing $\tau_x \eta$ and η_x, respectively, by η and η_0 as h and $a_{t,x} = K$. Noting $E^{\nu_\beta}[\frac{1}{g(\eta_0+1)}] = \frac{1}{\varphi(\beta)} E^{\nu_\beta}[1(\eta_0 \geq 1)]$ and $\tilde{h}(u_x) = 0$, by Theorem 2.6, $KL_G^{*,\nu_t^N} 1$ is replaced by

$$K \sum_x E^{\nu_{u_x}}[c^+ - c^-] \frac{\varphi'(u_x)}{\varphi(u_x)} (\eta_x - u_x).$$

Summarizing these and noting $\partial_t \varphi(u_x) = \varphi'(u_x) \partial_t u_x$ for $\varphi(u_x) = \varphi(u^N(t, \frac{x}{N}))$, $L_N^{*,\nu_t^N} 1 - \partial_t \log \psi_t^N$ is replaced, with an error given by Theorem 2.6, by

$$\sum_x \frac{\varphi'(u_x)}{\varphi(u_x)} \big(\Delta^N \varphi(u_x) + Kf(u_x) - \partial_t u_x\big)(\eta_x - u_x).$$

This vanishes if $u_x = u^N(t, \frac{x}{N})$ is the solution of (2.6).

(5) Completion of the proof. Finally, since $K = K(N) \leq \delta_0 (\log N)^{\frac{\sigma}{2}}$, we obtain

$$\partial_t H\big(\mu_t^N | \nu_t^N\big) \leq CK^{\frac{2}{\sigma}} H\big(\mu_t^N | \nu_t^N\big) + O(N^{d-\varepsilon_2}),$$

for $0 < \varepsilon_2 < \varepsilon_1$ in integrated form in t. Gronwall's inequality shows

$$H\big(\mu_t^N | \nu_t^N\big) \leq \big(H\big(\mu_0^N | \nu_0^N\big) + tO(N^{d-\varepsilon_2})\big) e^{CK^{\frac{2}{\sigma}} t}.$$

Note that $e^{CK^{\frac{2}{\sigma}} t} \leq N^{C\delta_0^{\frac{2}{\sigma}} t}$ from $K \leq \delta_0 (\log N)^{\frac{\sigma}{2}}$. Thus, taking $\delta_0 > 0$ small enough, Theorem 2.2 is shown.

(b) PDE part. The following proposition is a purely PDE result, which establishes the sharp interface limit for the solution $u^N(t)$ of (2.6) and leads to the MMC. Theorem 2.1 follows from (2.8) and this proposition.

Proposition 2.7. *Under our assumption, $u^N(t)$ converges to χ_{Γ_t} as $N \to \infty$, where χ_{Γ_t} is defined in (2.5) and the hypersurface Γ_t in $\mathbb{T}^d$ moves according to the MMC, $V = \lambda_0 \kappa$.*

The proof of Proposition 2.7 relies on the comparison theorem for the discrete PDE (2.6) due to the nondecreasingness of φ and consists of two parts: generation of interface and propagation of interface. In a short time, the reaction term $Kf(u^N)$ is dominant and the solution $u^N(t, \frac{x}{N})$ is pushed to one of the two stable points, α_1 and α_2, of f within the time $t = \frac{c}{K} \log K$, $c > 0$. This is called the generation of interface.

Once the interface is created, we can construct super- and subsolutions to (2.6) based on the traveling (standing) wave solution U_0: $\varphi(U_0)'' + f(U_0) = 0$ on $\mathbb{R}$ combing with the second-order term U_1 in the asymptotic expansion in K of the continuous PDE (2.3). By sandwiching the solution of (2.6) within super- and subsolutions, and studying the asymptotic behavior of these solutions as $K = K(N) \to \infty$, we obtain Proposition 2.7.

2.2. Other approaches

2.2.1. Ginzburg–Landau interface model

The Ginzburg–Landau $\nabla\phi$-interface model is an evolutional model of height functions of discretized interface. After characterizing all (tempered and shift-invariant) invariant measures of $\nabla\phi$-dynamics on $\mathbb{Z}^d$ as nonlinear version of massless Gaussian lattice free fields with long correlations, an anisotropic motion by mean curvature was derived under the hydrodynamic limit, see Funaki and Spohn [25] and Funaki [14]. Funaki [13] derived a PDE with an obstacle described by an evolutionary variational inequality from the Ginzburg–Landau interface model on a wall.

2.2.2. SPDE approach to stochastic MMC

An approach to stochastic MMC from SPDEs is also known. The sharp interface limit of the time-dependent Ginzburg–Landau model of nonconservative type, or equivalently the stochastic Allen–Cahn equation, was studied in [10,12,28]. In one dimension with space–time Gaussian white noise, the limit motion of a phase separation point is described by a stochastic differential equation [10]. In higher dimensions, stochastic MMC was derived in the limit in [12,28]. Chapter 4 of [16] gives a survey of related results. Physical background of these SPDEs is found in [38].

3. STEFAN PROBLEM

3.1. From two-component Glauber–Kawasaki dynamics

De Masi et al. [6] derived a system of diffusion equations with Stefan free boundary condition from two-component simple Kawasaki dynamics with relatively large mesoscopic annihilation effect when different types of particles meet. The annihilation effect leads to segregation in a competition–diffusion system at the macroscopic level.

We consider two-component Glauber–Kawasaki dynamics on $\mathbb{T}_N^d$, that is, the Markov process $(\eta_1^N(t), \eta_2^N(t)) = \{\eta_{1,x}^N(t), \eta_{2,x}^N(t)\}_{x\in\mathbb{T}_N^d}$ on $\mathcal{X}_N \times \mathcal{X}_N$ with generator $L_N = N^2 L_K^{(2)} + K L_G$, where $\mathcal{X}_N = \{0,1\}^{\mathbb{T}_N^d}$ in the present setting and

$$\begin{aligned}
\big(L_K^{(2)} f\big)(\eta_1,\eta_2) &= d_1\big(L_K f(\cdot,\eta_2)\big)(\eta_1) + d_2\big(L_K f(\eta_1,\cdot)\big)(\eta_2),\\
(L_K f)(\eta) &= \frac{1}{2}\sum_{x,y\in\mathbb{T}_N^d:|x-y|=1}\big\{f(\eta^{x,y}) - f(\eta)\big\}, \qquad \eta\in\mathcal{X}_N,\\
(L_G f)(\eta_1,\eta_2) &= \sum_{x\in\mathbb{T}_N^d}\eta_{1,x}\eta_{2,x}\big\{f\big(\eta_1^x,\eta_2^x\big) - f(\eta_1,\eta_2)\big\},
\end{aligned}$$

for $(\eta_1,\eta_2)\in\mathcal{X}_N\times\mathcal{X}_N$ and functions f on $\mathcal{X}_N\times\mathcal{X}_N$ (on $\mathcal{X}_N$ in the second line). Here, $d_1, d_2 > 0$, $\eta^{x,y}$ is the configuration $\eta = \{\eta_x\}_{x\in\mathbb{T}_N^d}$ with η_x and η_y exchanged, and η^x is η after a flip $\eta_x \leftrightarrow 1-\eta_x$ which happens at x. In particular, when two particles of different types meet, both of them disappear with high probability as $K = K(N)\to\infty$.

The macroscopic empirical measure $\alpha^N(dv;\eta)$ is defined for $\eta\in\mathcal{X}_N$ as in (2.1), and we set $\alpha_i^N(t,dv) = \alpha^N(dv;\eta_i^N(t))$ for $i = 1,2$ and $t\ge 0$.

Theorem 3.1 ([6]). *Let $1 \le K = K(N) \le \delta_0 (\log N)^{1/2}$ with small enough $\delta_0 > 0$, $K(N) \to \infty$, and assume proper conditions on the initial distribution of $(\eta_1^N(0), \eta_2^N(0))$ including those on the relative entropy as in Theorem 2.1. We additionally assume $e^{-c_1 K} \le u_i^N(0, \frac{x}{N}) \le c_2$ with $c_1 > 0$, $0 < c_2 < 1$ and their convergence to some $u_i(0, v)$ as $N \to \infty$. Then, $\alpha_i^N(t, dv)$ converges to $u_i(t, v)dv$ as $N \to \infty$ in probability for $i = 1, 2$. In the limit, $u_1(t, v)u_2(t, v) = 0$ a.e. holds and $w(t, v) := u_1(t, v) - u_2(t, v)$ is the unique weak solution of the equation*

$$\partial_t w = \Delta D(w), \tag{3.1}$$

where $D(w) = d_1 w$ for $w \ge 0$ and $= d_2 w$ for $w < 0$; cf. (3.4) for a formulation of the weak solution.

