ON THE UNIVERSALITY FROM INTERACTING PARTICLE SYSTEMS

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Dedicated to my mother.

ABSTRACT

In these notes, we review recent results for the limiting behavior of equilibrium fluctuations of interacting particle systems with one or several conserved quantities. Two main classes of models are considered. First, the weakly asymmetric simple exclusion process, a model with one conservation law, and whose fluctuations cross from the Edwards–Wilkinson (EW) universality class to the Kardar–Parisi–Zhang (KPZ) universality class. Second, we consider a class of Hamiltonian systems perturbed by a noise and conserving two quantities. In the case of an exponential potential, the transition occurs from diffusion to fractional $\frac{5}{3}$ behavior, while for a harmonic potential the fluctuations cross from diffusive to fractional $\frac{3}{2}$ behavior. We review two different methods which rigorously prove some of the aforementioned results.

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1. INTRODUCTION

Over the last years, there has been much progress in understanding the emergence of universality at the level of the macroscopic equations that rule the space-time evolution of the conserved quantities of 1-d interacting particle systems (IPS). To be concrete, we describe the simplest dynamics of an IPS, namely the exclusion process. In this process, particles evolve as 1-d continuous-time random walks, with the constraint that does not allow more than a particle per site at any given time. This means that after an exponential clock of parameter one, a particle jumps from x to y according to a transition probability $p(y-x)$. The number of particles in the system is conserved and one of the questions that one might ask is about starting from some configuration, how to figure out what is the typical configuration at any given time. A fundamental question in the IPS literature, known as *hydrodynamic limit*, is to describe the evolution of the distribution of the conserved quantities as a function of space and time in the thermodynamic limit, i.e., when the size of the system is taken to infinity. The hydrodynamic limit is nothing but a law of large numbers for the empirical measure associated with the conserved quantities of the system, in a suitable time-scale [20]. The limit is given by a partial differential equation (PDE) which can be parabolic, hyperbolic, or even of a fractional form.

Our focus on this article is to describe the limiting laws that appear when one looks at the deviations of the system from the hydrodynamical profile, therefore we are in the central limit theorem scaling. As expected, contrarily to the deterministic solution obtained in the law of large numbers, in this case, the fluctuations are described by some stochastic partial differential equation (SPDE), and the challenge is to derive and characterize it.

The explanation and characterization of anomalous behavior in 1-d nonequilibrium systems are challenging even when the interactions are on a finite size window. A way to characterize the behavior of systems that exhibit an anomalous behavior is by studying the dynamical structure function, describing the time-dependent fluctuations of the conserved quantities of the system. For systems with one conservation law, two universality classes can be obtained: the (Gaussian) universality class, whose scale-invariance is 1 : 2 : 4; and the superdiffusive KPZ universality class whose scale invariance is 1 : 2 : 3. The latter is conjectured to be the universal law for the fluctuations of models with some smoothing mechanism, slope-dependent growth speed, and short-range randomness, while the former should be universal for models without slope-dependence and therefore with Gaussian fluctuations.

For systems with more conservation laws, the situation is much more complicated and many other universality classes exist. To give some concrete examples, we start by considering the weakly asymmetric simple exclusion process (WASEP). By tuning the asymmetry, one can observe different limiting equations depending on whether the symmetry or the asymmetry is the dominant dynamics. In the case of nearest-neighbor jumps and with an asymmetry of order $O(\frac{1}{n^\kappa})$, where n is the scaling parameter, the crossover goes from a diffusive behavior (corresponding to the phase where the symmetry dominates, that is, $\kappa > \frac{1}{2}$) to a behavior given in terms of the stochastic Burgers equation (corresponding to the phase where both the symmetry and the asymmetry have the same impact, that is, for

$\kappa = \frac{1}{2}$). In the strong asymmetric regime, recent results show that the limiting behavior should be given in terms of the KPZ fixed point [22]. This has been rigorously proved only for totally asymmetric jumps and the current field, but the same behavior should be true for partially asymmetric jumps and even in the whole phase where the asymmetry dominates (corresponding to $\kappa \in [0, \frac{1}{2})$), see the upper dashed line in Figure 1.

The particle system described above has a unique conservation law—the number of particles—and therefore its analysis is much simpler when compared to systems with more conservation laws and whose hydrodynamic limit consists of a system of PDEs. Let us now describe the type of systems with several conservation laws that we focus on, namely the chains of oscillators. They consist of Hamiltonian systems that are perturbed by a conservative noise. In [7] it was introduced and studied from a numerical point of view, a class of Hamiltonian systems that present strong analogies with the standard chains of oscillators. These models, denoted by $(r(t), p(t))_{t\geqslant 0}$, can be described by considering two nonnegative potentials V and U and the equations of motion are given on x by

$$dp_x = \big(V'(r_{x+1}) - V'(r_x)\big)dt \quad \text{and} \quad dr_x = \big(U'(p_x) - U'(p_{x-1})\big)dt,$$

where p_x denotes the momentum of the particle x, q_x is its position, and $r_x = q_x - q_{x-1}$ is the deformation of the lattice at x. If we assume that $V = U$ and by mapping $\eta_{2x-1} = r_x$ and $\eta_{2x} = p_x$, the dynamics above can be rewritten as

$$d\eta_x(t) = \big(V'(\eta_{x+1}) - V'(\eta_{x-1})\big)dt.$$

With respect to the variables η, the energy of the system corresponds to $\sum_x V(\eta_x)$. To make the model mathematically tractable, the dynamics just described, which is purely deterministic, is perturbed by adding a noise that exchanges η_x with η_{x+1} at random exponentially distributed times, and this is done independently for each bond $\{x, x+1\}$. These models have two conserved quantities, the energy $\sum_x V(\eta_x)$ and the volume $\sum_x \eta_x$ and the analysis of their asymptotic behavior is much more intricate than for the case of models with just one conserved quantity as we described above. Examples of the Hamiltonian systems introduced above are the models of [1,3–6]. In those articles, it was studied the fluctuations of the conserved quantities, namely, the energy and the volume, starting the system from the invariant measure, which is of product form. By tuning the strength of the Hamiltonian dynamics by a factor $\frac{1}{n^\kappa}$ one can analyze the crossover fluctuations. The main problem when studying these Hamiltonian systems is that depending on the chosen potential, the volume and the energy can be linearly transported in the system, each one having its own velocity and living on its own time scale. This is the main problem in general when dealing with systems with more than one conservation law. However, there are cases in which the situation simplifies since the conserved quantities have the same velocity and they live on the same time scale, e.g., in [2], where multicomponent coupled equations have been obtained as scaling limits of the empirical measures of the conserved quantities for the multispecies zero-range process.

When dealing with systems with only one conservation law, there are not many doubts about the field that one should consider, *the field of the conserved quantity*. Nevertheless, in presence of more than one conservation law, since any linear combination of the conserved quantities is again conserved, the possibility on the choice of the fluctuation fields is wider.

In [26], with a focus on anharmonic chains of oscillators, it was developed the nonlinear fluctuating hydrodynamics theory for the equilibrium time-correlations of the conserved quantities of that model, see also [28] for previous results on the anomalous transport in 1-d Hamiltonian systems with an emphasis on the KPZ behavior. In those articles, there are analytical predictions, based on a mode-coupling approximation, for the form of the fluctuations of the conserved quantities. Depending on the value of the coupling constants many other universality classes pop up, besides the Gaussian and the KPZ, already seen in systems with only one conservation law. At the same time in [23], by analyzing coupled single-lane asymmetric simple exclusion processes, the authors obtained numerically some universality classes with several dynamical exponents for the two conserved quantities of the system. All the possible combinations of limits in that model are summarized in Table 1 of [23]. We apply in Sections 3.2 and 3.5 the strategy developed in [23,26] to compute the fluctuation fields for models of chains of oscillators with two different potentials, the exponential and the harmonic, respectively. We also describe in Section 3.3 an alternative way to compute these fields, based on the action of the generator on the conserved quantities of the system. Once the proper choice of the fluctuation fields is done, the next question is related to the predictions on the form of the fluctuations for those quantities. This means that one has to write down the equations for the time evolution of those fields and one has to close those equations in terms of those fields. But since each field evolves in a certain time scale (which is not necessarily the same for both), then one has to analyze the leading terms in the expansion of the equations in such a way that one can recover the limiting SPDE, and this can be a hard task. The strategy to do this, is to write down the instantaneous current of the system in terms of the field of the conserved quantities and this can be done by the so-called Boltzmann–Gibbs principle. This principle was introduced in [8] for systems with one conserved quantity and it states that any local field of the dynamics can be replaced (in a proper topology) by the fluctuation field of the conserved quantity. When one is looking at the fluctuation field of the conserved quantity of the WASEP described above, the aforementioned Boltzmann–Gibbs principle is sufficient to recover the SPDE satisfied by the limiting field in the regime where the symmetry dominates (i.e., $\kappa > \frac{1}{2}$), but when the asymmetry has the same impact as the symmetry, then the Boltzmann–Gibbs principle of [8] does not give any information about the limiting field. For that purpose a second-order Boltzmann–Gibbs principle has been derived in [12,14,15] which allows replacing any local field of the dynamics by the square of the fluctuation field of the conserved quantity of the system. By using this principle, it becomes simple to close the equation for the field of the conserved quantity and to recognize the SPDE satisfied by the limiting field. The proof for the second-order Boltzmann–Gibbs principle of [15] does not impose strong conditions on the underlying microscopic dynamics and allows obtaining the crossover fluctuations from a diffusive behavior to a behavior given by the stochastic Burgers equation, as described above for the WASEP, and, more generally, for any system which has an instantaneous current that can be written as a sum of polynomial functions. Therefore the application of this principle even to the Hamiltonian systems described above, allows getting information on the form of the fluctuations. But below the critical case, $\kappa = \frac{1}{2}$, nothing rigorous is known apart from the strong asymmetric regime and for a specific choice of the jump rates. We believe that the extension of this result to Hamiltonian systems will open a new way to obtain the fluctuations of the energy and the volume and to establish the precise dependence on the strength of the perturbing noise, at the level of the crossover from different SPDEs.

