MIXING TIME AND CUTOFF FOR ONE-DIMENSIONAL PARTICLE SYSTEMS

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ABSTRACT

We survey recent results concerning the total-variation mixing time of the simple exclusion process on the segment (symmetric and asymmetric) and a continuum analog, the simple random walk on the simplex with an emphasis on cutoff results. A Markov chain is said to exhibit cutoff if on a certain time scale, the distance to equilibrium drops abruptly from 1 to 0. We also review a couple of techniques used to obtain these results by exposing and commenting some elements of proof.

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1. A SHORT INTRODUCTION TO MARKOV CHAINS

1.1. Definition of a Markov chain

A stochastic process $(X_t)_{t\geq 0}$ indexed by $\mathbb{R}_+$ with values in a state-space Ω is said to be a *Markov process* if at each time $t \geq 0$, the distribution of the future $(X_{t+u})_{u\geq 0}$, conditioned on the past $(X_s)_{s\in[0,t]}$ is only determined by its present state X_t. This is equivalent to saying that for every bounded measurable function $F : \Omega^{\mathbb{R}_+} \to \mathbb{R}$, there exists $G : \Omega \to \mathbb{R}$ such that

$$\mathbb{E}\big[F\big[(X_{t+s})_{s\geq 0}\big] \,\big|\, (X_u)_{u\in[0,t]}\big] = G(X_t). \tag{1.1}$$

The assumption (1.1) can be interpreted as the *absence of memory* of the process and is called the *Markov property* (we refer to **[41, CHAPTER III]** for an introduction to Markov processes). *Markov chains* are Markov processes which are right continuous for the discrete topology on Ω, meaning that (X_t) always remains for some time in its current state before always jumping from it

$$\forall t \geq 0, \quad \inf\{s, X_{t+s} \neq X_t\} > 0.$$

Remark 1.1. The name Markov chains also (and perhaps more frequently) refers to discrete-time Markov processes, that is, processes indexed by $\mathbb{Z}_+$ rather than $\mathbb{R}_+$; see, for instance, **[39]**. Let us mention that all the continuous-time Markov chains mentioned in this paper are equivalent to discrete-time Markov chains in the sense that they can be obtained by composing a discrete-time Markov chain with a homogeneous Poisson process on $\mathbb{R}$, even when the considered state-space is infinite. In particular, they are càdlàg and do not display accumulation of jumps (a phenomenon called *explosion* see **[45, CHAPTER 4]**). We study these processes in continuous time rather than discrete mostly for practical and aesthetic reasons, but the results remain valid for the discrete-time version of the chains (and the adaptation of the proof from one setup to another is straightforward; see, for instance, **[25, APPENDIX B]**). While some references we refer to, such as **[46]**, mention only the discrete-time version of the chains, we always transpose the cited results in the continuous-time setup for a better presentation.

1.2. Markov semigroup, generator, invariant measures, and reversibility

The distribution of a Markov chain $(X_t)_{t\geq 0}$ is determined by two inputs:

(A) The distribution of its initial condition X_0, which is a probability distribution on Ω, which we denote by μ.

(B) The rules of evolution of the future given the present, that is, the mapping $(\Omega^{\mathbb{R}_+} \to \mathbb{R}) \to (\Omega \to \mathbb{R})$ that associates G to F in equation (1.1). It can be encoded in an operator acting on functions defined on Ω, the *generator* of the Markov chain.

Since, in the present paper, we are interested in statements which are valid for every initial distribution μ, when introducing examples of Markov chains, we are going to specify only their generator.

1.2.1. Finite state-space case

Let us start by defining the generator of a Markov chain in the simpler case when the state-space is finite (the reader can find in [31, CHAPTER 20] a more substantial introduction and proofs of the results mentioned in this section). An important intermediate step is the definition of a Markov semigroup $(P_t)_{t\geq 0}$ associated with the Markov chain. It is a sequence of $\Omega \times \Omega$ matrices that satisfy the semigroup property $P_{s+t} = P_s P_t$ (where matrix multiplication is considered) and such that for every $x, y \in \Omega$ and $s, t \geq 0$, when $\mathbb{P}[X_s = x] > 0$,

$$\mathbb{P}[X_{t+s} = y \mid X_s = x] = P_t(x, y). \tag{1.2}$$

Note that $(P_t)_{t\geq 0}$ jointly with the initial distribution fully determines the finite-dimensional distributions of the process since the iteration of (1.2) yields

$$\begin{aligned}\mathbb{P}[X_0 = x_0, X_{t_1} = x_1, X_{t_1+t_2} = x_2, \dots, X_{\sum_{i=1}^k t_i} = x_k] \\ = \mathbb{P}[X_0 = x_0] P_{t_1}(x_0, x_1) P_{t_2}(x_1, x_2) \cdots P_{s_k}(t_{k-1}, t_k).\end{aligned} \tag{1.3}$$

The semigroup property, together with our assumption that (X_t) is càdlàg, implies that there exists an $\Omega \times \Omega$ matrix $\mathcal{L}$ – the generator of the Markov chain – such that for all $t \geq 0$,

$$\forall t > 0, \quad P_t = e^{\mathcal{L}t} := \sum_{k=1}^{\infty} \frac{s^k}{k!} \mathcal{L}^k.$$

Note that when we have for $x, y \in \Omega$, $x \neq y$,

$$\begin{cases} \mathcal{L}(x, y) = \lim_{t\to 0} \frac{1}{t} P_t(x, y), \\ -\mathcal{L}(x, x) = \lim_{t\to 0} \frac{1}{t}(1 - P_t(x, x)), \end{cases} \tag{1.4}$$

then $\mathcal{L}(x, y)$ represents the rate at which our Markov chain jumps from x to y, while $-\mathcal{L}(x, x)$ corresponds to the rate at which the chain jumps away from x. In practice, when introducing the generator of a Markov chain, we simply write its action (by left multiplication) on $\mathbb{R}$-valued functions on Ω. That is,

$$\mathcal{L}f(x) := \sum_{y\in\Omega} \mathcal{L}(x, y) f(y) = \sum_{y\in\Omega\setminus\{x\}} \mathcal{L}(x, y)\big[f(y) - f(x)\big].$$

We focus on the case of *irreducible* Markov chains, that is, we assume that every state of Ω can be reached from any other state with a finite number of jumps. Formally, for each x, y, there exist $k \geq 1$ and a sequence $x_0, x_1, \dots, x_k$ with $x_0 = x$ and $x_k = y$ such that

$$\forall i \in [\![1, k]\!], \quad \mathcal{L}(x_{i-1}, x_i) > 0.$$

This condition immediately implies that $P_s(x, y) > 0$ for every $x, y \in \Omega$. If $\mathcal{L}$ is irreducible, and $\mathbb{P}_\mu$ denotes the law of the Markov chain with generator $\mathcal{L}$ and initial distribution μ, then there exists a unique probability π on Ω such that $\mathbb{P}_\pi(X_t = x) = \pi(x)$ for every π. Such a probability is called the *invariant distribution* of the Markov chain. Considering π as a (line) vector on Ω, this is equivalent to either of the two relations below

$$\begin{cases} \forall t > 0, \quad \pi P_t = \pi, \\ \pi \mathcal{L} = 0. \end{cases} \tag{1.5}$$

The convergence theorem for irreducible finite state-space Markov chains states (see, for instance, [31, THEOREMS 4.9 AND 20.1]) that the invariant probability measures π is also the limit distribution for X_t when $t \to \infty$. More precisely, for any probability μ on Ω, we have

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t = x) = \pi(x). \tag{1.6}$$

We want to investigate the quantitative aspect of this convergence. For the Markov chains in this paper, the stationary measure satisfies the so-called *detailed balance* condition

$$\forall x, y \in \Omega, \quad \pi(x)\mathcal{L}(x, y) = \pi(y)\mathcal{L}(y, x), \tag{1.7}$$

where we use the notation

$$[\![a, b]\!] := [a, b] \cap \mathbb{Z}. \tag{1.8}$$

It can be easily checked that (1.7) implies (1.5), but there are irreducible Markov chains for which the stationary probability does not satisfy (1.7). Markov chains for which the stationary measure satisfies (1.7) are called *reversible*.