The last sentence in the theorem means that u_i are solutions of diffusion equations $\partial_t u_i = d_i \Delta u_i$ on the regions $\{u_i > 0\}$ for $i = 1, 2$, and satisfy two-phase Stefan free boundary condition (cf., PDE literature [4] and references therein):

$$d_1 \partial_{\mathbf{n}} u_1 + d_2 \partial_{\mathbf{n}} u_2 = 0$$

at the free boundary $\Gamma_t := \{v \in \mathbb{T}^d; u_1 = u_2 = 0\}$, where $\mathbf{n}$ is the unit normal vector at Γ_t directed to the region $\{v \in \mathbb{T}^d; u_1 > 0\}$. Some extension of this theorem is given in [37].

For the proof, we apply again the relative entropy method to show that the microscopic system is close to $u_i^N(t, \frac{x}{N})$, which is determined as the solution of the system of the discretized hydrodynamic equation

$$\partial_t u_i^N\left(t, \frac{x}{N}\right) = d_i \Delta^N u_i^N\left(t, \frac{x}{N}\right) - K u_1^N\left(t, \frac{x}{N}\right) u_2^N\left(t, \frac{x}{N}\right), \quad i = 1, 2. \tag{3.2}$$

Then, we show convergence of the solution of (3.2) to that of the free boundary problem. Indeed, one can show two estimates, $\int_0^T \int_{\mathbb{T}^d} u_1^N(t, v) u_2^N(t, v) dt dv \le \frac{1}{K}$ and $\int_0^T \int_{\mathbb{T}^d} |\nabla^N u_i^N(t, v)|^2 dt dv \le \frac{1}{2d_i}$ for $i = 1, 2$, where $u_i^N(t, v)$ are the extensions on $\mathbb{T}^d$ of $u_i^N(t, \frac{x}{N})$ as step functions. These two estimates show the relative compactness of $\{u_i^N(t, v)\}_N$ in $L^2([0, T] \times \mathbb{T}^d)$. Take any limit $\{u_i\}$ of $\{u_i^N\}_N$. Then, one can show that $u_1 u_2 = 0$ a.e. from the first estimate and also that $w = u_1 - u_2$ is the weak solution of (3.1). Therefore, the uniqueness of the weak solution of (3.1) completes the proof.

3.2. From two-component Kawasaki dynamics with speed change and annihilation

Funaki [11] studied the derivation of the Stefan free boundary problems from Kawasaki dynamics with a speed change having two types of particles called water/ice (W/I). When W hits I, W is instantaneously killed, while I disappears after receiving ℓ hits of W. This models the effect of latent heat. We obtain a one-phase Stefan problem when I is immobile, and a two-phase Stefan problem when I is mobile. This model appears by first letting $K \to \infty$ for that discussed in Section 3.1 and, indeed, the two-phase Stefan problem obtained in the limit is the same (if we take $c^{\pm}(\eta) \equiv d_1, d_2$ and $\ell = 1$). The derivation of the two-phase Stefan problem in the repelling case is also possible in one dimension.

3.2.1. One-phase Stefan problem (Immobile Ice)

To record the number of hits by W particles, we label I particles by $-\ell,\dots,-1$ ($\ell\in\mathbb{N}$) and regard as different microscopic states for I. The I particles melt and disappear after they are hit ℓ times by W particles. Thus the configuration space is $\mathcal{X}_N := \{-\ell,\dots,1\}^{\mathbb{T}_N^d}$. For $\eta=\{\eta_x\}_{x\in\mathbb{T}_N^d}\in\mathcal{X}_N$, $\eta_x=1$ and 0 mean that the site x is occupied by a W particle or is vacant, respectively. The jump rate $c_{x,y}(\xi)$ of W particles is defined for $\xi\in\mathcal{X}^+ := \{0,1\}^{\mathbb{Z}^d}$, the I-disregarded configuration space, and $x,y\in\mathbb{T}_N^d$, $|x-y|=1$.

Then, the generator of our model is given by

$$L_N f(\eta)=\sum_{x,y\in\mathbb{T}_N^d:|x-y|=1} c_{x,y}(\eta^+)\bigl\{f(\eta^{x,y})-f(\eta)\bigr\}, \tag{3.3}$$

for functions f on $\mathcal{X}_N$, where $\eta^+ := \eta\vee 0$ denotes the I-disregarded configuration, while $\eta^{x,y}$ denotes the configuration after a W particle jumps from x to y, changing if $\eta_x=1$ and $\eta_y^+=0$, and $\eta^{x,y}=\eta$ (remaining unchanged) otherwise.

We assume that the jump rate $c_{x,y}$ satisfies "symmetry, spatial homogeneity, locality, positivity" and the "detailed balance condition" with respect to the local specification of a certain extreme canonical Gibbs measure ν_ρ, which exists uniquely for each density $\rho\in[0,1]$ and has the uniform mixing property. In addition, we assume the "gradient condition": There exist local functions $\{h_i\}_{1\le i\le d}$ of ξ such that the currents have the forms

$$c_{0,e_i}(\xi)(\xi_{e_i}-\xi_0)=h_i(\tau_{e_i}\xi)-h_i(\xi), \quad 1\le i\le d,\ \xi\in\mathcal{X}^+,$$

where $e_i\in\mathbb{Z}^d$, $|e_i|=1$, stands for the unit vector in the direction i. We also assume that the equilibrium means $P^+(\rho):=E^{\nu_\rho}[h_i]$, $\rho\in[0,1]$, are independent of i. One can see that $P^+(\rho)$ is nondecreasing and continuous in ρ.

Consider the macroscopic empirical measure $\alpha^N(t,dv)=\alpha^N(dv;\eta^N(t))$ of $\eta^N(t)=\{\eta_x^N(t)\}_{x\in\mathbb{T}_N^d}\in\mathcal{X}_N$, generated by N^2L_N, defined similarly as in (2.1). We assume that $\alpha^N(0)\to a(0,v)dv$ in probability as $N\to\infty$, where $a(0,v)\in[-\ell,1]$.

Theorem 3.2 ([11]). *For every $t>0$, $\alpha^N(t)$ converges to $a(t,v)dv$ in probability. The limit density $a(t,v)\in[-\ell,1]$ is a unique solution of the equation:*

$$\bigl\langle a(t),G\bigr\rangle=\bigl\langle a(0),G\bigr\rangle+\int_0^t\bigl\langle P\bigl(a(s)\bigr),\Delta G\bigr\rangle\,ds, \tag{3.4}$$

for every $G\in C^\infty(\mathbb{T}^d)$, where $\langle a,G\rangle=\int_{\mathbb{T}^d}a(v)G(v)\,dv$. The function P on $[-\ell,1]$ is defined by $P(a)=P^+(a)$ for $a\in[0,1]$ and $=P^+(0)$ for $a\in[-\ell,0]$.

Equation (3.4) is the weak (or enthalpy) formulation of the following one-phase Stefan problem for the density $u(t,v)\in[0,1]$ of W:

$$\begin{aligned}
&\partial_t u=\Delta P^+(u) && \text{on } \mathcal{L}(t),\\
&u(t,v)=0 \text{ and } \ell V=-\partial_{\mathbf{n}}P^+(u) && \text{at } \Sigma(t):=\partial\mathcal{L}(t),
\end{aligned}$$

where $\mathcal{L}(t):=\{v\in\mathbb{T}^d; u(t,v)>0\}$, $\mathbf{n}$ denotes the unit normal vector at $\Sigma(t)$ directed toward $\mathcal{L}(t)$, and V is the velocity of $\Sigma(t)$ in the direction $\mathbf{n}$. The speeds of loosing masses of W and I at $\Sigma(t)$ are given by $\partial_{\mathbf{n}}P^+(u)$ and $-V$, respectively. Since the loosing speed for W is ℓ times faster than that for I, we have the last Stefan free boundary condition.