The description of the universality classes from Gaussian to KPZ is for one component systems and in the case of multicomponent systems, as the Hamiltonian systems just mentioned, the scenario is much less understood. In the last years, this problem has attracted a lot of interest in both the physics and the mathematics communities, since up to very recently, there was no theoretical explanation and the numerical simulations were too controversial [10,21].

As mentioned above, in [23,26,27] with the so-called nonlinear fluctuating hydrodynamics theory, which has been developed during the last years, the authors proposed a rich and complex phase diagram of the universality classes for the aforementioned Hamiltonian systems. The richness of the diagram is explained by the nontrivial nonlinear couplings, occurring at different time scales, between the conserved quantities. This means, as mentioned above, that each conserved quantity will have its own velocity in a certain time scale and its own limiting SPDE ruling its fluctuations. This results have been, proved rigorously in the context of harmonic chains perturbed by a conservative noise (see [4,5,18]) and also for a 1-d infinite chain of coupled charged harmonic oscillators with a magnetic field [24]. The predictions in [26,27] are done starting from the macroscopic equation, which is assumed to be an Euler equation given by $\partial_t \varrho + \partial_x \langle j \rangle_\varrho = 0$, where $\varrho = (\varrho_1, \varrho_2, \varrho_3)$ is the vector whose ith ($i = 1, 2, 3$) component represents one of the conserved quantities of the system, j is the vector whose ith component represents the instantaneous current of the system for one of the conserved quantities and $\langle \cdot \rangle_\varrho$ represents the average with respect to the invariant state with parameter ϱ. By linearizing the equation, one arrives at $\partial_t \varrho + A \partial_x \varrho = 0$, and by adding a noise term and a dissipating term, one gets $\partial_t \varrho + \partial_x [A\varrho - D\partial_x \varrho + BW] = 0$, where W is an n-d white noise and A and D are matrices. For the dynamical correlation function given by $S(i,t) = \langle j_i(t), j_0(0) \rangle_\varrho$, one has $\sum_i S(i,t) = ACt$, where C is another matrix related to A and B. By taking certain ansatz for the matrices, one can predict many universality classes (only in the strong asymmetric regime). Besides the predictions not being mathematically rigorous, they bring up a new insight to approach the problem from the mathematical point of view: in order to study the fluctuations of systems with multicomponent conserved quantities, one has to look at a proper linear combination of the fields of the conserved quantities.

In [26,27] the authors give very detailed predictions about the correct time scales that one should see a nontrivial behavior for each one of the conserved quantities, and more than that, they also predicted what are the limiting processes that one is searching for. They did it for the models of chains of oscillators, but these models should have the same asymptotic behavior as the dynamics introduced above with only two conserved quantities. According to their predictions, one can get conserved quantities with a Gaussian behavior, or a KPZ behavior, or a fractional behavior given in terms of Lévy processes with a certain exponent that depends on the dynamics.

We highlight that the list of universality classes is not exhausted by those described above, as the EW [11], the KPZ, or those given by Lévy processes. Recently in [9], the authors analyzed a temperature-dependent model (that for zero temperature gives the classical ballistic deposition model) and the ∞-temperature version is a random interface, that does not belong to any of the universality classes mentioned above. Its scaling limit is given by the Brownian Castle, a renormalization fixed point, whose scale-invariance is given by $1 : 1 : 2$, distinct from both the EW or the KPZ classes.

Here we report two ways of rigorously obtaining some of the universality classes mentioned above. We present the methods for the WASEP and also for the model of chains of oscillators as described above for two different potentials. With this, we establish the existence of crossover lines, by tuning the parameter κ, in the phase diagram connecting some universality classes.

2. EXCLUSION PROCESS: A PROTOTYPE MODEL WITH ONE CONSERVATION LAW

We start by explaining in detail a model whose dynamics conserves one quantity, namely, the density of particles. Our prototype model is the exclusion process which we denote by $\eta(t)$. We consider the process evolving on the discrete torus $\mathbb{T}_n = \{0, 1, \dots, n-1\}$ and its dynamics can be described as follows. Each particle waits an exponential time of parameter 1 and then it jumps to a site according to a certain probability transition rate $p(\cdot)$. The exclusion rule dictates that the jump of a particle is performed if and only if the destination site is empty, otherwise nothing happens and the particle waits a new random time. The space state of this process is $\{0,1\}^{\mathbb{T}_n}$ and a configuration is denoted by $\eta = \{\eta_x \in \{0,1\} : x \in \mathbb{T}_n\}$. We denote the jump rate from the site x to the site y by $p(x,y) = p(y-x)$, and note that $p(\cdot)$ only depends on the size of the jump and not on the exact location where the jump is performed. When jumps are allowed only to nearest-neighbor sites the process is simple, so that $p(z) = 0$ if $|z| > 1$. We make the following choice $p(-1) = 1 - p(1)$ and $p(1) = p + \frac{E}{n^\kappa}$, where p, E, and κ are constants. If $E = 0$ (no dependence on κ) and $p = \frac{1}{2}$, we get the symmetric simple exclusion process (SSEP); if $E = 0$ and $p \neq \frac{1}{2}$, we get the asymmetric simple exclusion process (ASEP), and if $E \neq 0$ and $p = \frac{1}{2}$, the process is the WASEP. Observe that the parameter κ rules the strength of the asymmetry, that is, the higher the value of κ, the weaker the asymmetry. Its infinitesimal generator is given on functions $f : \{0,1\}^{\mathbb{T}_n} \to \mathbb{R}$ by

$$\mathscr{L}^{ex} f(\eta) = \sum_{x \in \mathbb{T}_n} \{p(1)\eta_x(1-\eta_{x+1}) + p(-1)\eta_{x+1}(1-\eta_x)\}\big(f(\eta^{x,x+1}) - f(\eta)\big), \quad (2.1)$$

where $\eta^{x,x+1}$ is the configuration obtained from the configuration η by swapping the occupation variables η_x and η_{x+1}:

$$\eta_y^{x,x+1} = \eta_{x+1}\mathbf{1}_{y=x} + \eta_x \mathbf{1}_{y=x+1} + \eta_y \mathbf{1}_{y \neq x,x+1}. \quad (2.2)$$

The system is speeded up in the time scale tn^a, where $a > 0$ is a constant. A simple computation shows that $\mathscr{L}^{ex}(\eta_x) = j_{x-1,x}(\eta) - j_{x,x+1}(\eta)$ where

$$j_{x,x+1}(\eta) = \eta_x(1-\eta_{x+1})\left(p + \frac{E}{n^\kappa}\right) - \eta_{x+1}(1-\eta_x)\left(1 - p - \frac{E}{n^\kappa}\right). \tag{2.3}$$

The system conserves one quantity—the *number of particles*, $\sum_{x\in\mathbb{T}_n} \eta_x$. The invariant measures are denoted by ν_ϱ and are, in fact, Bernoulli product measures of parameter ϱ:

$$\nu_\varrho(d\eta) = \prod_{x\in\mathbb{T}_n} \varrho^{\eta_x}(1-\varrho)^{1-\eta_x} \tag{2.4}$$

for $\varrho \in (0,1)$. In the case $E = 0$ and $p = \frac{1}{2}$, these measures are also reversible.

2.1. Hydrodynamic limit

The trajectories of the process live on the Skorohod space $\mathcal{D}([0,T], \{0,1\}^{\mathbb{T}_n})$ and $\mathbb{P}_{\mu_n}$ is the probability measure on that space induced by an initial measure μ_n and by the process $\{\eta(tn^a)\}_{t\geqslant 0}$. The expectation with respect to $\mathbb{P}_{\mu_n}$ is denoted by $\mathbb{E}_{\mu_n}$. We define the empirical measure associated to the density by

$$\pi^n(\eta, du) := \frac{1}{n}\sum_{x\in\mathbb{T}_n} \eta_x \delta_{\frac{x}{n}}(du),$$

where $\delta_{\frac{x}{n}}$ is a Dirac mass on $\frac{x}{n} \in \mathbb{T}$ and $\pi^n_t(\eta, du) := \pi^n(\eta(tn^a), du)$. The statement of the hydrodynamic limit can be rigorously stated as follows. We assume that the process starts from a probability measure μ_n for which a law of large numbers holds, i.e., the sequence of random measures $\pi^n_0(\eta, du)$ converges, in probability with respect to μ_n and when n is taken to infinity, to the deterministic measure $\varrho(0,u)du$, where the density $\varrho(0,u)$ is a measurable function. The claim in the hydrodynamic limit is that under the previous assumption, the same result holds at any time t, that is, the random measure $\pi^n_t(\eta, du)$ converges, in probability with respect to the distribution of the process at time t and when n is taken to infinity, to a deterministic measure $\varrho(t,u)du$. The function $\varrho(t,u)$ is the solution (usually in a weak sense) of a PDE, which is called, the *hydrodynamic equation* of the system. For the exclusion processes introduced above, one can get as hydrodynamic equations: for the SSEP and by rescaling time diffusively $a = 2$, the heat equation given by

$$\partial_t \varrho(t,u) = \frac{1}{2}\Delta\varrho(t,u).$$

For the WASEP with $\kappa = 1$ (resp. for the ASEP) and by rescaling time diffusively $a = 2$ (resp., in the hyperbolic scale $a = 1$), the viscous Burgers equation (resp., the inviscid Burgers equation),

$$\partial_t \varrho(t,u) = \frac{1}{2}\Delta\varrho(t,u) + (1-2E)\nabla F\big(\varrho(t,u)\big),$$
$$\partial_t \varrho(t,u) = (1-2E)\nabla F\big(\varrho(t,u)\big),$$

where $F(\varrho) = \varrho(1-\varrho)$. The last result is a Law of Large Numbers for the unique conserved quantity of the system, i.e., the density. Now the natural question that comes next is related to the fluctuations around the obtained hydrodynamical profile. Moreover, we could ask if there are equations that can be obtained for different dynamics which share common grounds. If so, what are their form and how do they relate? In these notes, we will explain two different methods that allow obtaining some answers in this direction.