1.2.2. Continuum state-space case

When our state-space is a continuum, the above description of the generator as a matrix cannot be used. In that case the semigroup associated to the Markov chain $(P_t)_{t\geq 0}$ is a sequence of probability kernels such that for every bounded measurable function f on Ω, and every s and t, we have[1]

$$\mathbb{E}\big[f(X_{t+s}) \mid X_s\big] = P_t f(X_s) \quad \text{with} \quad P_t f(x) := \int_\Omega f(y) P_t(x, \mathrm{d}y). \tag{1.9}$$

Informally, $P_t(x, A)$ is the probability that $X_{s+t} \in A$ given $X_s = x$. In analogy with (1.3), the semigroup $(P_t)_{t\geq 0}$, jointly with the initial distribution, determines fully the finite-dimensional distributions of $(X_t)_{t\geq 0}$. The generator of the Markov chain $\mathcal{L}$ can be defined in analogy with (1.4) by

$$\mathcal{L} f := \lim_{t\to 0} \frac{P_t f - f}{t}. \tag{1.10}$$

For a general Markov processes, the limit on the right-hand side in (1.10) may not exist for every bounded measurable f; the set of functions for which the limit (1.10) does exist is called the *domain* of the generator. In this paper, however, we are going to consider only Markov chains with uniformly bounded jump rates, so we will not have to worry about this. Conditions for the existence and uniqueness of a stationary probability distribution and for a convergence such as that in (1.6) in continuous state-space are very far from being as nice as in the finite case (see, for instance, [41, CHAPTER 3]). In this survey, we consider only chains for which the stationary measure exists and is unique. They also satisfy the continuum counterpart of (1.7), that is, the operator $\mathcal{L}$ is self-adjoint in $L_2(\pi)$.

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1 Strictly speaking, the relation (1.9) does not uniquely define $(P_t)_{t\geq 0}$, since one can modify $P_t(x, \cdot)$ for on a set of xs which is visited with probability zero but is is not a relevant issue for our discussion.

1.3. Total variation distance and mixing time

In order to quantify the convergence to equilibrium (1.6), we need a notion of distance on the set $M_1(\Omega)$ of probability measures on Ω, equipped with a σ-algebra (which is simply the power set $\mathcal{P}(\Omega)$ when Ω is finite). We consider the *total variation* distance, which quantifies how well two variables with different distributions can be coupled. Given $\alpha, \beta \in M_1(\Omega)$, the total variation distance between α and β is defined by

$$\|\alpha - \beta\|_{\mathrm{TV}} := \sup_{A \subset \Omega} \big|\alpha(A) - \beta(A)\big|,$$

where the supremum is taken over measurable sets. The following equivalent characterizations of the total variation distance helps to better grasp the notion. It is a sort of L_1 distance which measures how well two random variables can be coupled.

Proposition 1.2. *If Ω is finite or countable then we have*

$$\|\alpha - \beta\|_{\mathrm{TV}} := \frac{1}{2} \sum_{x \in \Omega} \big|\alpha(x) - \beta(y)\big|$$

If ν is a measure on Ω such that both α and β are absolutely continuous with respect to ν then

$$\|\alpha - \beta\|_{\mathrm{TV}} := \frac{1}{2} \int_{\Omega} \left| \frac{\mathrm{d}\alpha}{\mathrm{d}\nu} - \frac{\mathrm{d}\beta}{\mathrm{d}\nu} \right| \nu(\mathrm{d}x).$$

We have

$$\|\alpha - \beta\|_{\mathrm{TV}} := \min_{\substack{X_1 \sim \alpha \\ X_2 \sim \beta}} \mathbf{P}[X_1 = X_2]$$

where the minimum is taken over the set of all probability distribution $\mathbf{P}$ on $\Omega \times \Omega$ which have marginal laws α and β.

The total variation distance to equilibrium of the Markov chain with generator $\mathcal{L}$ and stationary measure π at time t is given by

$$d(t) := \sup_{\mu \in M_1(\Omega)} \big\| \mathbb{P}_\mu(X_t \in \cdot) - \pi \big\|_{\mathrm{TV}},$$

where $\mathbb{P}_\mu$ is the law of the Markov chain with generator $\mathcal{L}$ and initial measure μ. A standard coupling argument is sufficient to show that $d(t)$ is nondecreasing as a function of t. Given $\varepsilon \in (0, 1)$, the mixing time associated to the threshold ε, or ε-mixing time of the Markov chain X_t, is given by

$$T_{\mathrm{mix}}(\varepsilon) := \inf\big\{t > 0 : d(t) \le \varepsilon\big\} = \sup\big\{t > 0 : d(t) > \varepsilon\big\}.$$

It indicates how long it takes, for a Markov chain starting from an arbitrary initial condition, to get close to its equilibrium measure. Note that when Ω is finite and the chain is irreducible, (1.6) guarantees that $\lim_{t \to \infty} d(t) = 0$ so that $T_{\mathrm{mix}}(\varepsilon) < \infty$ for all ε. For chains with a continuum state space, it is relevant to study the mixing time in the form defined above only if there is a unique stationary probability measure $\lim_{t \to \infty} d(t) = 0$.

Remark 1.3. In the case when $d(t) \nrightarrow 0$, some relevant variant of the mixing time can be defined by considering a restriction on the initial condition, for instance, by restricting x to a compact subset of Ω; see, e.g., [5,13].

1.4. Organization of the paper

The main object of this paper is to survey some results and methods concerning the mixing time of some Markovian one-dimensional particle systems (with discrete and continuum state-space). In Section 2 we introduce these processes. In Section 3 we expose some results obtained with coauthors in the past decade, and propose a short survey of related research. In Section 4 we review a couple of pivotal ideas, which first appeared in [46] (in a slightly different form) and show how they can be combined to obtain (nonoptimal) upper bounds on the mixing time. In Section 5, we discuss the technical refinements that are required to improve these bounds to get optimal results.

Remark 1.4. In both Sections 4 and 5, we have made the choice to focus exclusively on upper-bound estimates for the mixing time. For the theorems presented in this survey – and in most instances of mixing-time problems – this is generally thought to be the hardest part of the results.

Some comments on notation. In the remainder of the paper, we always use the letter π (with superscripts and subscripts to underline the dependence on parameters) to denote the equilibrium measure of each of the considered Markov chain, so that the meaning of, say, π_N or $\pi_{N,k}$ will depend on the context. When several Markov chains with different initial distributions are considered, we may use a superscript to underline the initial distribution (for instance, (X_t^π) denotes a Markov chain starting from the stationary distribution). If the initial distribution is a Dirac mass δ_x with $x \in \Omega$, we write X_t^x rather than $X_t^{\delta_x}$.

2. ONE-DIMENSIONAL PARTICLE SYSTEMS AND INTERFACE MODELS

The Markov chains introduced in this section model the motion of particles in a one-dimensional space. In each instance, we do not introduce a single chain but rather a sequence of chains, which are indexed by one or two parameters, which correspond to the size of the system and/or the number of particles. We want to understand the evolution of the mixing time when these parameters diverge to infinity.

2.1. The interchange process on a segment

The symmetric interchange process on a segment. For $N \geq 2$, we let $\mathcal{S}_N$ denote the symmetric group, that is, the set of permutations on N elements. For $i \neq j$, we let $\tau_{i,j}$ denote the transposition which exchanges the position of i and j. We define the (symmetric) interchange process on the segment $[\![1, N]\!]$ (recall (1.8)) as the Markov chain on $\mathcal{S}_N$ with generator

$$\mathcal{L}^{(N)} f(\sigma) := \frac{1}{2} \sum_{i=1}^{N-1} \big[f(\sigma \circ \tau_{i,i+1}) - f(\sigma) \big].$$

It takes little effort to check that the Markov chain described above is irreducible, and that the uniform probability on $\mathcal{S}_N$ satisfies the detailed balanced condition (1.7). A more intuitive description of the process, which we denote by (σ_t), can be obtained using equation (1.4): it jumps away from its current stat with rate $(N-1)/2$ (that is, the times between consecutive

jumps are IID exponential variables of mean $2/(N-1)$), and when it jumps, it chooses uniformly among the permutations obtained by composing on the right with a transposition of the form $\tau_{i,i+1}$ for $i \in [\![1, N]\!]$, or in other words, it interchanges the value of two randomly chosen consecutive coordinates.

An alternative description is that σ_t is *updated* with a rate $N-1$ (which is twice the previous rate). At an update time t, one coordinate $i \in [\![1, N-1]\!]$ is chosen uniformly at random, and σ_t is resampled by choosing uniformly at random in the set $\Theta(i, \sigma_{t-})$, where σ_{t-} is used to denote the left limit at t and

$$\Theta(i,\sigma) := \{\sigma' \in \mathcal{S}_N : \forall j \in [\![1,N]\!] \setminus \{i, i+1\}, \sigma'(j) = \sigma(j)\} = \{\sigma, \sigma \circ \tau_{i,i+1}\}.$$

Note that with this description, at each update, the value of σ_t remains unchanged with probability $1/2$. This second description might seem initially less natural than the first, but it turns out to be more convenient to construct monotone couplings, see Section 4.1.

Remark 2.1. The process described above is one of many examples of random walks on $\mathcal{S}_N$. This family of processes has attracted attention since the origin of the study of mixing times, due to the connection it has with the problem of card shuffling (see **[31, CHAPTER 8]** and the references therein). The symmetric interchange process, which we have considered here on the segment can be generalized: the study of the mixing properties for the interchange process on an arbitrary graph has been an active field of research; see, for instance, **[7,18,35]** and the references therein.