3.2.2. Two-phase Stefan problem (Mobile Ice)

We make I particles with label $-\ell$ active. They perform Kawasaki dynamics with jump rates different from those for W particles. The particles with labels $-\ell+1,\dots,-1$ remain immobile and are regarded as those in intermediate states between I and W. One can determine the dynamics by properly introducing jump rates $c^+_{x,y}(\xi)$ and $c^-_{x,y}(\xi)$, $\xi \in \mathcal{X}^+$ of W/I particles, both of which satisfy the conditions in Section 3.2.1 with different extreme canonical Gibbs measures ν^+_ρ, ν^-_ρ and functions $\{h^+_i\}$, $\{h^-_i\}$, respectively. We write $P^+(\rho) = E^{\nu^+_\rho}[h^+_i]$ and $P^-(\rho) = E^{\nu^-_\rho}[h^-_i]$, $\rho \in [0,1]$.

In this setting, one can derive the following two-phase Stefan problem, written in strong form, for the density $u_1(t,v) \in [0,1]$ of W and $u_2(t,v) \in [0,1]$ of I:

$$
\begin{aligned}
&\partial_t u_1 = \Delta P^+(u_1) \quad \text{on } \mathcal{L}_1(t), \quad \partial_t u_2 = \Delta P^-(u_2) \quad \text{on } \mathcal{L}_2(t),\\
&u_1 = u_2 = 0 \quad \text{and} \quad (\ell-1)V = -\partial_{\mathbf{n}} P^+(u_1) - \partial_{\mathbf{n}} P^-(u_2) \quad \text{at } \Sigma(t),
\end{aligned}
$$

where $\Sigma(t) := \partial\mathcal{L}_1(t) = \partial\mathcal{L}_2(t)$, $\mathbf{n}$ and V are the same as above and $\mathcal{L}_i(t) = \{u_i > 0\}$.

4. KPZ EQUATION

4.1. KPZ equation as singular SPDE

4.1.1. Scalar KPZ equation

The KPZ (Kardar, Parisi, and Zhang [42]) equation describes the motion of a growing interface with random fluctuation. It is an equation for the height function $h(t,v)$ of a curve (interface) in the plane:

$$
\partial_t h = \frac{1}{2}\partial_v^2 h + \frac{1}{2}(\partial_v h)^2 + \dot{W}(t,v), \quad v \in \mathbb{R} \text{ or } \mathbb{T}. \tag{4.1}
$$

We consider it in one-dimension on the whole line $\mathbb{R}$ or on a finite interval $\mathbb{T} = [0,1)$ under the periodic boundary condition; $\dot{W}(t,v)$ is a space–time Gaussian white noise with mean 0 and covariance structure

$$
E\big[\dot{W}(t,v_1)\dot{W}(s,v_2)\big] = \delta(t-s)\delta(v_1-v_2). \tag{4.2}
$$

This means that the noise is independent for different (t,v), and $\dot{W}(t,v)$ is realized only as a generalized function (distribution), cf. [16].

The KPZ equation attracts a lot of attention from viewpoints of integrable probability [1,41,46,47], singular ill-posed SPDEs [31,34,35], and microscopic interacting particle systems [3].

Equation (4.1) is ill-posed in a classical sense due to the conflict between nonlinearity and roughness of the noise. It is known that the linear SPDE

$$
\partial_t h = \frac{1}{2}\partial_v^2 h + \dot{W}(t,v), \quad v \in \mathbb{R} \text{ or } \mathbb{T}, \tag{4.3}
$$

obtained by dropping the nonlinear term has a continuous solution which is α-Hölder continuous in v for every $\alpha < \frac{1}{2}$. Therefore, one can imagine that the nonlinear term $(\partial_v h)^2$

in (4.1) is undefinable in the usual sense. Actually, it requires a renormalization. The following renormalized KPZ equation with compensator $\delta_v(v)$ $(=+\infty)$ would have a meaning

$$\partial_t h = \frac{1}{2}\partial_v^2 h + \frac{1}{2}\{(\partial_v h)^2 - \delta_v(v)\} + \dot{W}(t,v). \tag{4.4}$$

To see (4.4) heuristically, consider the linear stochastic heat equation for $Z = Z(t,v)$:

$$\partial_t Z = \frac{1}{2}\partial_v^2 Z + Z\dot{W}(t,v), \quad v \in \mathbb{R} \text{ or } \mathbb{T}, \tag{4.5}$$

with a multiplicative noise defined in Itô's sense. It is known that (4.5) is well-posed in the mild or generalized functions' sense and the strong comparison principle holds: $Z(t,v) > 0$ for all $t > 0$ if it holds at $t = 0$. In particular, we can define the so-called Cole–Hopf solution

$$h_{\mathrm{CH}}(t,v) := \log Z(t,v). \tag{4.6}$$

Then, applying Itô's formula for (4.6) and noting $dW(t,v_1)dW(t,v_2) = \delta(v_1 - v_2)dt$ which follows from (4.2), we obtain the renormalized KPZ equation (4.4) for h_{CH} from (4.5). Note that $-\frac{1}{2}\delta_v(v)$ arises as an Itô correction term. To give a meaning to (4.4), we need to introduce approximations, see Section 4.1.3.

4.1.2. Coupled KPZ equation

One can extend the KPZ equation (4.1) to the equation for the system with n-components $h(t,v) = (h^i(t,v))_{i=1}^n$ on $\mathbb{T}$ (or $\mathbb{R}$):

$$\partial_t h^i = \frac{1}{2}\partial_v^2 h^i + \frac{1}{2}\Gamma^i_{jk}\partial_v h^j \partial_v h^k + \dot{W}^i(t,v), \quad 1 \le i \le n. \tag{4.7}$$

We use Einstein's convention to omit the sum over j and k for the second term, and $(\dot{W}^i(t,v))_{i=1}^n$ is a family of n independent space–time Gaussian white noises.

The coupling constants Γ^i_{jk} always satisfy the bilinear condition, $\Gamma^i_{jk} = \Gamma^i_{kj}$ for all i,j,k, and we sometimes assume the trilinear condition (T), namely $\Gamma^i_{jk} = \Gamma^i_{kj} = \Gamma^k_{ji}$ for all i,j,k.

The coupled KPZ equation is ill-posed. We need to introduce approximations with smooth noises and renormalizations. Equation (4.7) appears in the study of nonlinear fluctuating hydrodynamics [49,50] for a system with n-conserved quantities by taking second-order terms into account. We will discuss this for the interacting particle system in Section 4.2.3.

We also consider the coupled KPZ equation with constant drifts c^i as in [49,50]:

$$\partial_t h^i = \frac{1}{2}\partial_v^2 h^i + \frac{1}{2}\Gamma^i_{jk}\partial_v h^j \partial_v h^k + c^i \partial_v h^i + \dot{W}^i(t,v), \quad 1 \le i \le n. \tag{4.8}$$

We may assume $c^i = 0$ and reduce to (4.7) (with new space–time Gaussian white noises) by considering $\tilde{h}^i(t,v) := h^i(t, v - c^i t)$.

4.1.3. Two approximations, local and global well-posedness and invariant measure

We now discuss the approximations for (4.4) in the framework of the coupled KPZ equation. Indeed, one can introduce two types of approximations: one is simple, the other is suitable to find invariant measures. We replace the noise by a smeared one obtained by convoluting with the nonnegative symmetric kernel $\eta^\varepsilon(v) := \frac{1}{\varepsilon}\eta(\frac{v}{\varepsilon})$, which converges to $\delta_0(v)$ as $\varepsilon \downarrow 0$.

The first approximation is simple in the sense that we replace only the noise and introduce the renormalizations. It is given as follows. For $h^i = h^{\varepsilon,i}$, $\varepsilon > 0$,

$$\partial_t h^i = \frac{1}{2}\partial_v^2 h^i + \frac{1}{2}\Gamma^i_{jk}\big(\partial_v h^j \partial_v h^k - c^\varepsilon \delta^{jk} - B^{\varepsilon,jk}\big) + \dot{W}^i * \eta^\varepsilon(t,v), \tag{4.9}$$

where δ^{jk} is Kronecker's δ, $c^\varepsilon = \frac{1}{\varepsilon}\|\eta\|^2_{L^2(\mathbb{R})} - 1$ (on $\mathbb{T}$, while -1 is unnecessary on $\mathbb{R}$), and $B^{\varepsilon,jk}$ ($= O(\log \frac{1}{\varepsilon})$ in general) are renormalization factors. The renormalizations $B^{\varepsilon,jk}$ and $\tilde{B}^{\varepsilon,jk}$ introduced in (4.10) below are unnecessary under condition (T), especially, in the scalar-valued case, see Theorem 4.2(1).