2.2. Fluctuations

We consider the system starting from the invariant measure, which, for the model under investigation, is of product form and homogeneous, see (2.4). We define the empirical field associated to the conserved quantity—the *density fluctuation field*—which is the linear functional acting on functions f as

$$\mathcal{Y}_t^n(f) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{T}_n} f\left(\frac{x}{n}\right) \overline{\eta}_x(tn^a),$$

where $\overline{\eta}_x := \eta_x(tn^a) - \varrho$. The last identity is obtained by first integrating f with respect to the density empirical measure $\pi_t^n(\eta, du)$, then removing its mean (with respect to the invariant state) and then multiplying it by $\sqrt{n}$. The question that arises now is to understand the limit in distribution, as $n \to +\infty$, of $\mathcal{Y}_t^n$ that we denote by $\mathcal{Y}_t$. For the exclusion processes introduced above, one can get for the SSEP and rescaling time diffusively $a = 2$, the Ornstein–Uhlenbeck (OU) equation given by

$$d\mathcal{Y}_t = \frac{1}{2} \Delta \mathcal{Y}_t dt + \sqrt{F(\varrho)} \nabla \mathcal{W}_t.$$

For the WASEP with $\kappa > \frac{1}{2}$ and rescaling time diffusive $a = 2$, one can get exactly the same OU equation as in the symmetric case, while for $\kappa = \frac{1}{2}$ and still rescaling time diffusively $a = 2$, the KPZ equation (introduced in [19]) or its companion, namely the stochastic Burgers (SB) equation, respectively, for the height fluctuation field and the density fluctuation field,

$$d\mathcal{h}_t = \frac{1}{2} \Delta \mathcal{h}_t dt + 4E(\nabla \mathcal{h}_t)^2 dt + \sqrt{F(\varrho)} \mathcal{W}_t,$$
$$d\mathcal{Y}_t = \frac{1}{2} \Delta \mathcal{Y}_t dt + 4E \nabla \mathcal{Y}_t^2 dt + \sqrt{F(\varrho)} \nabla \mathcal{W}_t.$$

Above $\mathcal{W}_t$ is a space-time white-noise. Last results were proved in [14] and [13]. For the case $E = 0$, $p \neq 1/2$ and taking the system in the hyperbolic time scale $a = 1$, one can get

$$d\mathcal{Y}_t = (1 - 2\varrho)(1 - 2p) \nabla \mathcal{Y}_t dt.$$

Observe that in the last equation if we consider $\varrho = \frac{1}{2}$, we get a trivial evolution for the density field. The same result would be obtained if, instead of choosing $\varrho = \frac{1}{2}$, we redefine the field in a frame with the velocity $(1 - 2\varrho)n^{a-1}$. To simplify the presentation we consider $\varrho = \frac{1}{2}$ in what follows. Therefore, to get a nontrivial behavior, one has to speed up the time and in that case, and for the choice $a = \frac{3}{2}$, the limiting field should be given in terms of the KPZ fixed point, see [22]. In [12] it was proved that, up to the time scale $a = \frac{4}{3}$, there is no evolution of the field; beyond that time scale, the limit of this field is not known yet, but it should be given in terms of the KPZ fixed point. The results in [12] applied to the WASEP show that below the line $a = \frac{4}{3}(\kappa + 1)$ there is no time evolution of the limiting field, but the trivial evolution should go up to the line $a = \kappa + \frac{3}{2}$. Last results are summarized in Figure 1.

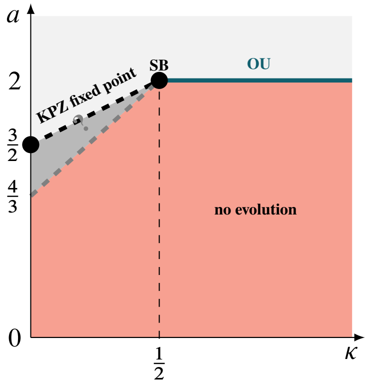

FIGURE 1
Density fluctuations.

The starting point to prove the latter results (that we now restrict to WASEP with $\varrho = \frac{1}{2}$) is to use Dynkin's formula, so that for $f \in C^2(\mathbb{T})$, where $\mathbb{T}$ denotes the one-dimensional torus,

$$\mathcal{M}_t^n(f) = \mathcal{Y}_t^n(f) - \mathcal{Y}_0^n(f) - \int_0^t (\partial_s + n^a \mathcal{L}^{ex}) \mathcal{Y}_s^n(f) ds, \tag{2.5}$$

is a martingale with respect to the natural filtration of the process. We say that it is the martingale associated to the field $\mathcal{Y}_t^n(f)$. A simple computation shows that

$$\mathcal{M}_t^n(f) = \mathcal{Y}_t^n(f) - \mathcal{Y}_0^n(f) - \frac{n^a}{2n^2} \mathcal{S}_t^n(f) - \frac{E n^a}{n^{\frac{3}{2}+\kappa}} \mathcal{A}_t^n(f).$$

Above, the contribution of the symmetric and asymmetric parts of the dynamics are respectively given by

$$\mathcal{S}_t^n(f) = \int_0^t \mathcal{Y}_s^n(\Delta_n f) ds \quad \text{and} \quad \mathcal{A}_t^n(f) = \int_0^t \sum_{x \in \mathbb{T}_n} \nabla_n f\left(\frac{x}{n}\right) \overline{\eta}_x(sn^a) \overline{\eta}_{x+1}(sn^a) ds,$$

where Δ_n and ∇_n, respectively, denote the discrete Laplacian and discrete derivative

$$\Delta_n f\left(\frac{x}{n}\right) = n^2 \left\{ f\left(\frac{x+1}{n}\right) - 2f\left(\frac{x}{n}\right) + f\left(\frac{x-1}{n}\right) \right\} \quad \text{and}$$
$$\nabla_n f\left(\frac{x}{n}\right) = n \left\{ f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right) \right\}.$$

A simple computation shows that the quadratic variation is given by

$$\langle \mathcal{M}^n(f) \rangle_t = \int_0^t \frac{n^a}{n^3} \left(\frac{1}{2} + \frac{E}{n^\kappa} \right) \sum_{x \in \mathbb{T}_n} \left(\nabla_n f\left(\frac{x}{n}\right) \right)^2 \left(\eta_x(sn^a) - \eta_{x+1}(sn^a) \right)^2 ds,$$

so that if $a = 2$ we get $\lim_{n\to\infty} \mathbb{E}_{\nu_\varrho}[(\mathcal{M}_t^n(f))^2] = tF(\varrho) \int (\nabla f(u))^2 du$, while for $a < 2$ it vanishes. To close the equations for the density fluctuation field, one just has to analyze the integral terms in the martingale above. The term coming from the symmetric part of the dynamics is simple since it is already written in terms of the fluctuation field $\mathcal{Y}_s^n$. A simple computation shows that the variance of that term is of order $O(n^{2(a-2)})$, so that it converges if $a = 2$, but for $a < 2$ it vanishes. The most complicated term is that coming from the asymmetric part of the dynamics, namely, the term $\mathcal{A}_t^n(f)$. Our tool to analyze the variance of this term is given in terms of a $\mathcal{H}_{-1}$-norm estimate, stated in Theorem 4 of [3] as

$$\mathbb{E}_{\nu_\varrho}\left[\left(\int_0^t \sum_{x\in\mathbb{T}_n} \nabla_n f\left(\frac{x}{n}\right)\overline{\eta}_x(sn^a)\overline{\eta}_{x+1}(sn^a)\,ds\right)^2\right] \leqslant \frac{C_f tn}{\sqrt{n^a}}. \tag{2.6}$$

From the latter result, the variance of the term involving $\mathcal{A}_t^n(f)$ is of order $O(n^{\frac{3}{2}a-2(\kappa+1)})$, so that it vanishes when $a < \frac{4}{3}(\kappa+1)$. In the diffusive time scale, that term vanishes for any $\kappa > \frac{1}{2}$, while for $\kappa = \frac{1}{2}$ we can use the second-order Boltzmann–Gibbs principle proved in [14,15]. This principle states that for any $t \in [0,T]$, any positive integer n, and any $\varepsilon \in (0,1)$, it holds

$$\mathbb{E}_{\nu_\varrho}\left[\left(\mathcal{A}_t^n(f) - \int_0^t \frac{1}{n}\sum_{x\in\mathbb{T}_n}\nabla_n f\left(\frac{x}{n}\right)\left(\mathcal{Y}_s^n\left(\iota_\epsilon\left(\frac{x}{n}\right)\right)\right)^2 ds\right)^2\right] \leqslant C_f T\left(\varepsilon + \frac{t}{\varepsilon^2 n}\right), \tag{2.7}$$

where for $u \in (0,1)$ and $y \in (0,1)$ we have $\iota_\epsilon(u)(y) = \mathbf{1}_{(u,u+\epsilon]}(y)$. From the latter results we see that for $a < \inf(\frac{4}{3}(\kappa+1), 2)$ we have $\mathcal{Y}_t^n(f) = \mathcal{Y}_0^n(f)$ plus terms that vanish in the $\mathbb{L}^2(\mathbb{P}_{\nu_\varrho})$-norm as $n \to +\infty$, so that the limit field has a trivial evolution. We note that in fact last result should be true for $a < \inf(\frac{3}{2}+\kappa, 2)$ but this has not been proved, yet. When $a = 2$ and $\kappa > \frac{1}{2}$, we see that