The biased interchange process. We consider a variant of the process which induces a bias towards more "ordered" permutations, that is, favors moves which drive the chain "closer" to the identity permutation. The set $\Theta(i,\sigma)$ is composed of two elements. We let $\sigma^{(i,+)}$ be the element of $\Theta(i,\sigma)$ such that $\sigma^{(i,+)}(i) < \sigma^{(i,+)}(i+1)$ and let $\sigma^{(i,-)}$ denote the element of $\Theta(i,\sigma)$ such that $\sigma^{(i,-)}(i) > \sigma^{(i,-)}(i+1)$ (intuitively, $\sigma^{(i,+)}$ is the permutation which is more ordered). Letting $p \in (1/2, 1)$ ($p = 1/2$ corresponds to the symmetric case considered above, the case $p \in (0, 1/2)$ is equivalent to $p \in (1/2, 1)$ after reverting the order of the coordinates) and setting $q := 1-p$, we define the generator of the biased interchange process of the segment

$$\mathcal{L}_N^{(p)} f(\sigma) := \sum_{i=1}^{N-1} p\big[f(\sigma^{(i,+)}) - f(\sigma)\big] + q\big[f(\sigma^{(i,-)}) - f(\sigma)\big].$$

The introduction of a bias drastically modifies the stationary distribution. We let $D(\sigma)$ denote the minimal number of transpositions of type $\tau_{i,i+1}$ which we need to compose to obtain σ – it corresponds to the distance between σ and the identity permutation in the Cayley graph generated by the nearest-neighbor transpositions $(\tau_{i,i+1})_{i=1}^{N-1}$ (see **[33, SECTION 3.4]** for an introduction to Cayley graphs). We have

$$D(\sigma) = \sum_{1 \le i < j \le N} \mathbf{1}_{\{\sigma(i) > \sigma(j)\}}.$$

Setting $\lambda := p/q$ ($\lambda > 1$), the probability measure $\pi_N^{(p)}$ defined by

$$\pi_N^{(p)}(\sigma) = \frac{\lambda^{-D(\sigma)}}{\sum_{\sigma' \in \mathcal{S}_N} \lambda^{-D(\sigma')}}$$

satisfies the detailed balance condition for $\mathcal{L}_N^{(p)}$. As $\lambda > 1$, the measure $\pi_N^{(p)}$ concentrates most of its mass in a small neighborhood of the identity (more precisely, $D(\sigma)$ is typically of order N under $\pi_N^{(p)}$, while it is of order N^2 under the uniform measure).

2.2. The exclusion process on the segment

This Markov chain models the evolution of particles diffusing on a segment and subject to *exclusion*: each site can host at most one particle. Let N denote the length of the segment. A particle configuration is encoded by a sequence of 0 and 1 on the segment, ones and zeros respectively indicating the presence/absence of particle at a site. The space of configurations with a fixed number of particles k is defined by

$$\Omega_{N,k} := \left\{ \xi, [\![1, N]\!] \to \{0, 1\} : \sum_{i=1}^{N} \xi(i) = k \right\}.$$

Given $\xi \in \Omega_{N,k}$ and distinct $i, j \in [\![1, N]\!]$, we set $\xi^{(i,j)} = \xi \circ \tau_{i,j}$ and define the generator of the *Symmetric Simple Exclusion Process* (or SSEP) to be

$$\mathcal{L}_{N,k} f(\xi) := \frac{1}{2} \sum_{i=1}^{N-1} \big[f(\xi \circ \tau_{i,i+1}) - f(\xi) \big].$$

An intuitive way to describe the above Markov chain is to say that each particle particle performs an independent, continuous-time nearest neighbor random walk with jump rate $1/2$ to the left and to the right, but that any jump which would result in either a particle moving out of the segment (that is, a jump to the site 0 or $N + 1$) or two particles occupying the same site (that is, a jump of a particle to an already occupied site) are canceled (see Figure 1). The uniform probability on $\Omega_{N,k}$ satisfies the detailed balance condition (1.7).

Given $p \in (1/2, 1)$, we can also define the *Asymmetric Simple Exclusion Process* (or ASEP) which is a similar process on $\Omega_{N,k}$, but where the particles perform a random walk with respective jump rates p and q to the right and to the left. The corresponding generator is

$$\begin{aligned} \mathcal{L}_{N,k}^{(p)} f(\xi) := & \sum_{i=1}^{N-1} p \mathbf{1}_{\{\xi(i) > \xi(i+1)\}} \big[f(\xi \circ \tau_{i,i+1}) - f(\xi) \big] \\ & + \sum_{i=1}^{N-1} q \mathbf{1}_{\{\xi(i) < \xi(i+1)\}} \big[f(\xi \circ \tau_{i,i+1}) - f(\xi) \big]. \end{aligned} \tag{2.1}$$

Here also the introduction of the bias yields a modification of the stationary probability. The probability which satisfies the detailed balance condition is given by (recall that $\lambda = p/q$)

$$\pi_{N,k}^{(p)}(\xi) := \frac{\lambda^{-A(\xi)}}{\sum_{\xi' \in \Omega_{N,k}} \lambda^{-A(\xi')}}$$

where $A(\xi) := \sum_{i=1}^{N}(N-i)\xi(i) - \frac{k(k-1)}{2}$ denotes the (minimal) number of particle moves that separates ξ from the configuration $\mathbf{1}_{[\![N-k+1,N]\!]}$ with all particles packed to the right of the segment. Note that $A(\xi)$ is typically of order 1 under $\pi_{N,k}^{(p)}$ whereas it is of order N^2 under the uniform measure.

2.3. The corner-flip dynamics

We consider a Markov chain that models the motion of an interface, which is subject only to local moves. The one-dimensional interface is the graph of a one-dimensional nearest-neighbor path which belongs to the state space

$$\Xi_{N,k} := \big\{\zeta,\ [\![0,N]\!] \to \mathbb{Z} : \zeta(0) = 0, \zeta(N) = N - 2k, \\ \forall i \in [\![0,N-1]\!], |\zeta(i+1) - \zeta(i)| = 1\big\}. \tag{2.2}$$

We introduce a Markov chain on $\Xi_{N,k}$ that only changes the coordinates of ζ one at a time. Given $\zeta \in \Xi_N$ and $i \in [\![1,N-1]\!]$, we introduce $\zeta^{(i)}$ to be the element of Ω_N for which only the coordinate at i has been changed (see Figure 1):

$$\begin{cases} \zeta^{(i)}(j) := \zeta(j) & \text{if } j \neq i, \\ \zeta^{(i)}(i) := \zeta(i) - 2 & \text{if } \zeta(i+1) = \zeta(i-1) := \zeta(i) - 1, \\ \zeta^{(i)}(i) := \zeta(i) + 2 & \text{if } \zeta(i+1) = \zeta(i-1) := \zeta(i) + 1, \\ \zeta^{(i)}(i) := \zeta(i) & \text{if } |\zeta(i+1) - \zeta(i-1)| = 2. \end{cases}$$

The generator of the symmetric corner flip dynamics is given by

$$\mathfrak{L}_{N,k} := \frac{1}{2}\sum_{i=1}^{N-1}\big[f(\zeta^{(i)}) - f(\zeta)\big].$$

A way to visualize this dynamics is to say that each "corner" displayed by the the graph of ζ is flipped with rate $1/2$. The uniform measure on $\Xi_{N,k}$ satisfies the detailed balance condition (1.7).

We can also define an asymmetric version of the dynamics which favors flipping the corners in one direction. Given $\zeta \in \Xi_N$ and $i \in [\![i, i+1]\!]$, we define $\zeta^{(i,\pm)}$ to be respectively the "highest" and "lowest" path in the set $\{\zeta^{(i)}, \zeta\}$ (the set is possibly a singleton, so that we may have $\zeta^{(i,+)} = \zeta^{(i,-)}$)

$$\begin{cases} \zeta^{(i,\pm)}(j) := \zeta(j) & \text{if } j \neq i, \\ \zeta^{(i,+)}(i) := \max\big(\zeta(i), \zeta^{(i)}(i)\big), \\ \zeta^{(i,-)}(i) := \min\big(\zeta(i), \zeta^{(i)}(i)\big), \end{cases}$$

and define the generator of the asymmetric corner-flip dynamics as

$$\mathfrak{L}_{N,k}^{(p)} := \sum_{i=1}^{N-1} p\big[f(\zeta^{(i,+)}) - f(\zeta)\big] + q\big[f(\zeta^{(i,-)}) - f(\zeta)\big].$$