The second approximation, which is suitable to find the invariant measure, is given as follows. For $\tilde{h}^i = \tilde{h}^{\varepsilon,i}$, $\varepsilon > 0$,

$$\partial_t \tilde{h}^i = \frac{1}{2}\partial_v^2 \tilde{h}^i + \frac{1}{2}\Gamma^i_{jk}\big(\partial_v \tilde{h}^j \partial_v \tilde{h}^k - c^\varepsilon \delta^{jk} - \tilde{B}^{\varepsilon,jk}\big) * \eta_2^\varepsilon + \dot{W}^i * \eta^\varepsilon(t,v), \tag{4.10}$$

with renormalization factors c^ε as above and $\tilde{B}^{\varepsilon,jk}$ ($= O(\log \frac{1}{\varepsilon})$), where $\eta_2^\varepsilon = \eta^\varepsilon * \eta^\varepsilon$. This approximation for the scalar KPZ equation was introduced in Funaki and Quastel **[23]** and the idea behind (4.10) is the fluctuation–dissipation relation. Renormalization factor c^ε comes from the second-order terms in the related Wiener–Itô chaos expansion, while $B^{\varepsilon,jk}$ and $\tilde{B}^{\varepsilon,jk}$ are from the fourth-order terms. For the solution of (4.10) (with $\tilde{B} = 0$), **[15]** showed on $\mathbb{R}$, under condition (T), the infinitesimal invariance of the distribution of $B * \eta^\varepsilon(v), v \in \mathbb{R}$, where B is the $\mathbb{R}^n$-valued two-sided Brownian motion, cf. Theorem 4.2(2).

When $n = 1$ and $\Gamma = 1$, the solution of (4.9) with $B^\varepsilon = 0$ converges as $\varepsilon \downarrow 0$ to the Cole–Hopf solution h_{CH} of the KPZ equation, while **[23]** showed on $\mathbb{R}$ that the solution of (4.10) with $\tilde{B}^\varepsilon = 0$ converges to $h_{\mathrm{CH}} + \frac{1}{24}t$, see also **[39]**. This was shown based on the Boltzmann–Gibbs principle, which follows by Kipnis–Varadhan estimate (cf. **[44, 45]**), see (4.19). This estimate is sometimes called Itô–Tanaka trick.

The method of **[23]** is based on the Cole–Hopf transform, but it is not available for the coupled equation in general. Instead, the paracontrolled calculus due to Gubinelli et al. **[31]** is applicable. Funaki and Hoshino **[19]** showed the following three results on $\mathbb{T}$.

First is the convergence of h^ε and $\tilde{h}^\varepsilon$ and local-in-time well-posedness of coupled KPZ equation (4.7). Let $\mathcal{C}^\alpha = (\mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}))^n$, $\alpha \in \mathbb{R}$ be an $\mathbb{R}^n$-valued Hölder–Besov space on $\mathbb{T}$.

Theorem 4.1 (**[19]**). (1) *Assume $h_0 \in \mathcal{C}^\delta$, $\delta \in (0, \frac{1}{2})$, then a unique solution h^ε of (4.9) exists up to survival time $T^\varepsilon \in (0, \infty]$. With a proper choice of $B^{\varepsilon,jk}$, there exists $T_{sur} > 0$ such that $T_{sur} \le \liminf_{\varepsilon\downarrow 0} T^\varepsilon$ holds, and h^ε converges in probability as $\varepsilon \downarrow 0$ to some h in $C([0,T],\mathcal{C}^\delta) \cap C((0,T],\mathcal{C}^\alpha)$ for every $\alpha < \frac{1}{2}$ and $0 < T < T_{sur}$.*

(2) *Similar result holds for the solution $\tilde{h}^\varepsilon$ of (4.10) with some limit $\tilde{h}$. Under proper choices of $B^{\varepsilon,jk}$ and $\tilde{B}^{\varepsilon,jk}$, we can actually make $h = \tilde{h}$.*

Second, under the trilinear condition (T), we have the invariance of Wiener measure on $\mathbb{T}$ in the following sense and can explicitly compute the difference of two limits.

Theorem 4.2 ([19]). *Assume the trilinear condition* (T). *Then,*

(1) $B^{\varepsilon,jk}, \tilde{B}^{\varepsilon,jk} = O(1)$ *as* $\varepsilon \downarrow 0$ *so that the solutions of* (4.9) *with* $B^{\varepsilon,jk} = 0$ *and* (4.10) *with* $\tilde{B}^{\varepsilon,jk} = 0$ *converge. In the limit, we have* $\tilde{h}^i(t,v) = h^i(t,v) + c^i t$, $1 \le i \le n$, *where*

$$c^i = \frac{1}{24} \sum_{j,k,k_1,k_2} \Gamma^i_{jk} \Gamma^j_{k_1 k_2} \Gamma^k_{k_1 k_2}.$$

(2) *Moreover, the distribution of* $\{\partial_v B\}_{v \in \mathbb{T}}$, *where* B *is the* $\mathbb{R}^n$*-valued periodic Brownian motion on* $\mathbb{T}$, *is the unique invariant (probability) measure for the tilt process* $u = \partial_v h$. *Or, one can say that the periodic Wiener measure on the quotient space* $\mathcal{C}^\alpha/\sim$, $\alpha < \frac{1}{2}$, *where "*$\sim$*" is defined by* $h \sim h + c$ *for* $c \in \mathbb{R}$, *is invariant for* h. *The uniqueness of invariant measures does not hold on* $\mathbb{R}$. *Wiener measures with constant drifts are all invariant on* $\mathbb{R}$, *see* [23].

Third is the global-in-time well-posedness (existence and uniqueness of solutions) of (4.7) in paracontrolled sense under (T). Indeed, assuming (T), we take the initial value $h(0)$ as $h(0,0) = 0$ and $u_0 := \partial_v h(0) \overset{\text{law}}{=} \{\partial_v B\}_{v \in \mathbb{T}}$ (i.e., stationary). Then, one can show the uniform bound for $u = \partial_v h$, namely that $E[\sup_{t \in [0,T]} \|u(t;u_0)\|^p_{\mathcal{C}^{\alpha-1}}] < \infty$ for every $T > 0$, $p \ge 1$, $\alpha < \frac{1}{2}$. This implies the global-in-time existence of the solution for a.a.-u_0. Combing this with the strong Feller property of $\partial_v h$ for h in (4.7) on $\mathcal{C}^{\alpha-1}$, $\alpha \in (0, \frac{1}{2})$ shown by [36], we obtain the global existence for $u = \partial_v h$ for all given u_0.

For $u \equiv u^\varepsilon = (u^i)_i = (\partial_v h^i)_i$ for h^i in (4.9), we have

$$\sum_{i,j,k} \Gamma^i_{jk} \int_{\mathbb{T}} u^i \partial_v (u^j u^k) dv = 0$$

under (T). This shows an a priori estimate and global well-posedness for (4.9) at least if $h(0) \in H^1(\mathbb{T})$. Therefore, Theorem 4.1(1) holds globally in time if $h(0) \in H^1(\mathbb{T})$.

The example given in [9] with $n = 2$ does not satisfy (T), but the logarithmic renormalization term is unnecessary, and one can show the existence of an invariant measure. The role of the trilinear condition (T) is discussed further in [18].

4.1.4. Proof of Theorems 4.1 and 4.2

We think of the Ornstein–Uhlenbeck part (as in (4.3) for the scalar-valued case) as the leading term and of the nonlinear term as its perturbation. This leads to an expansion of the equation. We introduce finitely many driving terms $\mathbb{H}$, which involve renormalizations, and show that, once these terms are determined, the rest of the equation is solvable in the framework of the paracontrolled calculus. In particular, we can show the local-in-time solvability and continuity of the solutions in $\mathbb{H}$.