$$\mathcal{M}_t^n(f) = \mathcal{Y}_t^n(f) - \mathcal{Y}_0^n(f) - \int_0^t \mathcal{Y}_s^n(\Delta_n f)ds$$

plus terms that vanish in the $\mathbb{L}^2(\mathbb{P}_{\nu_\varrho})$-norm as $n \to +\infty$, so that we get the OU equation; while for $\kappa = \frac{1}{2}$, we get

$$\begin{aligned}\mathcal{M}_t^n(f) = \mathcal{Y}_t^n(f) - \mathcal{Y}_0^n(f) - \int_0^t \mathcal{Y}_s^n(\Delta_n f)ds \\ + \int_0^t \frac{1}{n}\sum_{x\in\mathbb{T}_n}\nabla_n f\left(\frac{x}{n}\right)\left(\mathcal{Y}_s^n\left(\iota_\epsilon\left(\frac{x}{n}\right)\right)\right)^2 ds\end{aligned}$$

plus terms that vanish in the $\mathbb{L}^2(\mathbb{P}_{\nu_\varrho})$-norm as $n \to +\infty$, so that we get the SB equation. In fact, with a little more effort one can get an energy solution to the SB as introduced in [14] and [15], for which the uniqueness as been proved in [17] by using paracontrolled calculus, see [16]. In the next subsection we analyze the same problem for a model with two conservation laws.

3. A PROTOTYPE MODEL WITH TWO CONSERVATION LAWS

Fix once and for all a positive real parameter $b > 0$. We start with the 1-d potential $V_b : \mathbb{R} \to [0, +\infty)$ defined by $V_b(u) = e^{-bu} - 1 + bu$ and we consider the Markov process $\{\eta_x(t)\}_{t\geqslant 0}$ with state space $\Omega_n := \mathbb{R}^{\mathbb{T}_n}$, whose infinitesimal generator is denoted by $\mathcal{L}$ and is given by $\mathscr{L} = \alpha_n \mathscr{A}_b + \gamma \mathcal{S}$, where $\gamma > 0$, $\alpha_n = \alpha n^{-\kappa}$, $\alpha \in \mathbb{R}$, and $\kappa > 0$. The operators $\mathscr{A}_b$ and $\mathcal{S}$ act on differentiable functions $f : \Omega_n \to \mathbb{R}$ as follows:

$$(\mathscr{A}_b f)(\eta) = \sum_{x \in \mathbb{T}_n} \big(V_b'(\eta_{x+1}) - V_b'(\eta_{x-1})\big)(\partial_{\eta_x} f)(\eta)$$

and

$$(\mathcal{S} f)(\eta) = \sum_{x \in \mathbb{T}_n} \big(f(\eta^{x,x+1}) - f(\eta)\big). \tag{3.1}$$

The system under investigation is a Hamiltonian system that is perturbed by a stochastic noise generated by $\mathcal{S}$. Above, the configuration $\eta^{x,x+1}$ is given in (2.2). The interested reader can find more details on these models in [3,7,27].

Observe that all the objects defined above should be indexed on the scaling parameter n, but in order to simplify notation we will omit the dependence on it. The parameter $\alpha_n = \alpha n^{-\kappa}$ regulates the intensity of the asymmetry in the system in terms of the scaling parameter n. The role of the parameter γ is to regulate the intensity of the stochastic noise. The system will be speed up in the time scale tn^a with $a > 0$. From the expression of the potential $V_b(u)$, we see that if $\xi_x = e^{-b\eta_x}$ then $V_b(\eta_x) = \xi_x - 1 + b\eta_x$. A simple computation shows that

$$\mathscr{L}\big(V_b(\eta_x)\big) = j^e_{x-1,x}(\eta) - j^e_{x,x+1}(\eta), \quad \mathscr{L}(\eta_x) = j^v_{x-1,x}(\eta) - j^v_{x,x+1}(\eta), \tag{3.2}$$

where

$$\begin{aligned} j^e_{x,x+1}(\eta) &= -\alpha_n b^2 \xi_x \xi_{x+1} + \alpha_n b^2(\xi_x + \xi_{x+1}) - \gamma \nabla\big(V_b(\eta_x)\big), \\ j^v_{x,x+1}(\eta) &= \alpha_n b(\xi_x + \xi_{x+1}) - \gamma \nabla \eta_x, \end{aligned} \tag{3.3}$$

and $\nabla \tau_x f(\eta) = \tau_{x+1} f(\eta) - \tau_x f(\eta)$, where $\tau_x f(\eta) = f(\tau_x \eta)$. The same is true for ξ_x, that is, $\mathscr{L}(\xi_x) = j^\xi_{x-1,x}(\eta) - j^\xi_{x,x+1}(\eta)$, where

$$j^\xi_{x,x+1}(\eta) = -\alpha_n b^2 \xi_x \xi_{x+1} - \gamma \nabla \xi_x. \tag{3.4}$$

We have two conserved quantities, namely *energy* and *volume*, $\sum_{x \in \mathbb{T}_n} V_b(\eta_x)$ and $\sum_{x \in \mathbb{T}_n} \eta_x$, see [7] where it is proved that, in some sense, they are the only conserved quantities. Observe that any linear combination (plus constants) of energy and volume is also conserved, as the quantity $\sum_{x \in \mathbb{T}_n} \xi_x$ which will be very relevant in what follows. The invariant measures of the process are denoted by $\mu_{\bar\beta,\bar\lambda}$ and are explicitly given by

$$\mu_{\bar\beta,\bar\lambda}(d\eta) = \prod_{x \in \mathbb{T}_n} \bar{Z}^{-1}(\bar\beta, \bar\lambda) \exp\big\{-\bar\beta e^{-b\eta_x} - \bar\lambda \eta_x\big\} d\eta_x, \tag{3.5}$$

for $\bar\beta, \bar\lambda > 0$, where $\bar{Z}(\bar\beta, \bar\lambda) = \Gamma(\bar\lambda/b)/(b\bar\beta^{\bar\lambda/b})$ is the normalization constant. Let us denote by $E_{\mu_{\bar\beta,\bar\lambda}}$ the expectation with respect to $\mu_{\bar\beta,\bar\lambda}$. We denote by $e := e(\bar\beta, \bar\lambda)$, $v := v(\bar\beta, \bar\lambda)$, and $\rho = \rho(\bar\beta, \bar\lambda)$ the averages of the quantities $V_b(\eta_x)$, η_x, and ξ_x with respect to $\mu_{\bar\beta,\bar\lambda}$, that is,

$$e = E_{\mu_{\bar\beta,\bar\lambda}}\big[V_b(\eta_x)\big], \quad v = E_{\mu_{\bar\beta,\bar\lambda}}[\eta_x], \quad \text{and} \quad \rho = E_{\mu_{\bar\beta,\bar\lambda}}[\xi_x]. \tag{3.6}$$

With the notations that we have just introduced, we see that

$$\begin{aligned} E_{\mu_{\bar\beta,\bar\lambda}}\big[j^e_{x,x+1}\big] &= -\alpha_n b^2(e-bv)^2+\alpha_n b^2 = -\alpha_n b^2\rho(1-2\rho),\\ E_{\mu_{\bar\beta,\bar\lambda}}\big[j^v_{x,x+1}\big] &= 2\alpha_n b(1+e-bv) = 2\alpha_n b\rho,\\ E_{\mu_{\bar\beta,\bar\lambda}}\big[j^\xi_{x,x+1}\big] &= -\alpha_n b^2\rho^2. \end{aligned} \tag{3.7}$$

Note that $\rho = \frac{\bar\lambda}{b\bar\beta}$ and $\tau^2 = \frac{\bar\lambda}{b\bar\beta^2}$.

3.1. Hydrodynamic limits

Now we describe the space-time evolution of the relevant quantities of the system. Therefore, for any configuration $\eta \in \Omega_n$, we define the empirical measures associated to the energy and the volume as $\pi^{n,e}(\eta, du)$ and $\pi^{n,v}(\eta, du)$ in $\mathbb{R}$ by

$$\pi^{n,e}(\eta,du) = \frac{1}{n}\sum_{x\in\mathbb{T}_n} V_b(\eta_x)\delta_{\frac{x}{n}}(du) \quad \text{and} \quad \pi^{n,v}(\eta,du) = \frac{1}{n}\sum_{x\in\mathbb{T}_n}\eta_x\delta_{\frac{x}{n}}(du),$$

and let $\pi^{n,\cdot}_t(\eta,du) := \pi^{n,\cdot}(\eta(tn^a),du)$. In [7], for $a=1$ and in the strong asymmetric regime, if $e_0:\mathbb{R}\to\mathbb{R}$ and $v_0:\mathbb{R}\to\mathbb{R}$ are measurable functions and if $\pi^{n,e}_0(\eta,du)\to^w_{n\to+\infty} e_0(u)du$ and $\pi^{n,v}_0(\eta,du)\to^w_{n\to+\infty} v_0(u)du$, where the convergence is in the weak sense and with respect to an initial measure μ_n, then the same result is true for any $t\in[0,T]$ (before the appearance of shocks), namely $\pi^{n,e}_t(\eta,du)\to^w_{n\to+\infty} e(t,u)du$ and $\pi^{n,v}_t(\eta,du)\to^w_{n\to+\infty} v(t,u)du$, where

$$\begin{cases} \partial_t e(t,u) - \alpha b^2\partial_u((e(t,u)-bv(t,u))^2) = 0,\\ \partial_t v(t,u) + 2\alpha b\partial_u(e(t,u)-bv(t,u)) = 0, \end{cases} \tag{3.8}$$

with initial conditions e_0 and v_0, respectively.