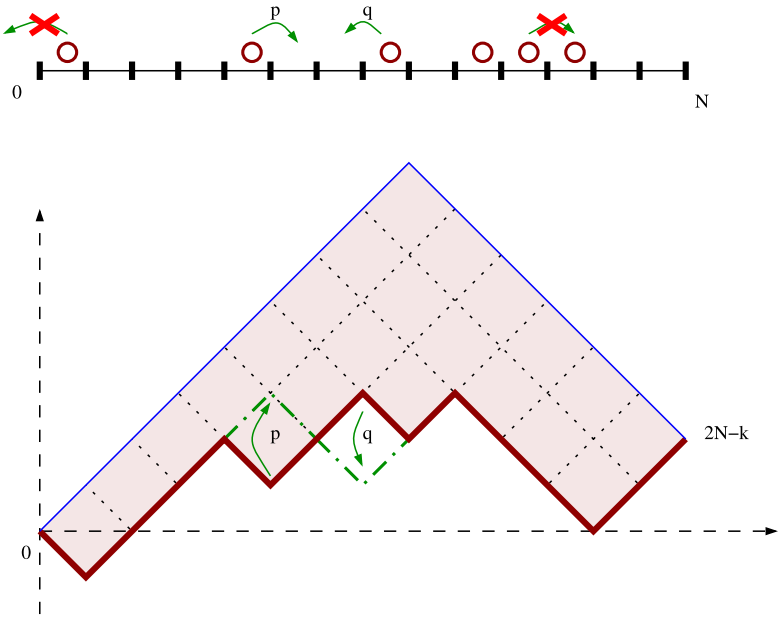

FIGURE 1

Graphical representation of the exclusion process and of the corner-flip dynamics with $k = 6$ and $N = 14$. Particles represented by circles, jump to the right with rate p and to the left with rate $q = 1 - p$ ($p = 1/2$ in the symmetric case), jumps are canceled if a particle tries to jump to an already occupied site. After applying the transformation h given in (2.5), we obtain the corner-flip dynamics, where each downward pointing corner on our interface is flipped up with rate p and each upward pointing corner is flipped down with rate q. The quantity $A(\zeta)$ which is the number of up-flips that need to be performed to reach the maximal configuration $\wedge$ (represented as a thin solid line of the figure) is equal to 22.

For the asymmetric corner flip, the reversible measure $\pi_{N,k}^{(p)}$ is defined by

$$\pi_{N,k}^{(p)}(\zeta) := \frac{\lambda^{-A(\zeta)}}{\sum_{\zeta' \in \Xi_{N,k}} \lambda^{-A(\zeta')}},$$

where $A(\zeta)$ denotes the halved geometric area lying between ζ and the highest path in $\Omega_{N,k}$ defined by

$$\wedge(i) = \min\bigl(i, 2(N-k) - i\bigr), \tag{2.3}$$

that is,

$$A(\zeta) := \frac{1}{2} \sum_{i=1}^{N-1} \bigl(\wedge(i) - \zeta(i)\bigr).$$

2.4. Random walk on the simplex

Let us finally consider a Markov chain for which the state-space is a continuum. We let $\mathfrak{X}_N$ denote the $(N-1)$-dimensional simplex defined by

$$\mathfrak{X}_N := \bigl\{(x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} : 0 \le x_1 \le \cdots \le x_{N-1} \le N\bigr\}.$$

We introduce a dynamics which is a continuum analog of the symmetric exclusion, the coordinates $x_1, \dots, x_{N-1}$ can be thought as the positions of $N-1$ particles on the segment $[0, N]$. The generator of the dynamics is given by

$$\mathrm{L}_N f(x) = \sum_{i=1}^{N-1} \int_0^1 \big[f(x^{(u,i)}) - f(x) \big] \mathrm{d}u,$$

for $f : \mathfrak{X}_N \to \mathbb{R}$ bounded and measurable, where $x^{(u,i)} \in \mathfrak{X}_N$ is defined by

$$\begin{cases} x_j^{(u,i)} := x_j & \text{for } j \neq i, \\ x_i^{(u,i)} := u x_{i+1} + (1-u) x_{i-1}, \end{cases}$$

with the convention that $x_0 = 0$ and $x_N = N$. In words, at rate one, each coordinate is resampled uniformly on its possible range of values, which is the segment $[x_{i-1}, x_{i+1}]$ (see Figure 2).

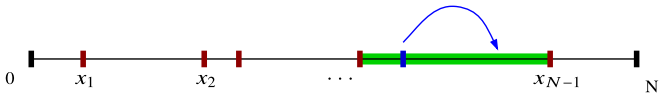

FIGURE 2

Graphical representation of the random walk on the simplex ($N = 7$). When the position of a particle is updated (x_5 on the picture), it is resampled uniformly in the interval delimited by the neighboring particles (that is, $[x_4, x_6]$), with the convention that $x_0 = 0$ and $x_N = N$.

The uniform probability on $\mathfrak{X}_N$, π_N, defined by

$$\pi_N(\mathrm{d}x) := \frac{(N-1)!}{N^{N-1}} \mathbf{1}_{\{x \in \mathfrak{X}_N\}} \mathrm{d}x_1 \cdots \mathrm{d}x_{N-1},$$

is stationary for L_N, and the generator is self-adjoint in $L^2(\pi_N)$.

Let us present an explicit construction of the Markov chain $(\mathbf{X}_t^x)_{t \geq 0}$ with initial distribution δ_x using auxiliary random variables. Such a construction is referred to as *a graphical construction* and turns out to be very convenient to work with (see, for instance, Section 4.1 below). It follows the following steps:

(i) To each coordinate $i \in [\![1, N-1]\!]$, we associate an independent rate-1 Poisson clock process $(\mathcal{T}_n^{(i)})_{n \geq 1}$ (the increments of $\mathcal{T}^{(i)}$ are i.i.d. exponential variables of mean 1) and a sequence of uniform random variables on $[0, 1]$, $(U_n^{(i)})_{n \geq 1}$.

(ii) We set $\mathbf{X}^x(0) = x$. The process is càdlàg and $(\mathbf{X}^x(t))_{t \geq 0}$ remains constant on each open interval of the set $(0, \infty) \setminus \{\mathcal{T}_n^{(i)}, n \geq 1, i \in [\![1, N-1]\!]\}$.

(iii) At time $t = \mathcal{T}_n^{(i)}$, we determine $\mathbf{X}^x(t)$ from $\mathbf{X}^x(t_-)$ (the left limit at t) by setting

$$X_i(t) := U_n^{(i)} X_{i+1}(t_-) + \big(1 - U_n^{(i)}\big) X_{i-1}(t).$$

The other coordinates are unchanged, $X_j(t) = X_j(t_-)$ for $j \neq i$.

The reader can check that, for any bounded measurable function f and every $x \in \mathfrak{X}_N$,

$$\lim_{t\to 0} \frac{\mathbb{E}[f(\mathbf{X}^x(t))] - f(x)}{t} = \mathrm{L}_N f(x). \tag{2.4}$$

2.5. Correspondences

The Markov chains presented in Sections 2.1, 2.2, and 2.3 are very much related to each other. Let us first describe the correspondence between the particle system and discrete interfaces. Let us consider $\zeta : \Omega_{N,k} \to \Xi_{N,k}$ defined by

$$h(\xi)(x) := \sum_{y=1}^{x} \big(1 - 2\xi(x)\big). \tag{2.5}$$

It is immediate to check that $h(\xi) \in \Xi_{N,k}$ for every $\xi \in \Omega_{N,k}$ and that h is a bijection (we have $h^{-1}(\zeta)(x) = \frac{1+\zeta(x-1)-\zeta(x)}{2}$). Furthermore, we have $\mathfrak{L}_{N,k}^{(p)} \circ h = h \circ \mathcal{L}_{N,k}^{(p)}$ and, as a consequence, if $(\eta_t)_{t\geq 0}$ is a Markov chain with generator $\mathcal{L}_{N,k}^{(p)}$, then its image $h(\eta_t)_{t\geq 0}$ is a Markov chain on $\Xi_{N,k}$ with generator $\mathfrak{L}_{N,k}^{(p)}$ (this is, of course, also true in the symmetric case, when $p = 1/2$).

The corner-flip representation of the exclusion process can be convenient for reasoning since it allows for a better visual representation of an order relation which is conserved by the dynamics (see Section 4.1).