4.2. From interacting particle systems

Bernardin et al. [2] derived the coupled KPZ equation (4.8) with drifts from a microscopic interacting particle system called multispecies zero-range process with weak asymmetry (WA). The derivation of scalar KPZ(–Burgers) equation from particle systems was studied in [3] (WA simple exclusion process), [29] (WA exclusion process with speed change), and [30] (WA zero-range process).

4.2.1. n-species zero-range process on $\mathbb{T}_N$

To derive an n-component system in the limit, we need to consider a system with n-conserved quantities at the microscopic level. We consider n-species zero-range process on $\mathbb{T}_N = \{1, 2, \dots, N\}$ with periodic boundary condition, namely, particles of n-types, which perform random walks on $\mathbb{T}_N$ and interact only at the same sites. The corresponding macroscopic space is $\mathbb{T} = [0, 1)$. Compared to the model discussed in Section 2.1.1, our system has multiple species, but is limited in one-dimension and is without Glauber part. A configuration of particles is denoted by $\boldsymbol{\eta} = (\eta^i)_{i=1}^n = (\{\eta_x^i\}_{x\in\mathbb{T}_N})_{i=1}^n \in \mathcal{X}_N^n$, where $\mathcal{X}_N = \mathbb{Z}_+^{\mathbb{T}_N}$ is the configuration space of single-species particles and $\eta_x^i \in \mathbb{Z}_+$ denotes the number of particles of the ith species at $x \in \mathbb{T}_N$.

We introduce a weak asymmetry (WA) in jump rates. Once a jump happens, the probabilities of a jump of the ith particles to the right or the left are given by $p_i^N(\pm 1) = \frac{1}{2} \pm c^{i,N}$ ($+$ for right, $-$ for left) with small $c^{i,N}$. As we will see, $c^{i,N} = \frac{c^i}{N}$, i.e., $O(\frac{1}{N})$ for the hydrodynamic limit and linear fluctuation (see Section 4.2.2), while $c^{i,N} = \frac{c}{\sqrt{N}} + \frac{c^i}{N}$, i.e., $O(\frac{1}{\sqrt{N}})$ for KPZ nonlinear fluctuation (see Section 4.2.3). Note that the constant c in leading order is common for all i.

We consider the Markov process $\boldsymbol{\eta}^N(t) = \{\eta_x^{N,i}(t)\}_{x,i}$ on $\mathcal{X}_N^n$ with the generator

$$L_N f(\boldsymbol{\eta}) = N^2 \sum_{x\in\mathbb{T}_N, 1\le i\le n, e=\pm 1} p_i^N(e) g_i(\boldsymbol{\eta}_x)\{f(\boldsymbol{\eta}^{x,x+e;i}) - f(\boldsymbol{\eta})\},$$

for functions f on $\mathcal{X}_N^n$, where $\boldsymbol{\eta}_x = (\eta_x^i)_{i=1}^n$ and $\boldsymbol{\eta}^{x,y;i}$ stands for the configuration $\boldsymbol{\eta}$ after one ith particle jumps from x to y (which is possible only when $\eta_x^i \ge 1$). The diffusive time change N^2 is introduced. The jump rate g_i of the ith particles has the zero-range property, that is, it is a function on $\mathbb{Z}_+^n$, which is the configuration space at a single site, so that $g_i = g_i(\mathbf{k})$ for $\mathbf{k} = (k_i)_{i=1}^n \in \mathbb{Z}_+^n$. In particular, interaction occurs only at the same sites. We assume that the jump rates $\{g_i(\mathbf{k})\}_{1\le i\le n, \mathbf{k}\in\mathbb{Z}_+^n}$ satisfy the conditions of "nondegeneracy, linear growth, nontriviality of Dom_Z (defined below)" and the "detailed balance condition" with respect to product measures, $\frac{g_i(\mathbf{k})}{g_i(\mathbf{k}_{j,-})} = \frac{g_j(\mathbf{k})}{g_j(\mathbf{k}_{i,-})}$ for all $i \ne j$ and $\mathbf{k} \in \mathbb{Z}_+^n$ with $k_i, k_j \ge 1$, where $\mathbf{k}_{j,-} = (k_1, \dots, k_{j-1}, k_j - 1, k_{j+1}, \dots, k_n)$, see [2] for details.

The invariant measures, or equilibrium states of $\boldsymbol{\eta}^N(t)$, are superpositions of the product measures $\{\bar{\nu}_{\boldsymbol{\varphi}} := p_{\boldsymbol{\varphi}}^{\otimes\mathbb{T}_N}\}$ with one-site marginal

$$p_{\boldsymbol{\varphi}}(\mathbf{k}) = \frac{1}{Z_{\boldsymbol{\varphi}}} \frac{\boldsymbol{\varphi}^{\mathbf{k}}}{\mathbf{g}(\mathbf{k})!}, \quad Z_{\boldsymbol{\varphi}} = \sum_{\mathbf{k}\in\mathbb{Z}_+^n} \frac{\boldsymbol{\varphi}^{\mathbf{k}}}{\mathbf{g}(\mathbf{k})!}.$$

Here $\boldsymbol{\varphi} = (\varphi_i)_{i=1}^n$ are nonnegative parameters, called fugacity, $\boldsymbol{\varphi}^{\mathbf{k}} = \varphi_1^{k_1} \cdots \varphi_n^{k_n}$, and

$$\mathbf{g}(\mathbf{k})! = \prod_{\ell=1}^{|\mathbf{k}|} g_{i(\ell)}(\mathbf{k}_\ell),$$

with $|\mathbf{k}| = k_1 + \cdots + k_n$, is a product along an increasing path $\mathbf{k}_0 = \mathbf{0} \to \cdots \to \mathbf{k}_\ell \to \cdots \to \mathbf{k}_{|\mathbf{k}|} = \mathbf{k}$ connecting $\mathbf{0}$ and $\mathbf{k}$ in $\mathbb{Z}_+^n$ such that $|\mathbf{k}_\ell| = \ell$, $0 \le \ell \le |\mathbf{k}|$, where $i(\ell)$ is the coordinate increased by 1 from $\mathbf{k}_{\ell-1}$ to $\mathbf{k}_\ell$. We set $\mathrm{Dom}_Z := \{\boldsymbol{\varphi} \in (0,\infty)^n; Z_{\boldsymbol{\varphi}} < \infty\}$. Note that, by the detailed balance condition, $\mathbf{g}(\mathbf{k})!$ does not depend on the choice of the increasing path $\{\mathbf{k}_\ell\}$, so is well-defined.

As in Section 2.1.1, we change the parameter from fugacity $\boldsymbol{\varphi}$ to density $\mathbf{a} = (a^i)_{i=1}^n$ of particles. Namely, define the map $R : \boldsymbol{\varphi} \mapsto \mathbf{a} = (a^i(\boldsymbol{\varphi}))_{i=1}^n$ by

$$a^i \equiv a^i(\boldsymbol{\varphi}) := E^{\bar{\nu}_{\boldsymbol{\varphi}}}\big[\eta_0^i\big], \quad 1 \le i \le n, \tag{4.11}$$

which is defined on $\mathrm{Dom}_R := \{\boldsymbol{\varphi} \in \mathrm{Dom}_Z; a^i(\boldsymbol{\varphi}) < \infty, 1 \le i \le n\}$ and denote $\nu_{\mathbf{a}} := \bar{\nu}_{\boldsymbol{\varphi}}$. The correspondence $\boldsymbol{\varphi} \leftrightarrow \mathbf{a}$ is one-to-one. We accordingly have a family of invariant measures $\{\nu_{\mathbf{a}}\}_{\mathbf{a}}$ parametrized by density $\mathbf{a} \in [0,\infty)^n$.

4.2.2. Hydrodynamic limit and linear fluctuation

We discuss the hydrodynamic limit (LLN) and the equilibrium linear fluctuation problem (CLT).