Our interest in these notes is to go beyond the hydrodynamic limit and ask about the form of the fluctuations around the hydrodynamical profile. The study of nonequilibrium fluctuations is usually very intricate, since it requires knowledge on the correlations of the system, therefore we restrict ourselves to the equilibrium scenario. So from now on, we assume that our Markov process starts from the invariant measure $\mu_{\bar\beta,\bar\lambda}$.

For systems with only one conservation law, there is no ambiguity in the choice of the fields that one should look at. When systems have more than one conserved quantity, and whose evolution is coupled, as is our case here, we have to be careful when we define those fields. In the next section, we apply the mode-coupling theory explained in detail in [27] and compute the correct fields that one should look at. We also explain in Section 3.3 another way of computing those fields just by employing Dynkin's formula.

3.2. Predictions from mode coupling theory

The system under investigation has two conserved quantities and it is possible to have an estimate on the scaling exponent and the form of the liming fluctuations by using the nonlinear fluctuating hydrodynamics theory. This is a macroscopic theory that requires only the knowledge of the hydrodynamic equations (3.8) in the hyperbolic time scale, we refer to [23, 26, 27]. Nevertheless, observe that this theory has only been developed in the strong asymmetric regime corresponding to $\kappa = 0$. Let us fix the quantities ξ_x and η_x, and recall (3.7). Assume $\rho \neq 0$. Consider the column flux matrix

$$j = \begin{pmatrix} E_{\mu_{\bar\beta,\bar\lambda}}[j^{\xi}_{x,x+1}] \\ E_{\mu_{\bar\beta,\bar\lambda}}[j^{v}_{x,x+1}] \end{pmatrix} = \begin{pmatrix} -\alpha_n b^2\rho^2 \\ 2\alpha_n b\rho \end{pmatrix}. \tag{3.9}$$

The Jacobian is thus given by

$$J = \begin{pmatrix} \partial_\rho E_{\mu_{\bar\beta,\bar\lambda}}[j^{\xi}_{x,x+1}] & \partial_v E_{\mu_{\bar\beta,\bar\lambda}}[j^{\xi}_{x,x+1}] \\ \partial_\rho E_{\mu_{\bar\beta,\bar\lambda}}[j^{v}_{x,x+1}] & \partial_v E_{\mu_{\bar\beta,\bar\lambda}}[j^{v}_{x,x+1}] \end{pmatrix} = \begin{pmatrix} -2\alpha_n b^2\rho & 0 \\ 2\alpha_n b & 0 \end{pmatrix}. \tag{3.10}$$

Now observe that the eigenvalues of last matrix are given by $v_1 := -2\alpha_n b^2\rho$ and $v_2 = 0$. This corresponds to the velocity that we should take in order to see the evolution of the fields. The corresponding eigenvectors are given by $\tau_1 = \left(\begin{smallmatrix} 1 \\ -\frac{1}{b\rho}\end{smallmatrix}\right)$ and $\tau_2 = \left(\begin{smallmatrix} 0 \\ \mathrm{c}\end{smallmatrix}\right)$, where c is a constant. To obtain the linear combination of the fields that one should look at, we need to find the matrix R that diagonalizes J, that is, $RJR^{-1} = \left(\begin{smallmatrix} v_1 & 0 \\ 0 & v_2\end{smallmatrix}\right)$. Observe that R^{-1} is the matrix whose columns are the eigenvectors of J so that

$$R^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{b\rho} & \mathrm{c} \end{pmatrix} \quad \text{and} \quad R = \frac{1}{\mathrm{c}}\begin{pmatrix} \mathrm{c} & 0 \\ \frac{1}{b\rho} & 1 \end{pmatrix}. \tag{3.11}$$

The free constant c is determined by the equation $RKR^{-1} = \mathbb{I}$, where $\mathbb{I}$ denotes the identity matrix. The matrix K is a symmetric matrix and it is called the compressibility matrix. According to the nonlinear fluctuating hydrodynamic theory, the quantities that we should look at are given by the identity $(\mathscr{U}_1, \mathscr{U}_2) = R(\overline{\xi}_x, \overline{\eta}_x)$, which gives $\mathscr{U}_1 = \overline{\xi}_x$ and $\mathscr{U}_2 = \frac{1}{\mathrm{c}}\frac{\overline{\xi}_x}{b\rho} + \overline{\eta}_x$. Therefore, the quantities $\mathscr{U}_1$ and $\mathscr{U}_2$ are the conserved quantities that we should look at and on a frame with velocity v_1 and v_2, respectively. Now let us see the predictions on the form of the fluctuations for each one of these quantities. Given the two entries of the matrix (3.10), we look now at the corresponding Hessians:

$$\mathscr{H}^1 = -2\alpha_n b^2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathscr{H}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.12}$$

The coupling constants, which are determined by the above matrices, are given on $i \in \{1,2\}$ by $G^i = \frac{1}{2}\sum_{j=1}^{2} R_{i,j}[(R^{-1})^{\dagger}\mathscr{H}^j R^{-1}]$ where $R_{i,j}$ is the entry of the matrix R. A simple computation shows that

$$\left[(R^{-1})^{\dagger}\mathscr{H}^1 R^{-1}\right] = -\alpha_n b^2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and from this we get

$$G^1 = -\alpha_n b^2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad G^2 = -\frac{\alpha_n b\mathrm{c}}{\rho}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From Section 2.2 of [27], we obtain for the strong asymmetric regime ($\kappa = 0$), since $G^1_{1,1} = 1$, $G^2_{2,2} = 0$, and $G^2_{1,1} = 1$, that the equilibrium fluctuations of the quantity $\mathscr{U}_1$ should be in the KPZ universality class, while the fluctuations of the quantity $\mathscr{U}_2$ should be described by a Lévy process with exponent $\frac{5}{3}$.

3.3. Choice of the fluctuations fields

Suppose the system starts from the invariant measure $\mu_{\bar\beta,\bar\lambda}$ as defined in (3.5). Recall also (3.6). In the same spirit as we defined above, we present now an alternative way to obtain the fields that we need to look at. Let us define on $f \in C^2(\mathbb{T})$ the quantities:

$$\mathcal{X}_t^n(f) = \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{T}_n} f\left(\frac{x}{n}\right)\big(\xi_x(tn^a) - \rho\big) \quad \text{and}$$
$$\mathcal{V}_t^n(f) = \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{T}_n} f\left(\frac{x}{n}\right)\big(\eta_x(tn^a) - v\big).$$

At this point it is important to recall (3.3) and (3.4) and to center them with respect to the invariant measure $\mu_{\bar\beta,\bar\lambda}$. We see that the centered currents become

$$j^{\xi}_{x,x+1}(\eta) = -\alpha_n b^2\bar\xi_x\bar\xi_{x+1} - \alpha_n b^2\rho\bar\xi_x - \alpha_n b^2\rho\bar\xi_{x+1} - \alpha_n b^2\rho^2 - \gamma\nabla\bar\xi_x, \tag{3.13}$$
$$j^{v}_{x,x+1}(\eta) = \alpha_n b(\bar\xi_x + \bar\xi_{x+1} + 2\rho) - \gamma\nabla\bar\eta_x. \tag{3.14}$$

Our starting point is Dynkin's formula, which allows us to associate the martingales to each field $\mathcal{X}_t^n(f)$ and $\mathcal{V}_t^n(f)$ as in (2.5). The important terms to analyze are the time integrals. Observe that since our test functions are time-independent the contributions from the terms ∂_s are null. Let us now check the contribution from the action of the generator. We start with the field $\mathcal{X}_t^n$. Note that from (3.13), since $\sum_{x\in\mathbb{T}_n}\nabla_n f(\frac{x}{n}) = 0$, and from a summation by parts, we have that

$$\begin{aligned} n^a\mathcal{L}\mathcal{X}_s^n(f) = {} & \frac{\gamma}{n^{2-a}}\mathcal{X}_s^n(\Delta_n f) + \frac{\alpha_n b^2\rho}{n^{2-a}}\mathcal{X}_s^n(\Delta_n f) + \frac{2b^2\rho\alpha_n}{n^{1-a}}\mathcal{X}_s^n(\nabla_n f) \\ & - \frac{b^2\alpha_n}{n^{3/2-a}}\sum_{x\in\mathbb{T}_n}\nabla_n f\left(\frac{x}{n}\right)\bar\xi_x(sn^a)\bar\xi_{x+1}(sn^a). \end{aligned} \tag{3.15}$$

At this point we stop with the computations for the ξ field and observe that by similar computations we obtain for the volume field

$$n^a\mathcal{L}\mathcal{V}_s^n(f) = \frac{\gamma}{n^{2-a}}\mathcal{V}_s^n(\Delta_n f) + \frac{2b\alpha_n}{n^{1-a}}\mathcal{X}_s^n(\nabla_n f) - \frac{b\alpha_n}{n^{2-a}}\mathcal{X}_s^n(\Delta_n f). \tag{3.16}$$

As one can see from the expansions above, the evolution of the $\mathcal{X}_t^n$ field is independent of the evolution of the $\mathcal{V}_t^n$ field but for the volume field, this is not the case. Let us now explain how to get a linear combination of the fields that one should focus on to have both fields drifting at the same velocity $\mathfrak{v}_n$.