Another useful – although not bijective – correspondence is that between the interchange and exclusion processes. Given $k \in [\![1, N-1]\!]$, we define $\xi^{(k)} : \mathcal{S}_N \to \Omega_{N,k}$ as

$$\xi^{(k)}(\sigma) = \mathbf{1}_{[\![N-k+1,N]\!]} \circ \sigma.$$

Since $\mathcal{L}_{N,k}^{(p)} \circ \xi^{(k)} = \xi^{(k)} \circ \mathcal{L}_N^{(p)}$, if $(\sigma_t)_{t\geq 0}$ is a Markov chain on $\mathcal{S}_N$ with generator $\mathcal{L}_N^{(p)}$ then $(\xi^{(k)}(\sigma_t))_{t\geq 0}$ is a Markov chain on $\Omega_{N,k}$ with generator $\mathcal{L}_{N,k}^{(p)}$. The whole sequence of projections $(\xi^{(k)}(\sigma))_{k=1}^{N-1}$ allows recovering σ since we have

$$\sigma(i) = N - k + 1 \quad \Leftrightarrow \quad \xi^{(k)}(i) - \xi^{(k-1)}(i) = 1. \tag{2.6}$$

3. REVIEW OF MIXING-TIME RESULTS FOR ONE-DIMENSIONAL PARTICLE SYSTEMS

3.1. The cutoff phenomenon

Let us now survey a few results concerning the mixing time of the Markov chains introduced in the previous section. For all of these processes, an asymptotic equivalent to the mixing time $T_{\mathrm{mix}}^{(N)}(\varepsilon)$ (or $T_{\mathrm{mix}}^{(N,K)}(\varepsilon)$ if we have several parameters) is obtained in the limit when the parameter N (or N and k) tends to infinity. A striking common feature of all these results is that, in the asymptotic equivalent of $T_{\mathrm{mix}}^{(N)}(\varepsilon)$, there is no dependence on ε ($T_{\mathrm{mix}}^{(N)}(\varepsilon)$ depends on ε but this dependence only appear in higher order terms). In particular, we have for any $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{T_{\mathrm{mix}}^{(N)}(\varepsilon)}{T_{\mathrm{mix}}^{(N)}(1-\varepsilon)} = 1.$$

This mean that on a certain time scale, the distance to equilibrium drops abruptly from 1 to 0. This phenomenon is known as a cutoff and is believed to hold for a wide class of Markov chains (we refer to [12] and [31, CHAPTER 18], and the references therein for a historical introduction to cutoff). Cutoff is delicate to prove most of the time. For many Markov chains, while a short argument allows identifying the mixing time up to a constant multiplicative factor (cf. Section 4), much more effort is usually needed to obtain asymptotically matching upper and lower bounds.

3.2. Mixing time results

The SSEP and the interchange process. We group the results concerning the exclusion and interchange processes as their proofs share a lot of common ideas. We start with the symmetric case. The lower bounds in the result below have been proved by Wilson in [46] while the upper bounds have been obtained by the author in [25].

Theorem 3.1. *For any sequence k_N such that $\lim \min(k_N, N - k_N) = \infty$, the mixing time of the symmetric exclusion process on $[\![1, N]\!]$ with k_N particles satisfies, for any $\varepsilon \in (0, 1)$,*

$$\lim_{N\to\infty} \frac{T_{\text{mix}}^{\text{SSEP},(N,k_N)}(\varepsilon)}{N^2 \log[\min(k_N, N - k_N)]} = \frac{1}{\pi^2}. \tag{3.1}$$

For the interchange process on $[\![1, N]\!]$, we have, for any $\varepsilon \in (0, 1)$,

$$\lim_{N\to\infty} \frac{T_{\text{mix}}^{\text{IP},(N)}(\varepsilon)}{N^2 \log N} = \frac{1}{\pi^2}.$$

In view of the correspondence discussed in Section 2.5, the first part of the result, that is, (3.1), is also valid for the corner-flip dynamics introduced in Section 2.3.

The random walk on the simplex. The process is in a sense very similar to the simple exclusion process with a positive density of particles. However, the methods developed in [25, 46] – and, more generally, many of the techniques concerning upper bounds for the mixing time – rely on the fact that the state-space is discrete. The following, proved in [4], is one of a few cutoff results that have been proved for a Markov chain evolving in a continuum (see [19] for another example).

Theorem 3.2. *For the random walk on the simplex $\mathfrak{X}_N$, we have, for any $\varepsilon \in (0, 1)$,*

$$\lim_{N\to\infty} \frac{T_{\text{mix}}^{\text{RW},(N)}(\varepsilon)}{N^2 \log N} = \frac{1}{\pi^2}.$$

Upper and lower bounds of the right order - that is $N^2 \log N$ - but without the right constant factor has been proved prior to the above theorem in [38] (see also [37] for similar results in a periodic setting).

The ASEP and the biased interchange process. The introduction of a bias has the effect of making the system mix considerably faster: a time of order N is required for mixing instead of $N^{2+o(1)}$ in the symmetric case (this was proved in [2]). In [22] jointly with C. Labbé, we were able to identify the sharp asymptotics of the mixing time, proving cutoff both for the ASEP and for the biased interchange process.

Theorem 3.3. *For any $p \in (1/2, 1]$ and any sequence (k_N) such that*

$$\forall N \geq 2, k_N \in [\![1, N-1]\!] \quad \textit{and} \quad \lim_{N\to\infty} \frac{k_N}{N} = \alpha \in [0,1],$$

the mixing time of the asymmetric exclusion process with k_N particles satisfies, for every $\varepsilon \in (0,1)$,

$$\lim_{N\to\infty} \frac{T_{\text{mix}}^{\text{ASEP},(p,N,k_N)}(\varepsilon)}{N} = \frac{(\sqrt{\alpha} + \sqrt{1-\alpha})^2}{2p-1}.$$

For the biased interchange process on a segment, we have, for every $\varepsilon \in (0,1)$,

$$\lim_{N\to\infty} \frac{T_{\text{mix}}^{\text{BIP},(p,N)}(\varepsilon)}{N} = \frac{2}{2p-1}.$$

Note that the expression for the mixing time in the above result diverges when p tends to $1/2$. In [23,30] the crossover regime between the symmetric and asymmetric case is investigated. The right order of magnitude for the mixing time is established in [30], while [23] proves cutoff results.

3.3. Review of related works

Cutoff window and profile. The results above concern the first-order asymptotics of the mixing time. However, one can aim for results with a finer precision. For instance, one can try to estimate the order of magnitude of $T_{\text{mix}}^{(N)}(\varepsilon) - T_{\text{mix}}^{(N)}(1-\varepsilon)$ (say, for a fixed $\varepsilon \in (0, 1/2)$, this could theoretically depend on the value of ε, but in practice it does not for most chains), a quantity called the width of *the cutoff window*. One can further refine the picture and look for the limit of the distance to equilibrium $d^{(N)}(t)$ after recentering the picture at $t = T_{\text{mix}}^{(N)}(1/2)$ and rescaling time by the cutoff window width. This is called *the cutoff profile*, and is the finest degree of description of convergence to equilibrium. For the SSEP on the circle – which is the closest cousin for the exclusion SSEP on the segment – the cutoff window of order N^2 and the profile have been identified in [24,26]. In the asymmetric case, the cutoff window of order $N^{1/3}$ and the profile have been identified in [3] (in the case where the density of particles is positive).

The exclusion process with an open boundary condition. We have considered the above dynamics where the number of particles is conserved. It is possible to consider the case of open boundaries, where particles can enter and exit the segment on the left and on the right. In that case, the equilibrium and dynamical behavior of the system depends a lot on the value chosen for the exit and entrance rate of the particles at the left and right boundary. Mixing-time results for the exclusion of the segment with a variety of boundary conditions are proved in [15], where several open questions and conjectures are also displayed. One of these conjectures is solved in [43], where it is shown that in the maximal current phase, for the totally asymmetric exclusion process (TASEP), the mixing time in that case is of order $N^{3/2}$. A similar result is predicted to hold for the asymmetric exclusion process on the circle, and the corresponding lower bound on the mixing time can be deduced from the results in [1].

The exclusion process in a random environment. Another variant of the process has been considered where the bias that each particle feels depends on the site at which it lies. That

is, p varies with i. In [29, 42] the case of i.i.d. random biases has been considered. This is a multiparticle version of the the classical Random Walk in a Random Environment (RWRE) (see, e.g., [20, 44] for seminal references and [14] for a study of the mixing time of RWRE). The works [29, 42] show that the presence of inhomogeneities in the environment can slow down the convergence to equilibrium.

The exclusion process in higher dimensions. The symmetric exclusion process on a higher-dimensional rectangle or torus has also been investigated. Proving result beyond dimension one turns out to be more difficult since monotonicity (in the sense of Section 4.1), which is a tool of crucial importance, cannot be used. It has been shown in [34] that the exclusion process in that case continues with a mixing time of order $N^2 \log k$ (see also [36, 48] for earlier functional inequalities, which implies that the mixing time is of order $N^2 \log N$ when there is a density of particles).

More general interfaces. The mixing of one-dimensional interfaces has been studied well beyond the case of the corner-flip dynamics. In [6, 8, 10, 27, 28, 47], the case of interfaces interacting with a substrate has been considered. The references [8, 9, 11, 17] investigate the mixing time of higher-dimensional interfaces. In [5], interfaces with real-valued height functions are considered beyond the case of the random walk on the simplex. Let us finally mention [13] which proves a cutoff for Gaussian interfaces (the lattice free field) in arbitrary dimension.