Hydrodynamic limit. Recall that the weak asymmetry is $O(\frac{1}{N})$, i.e., $p_i^N(\pm 1) = \frac{1}{2} \pm \frac{c^i}{N}$ and c^i may be different for different species. We consider an $\mathbb{R}^n$-valued macroscopic empirical measure $X_t^N = (X_t^{N,i})_{i=1}^n$ on $\mathbb{T}$ defined as in (2.1) taking $\eta_x^{N,i}(t)$ for η_x:

$$X_t^{N,i}(dv) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_x^{N,i}(t) \delta_{\frac{x}{N}}(dv), \quad v \in \mathbb{T}, \ 1 \le i \le n.$$

One can show that, multiplied by a test function $G \in C^\infty(\mathbb{T})$ as in (2.2), X_t^N converges as $N \to \infty$ to $\mathbf{a}(t,v)dv = (a^i(t,v)dv)_{i=1}^n$ in probability and the limit density $a^i(t,v)$ is the solution of the system of nonlinear PDEs:

$$\partial_t a^i = \frac{1}{2} \partial_v^2 \varphi_i(\mathbf{a}) - 2c^i \partial_v \varphi_i(\mathbf{a}), \quad v \in \mathbb{T}, \ 1 \le i \le n, \tag{4.12}$$

where $\varphi_i(\mathbf{a}) := E^{\nu_{\mathbf{a}}}[g_i(\boldsymbol{\eta}_0)]$. This is a multispecies version of (2.3) with $K = 0$ and weak asymmetry. The diffusion matrix is parabolic in the sense that $\sum_{ij} \frac{\partial \varphi_i}{\partial a^j} \xi_i \xi_j \ge 0$ for any $(\xi_i) \in \mathbb{R}^n$.

The hydrodynamic equation (4.12) can be heuristically derived as follows. By Dynkin's formula, we have

$$\big\langle X_t^{N,i}, G\big\rangle = \big\langle X_0^{N,i}, G\big\rangle + \int_0^t L_N X_s^{N,i}(G) ds + M_t^{N,i}(G) \tag{4.13}$$

and

$$L_N X^{N,i}(G) = \frac{1}{2N} \sum_{x \in \mathbb{T}_N} g_i(\boldsymbol{\eta}_x) \Delta^N G\Big(\frac{x}{N}\Big)$$

$$+\frac{c^i}{N}\sum_{x\in\mathbb{T}_N} g_i(\boldsymbol{\eta}_x)\left\{\nabla^N G\left(\frac{x}{N}\right)+\nabla^N G\left(\frac{x-1}{N}\right)\right\},$$

where $\nabla^N G(\frac{x}{N}) = N(G(\frac{x+1}{N}) - G(\frac{x}{N}))$ and we recall (2.7) for Δ^N. For martingale terms, $\lim_{N\to\infty} E[M_t^{N,i}(G)^2] = 0$ hold. By local ergodicity in local equilibria, one can replace $g_i(\boldsymbol{\eta}_x)$ by its local average $\varphi_i(\mathbf{a}(t,\frac{x}{N}))$ and obtain the weak form of (4.12) for $a^i(t,v)$ in the limit.

Linear fluctuation. Keeping the weak asymmetry the same as above, we discuss equilibrium fluctuation so that we assume $\boldsymbol{\eta}^N(0) \overset{\text{law}}{=} \nu_{\mathbf{a}_0}$ for any fixed $\mathbf{a}_0 = (a_0^i)_{i=1}^n \in (0,\infty)^n$. Consider the fluctuation field $Y_t^N = (Y_t^{N,i})_{i=1}^n$ around $\mathbf{a}_0$ defined by

$$Y_t^{N,i}(dv) = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{T}_N}\left(\eta_x^{N,i}(t) - a_0^i\right)\delta_{\frac{x}{N}}(dv), \quad v\in\mathbb{T},\ 1\le i\le n. \tag{4.14}$$

The limit $Y_t = (Y_t^i)_{i=1}^n$ is the solution of linear SPDE (Ornstein–Uhlenbeck process)

$$\partial_t Y^i = \frac{1}{2}\sum_{j=1}^n \partial_{a^j}\varphi_i(\mathbf{a}_0)\partial_v^2 Y^j - 2c^i\sum_{j=1}^n \partial_{a^j}\varphi_i(\mathbf{a}_0)\partial_v Y^j + \sqrt{\varphi_i(\mathbf{a}_0)}\partial_v \dot{W}^i, \tag{4.15}$$

where $\dot{W} = (\dot{W}^i(t,v))_{i=1}^n$ is a family of n independent space–time Gaussian white noises. The coefficient $\partial_{a^j}\varphi_i(\mathbf{a}_0)$ arises as a linearization of $\varphi_i(\mathbf{a})$ in equation (4.12) around $\mathbf{a}_0$,

$$\begin{aligned}\varphi_i(\mathbf{a}) &= \varphi_i(\mathbf{a}_0) + \sum_{j=1}^n \partial_{a^j}\varphi_i(\mathbf{a}_0)\left(a^j - a_0^j\right) + \cdots \\ &\cong \varphi_i(\mathbf{a}_0) + \frac{1}{\sqrt{N}}\sum_{j=1}^n \partial_{a^j}\varphi_i(\mathbf{a}_0)\cdot Y^j + \cdots.\end{aligned}$$

To make this replacement rigorous, we need to establish the first-order Boltzmann–Gibbs principle. The limit noise $(\sqrt{\varphi_i(\mathbf{a}_0)}\partial_v\dot{W}^i)_i$ is obtained by computing quadratic and cross-variations of the martingale terms $\tilde{M}_t^{N,i}(G) = \sqrt{N}M_t^{N,i}(G)$ of $\langle Y_t^{N,i}, G\rangle$. Indeed, we have

$$\begin{aligned}\frac{d}{dt}\langle\tilde{M}^{N,i}(G)\rangle_t &= N\left(L_N\langle\eta^{N,i}(t),G\rangle^2 - 2\langle\eta^{N,i}(t),G\rangle L_N\langle\eta^{N,i}(t),G\rangle\right) \\ &= \frac{1}{N}\sum_{x\in\mathbb{T}_N} g_i\left(\eta_x^N(t)\right)\left(\nabla^N G\left(\frac{x}{N}\right)\right)^2 + O\left(\frac{1}{N}\right) \to \varphi_i(\mathbf{a}_0)\|G'\|_{L^2(\mathbb{T})}^2,\end{aligned}$$

as $N\to\infty$, since $\boldsymbol{\eta}^N(t) \overset{\text{law}}{=} \nu_{\mathbf{a}_0}$ for all $t\ge 0$, while $\langle\tilde{M}^{N,i}(G_1),\tilde{M}^{N,j}(G_2)\rangle_t = 0$ for $i\ne j$.

See Section 2.2 of [16] for nonequilibrium fluctuation. The class of models having Ornstein–Uhlenbeck scaling limit is sometimes called Edwards–Wilkinson university class.

4.2.3. Nonlinear fluctuation leading to coupled KPZ–Burgers equation

Now the weak asymmetry is $O(\frac{1}{\sqrt{N}})$, i.e., $p_i^N(\pm 1) = \frac{1}{2} \pm (\frac{c}{\sqrt{N}} + \frac{c^i}{N})$, which is larger than before. Note that the leading constant c is common to have the common moving frame. Compared to Section 4.2.2, c^i are replaced by $c\sqrt{N}+c^i$ so that equation (4.12) for the density of the ith particles will look like

$$\partial_t a^i = \frac{1}{2}\partial_v^2\varphi_i(\mathbf{a}) - 2(c\sqrt{N}+c^i)\partial_v\varphi_i(\mathbf{a}) + \frac{1}{\sqrt{N}}(\text{noise}). \tag{4.16}$$

We consider the fluctuation field under equilibrium, i.e., $\boldsymbol{\eta}^N(0) \overset{\text{law}}{=} \nu_{\mathbf{a}_0}$ for some $\mathbf{a}_0$. This time, $\mathbf{a}_0$ should be chosen properly. To cancel the diverging factor $2c\sqrt{N}$ in (4.16), we introduce the moving frame with speed $2c\lambda\sqrt{N}$ at the macroscopic level with a suitably chosen $\lambda = \lambda(\mathbf{a}_0)$ to the fluctuation field so that (4.14) is modified to become

$$Y_t^{N,i}(dv) = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{T}_N}\big(\eta_x^{N,i}(t)-a_0^i\big)\delta_{\frac{x}{N}-2c\lambda\sqrt{N}t}(dv), \quad v\in\mathbb{T},\ 1\le i\le n. \tag{4.17}$$

The frame should have common speed for all i and this gives the restriction on the choice of $\mathbf{a}_0$. Indeed, we need to assume the frame condition [FC], namely $\partial_{a^j}\varphi_i(\mathbf{a}_0) = -\lambda\delta^{ij}$ for $\mathbf{a}_0$ and λ. Then, our main result for the nonlinear fluctuation is stated as follows.