Let us redefine the fields above, by considering a test function which is time dependent and given by a translation with a velocity $\mathfrak{v}_n$ and let $\mathcal{Z}_t^{n,\mathfrak{u}}(f)$ be the field corresponding to the variable $\bar\zeta_x^{\mathfrak{u}} = \bar\xi_x + \mathfrak{u}\bar\eta_x$, that is,

$$\mathcal{Z}_t^{n,\mathfrak{u}}(f) = \mathcal{X}_t^n(T^-_{\mathfrak{v}_n t}f) + \mathfrak{u}\mathcal{V}_t^n(T^-_{\mathfrak{v}_n t}f),$$

where $u \in \mathbb{R}$ and $T^{-}_{\mathfrak{v}_n t} f(\frac{x}{n}) = f(\frac{x}{n} - \mathfrak{v}_v t)$. A simple computation based on (3.15) and (3.16) shows that

$$n^a \mathscr{L}\mathfrak{X}^{n,\mathfrak{u}}_s(f) = \frac{\gamma}{n^{2-a}} \mathfrak{X}^{n,\mathfrak{u}}_s(\Delta_n f)$$
$$+ \frac{n^a \alpha_n b}{n}(-2b\rho + 2\mathfrak{u})\mathcal{X}^n_s(\nabla_n T^{-}_{\mathfrak{v}_n s} f) - \frac{\alpha_n(\mathfrak{u}b - b^2\rho)}{n^{2-a}}\mathcal{X}^n_s(\Delta_n T^{-}_{\mathfrak{v}_n s} f)$$
$$- \frac{b^2\alpha_n}{n^{3/2-a}} \sum_{x\in\mathbb{T}_n} \nabla_n T^{-}_{\mathfrak{v}_n s} f\left(\frac{x}{n}\right)\bar{\xi}_x(sn^a)\bar{\xi}_{x+1}(sn^a).$$

Observe also that since now the test functions are time-dependent, we get a contribution from the term $\partial_s \mathfrak{X}^{n,\mathfrak{u}}_s(f)$ given by $\partial_s \mathfrak{X}^{n,\mathfrak{u}}_s(f) = \mathfrak{v}_n \mathfrak{X}^{n,\mathfrak{u}}_s(\nabla f)$. To find the velocity $\mathfrak{v}_n$, we only look at the degree-one terms in the last display. Observe that both the rightmost term in the second line of the latter display and the rightmost term on the first line of the same display have a smaller variance when compared to the leftmost term on the second line of that display. Therefore the latter is the term that one has to get rid of. By combining that term with the contribution from $\partial_s \mathfrak{X}^{n,\mathfrak{u}}_s(f)$, we see that to find the constants $\mathfrak{u}$ and $\mathfrak{v}_n$ we have to solve the system of equations:

$$\begin{cases} \frac{n^a \alpha_n b}{n}(-2b\rho + 2\mathfrak{u}) = \mathfrak{v}_n, \\ 0 = \mathfrak{v}_n \mathfrak{u}. \end{cases} \tag{3.17}$$

The latter system is obtained by equating the coefficients in front of the quantities $\bar{\xi}_x$ and $\bar{\eta}_x$ in the expression

$$\frac{n^a \alpha_n b}{n}(-2b\rho + 2\mathfrak{u})\mathcal{X}^n_s(\nabla_n T^{-}_{\mathfrak{v}_n s} f) + \mathfrak{v}_n \mathfrak{X}^{n,\mathfrak{u}}_s(\nabla f).$$

The system (3.17) has two solutions:

(I) $\mathfrak{u} = 0$, which gives $\mathfrak{v}_n = -bn^{a-1}\alpha_n \mathfrak{v}$, where $\mathfrak{v} = 2b\rho$. In this case, we should consider the field associated to the quantity $\mathcal{U}_1 = \bar{\xi}_x$ in a moving time-dependent frame.

(II) $\mathfrak{v}_n = 0$, which gives $\mathfrak{u} = b\rho$. In this case, we should consider the field associated to the quantity $\mathcal{U}_2 = \bar{\xi}_x + b\rho\bar{\eta}_x$ and with no velocity since $\mathfrak{v}_n = 0$.

Observe that these results match the predictions from the previous subsection. Now that the fields are fixed, let us see what we can say in these two cases.

3.4. Limiting equations

From the computations of the last subsections, we know exactly the linear combination of the conserved quantities that we should look at and the corresponding velocities. Now we explain what we can rigorously prove for each one of them.

3.4.1. Case (I)

In this case $\mathfrak{v}_n = -2b^2\rho n^{a-1}\alpha_n$. Since $\mathfrak{u} = 0$, we have that

$$\mathfrak{X}^{n,0}_t(f) = \mathcal{X}^n_t(T^{-}_{\mathfrak{v}_n t} f) = \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{T}_n} (T^{-}_{\mathfrak{v}_n t} f)\left(\frac{x}{n}\right)\bar{\xi}_x(tn^a).$$

Therefore,

$$
\begin{aligned}
(\partial_s + n^a\mathscr{L})\mathfrak{X}_s^{n,0}(f) = {} & \frac{\gamma}{n^{2-a}}\mathfrak{X}_s^{n,0}(\Delta_n f) + \frac{\alpha_n b^2\rho}{n^{2-a}}\mathfrak{X}_s^{n,0}(\Delta_n f) \\
& - 2b^2\rho n^{a-1}\alpha_n \mathfrak{X}_s^{n,0}(\nabla_n f) - \upsilon_n \mathfrak{X}_s^{n,0}(\nabla f) \\
& - \frac{b^2\alpha_n}{n^{3/2-a}} \sum_{x\in\mathbb{T}_n} \nabla_n T^{-}_{\mathfrak{v}_n s} f\left(\frac{x}{n}\right)\bar{\xi}_x(sn^a)\bar{\xi}_{x+1}(sn^a).
\end{aligned} \tag{3.18}
$$

By a Taylor expansion on f and the choice of $\mathfrak{v}_n$, the second line above vanishes as $n \to +\infty$. From (2.6), we see that the term in the last line of (3.18) has a variance of order $O(\alpha_n^2 n^{3a/2-2})$ so that for $a < \frac{4}{3}(\kappa+1)$ the $\mathbb{L}^2(\mathbb{P}_{\mu_{\bar\beta,\bar\lambda}})$-norm of that term vanishes as $n \to +\infty$. Moreover, if $a < 2$ (resp. $a < 2+\kappa$), the $\mathbb{L}^2(\mathbb{P}_{\mu_{\bar\beta,\bar\lambda}})$-norm of the first (resp. second) term on the right-hand side of the first line in (3.18) vanishes as $n \to +\infty$. In the case $a = 2$ and $\kappa = 1/2$, we can treat the term in the last line of (3.18) by using (2.7). From that result, the last line of (3.18) can be written, for n sufficiently big and ε sufficiently small, as

$$
b^2\alpha \int_0^t \frac{1}{n}\sum_{x\in\mathbb{T}_n} \nabla_n f\left(\frac{x}{n}\right)\left(\mathfrak{X}_s^{n,0}\left(\iota_\epsilon\left(\frac{x}{n}\right)\right)\right)^2 ds. \tag{3.19}
$$

Let us denote the martingale associated to $\mathfrak{X}_t^{n,0}(f)$ by $\mathscr{M}_t^{n,0}(f)$. We note that its quadratic variation is given by

$$
\begin{aligned}
\langle \mathscr{M}^{n,0}(f)\rangle_t &= \int_0^t \left\{ n^a\mathscr{L}\big(\mathfrak{X}_s^{n,0}(f)\big)^2 - 2\mathfrak{X}_s^{n,0}(f) n^a \mathscr{L}\mathfrak{X}_s^{n,0}(f)\right\} ds \\
&= \gamma \int_0^t \frac{n^a}{n}\sum_{x\in\mathbb{T}_n}\left(f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right)\right)^2 \big(\xi_{x+1}(sn^a) - \xi_x(sn^a)\big)^2 ds.
\end{aligned} \tag{3.20}
$$

From simple computations, we see that if $a = 2$ and $\kappa > \frac{3}{4}a - 1$ then

$$
\mathscr{M}_t^{n,0}(f) = \mathfrak{X}_t^{n,0}(f) - \mathfrak{X}_0^{n,0}(f) - \gamma\int_0^t \mathfrak{X}_s^{n,0}(\Delta_n f) ds
$$

plus a term that vanishes in $\mathbb{L}^2(\mathbb{P}_{\mu_{\bar\beta,\bar\lambda}})$ as $n \to +\infty$. Moreover, the quadratic variation of the martingale satisfies

$$
\lim_{n\to+\infty} \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}}\big[\langle \mathscr{M}^{n,0}(f)\rangle_t\big] = 2t\gamma\tau^2\|\nabla f\|_0^2,
$$

where τ^2 is the variance of ξ_x with respect to $\mu_{\bar\beta,\bar\lambda}$ and $\|f\|_0^2$ denotes the $\mathbb{L}^2$-norm of f. Then $(\mathfrak{X}_t^{n,0})_n$ converges to the solution of the OU equation

$$
d\mathfrak{X}_t^0 = \gamma\Delta\mathfrak{X}_t^0 dt + \sqrt{2\gamma\tau^2}\nabla\mathscr{W}_t.
$$

Now, for $a < 2$ and $\kappa > \frac{3}{4}a - 1$, we have $\mathscr{M}_t^{n,0}(f) = \mathfrak{X}_t^{n,0}(f) - \mathfrak{X}_0^{n,0}(f)$ plus a term that vanishes in $\mathbb{L}^2(\mathbb{P}_{\mu_{\bar\beta,\bar\lambda}})$ as $n \to +\infty$. Moreover, the quadratic variation of the martingale satisfies

$$
\lim_{n\to+\infty} \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}}\big[\langle \mathscr{M}^{n,0}(f)\rangle_t\big] = 0. \tag{3.21}
$$

Then $\mathfrak{X}_t^0$ has a trivial evolution given by $\mathfrak{X}_t^0 = \mathfrak{X}_0^0$, so that $d\mathfrak{X}_t^0 = 0$. If $a = 2$ and $\kappa = \frac{3}{4}a - 1 = \frac{1}{2}$, then

$$\mathcal{M}_t^{n,0}(f) = \mathfrak{X}_t^{n,0}(f) - \mathfrak{X}_0^{n,0}(f) - \gamma \int_0^t \mathfrak{X}_s^{n,0}(\Delta_n f) ds + b^2\alpha \int_0^t \frac{1}{n} \sum_{x \in \mathbb{T}_n} \nabla_n f\left(\frac{x}{n}\right)\left(\mathfrak{X}_s^{n,0}\left(\iota_\epsilon\left(\frac{x}{n}\right)\right)\right)^2 ds.$$

Moreover, the quadratic variation of the martingale satisfies (3.21), so that $\mathfrak{X}_s^0$ is solution of the stochastic Burgers equation

$$d\mathfrak{X}_t^0 = \gamma\Delta\mathfrak{X}_t^0 dt + b^2\alpha\nabla(\mathfrak{X}_t^0)^2 dt + \sqrt{2\gamma\tau^2}\nabla\mathcal{W}_t.$$

The evolution of this quantity should be as described in Figure 1.