4. A FEW TECHNICAL TOOLS USED TO PROVE THESE RESULTS

We review of a few key ingredients used in the proof of the results presented in the previous section. More precisely, to illustrate these techniques, we present a proof of nonoptimal results concerning the mixing time of the simple exclusion process on the segment (symmetric and asymmetric), or rather, its corner-flip representation. Although the presentation slightly differs, the argument found below is in spirit very similar to that found in [46, SECTION 3]. The reasoning can be applied without much change to the interchange process (see Remark 4.7) but, for clarity and conciseness, we limit the exposition of details to the case of the exclusion process. We discuss in Section 5 which additional ideas are needed to improve on this nonoptimal result.

4.1. Order preservation

Let $\leq$ be a partial order relation on Ω. Given $\alpha, \beta \in M_1(\Omega)$, we say that α is stochastically dominated by β (for the order $\leq$), and write $\alpha \preccurlyeq \beta$, if one can construct – on the same probability space – a pair of Ω-valued variables Z_α and Z_β with respective distributions α and β such that we have $Z_\alpha \leq Z_\beta$ with probability one. A Markov chain with generator $\mathcal{L}$ is said to be *order preserving* or *attractive* if its semigroup preserves stochastic ordering, that is, for any $t > 0$,

$$\alpha \preccurlyeq \beta \Rightarrow \alpha P_t \preccurlyeq \beta P_t.$$

An equivalent way of saying this is that the dynamic is order preserving if, for any $x, y \in \Omega$ such that $x \preccurlyeq y$, one can couple two Markov chains $(X_t^x)_{t\geq 0}$ and $(X_t^y)_{t\geq 0}$ with respective

initial conditions x and y in such a way that

$$\forall t \geq 0, \quad X_t^x \leq X_t^y.$$

Order preservation for the corner-flip dynamics. We define $\leq$ on $\Xi_{N,k}$ to simply be the coordinatewise order, that is,

$$\zeta \leq \zeta' \quad \Leftrightarrow \quad \forall i \in [\![1, 2N-1]\!], \quad \zeta(i) \leq \zeta'(i). \tag{4.1}$$

To show that the corner-flip dynamics on $\Xi_{N,k}$ is order preserving, we use a construction which is similar to that presented above in equation (2.4), using clock processes $(\mathcal{T}_n^{(i)})_{i\in[\![1,N-1]\!],n\geq 0}$ (independent Poisson processes with mean-1 interarrival law) and accessory variables $(U_n^{(i)})_{i\in[\![1,N-1]\!],n\geq 0}$ which are i.i.d. uniform variables on the interval $[0, 1)$. The clock processes $(\mathcal{T}^{(i)})_{n\geq 0}$ determine when the updates of coordinate i are performed, and the variables $U_n^{(i)}$ are used to determine whether the corner should be flipped up or down. Given $\zeta \in \Xi_{N,k}$, we construct (h_t^ζ) as the unique càdlàg process which satisfies:

(i) $h_0^\zeta = \zeta$,

(ii) $(h_t^\zeta)_{t\geq 0}$ remains constant on the intervals of $\mathbb{R}_+ \setminus (\mathcal{T}_n^{(i)})_{i\in[\![1,N-1]\!],n\geq 0}$;

(iii) If $t = \mathcal{T}_n^{(i)}$ and $h_{t-}^\zeta = \xi$, then

(A) if $U_n^{(i)} \in [1-p, 1)$ set $h_t = \xi^{(i,+)}$,

(B) if $U_n^{(i)} \in [0, 1-p)$ set $h_t = \xi^{(i,-)}$.

Since the sets $\{\mathcal{T}_n^{(i)}\}_{i\in[\![1,N-1]\!],n\geq 0}$ display no accumulation points, (h_t^ζ) can be constructed by performing the updates sequentially. We can use this construction (using the same $\mathcal{T}$ and U) to obtain a collection of processes (h_t^ζ), $\zeta \in \Xi_{N,k}$, constructed on the same probability space, such that

$$\zeta \leq \zeta' \quad \Rightarrow \quad \forall t \geq 0, \quad h_t^\zeta \leq h_t^{\zeta'}. \tag{4.2}$$

The validity of (4.2) follows from the fact that each update is order preserving, which holds true because, for any fixed i, the applications $\zeta \mapsto \zeta^{(i,\pm)}$ are order preserving. A coupling such as that presented above, where chains starting from all initial conditions are constructed on a common probability space, is called a *grand coupling*. This type of construction using an auxiliary variable is called the *graphical construction* and is quite common for interacting particle or spin systems (see, for instance, **[32, CHAPTER III.6]**).

Remark 4.1. For the interchange process, we can use the order which corresponds to (4.1) after applying the correspondences of Section 2.5, and a similar construction allows obtaining a monotone grand coupling. An analogous construction also provides a monotone grand coupling for the random walk on the simplex.

4.2. Connection with the discrete heat equation

Let us expose first how the evolution of the mean of simple observables – the height function in the symmetric case, the exponential of the height in the asymmetric case – can be described by a simple system of linear equations.

4.2.1. The symmetric case

Given $\zeta \in \Xi_{N,k}$, we define $u^\zeta(t,\cdot)$ to be the recentered mean height of the interface at time t for the corner-flip dynamics with initial condition ζ,

$$u^\zeta(t,i) := \mathbb{E}\big[h_t^\zeta(i)\big]. \tag{4.3}$$

For a real-valued function f defined on $[\![1, N-1]\!]$, we define $\Delta_D f$ (Δ_D being the discrete Laplace operator with Dirichlet boundary condition) by

$$\Delta_D f(i) := f(i+1) + f(i-1) - 2f(i) \quad \text{for } i \in [\![1, N-1]\!], \tag{4.4}$$

with the convention that $f(0) = 0$ and $f(N) = N - 2k$. The function u^ζ is the unique solution of the following system of differential equations that can be considered as a partial differential equation where the space variable is discrete (Δ_D acts on the second variable)

$$\partial_t u(t,i) = \frac{1}{2}\Delta_D u(t,i), \quad \forall i \in [\![1, N-1]\!]. \tag{4.5}$$

Setting $U^{(i)}(\zeta) := \zeta(i)$, equation (4.5) is deduced from the identity (that can be checked from the definition of the generator), namely

$$\mathfrak{L}_{N,k} U^{(i)}(\zeta, i) = \frac{1}{2}\Delta_D \zeta(i).$$

More precisely, (4.5) is obtained by combining (1.10), the Markov property, the above identity, and the fact that Δ_D, being an affine transformation, commutes with the expectation, as follows:

$$\partial_t u(t,i) = \mathbb{E}\big[\mathfrak{L}_{N,k} U^{(i)}\big(h_t^\zeta\big)\big] = \mathbb{E}\big[\Delta_D h_t^\zeta(i)\big] = \Delta_D\big(\mathbb{E}\big[h_t^\zeta\big]\big)(i) = \Delta_D u(t,i). \tag{4.6}$$

The fact that u^ζ does not satisfy the zero boundary condition is not a problem since, in computations, we consider the difference $u^\zeta - u^{\zeta'}$ which displays the zero boundary condition. The Dirichlet Laplacian with the zero boundary condition $\Delta_D^{(0)}$ is a linear operator that can easily be diagonalized. The family $(\overline{\sin}^{(j)})_{j=1}^{N-1}$ defined by $\overline{\sin}^{(j)}(i) := \sin(\frac{ij\pi}{N})$ forms a base of eigenvectors of $\Delta_D^{(0)}$ in $\mathbb{R}^N$, and we have

$$\Delta_D^{(0)} \overline{\sin}^{(j)} = -2\gamma_N^{(j)} \overline{\sin}^{(j)} \quad \text{where } \gamma_N^{(j)} = 1 - \cos\left(\frac{j\pi}{N}\right). \tag{4.7}$$

Using Parceval's inequality, we obtain the following contractive estimates, which we use to bound the mixing time.

Lemma 4.2. *If $u : [0,\infty) \times [\![1, N-1]\!]$ satisfies $\partial_t u = \Delta_D^{(0)} u$, then we have, for any $t \geq 0$,*

$$\sum_{i=1}^{N-1} u(t,i)^2 \leq e^{-2\gamma_1^{(N)} t} \sum_{i=1}^{N-1} u(0,i)^2.$$

4.2.2. The asymmetric case

When $p \neq 1/2$, the quantity $\mathfrak{L}_{N,k}^{(p)} U^{(i)}(\zeta)$ cannot be expressed as a linear combination of $U^{(j)}(\zeta)$, $j \in [\![1, N]\!]$ so that there is no way to recover a linear system analogous to (4.5) for the averaged heights.