Theorem 4.3 ([2]). *Assume the frame condition* [FC]. *Then,* $Y_t^N = (Y_t^{N,i})_{i=1}^n$ *converges to* $Y_t = (Y_t^i)_{i=1}^n$ *in law on the Skorohod space* $D([0,T],\mathcal{S}'(\mathbb{T})^n)$ *with dual of* $\mathcal{S}(\mathbb{T}) = C^\infty(\mathbb{T})$. *The limit* Y_t *is the unique stationary martingale solution of the coupled KPZ–Burgers equation*

$$\begin{aligned}\partial_t Y^i =& \frac{1}{2}\partial_{a^i}\varphi_i(\mathbf{a}_0)\partial_v^2 Y^i + \Gamma^i_{jk}(\mathbf{a}_0)\partial_v(Y^jY^k) \\ &- 2c^i\partial_{a^i}\varphi_i(\mathbf{a}_0)\partial_v Y^i + \sqrt{\varphi_i(\mathbf{a}_0)}\partial_v\dot{W}^i, \quad v\in\mathbb{T},\end{aligned} \tag{4.18}$$

where $(\dot{W}^i)_{i=1}^n$ *is the same as in* (4.15) *and* $\Gamma^i_{jk}(\mathbf{a}_0) = -c\partial_{a^j}\partial_{a^k}\varphi_i(\mathbf{a}_0)$. *We use Einstein's convention for the second term. Observe that if* $c=0$ *and under* [FC] *then* (4.18) *is the same as* (4.15) *for the linear fluctuation.*

The reason to have the limit noises in (4.18) is the same as (4.15); note that they have the same distribution under the shift by moving frame. We give a heuristic reason to have the nonlinear drift term in the limit. The main idea is the combination of the averaging effect due to ergodicity under $\nu_{\mathbf{a}_0}$ and Taylor expansion, now up to the second-order terms. Noting $a^i = a_0^i + \frac{1}{\sqrt{N}}Y^i + \cdots$ and the moving frame in (4.17), in (4.16) we will have

$$\partial_t a^i = \frac{1}{\sqrt{N}}\partial_t Y^i + 2c\lambda\sqrt{N}\partial_v a^i + \cdots = \frac{1}{\sqrt{N}}\partial_t Y^i + 2c\lambda\partial_v Y^i + \cdots,$$

$$\varphi_i(\mathbf{a}) = \varphi_i(\mathbf{a}_0) + \frac{1}{\sqrt{N}}\sum_{j=1}^n \partial_{a^j}\varphi_i(\mathbf{a}_0)\cdot Y^j + \frac{1}{2N}\sum_{j,k=1}^n \partial_{a^j}\partial_{a^k}\varphi_i(\mathbf{a}_0)\cdot Y^jY^k + \cdots.$$

We put these expansions into (4.16) and multiply it by $\sqrt{N}$. Then, noting $\partial_v\varphi_i(\mathbf{a}_0) = 0$ and observing that the diverging terms of order $O(\sqrt{N})$ exactly cancel by the frame condition [FC], we will obtain (4.18).

More precisely, in Dynkin's formula (4.13), by the averaging effect, we replace

$$g_i(\boldsymbol{\eta}_x) \cong E^{\nu_{\mathbf{a}_0+\frac{1}{\sqrt{N}}Y_t(\frac{x}{N}-2c\lambda\sqrt{N}t)}}[g_i(\boldsymbol{\eta}_x)] = \varphi_i\bigg(\mathbf{a}_0 + \frac{1}{\sqrt{N}}Y_t\Big(\frac{x}{N}-2c\lambda\sqrt{N}t\Big)\bigg)$$

and then make the above expansion. This procedure will be made rigorous by the second-order Boltzmann–Gibbs principle. We give a slightly more detailed outline of the proof.

Proof of Theorem 4.3. For the proof, we need to establish the second-order Boltzmann–Gibbs principle (Theorem 4.4), which is a combination of averaging due to ergodicity and Taylor expansion. It guarantees the replacement under space–time average of a function $f = f(\boldsymbol{\eta})$, whose ensemble average and its first derivatives vanish at $\mathbf{a}_0$, by quadratic function of $\eta^i - a^i$, in equilibrium $\nu_{\mathbf{a}_0}$. For the identification of the limit, we use the uniqueness of stationary coupled KPZ–Burgers martingale solutions obtained in [32]. In the limit SPDE, the drift term with c^i can be killed by the spatial shift as noted below (4.8) (at the level of KPZ equation) so that we assume $c^i = 0$ for simplicity. We also show the tightness of $\{Y_t^N\}_N$ in the uniform topology in $D([0,T], \mathcal{S}'(\mathbb{T})^n)$.

For $\zeta = (\zeta_x)$, the sample average of ζ around x in size $\ell \geq 1$ is defined by $\zeta_x^{(\ell)} := \frac{1}{2\ell+1}\sum_{|y|\leq \ell} \zeta_{x+y}$. The ensemble average of f is denoted by $\langle f\rangle(\mathbf{a}) = E^{\nu_{\mathbf{a}}}[f]$.

Theorem 4.4. *Let $f = f(\boldsymbol{\eta}) \in L^5(\nu_{\mathbf{a}_0})$ be a local function supported on sites $|y| \leq \ell_0$ such that $\langle f\rangle(\mathbf{a}_0) = 0$ and $\partial_{a^i}\langle f\rangle(\mathbf{a}_0) = 0$ for all i. Then, there exists $C = C(\ell_0) > 0$ such that for $T > 0$, $\ell \geq \ell_0$, and $\psi : \mathbb{T}_N \to \mathbb{R}$, we have*

$$
\begin{aligned}
E^{\nu_{\mathbf{a}_0}}\Bigg[\sup_{0\leq t\leq T}\Bigg(\int_0^t ds \sum_{x\in\mathbb{T}_N} \psi_{x-[cs]}\Bigg(f\big(\tau_x \boldsymbol{\eta}^N(s)\big) - \frac{1}{2}\sum_{j,k=1}^n \partial_{a^j}\partial_{a^k}\langle f\rangle(\mathbf{a}_0) \\
\times \Big\{\big((\eta^{N,j})_x^{(\ell)}(s) - a_0^j\big)\big((\eta^{N,k})_x^{(\ell)}(s) - a_0^k\big) - \frac{V_{jk}(\mathbf{a}_0)}{2\ell+1}\Big\}\Bigg)\Bigg)^2\Bigg] \\
\leq C\|f\|_{L^5(\nu_{\mathbf{a}_0})}^2 \Bigg(\frac{T\ell}{N}\|\psi\|_{L^2(\mathbb{T}_N)}^2 + \frac{T^2N^2}{\ell^3}\|\psi\|_{L^1(\mathbb{T}_N)}^2\Bigg),
\end{aligned}
$$

where $(V_{jk}(\mathbf{a}_0)) = \mathrm{cov}(\nu_{\mathbf{a}_0})$ and $\|\psi\|_{L^p(\mathbb{T}_N)} = (\frac{1}{N}\sum_{x\in\mathbb{T}_N}|\psi_x|^p)^{1/p}$, $p \geq 1$.