3.4.2. Case (II)

In this case $\mathfrak{u} = b\rho$ and $\mathfrak{v}_n = 0$. Then

$$n^a \mathcal{L}\mathfrak{X}_s^{n,\mathfrak{u}}(f) = \frac{\gamma}{n^{2-a}}\mathfrak{X}_s^{n,\mathfrak{u}}(\Delta_n f) - \frac{b^2\alpha_n}{n^{3/2-a}} \sum_{x \in \mathbb{T}_n} \nabla_n f\left(\frac{x}{n}\right)\bar{\xi}_x(sn^a)\bar{\xi}_{x+1}(sn^a).$$

Doing the same analysis as above, we see that the first term on the right-hand side of the latter display vanishes, as $n \to +\infty$, if $a < 2$ and the last term has a variance of order $O(\alpha_n^2 n^{3a/2-2})$, so that for $a < \frac{4}{3}(\kappa + 1)$ the $\mathbb{L}^2(\mathbb{P}_{\mu_{\bar{\beta},\bar{\lambda}}})$-norm of that term vanishes as $n \to +\infty$. This means that for this quantity we can show that its behavior is diffusive if $\kappa > \frac{1}{2}$ and $a = 2$, and trivial if $a < \inf(\frac{4}{3}(\kappa + 1), 2)$.

We note that in [1] the second quantity that was analyzed was the joint field for both quantities $\bar{\xi}_x$ and $\bar{\eta}_x$, and all the limiting behavior was derived rigorously. Nevertheless, as we have seen above, the second quantity that one should look at is $\bar{\zeta}_x^{\mathfrak{b}\rho} = \bar{\xi}_x + b\rho\bar{\eta}_x$, and only partial results are proved. According to the nonlinear fluctuating hydrodynamics theory developed in [23, 26, 28], one should get for $\kappa = 0$ a fractional behavior given by a Lévy $\frac{5}{3}$ and this should persist up to $\kappa < 1/3$, and for $\kappa > \frac{1}{2}$ one should see a diffusive behavior [25]. The predictions from mode-coupling theory in the weak asymmetric regime are a bit controversial so that the regime $\kappa \in [\frac{1}{3}, \frac{1}{2}]$ is still unclear [25]. In the next section, we analyze one potential for which we can prove rigorously all possible limits.

3.5. The harmonic case

Let us now consider the same model as in the beginning of this section but with the potential $V(x) = \frac{x^2}{2}$ so that $\mathcal{L} = \alpha_n\mathcal{A} + \gamma\mathcal{S}$, where $\gamma > 0$, $\alpha_n = \alpha n^{-\kappa}$, $\alpha \in \mathbb{R}$, $\kappa > 0$,

$$(\mathcal{A}f)(\eta) = \sum_{x \in \mathbb{T}_n} (\eta_{x+1} - \eta_{x-1})(\partial_{\eta_x} f)(\eta),$$

and the operator $\mathcal{S}$ is defined in (3.1). The translation invariant stationary measures $\mu_{v,\beta}$ are explicitly given by the product of Gaussian measures

$$\mu_{v,\beta}(d\eta) = \prod_{x \in \mathbb{T}_n} \left(\frac{\beta}{2\pi}\right)^{1/2} \exp\left\{-\frac{\beta}{2}(\eta_x - v)^2\right\} d\eta_x.$$

In this case the system also conserves two quantities, the energy $\sum_x \eta_x^2$ and the volume $\sum_x \eta_x$. Note that the average with respect to $\mu_{v,\beta}$ of η_x and η_x^2 is equal to v and $v^2 + \frac{1}{\beta}$, respectively. If we repeat the computations of Section 3.2 applied to this potential, we see that

$$j^e_{x,x+1}(\eta) = -2\alpha_n \eta_x \eta_{x+1} - \gamma \nabla \eta_x^2, \tag{3.22}$$

$$j^v_{x,x+1}(\eta) = \alpha_n(\eta_x + \eta_{x+1}) - \gamma \nabla \eta_x. \tag{3.23}$$

In this case the Jacobian matrix is given by

$$J = \begin{pmatrix} 0 & -4\alpha_n v \\ 0 & 2\alpha_n \end{pmatrix}$$

with eigenvalues $v_1 = 2\alpha_n$ and $v_2 = 0$, and the corresponding eigenvectors $\tau_1 = \binom{-2v\mathfrak{d}}{\mathfrak{d}}$ and $\tau_2 = \binom{\mathfrak{e}}{0}$, where $\mathfrak{d}$ and $\mathfrak{e}$ are constants. Moreover,

$$R^{-1} = \begin{pmatrix} -2v\mathfrak{d} & \mathfrak{e} \\ \mathfrak{d} & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & \frac{1}{\mathfrak{d}} \\ \frac{1}{\mathfrak{e}} & \frac{2v}{\mathfrak{e}} \end{pmatrix}.$$

From this, we see that the quantities that we should analyze are

$$(\mathcal{U}_1, \mathcal{U}_2) = R(\overline{\xi}_x, \overline{\eta}_x),$$

with $\mathcal{U}_1 = \frac{\overline{\eta}_x}{\mathfrak{d}}$ and $\mathcal{U}_2 = \frac{2v\overline{\eta}_x}{\mathfrak{e}} + \frac{\overline{\eta_x^2}}{\mathfrak{e}}$. Note that for $v = 0$ we simply get $(\mathcal{U}_1, \mathcal{U}_2)$ as the volume and energy. By computing the Hessian matrices associated with the currents, we see that the predictions tell us that in the strong asymmetric regime ($\kappa = 0$) we should have $\mathcal{U}_1$ diffusive and $\mathcal{U}_2$ Lévy with exponent $\frac{3}{2}$. In the case $v = 0$ last result was proved in [4]. For the volume, i.e. the quantity $\mathcal{U}_1$, when we take the fluctuation field with velocity zero, we get a process that is linearly transported in time, see the line in red colour in Figure 2, while if we take it with the velocity v_1 we get an OU without drift, see the line in green colour in Figure 2 below. For $\mathcal{U}_2$ with velocity $v = 0$, i.e. the energy (recall that $v_2 = 0$) we have the results summarized in Figure 3.

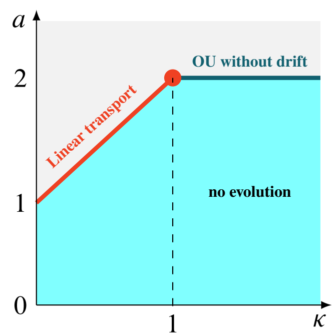

FIGURE 2
$\mathcal{U}_1$ fluctuations.

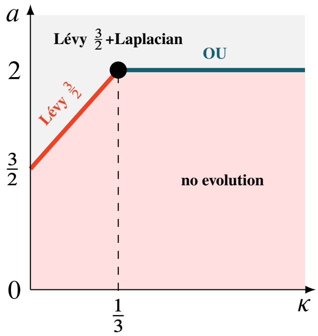

FIGURE 3
$\mathcal{U}_2$ fluctuations.

In Figure 2, the line in red colour corresponds to $a = \kappa + 1$; while in Figure 3 the line where we see the Lévy process with exponent $\frac{3}{2}$ is given by $a = \frac{3}{2}(\kappa + 1)$. Last results were proved in [4,5]. When the volume is taken with velocity v_1 then the line in green colour reaches the vertical line corresponding to $\kappa = 0$. Let us comment a bit on the proof of this result. We focus on the energy and note that $v_2 = 0$, so that the associated fluctuation field is given by

$$\mathcal{E}_t^n(f) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{T}_n} f\left(\frac{x}{n}\right) \overline{\eta_x^2}(tn^a).$$

Moreover, by assuming that $v = 0$, we get the following action of the generator:

$$(\partial_s + n^a \mathcal{L})\mathcal{E}_s^n(f) = \gamma n^{a-2} \mathcal{E}_s^n(\Delta_n f) - 2\alpha n^{a-\kappa-3/2} \sum_{x \in \mathbb{T}_n} \nabla_n f\left(\frac{x}{n}\right) \eta_x(sn^a)\eta_{x+1}(sn^a). \tag{3.24}$$