However, we can obtain something similar for the evolution of an averaged quantity related to the heights. The key idea which can be traced back to [16] (where it is used to derive hydrodynamic limits) is to apply the so-called discrete Cole–Hopf transform. We consider exponentials of heights rather the than heights themselves. Recalling that $\lambda = p/q$, we define

$$V(\zeta, i) := \lambda^{\frac{1}{2}\zeta(i)} \quad \text{and} \quad v^{\zeta}(t, i) := \mathbb{E}^{(p)}\big[V\big(h_t^{\zeta}\big)(i)\big].$$

Setting $\varrho := (\sqrt{p} - \sqrt{q})^2$, it can be checked from the definition of the generator that for every ζ and $i \in [\![1, N-1]\!]$, we have

$$\mathfrak{L}_{N,k}^{(p)} V(\zeta, i) = \sqrt{pq}\,\Delta_D V(\zeta, i) - \varrho V(\zeta, i), \tag{4.8}$$

where this time Δ_D denotes the Dirichlet Laplacian defined as in (4.4) but with the boundary condition $f(0) = 1$ and $f(N) = \lambda^{\frac{N}{2}-k}$ (we refer to [22, SECTION 3.3] for details on the computation leading to (4.8)). In (4.8) note that $\mathfrak{L}_{N,k}^{(p)}$ acts on the first coordinate while Δ_D acts on the second. As in (4.6), we obtain from (4.8) that v^{ζ} satisfies

$$\partial_t v(t, i) := (\sqrt{pq}\,\Delta_D - \varrho) v(t, i), \quad \forall i \in [\![1, N-1]\!]. \tag{4.9}$$

Again, the nonzero boundary condition for Δ_D here is of no importance since in practice we are going to consider the difference $v^{\zeta} - v^{\zeta'}$. As in the symmetric case, the diagonalization of the operator with the zero boundary condition $\Delta_D^{(0)}$ yields the following estimate.

Lemma 4.3. *If v satisfies $\partial_t v = \sqrt{pq}\,\Delta_D^{(0)} v - \varrho v$ then we have*

$$\sum_{i=1}^{N-1} v(t, i)^2 \le e^{-2(\gamma_1^{(N)} + \varrho)t} \sum_{i=1}^{N-1} v(0, i)^2.$$

4.3. Using the heat equations to obtain bounds on the mixing time

Let $(h_t^{(1)})$ and $(h_t^{(2)})$ be two *ordered* corner flip dynamics, that is, such that $h_t^{(1)} \le h_t^{(2)}$ for all t. Using only Lemmas 4.2, 4.3, and order preservation, we can control the coupling time of $(h_t^{(1)})$ and $(h_t^{(2)})$ defined by

$$\tau := \inf\big\{t > 0 : h_t^{(1)} = h_t^{(2)}\big\}. \tag{4.10}$$

Proposition 4.4. *If $(h_t^{(1)})$ and $(h_t^{(2)})$ are two ordered symmetric corner flip dynamics then, for any $t > 0$, we have*

$$\mathbb{P}[\tau > t] \le k(N-1) e^{-\gamma_1^{(N)} t}.$$

If $(h_t^{(1)})$ and $(h_t^{(2)})$ are two ordered asymmetric corner flip dynamics with parameter p, then we have

$$\mathbb{P}[\tau > t] \le k(N-1) \lambda^{N/2-1} e^{-\varrho t}.$$

From these coupling estimates, we can derive upper estimates on the mixing time.

Corollary 4.5. *We have*

$$T_{\text{mix}}^{\text{SSEP},(N,k_N)} \le \frac{1}{\gamma_1^{(N)}} \log\left(\frac{2k(N-1)}{\varepsilon}\right),$$

$$T_{\text{mix}}^{\text{ASEP},(p,N,k_N)} \le \frac{1}{\varrho}\left[\left(\frac{N}{2} - 1\right)\log\lambda + \log\left(\frac{2k(N-1)}{\varepsilon}\right)\right].$$

Remark 4.6. Replacing $\gamma_1^{(N)}$ by an asymptotic equivalent ($\frac{\pi^2}{2N^2}$), we find that the upper bound on the SSEP mixing time is $\frac{2N^2}{\pi^2}(\log N + \log k)(1 + o(1))$ which is, in the best case, a factor of 4 away from the estimate given in Theorem 3.1. For the ASEP, our upper bound is asymptotically equivalent to $\frac{\log \lambda}{2\varrho} N$. Since we have, for every $p \in (1/2, 1)$,

$$\frac{\log \lambda}{2\varrho} > \frac{2}{2p-1} = \max_{\alpha \in [0,1]} \frac{(\sqrt{\alpha} + \sqrt{1-\alpha})^2}{2p-1},$$

in this case again the estimate is not sharp. The reason why the bounds in Corollary 4.5 are not sharp is further discussed in Section 5.

Proof of Corollary 4.5. Using the correspondence of Section 2.5, we can reason with the corner flip dynamics since it has the same mixing time. In order to prove an upper bound on the mixing time, one must bound from above the distance between $\mathbb{P}[h_t^\zeta \in \cdot]$ and the stationary measure π for an arbitrary $\zeta \in \Xi_{N,k}$. In order to transform this into a coupling problem, note that $\pi = \mathbb{P}[h_t^\pi \in \cdot]$ where, with some abuse of notation, we let h_t^π denote a Markov chain with initial condition π.

Let us consider now three different dynamics, h_t^ζ, $h_t^\wedge$, and h_t^π, with respective initial conditions ζ, $\wedge$ (defined in (2.3)), and stationary distribution. They are constructed on the same probability space and coupled in such a way that, for all $t \geq 0$ (Section 4.1 gives such a coupling),

$$h_t^\pi \leq h_t^\wedge \quad \text{and} \quad h_t^\zeta \leq h_t^\wedge.$$

Using (1.2), stationarity, and union bound, we have

$$\big\| \mathbb{P}\big[h_t^\zeta \in \cdot\big] - \pi \big\|_{\mathrm{TV}} \leq \mathbb{P}\big[h_t^\zeta \neq h_t^\pi\big] \leq \mathbb{P}\big[h_t^\zeta \neq h_t^\wedge\big] + \mathbb{P}\big[h_t^\pi \neq h_t^\wedge\big] = \mathbb{P}[\tau_1 > t] + \mathbb{P}[\tau_2 > t],$$

where we have set

$$\tau_1 := \inf\big\{t : h_t^\zeta \neq h_t^\wedge\big\} \quad \text{and} \quad \tau_2 := \inf\big\{t : h_t^\pi \neq h_t^\wedge\big\}. \tag{4.11}$$

The tail distributions of τ_1 and τ_2 can be estimated using Proposition 4.4, and we obtain (let us now for the first time highlight the difference in p)

$$\begin{cases} \big\| \mathbb{P}\big[h_t^\zeta \in \cdot\big] - \pi \big\|_{\mathrm{TV}} \leq 2(N-1)k e^{-\gamma_1^{(N)} t} & \text{in the symmetric case,} \\ \big\| \mathbb{P}\big[h_t^\zeta \in \cdot\big] - \pi \big\|_{\mathrm{TV}} \leq 2(N-1)k \lambda^{N/2-1} e^{-\varrho t} & \text{in the asymmetric case.} \end{cases} \tag{4.12}$$

The reader can then check that the value of t which makes the right-hand side in (4.12) equal to ε is the claimed upper bound on the mixing time. ∎

Proof of Proposition 4.4. Let us start with the symmetric case. We set

$$h_t^{(1,2)}(i) := h_t^{(2)}(i) - h_t^{(1)}(i) \quad \text{and} \quad u^{(1,2)}(t,i) := \mathbb{E}\big[\big(h_t^{(2)} - h_t^{(1)}\big)(i)\big].$$

Since $h_t^{(1)} \leq h_t^{(2)}$, we have $h_t^{(1,2)}(i) \geq 0$ for all i and, if $h_t^{(1)} \neq h_t^{(2)}$, the inequality must be strict for at least one value of i. Since the minimal discrepancy between two values of $\zeta(i)$ is 2, this implies that

$$\mathbb{P}[\tau > t] = \mathbb{P}\big[h_t^{(1)} \neq h_t^{(2)}\big] = \mathbb{P}\left[\sum_{i=1}^{N-1} h_t^{(1,2)}(i) \geq 2\right] \leq \frac{1}{2} \sum_{n=1}^{N-1} u^{(1,2)}(t,i). \tag{4.13}$$

Combining Cauchy–Schwarz inequality with Lemma 4.2 – from (4.5) we know that $u^{(1,2)}$ satisfies the assumption – we have

$$\sum_{i=1}^{N-1} u^{(1,2)}(t,i) \le \left((N-1)\sum_{i=1}^{N-1} u^{(1,2)}(t,i)^2\right)^{1/2} \le e^{-\gamma_1^{(N)}t}\left((N-1)\sum_{i=1}^{N-1} u^{(1,2)}(0,i)^2\right)^{1/2},$$

and we can conclude using the fact $u^{(1,2)}(0,i) \le 2k$ since $2k$ is a bound for the maximal height difference between two elements in $\Xi_{N,k}$.