Proof. We apply Kipnis–Varadhan estimate to reduce the dynamic problem to a static one (bound by H^{-1}-norm): roughly for a mean-zero function F,

$$
E^{\nu_{\mathbf{a}_0}}\Bigg[\sup_{0\leq t\leq T}\bigg(\int_0^t F\big(\boldsymbol{\eta}(s)\big)ds\bigg)^2\Bigg] \leq CTE^{\nu_{\mathbf{a}_0}}\big[F\cdot\big(-L_N^{\mathrm{sym}}\big)^{-1}F\big], \tag{4.19}
$$

where L_N^{sym} is the symmetric part of the generator L_N of $\boldsymbol{\eta}(t)$. To estimate the H^{-1}-norm by L^2-norm, we apply the spectral gap estimate of the operator $-L_N^{\mathrm{sym}}$, but this works only on a bounded region and depends on the size of this region. Let $L_{\mathbf{k},\ell}^{\mathrm{sym}}$ be the symmetrized generator on $\Lambda_\ell = \{x; |x| \leq \ell\}$ with particles numbered $\mathbf{k}$ on Λ_ℓ, and let $W(\mathbf{k},\ell)$ be the inverse of the spectral gap of $-L_{\mathbf{k},\ell}^{\mathrm{sym}}$. Then, one can show $E^{\nu_{\mathbf{a}}}[W(\mathbf{k},\ell)^2] \leq C\ell^4$. We need some assumption on $(g_i)_{i=1}^n$ to show this. So, we need to confine ourselves to a bounded region of size ℓ by conditioning, that is, under the canonical ensemble.

Then we give static estimates. More precisely, we give a decay estimate for the canonical average as $\ell \to \infty$ to get grandcanonical average called the equivalence of ensembles (shown by applying the local CLT, see the first "$\sim$" below) and also Taylor expansion (see the second "∼"). To give some feeling, for $y \in \mathbb{R}^n$, we present

$$E^{\nu_{\mathbf{a}_0}}\big[f(\boldsymbol{\eta})|\boldsymbol{\eta}^{(\ell)} = y\big] = \frac{E^{\nu_{\mathbf{a}_0}}[f(\boldsymbol{\eta}) \cdot 1_{\{\boldsymbol{\eta}^{(\ell)}=y\}}]}{\nu_{\mathbf{a}_0}(\boldsymbol{\eta}^{(\ell)} = y)} \sim E^{\nu_y}\big[f(\boldsymbol{\eta})\big]$$
$$\sim \langle f\rangle(\mathbf{a}_0) + \nabla_{\mathbf{a}}\langle f\rangle(\mathbf{a}_0) \cdot (y - \mathbf{a}_0) + \frac{1}{2}\big(y - \mathbf{a}_0, D_{\mathbf{a}}^2\langle f\rangle(\mathbf{a}_0)(y - \mathbf{a}_0)\big) + \cdots .$$

This leads to (the static version of) Theorem 4.4 by taking $y = \boldsymbol{\eta}^{(\ell)}$. ■

To characterize the limit, we apply the martingale problem approach called energy solution, especially, its uniqueness. The authors of [32] established the uniqueness of stationary energy solutions satisfying Yaglom reversibility, that is, the time reversed process has the negative nonlinear drift compared to the original process.

Indeed, the coupled KPZ–Burgers equation for $Y^i = \partial_v h^i$ for h satisfying (4.7) in canonical form is written as

$$\partial_t Y^i = \frac{1}{2}\partial_v^2 Y^i + \frac{1}{2}\Gamma_{jk}^i \partial_v (Y^j Y^k) + \partial_v \dot{W}^i(t, v), \quad v \in \mathbb{T},\ 1 \le i \le n. \tag{4.20}$$

Its formal generator is given by $\mathscr{L} = \mathscr{L}_0 + \mathscr{A}$, where

$$\mathscr{L}_0\Phi(Y) = \frac{1}{2}\sum_i \bigg(\int_{\mathbb{T}} \partial_v^2 D_{Y^i(v)}^2 \Phi\, dv + \int_{\mathbb{T}} \partial_v^2 Y^i(v) \cdot D_{Y^i(v)}\Phi\, dv\bigg),$$
$$\mathscr{A}\Phi(Y) = \frac{1}{2}\sum_{i,j,k} \Gamma_{jk}^i \int_{\mathbb{T}} \partial_v\big(Y^j(v)Y^k(v)\big) D_{Y^i(v)}\Phi\, dv,$$

for $\Phi = \Phi(Y)$ and D, D^2 denoting the Fréchet derivatives. The authors of [32] gave the precise definition of $\mathscr{L}$ and its domain $\mathscr{D}(\mathscr{L})$. Then they showed that the Kolmogorov backward equation $\partial_t\Psi = \mathscr{L}\Psi$ is solvable in the paracontrolled sense in $\Psi = \Psi(t, Y) \in \mathscr{D}(\mathscr{L})$ for a wide class of initial values $\Psi(0) = \Psi_0$. In particular, this shows the uniqueness for the $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$-martingale problem as follows: $\Phi(t, Y_t) - \Phi(0, Y_0) - \int_0^t (\partial_s\Phi + \mathscr{L}\Phi)(s, Y_s)ds$ is a martingale for $\Phi(t, \cdot) \in \mathscr{D}(\mathscr{L})$. Take $\Phi(t, Y) = \Psi(T - t, Y)$, $t \in [0, T]$, with the solution Ψ of the Kolmogorov equation. Then, $\Psi(T - t, Y_t) - \Psi(T, Y_0)$ is a martingale. Take $t = T$, and we have $E_{Y_0}[\Psi_0(Y_T)] = \Psi(T, Y_0)$. This shows the uniqueness.

They further showed that the stationary solution of the cylinder function martingale problem, that is, instead of $\Phi \in \mathscr{D}(\mathscr{L})$ the martingale property holds for tame functions $\Phi(Y) = f(\langle Y, \psi_1\rangle, \ldots, \langle Y, \psi_n\rangle)$ satisfying Kipnis–Varadhan estimate, is a solution of the $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$-martingale problem.

The proof of Theorem 4.3 is completed by combining all these arguments. ■

The coupling constants $\Gamma_{jk}^i(\mathbf{a}_0)$ in our coupled KPZ–Burgers equation (4.18) satisfy the trilinear condition (T) after rewriting it in a canonical form (4.20) by a proper change of the time and magnitude taking $c^i = 0$. The scaling limit of Y^N under the product measure $\nu_{\mathbf{a}_0}$ is the "spatial Gaussian white noise" (at Burgers' level), so that this is consistent in view of Theorem 4.2(2).

4.3. Related results

4.3.1. Stochastic eight-vertex model

Funaki et al. [22] introduced the stochastic eight-vertex model motivated by the eight-vertex model in statistical mechanics. It is a totally asymmetric discrete time particle system on $\mathbb{Z}$ with jumps to one of the consecutive vacant sites on the right. Moreover, a Glauber-type mechanism, that is, creation of pair of particles and annihilation of colliding two particles, is allowed. A new type of KPZ–Burgers equation is obtained in the scaling limit for a properly defined fluctuation field:

$$\partial_t Y = \frac{\nu_*}{2}\partial_v^2 Y - \frac{\kappa_*}{2}\partial_v Y^2 - \frac{\lambda_*}{2} Y + \sqrt{D_1}\partial_v \dot{W}^1(t,v) + \sqrt{D_2}\dot{W}^2(t,v), \quad v \in \mathbb{R}, \tag{4.21}$$

with some constants $\nu_*, \kappa_*, \lambda_*, D_1, D_2 > 0$ satisfying the Einstein relation and two independent space–time Gaussian white noises $\dot{W}^1$ and $\dot{W}^2$.

4.3.2. Related singular quasilinear SPDE

Under the hydrodynamic limit for a zero-range process on $\mathbb{T}_N$ in a random environment, first for smeared one and then removing it, we obtain the singular quasilinear SPDE for the limit density $u = u(t,v)$ of particles:

$$\partial_t u = \partial_v^2 \varphi(u) + \partial_v\{\varphi(u)\dot{W}(v)\}, \quad v \in \mathbb{T}, \tag{4.22}$$

where $\dot{W}(v)$ is the spatial Gaussian white noise; compare with (4.12) taking $c^i = c^i(v)$ and moving it in the inside of ∂_v. In Funaki and Xie [27], after proving the global-in-time well-posedness in paracontrolled sense for (4.22) (i.e., $u \in C([0,\infty), \mathcal{C}^{\alpha-1})$, $\alpha \in (\frac{13}{9}, \frac{3}{2})$), the asymptotic behavior of the solution $u(t)$ as $t \to \infty$ was studied at least when $W(v)$ is nearly periodic. The equation has the conserved quantity $m = \int_{\mathbb{T}} u(t,v)dv$ and the limit as $t \to \infty$ is uniquely determined for each $m \in \mathbb{R}$. In Funaki et al. [20], a more general SPDE with second $\varphi(u)$ replaced by another function $\chi(u)$ was studied.

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