From (2.6) we know that the second term on the right-hand side of the last display vanishes, as $n \to +\infty$, for $a < \frac{4}{3}(\kappa + 1)$. Let us now explain how to prove that, in fact, the last term vanishes for $a < \frac{3}{2}(\kappa + 1)$, giving rise to the rosy area in Figure 3 above, and on the line $a = \frac{3}{2}(\kappa + 1)$ and for $\kappa \in [0, \frac{1}{3})$ we get the fractional behavior given by the Lévy process with exponent $\frac{3}{2}$. At this point we need to use the deterministic part of the dynamics given by the Hamiltonian $\mathcal{A}$. In order to do that, the idea is to rewrite the rightmost term of the last display in terms of the correlation field of the volume η_x, that we denote by $\mathbb{Q}_t^n$. This fields acts on functions $h : \mathbb{T}^2 \to \mathbb{R}$ as follows:

$$\mathbb{Q}_t^n(h) = \frac{1}{n} \sum_{x \neq y} h\left(\frac{x}{n}, \frac{y}{n}\right) \eta_x(tn^a)\eta_y(tn^a),$$

and since $v = 0$, the field is centered. Note that the definition of $\mathbb{Q}_t^n$ does not depend on the value of the function h on the diagonal $x = y$ because, when $x = y$, from $\eta_x \eta_y$ we would recover the energy η_x^2. With this notation, we can rewrite (3.24) as

$$(\partial_s + n^a \mathcal{L})\mathcal{E}_s^n(f) = \gamma n^{a-2} \mathcal{E}_s^n(\Delta_n f) - 2\alpha n^{a-\kappa-3/2} \mathbb{Q}_s^n(\nabla_n f \otimes \delta),$$

where $\nabla_n f \otimes \delta : \mathbb{T}_n^2 \to \mathbb{R}$ is a discrete approximation of the distribution $f'(x) \otimes \delta(x = y)$ and $\delta(x = y)$ is the δ of Dirac at the line $x = y$ and it is given by

$$(\nabla_n f \otimes \delta)\left(\frac{x}{n}, \frac{y}{n}\right) = \frac{n}{2}\nabla_n f\left(\frac{x}{n}\right)\mathbf{1}_{y=x+1} + \frac{n}{2}\nabla_n f\left(\frac{x-1}{n}\right)\mathbf{1}_{y=x-1}. \tag{3.25}$$

This means that for the energy we get

$$\mathcal{M}_t^{n,e}(f) = \mathcal{E}_t^n(f) - \mathcal{E}_0^n(f) - \int_0^t \gamma n^{a-2}\mathcal{E}_s^n(\Delta_n f) - 2\alpha n^{a-\kappa-3/2}\mathbb{Q}_s^n(\nabla_n f \otimes \delta) ds. \tag{3.26}$$

In order to close the equation for the energy field, we need to understand the behavior of the correlation field. A long computation (for details we refer the reader to Appendix A of [5]) shows that

$$\begin{aligned}\mathcal{M}_t^{n,q}(h) &= \mathbb{Q}_t^n(h) - \mathbb{Q}_0^n(h) \\ &\quad - \int_0^t \mathbb{Q}_s^n(\gamma n^{a-2}\Delta_n h + \alpha n^{a-\kappa-1}A_n h) - 2\alpha n^{a-\kappa-3/2}\mathcal{E}_s^n(\mathcal{D}_n h) \\ &\quad + 2\mathbb{Q}_s^n(n^{a-2}\tilde{\mathcal{D}}_n h) ds,\end{aligned}$$

where the operator $\Delta_n h$ is a discrete approximation of the 2-d Laplacian of h given by

$$\begin{aligned}\Delta_n h\left(\frac{x}{n}, \frac{y}{n}\right) &= n^2\left(h\left(\frac{x+1}{n}, \frac{y}{n}\right) + h\left(\frac{x-1}{n}, \frac{y}{n}\right) + h\left(\frac{x}{n}, \frac{y+1}{n}\right) + h\left(\frac{x}{n}, \frac{y-1}{n}\right)\right. \\ &\quad \left. - 4h\left(\frac{x}{n}, \frac{y}{n}\right)\right).\end{aligned}$$

Above $A_n h$ is a discrete approximation of the directional derivative $(-2, -2) \cdot \nabla h$ given by

$$A_n h\left(\frac{x}{n}, \frac{y}{n}\right) = n\left(h\left(\frac{x}{n}, \frac{y-1}{n}\right) + h\left(\frac{x-1}{n}, \frac{y}{n}\right) - h\left(\frac{x}{n}, \frac{y+1}{n}\right) - h\left(\frac{x+1}{n}, \frac{y}{n}\right)\right),$$

the operator $\mathcal{D}_n h$ is a discrete approximation of the directional derivative of h along the diagonal $x = y$ and it is given by $\mathcal{D}_n h(\frac{x}{n}) = n(h(\frac{x}{n}, \frac{x+1}{n}) - h(\frac{x-1}{n}, \frac{x}{n}))$, and the operator $\tilde{\mathcal{D}}_n$ is defined as follows:

$$\begin{aligned}\tilde{\mathcal{D}}_n h\left(\frac{x}{n}, \frac{y}{n}\right) &= n^2\left(\tilde{\mathcal{E}}_n h\left(\frac{x}{n}\right) - \frac{1-\kappa}{2}\tilde{\mathcal{F}}_n h\left(\frac{x}{n}\right)\right)\mathbf{1}_{y=x+1} \\ &\quad + n^2\left(\tilde{\mathcal{E}}_n h\left(\frac{y}{n}\right) - \frac{1-\kappa}{2}\tilde{\mathcal{F}}_n h\left(\frac{y}{n}\right)\right)\mathbf{1}_{y=x-1},\end{aligned}$$

with $\tilde{\mathcal{E}}_n h(\frac{x}{n}) = h(\frac{x}{n}, \frac{x+1}{n}) - h(\frac{x}{n}, \frac{x}{n})$ and $\tilde{\mathcal{F}}_n h(\frac{x}{n}) = h(\frac{x+1}{n}, \frac{x+1}{n}) - h(\frac{x}{n}, \frac{x}{n})$. Now we need to link the two equations above. To do so, we take h_n, as the symmetric function, solution of the Poisson equation

$$\gamma \Delta_n h\left(\frac{x}{n}, \frac{y}{n}\right) + \alpha n^{1-\kappa}\mathcal{A}_n h\left(\frac{x}{n}, \frac{y}{n}\right) = 2\alpha n^{1/2-\kappa}\nabla_n f \otimes \delta\left(\frac{x}{n}, \frac{y}{n}\right). \tag{3.27}$$

We get, by neglecting the martingales (since it can be shown that they vanish in $\mathbb{L}^2(\mathbb{P}_{\mu_{v,\beta}})$ as $n \to +\infty$ if $a < 2$), that

$$\begin{aligned}\mathcal{E}_t^n(f) - \mathcal{E}_0^n(f) &= \int_0^t \mathcal{E}_s^n(\gamma n^{a-2}\Delta_n f - 2\gamma n^{a-\kappa-3/2}\mathcal{D}_n h_n)\, ds \\ &\quad + \mathbb{Q}_0^n(h_n) - \mathbb{Q}_t^n(h_n) + 2\int_0^t \mathbb{Q}_s^n(n^{a-2}\tilde{\mathcal{D}}_n h_n)\, ds.\end{aligned} \tag{3.28}$$

Now we need to analyze each term at the right-hand side of the last display. By Fourier estimates, one can show that the discrete $\mathbb{L}^2$-norm of h_n vanishes as $n \to +\infty$, and from this and the Cauchy–Schwarz inequality we get that the $\mathbb{L}^2(\mathbb{P}_{\mu_{v,\beta}})$-norm of the second and third terms on the right-hand side of last display vanish, as $n \to \infty$. From long computations, one can also show the next result whose proof can be seen in [5].

Lemma 1. *Let h_n be the solution of the Poisson equation given in (3.27), $a = \inf(\frac{3}{2}(1+\kappa), 2)$ and $\kappa \in (0,1)$. For any $t > 0$, we have that*

$$\lim_{n\to\infty} \mathbb{E}_{\mu_{v,\beta}}\left[\left(\int_0^t \mathbb{Q}_s^n(n^{a-2}\tilde{\mathfrak{D}}_n h_n)\,ds\right)^2\right] = 0.$$

Finally, the remaining term has a nontrivial contribution to the limit which is given by the next lemma, whose proof can be found in [5].

Lemma 2. *If $a = \inf(\frac{3}{2}(1+\kappa), 2)$ and $\kappa \in (0, +\infty)$, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{x\in\mathbb{Z}} \left| \{\gamma n^{a-2}\Delta_n f - 2\alpha n^{a-\kappa-3/2}\mathfrak{D}_n h_n\}\left(\frac{x}{n}\right) - \mathbb{L}_{\alpha,\kappa} f\left(\frac{x}{n}\right)\right|^2 = 0, \qquad (3.29)$$

where $\mathbb{L}_{\alpha,\kappa} = \mathbf{1}_{\kappa\geqslant 1/3}\Delta + \alpha^{3/2}\mathbf{1}_{\kappa\leqslant 1/3}\mathcal{L}$, with $\mathcal{L} = -\frac{1}{\sqrt{2}}\{(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}\}$.

From last results, for $a < 2$ the limiting field satisfies $\mathcal{E}_t(f) - \mathcal{E}_0(f) = \int_0^t \mathcal{E}_s(\mathcal{L}f)\,ds$. For $a = 2$, we recover the latter identity plus the contribution from the martingale, which in the diffusive time scale does not vanish anymore. This proves the results for the energy in Figure 3. We observe that the previous method allowed us to extend the Gaussianity of the limit field for $\kappa > \frac{1}{3}$, for which the second-order Boltzmann–Gibbs principle would give the same result but only for $\kappa > \frac{1}{2}$. In this sense, this method allows us to reach areas of the phase diagram that we could not reach with the previous method. Nevertheless, it relies on the specific form of the dynamics and the fact that the equation for the quadratic field only involves terms of the energy field and the quadratic field itself, and this is not the case for the majority of the dynamics, some other mixtures of fields of the conserved quantities might appear. We note, however, that in [6], by perturbing the harmonic potential weakly by a quartic potential, the result obtained above for harmonic case persists up to some small critical value of the anharmonicity. There is still work to do in this direction, and we believe that one should analyze the action of the generator in those mixtures of the fields and keep track of the relevant quantities that give a nontrivial contribution to the limit. This study could give a way to prove rigorously the results predicted by mode-coupling theory for many other models.

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