For the asymmetric case, we apply the reasoning to the exponential of the heights

$$W(\zeta) := \sum_{i=1}^{N-1} V(\zeta,i) \quad \text{and} \quad W_t^{(1,2)} := W\big(h_t^{(2)}\big) - W\big(h_t^{(1)}\big). \tag{4.14}$$

Note that, since $\zeta(i) \ge -k$ for all ζ and i, the minimal positive value of $W_t^{(1,2)}$ is given by

$$\delta_{\min} := \min_{\substack{\zeta' \ge \zeta \\ \zeta' \ne \zeta}} W\big(\zeta'\big) - W(\zeta) = (\lambda-1)\lambda^{-k/2}.$$

Repeating the reasoning in (4.13) in the asymmetric case, we obtain that

$$\mathbb{P}[\tau > t] \le \frac{\mathbb{E}[W_t^{(1,2)}]}{\delta_{\min}}.$$

Now from (4.9), $v^{(1,2)}(t,i) := \mathbb{E}[V(h_t^{(2)},i) - V(h_t^{(1)},i)]$ satisfies the assumptions of Lemma 4.3. We obtain, using Cauchy–Schwarz inequality, that

$$\mathbb{E}\big[W_t^{(1,2)}\big] \le \sum_{i=1}^{N-1} v^{(1,2)}(t,i) \le e^{-\varrho t}\left((N-1)\sum_{i=1}^{N-1} v^{(1,2)}(0,i)^2\right)^{1/2}.$$

Now considering that the maximal possible height difference is $2k$ and that the maximal possible value of $\zeta(i)$ is always smaller than $N-k$, we have, for every $i \in [\![1, N-1]\!]$,

$$v^{(1,2)}(0,i) \le \max_{\zeta' \in \Xi_{N,k}} \lambda^{\frac{\zeta'(i)}{2}}\big(1-\lambda^{-k}\big) \le k(\lambda-1)\lambda^{\frac{N-k}{2}-1}.$$

Setting $\delta_{\max} := (\lambda-1)\lambda^{\frac{N-k}{2}-1}$, we obtain that $\sum_{i=1}^{N-1} v^{(1,2)}(0,i)^2 \le \delta_{\max}^2 (N-1)k^2$ so that

$$\mathbb{P}[\tau > t] \le \frac{\delta_{\max}}{\delta_{\min}}(N-1)ke^{-\varrho t},$$

which is the desired result. ∎

Remark 4.7. Note that the argument exposed in this section can also be used without changes for the interchange process. Indeed, the correspondences exposed in Section 2.5 allow us to associate, to the dynamics σ_t, $N-1$ corner-flip dynamics $(h_t^{(k)})$, $k = 1, \dots, N$, defined by

$$h_t^{(k)} = h \circ \xi^{(k)} \circ \sigma_t,$$

where the transformations h and $\xi^{(k)}$ are those of Section 2.5. The observation (2.6) guarantees that two dynamics $\sigma_t^{(1)}$ and $\sigma_t^{(2)}$ are coupled when all the corresponding corner-flip dynamics are coupled, so that the analog of Proposition 4.4 is valid for the interchange process on the segment, with the factor $k(N-1)$ replaced by $(N-1)^3$. The reader can refer to **[46, SECTION 3]** and **[22, SECTION 3.4]** for more details in the symmetric and asymmetric cases, respectively.

5. SHORTCOMINGS AND POSSIBLE IMPROVEMENTS OF THE REASONING ABOVE

5.1. Symmetric dynamics

As mentioned in Remark 4.6, the upper-bound on the SSEP mixing time is suboptimal, off by a factor of 4 in the case when k and $N-k$ are of order N. There are two separate reasons for which the method does not yield an optimal result, each being accountable for a multiplicative factor of 2. To illustrate this, let us mention [46, SECTION 8], where it is proved that for the monotone coupling inherited from the graphical construction (described in Section 4.1), the coupling time τ_1 in (4.11) is of order $\frac{2}{\pi^2} N^2 \log k$. This results shows that not only the method above is off by a factor of 2 to estimate the coupling time, but also, compared to Theorem 3.1, that this coupling time itself does allow for a sharp estimate on the mixing time. This means that in order to improve the bound on the mixing time, we have to design a monotone coupling that makes the value of the coupling time τ as small as possible.

This becomes particularly obvious when the random walk on the simplex is considered (recall Section 2.4). If one considers the monotone grand coupling based on the graphical construction presented in Section 2.4, then trajectories starting with different initial conditions *never* coalesce ($\tau = \infty$ almost surely). Hence for this model, there can be no equivalent of Proposition 4.4: any nontrivial estimate of τ must rely on specific features of the coupling beyond monotonicity.

In [4,5,23–26], refinements have been performed in order to obtain optimal estimates on the mixing time. This first one is the introduction of a coupling that is aimed at minimizing the coalescence time. The basic idea for the discrete model is to make the corner-flips performed by $h_t^{(1)}$ and $h_t^{(2)}$ less synchronized while preserving monotonicity so that the quantity

$$A(t) := \sum_{i=1}^{N-1} \big(h_t^{(2)} - h_t^{(1)}\big),$$

which is an integer-valued supermartingale, hits zero faster. Roughly speaking, this is achieved by having, at any given time, independent corner flips for coordinates at which $h_t^{(2)}(x) > h_t^{(1)}(x)$, and synchronized corner flips for coordinates at which $h_t^{(2)}(x) = h_t^{(1)}(x)$ (the couplings used in the continuous setup in [4,5] are based on an analogous intuition). The second key improvement is to use diffusion estimates in order to estimate the time when $A(t)$ hits 0, instead of relying on Markov's inequality. For the corner-flip dynamics, $A(t)$ is a time-changed random walk on $\mathbb{Z}_+$, and the hitting time of 0 can be precisely estimated if one has some control over its jump rate (see [24–26]). This idea was considerably improved in [4, 5, 23] where we need to estimate the hitting time of zero of a supermartingale which is not integer-valued. The improvement comes from reasoning in terms of martingale brackets instead of jump rate.

5.2. Asymmetric dynamics

Remark 4.6 also underlines that the result of the previous section is also suboptimal in the asymmetric case. The reason for this is that the quantity $W_t^{(2,1)}$ considered in (4.14)

is typically much smaller than its average (by a factor which is exponential in N). Since this quantity has very wild fluctuation, it is not possible to apply to it the same technique as in the symmetric case. The proof of Theorem 3.2 presented in [22] relies on two key ingredients:

(A) Hydrodynamic limits;

(B) The control of particle speed when the density is vanishing.

Hydrodynamic limits are an extensively studied topic for particle systems (see [21]). The hydrodynamic limit of a system is the limit obtained for the evolution of the particle density after rescaling time and space. It usually takes the form of the solution to partial differential equation. In the case of the asymmetric exclusion process, is has been established (see [40] where the result is proved in a much broader context) that the hydrodynamic limit – after rescaling time and space by N – is the solution of the equation

$$\partial_t \rho = (2p-1)\partial_x\big[\rho(1-\rho)\big]. \tag{5.1}$$

More precisely, for the exclusion on the segment, we have to consider some specific notion of a solution and boundary conditions (see [22, SECTION 5] for details). In this context, given any initial condition ρ_0, which satisfies

$$\forall x \in [0,1],\ 0 \le \rho_0(x) \le 1 \quad \text{and} \quad \int_{[0,1]} \rho(x) = \alpha,$$

(5.1) has a unique solution which stabilizes to the fixed point $\mathbf{1}_{[1-\alpha,1]}$ after a time $\frac{(\sqrt{\alpha}+\sqrt{1-\alpha})^2}{2p-1}$, indicating that at time $\frac{(\sqrt{\alpha}+\sqrt{1-\alpha})^2 N}{2p-1}$ the system is macroscopically at equilibrium.

What remains to check afterwards is whether around that time the system is also at equilibrium in the total variation sense, which is *a priori* a much finer statement. The important point is to verify that the position of the leftmost particle and rightmost empty site match the indication given by the macroscopic profile (that is, are both $(1-\alpha)N + o(N)$), and this is where the point (B) comes into play (we refer to [22, SECTION 6] for more details).

Once we have proved that both the density of particles and the position of the leftmost particle/rightmost empty site have reached their equilibrium, we still have not proved that the system is at equilibrium. However, this information implies that with the notation of Section 4.3, when $t = t_{\alpha,N} := \frac{(\sqrt{\alpha}+\sqrt{1-\alpha})^2}{2p-1}$, we have $W_t^{(1,2)} = \exp(o(N))\delta_{\min}$. Hence we can use, as a third step of our reasoning, the contraction estimate of Lemma 4.3 to show that a coupling must occur shortly after time $t_{\alpha,N}$.

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