# ULTRAMETRICITY IN SPIN GLASSES


## DMITRY PANCHENKO



### ABSTRACT

Ultrametricity of the Gibbs measure is a fundamental feature of the Parisi solution of the Sherrington–Kirkpatrick model of spin glasses. We will start by describing one origin of ultrametricity in a way that requires no special knowledge, and after that review some background and discuss some applications.



### MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 82B44; Secondary 60K35

### KEYWORDS

Spin glasses, Parisi ansatz, ultrametricity

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14





Proc. Int. Cong. Math. 2022, Vol. 6, pp. 4376–4392
DOI 10.4171/ICM2022/69

Published by EMS Press

# 1. INTRODUCTION

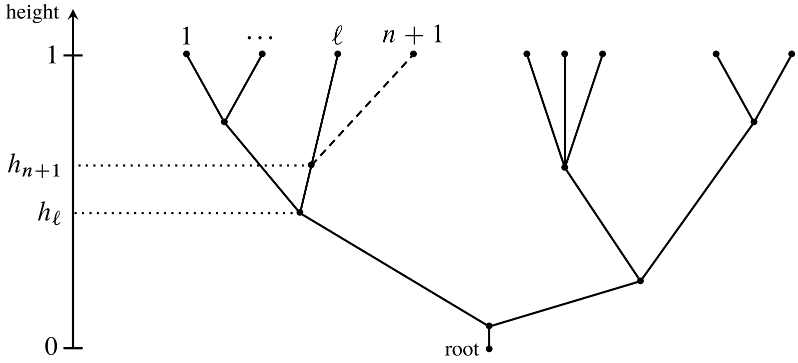


**FIGURE 1**
New leaf $n+1$ is attached to the path from a randomly chosen leaf $\ell \in \{1, \dots, n\}$ to the root. Because $h_{n+1} := \max(t_{n+1}, h_\ell) \geq h_\ell$, it is attached at or above the height $h_\ell$ where the leaf $\ell$ was originally attached to the tree.

Given a probability distribution $\zeta$ on [0, 1], let us generate a tree with the root at height 0 and countably many leaves at height 1 using the following simple sequential process (see Figure 1). The height of any point on the tree refers to its coordinate on the vertical axis (labeled "height" in the figure). In this process, $h_\ell$ will represent the height at which leaf $\ell$ was attached to the tree (during its turn). Proceed as follows:

(1) Attach leaf 1 by a new branch from the root and set $h_1 = 0$.

(2) For $n+1 = 2, 3, \dots$, repeat the following steps:

  (2a) Pick a leaf $\ell \in \{1, \dots, n\}$ uniformly at random.

  (2b) Generate a new random variable $t_{n+1}$ from the distribution $\zeta$.

  (2c) Attach the leaf $n+1$ to the path from the chosen leaf $\ell$ to the root at the height $h_{n+1} := \max(t_{n+1}, h_\ell)$, by adding a new branch.

For any two leaves $\ell, \ell' \geq 1$, let $R_{\ell,\ell'}$ be the height at which the paths from these leaves to the root meet. We will call $R := (R_{\ell,\ell'})_{\ell,\ell' \geq 1}$ the *overlap array*, since these quantities measure how much the paths overlap. We will denote by $R^n := (R_{\ell,\ell'})_{\ell,\ell' \leq n}$ the $n \times n$ block of overlaps corresponding to the first $n$ leaves. The array $R$ satisfies the following three properties, which we list from the most obvious to less obvious:

(a) The conditional distribution of the overlap $R_{1,n+1}$ given $R^n$ is equal to

$$\mathcal{L}\big(R_{1,n+1} \mid R^n\big) = \frac{1}{n}\zeta + \frac{1}{n}\sum_{\ell=2}^{n} \delta_{R_{1,\ell}}. \tag{1.1}$$

(b) The array $R$ is nonnegative definite, with diagonal elements $R_{\ell,\ell} = 1$ and off-diagonal elements $R_{\ell,\ell'} \in [0,1]$.

(c) The array $(R_{\ell,\ell'})$ is *weakly exchangeable*, which means that

$$(R_{\ell,\ell'})_{\ell,\ell'\le n} \stackrel{d}{=} (R_{\pi(\ell),\pi(\ell')})_{\ell,\ell'\le n}$$

for any $n \ge 2$ and any permutation $\pi$ of $\{1,\dots,n\}$.

Property (a) holds because, if in step (2a) we picked $\ell = 1$ then there would be no constraint on where new branch is attached and so $R_{1,n+1} = t_{n+1}$ had distribution $\zeta$, and if we picked $\ell \ge 2$ then $R_{1,n+1} = R_{1,\ell}$.

Property (b) can be seen in different ways, but one way to see it is to notice that, because of the tree structure, for any $q \in [0,1]$, the relation $\ell \sim_q \ell'$ on the leaves defined by

$$\ell \sim_q \ell' \iff R_{\ell,\ell'} \ge q \tag{1.2}$$

is an equivalence relation and, therefore, the array $(\mathrm{I}(R_{\ell,\ell'} \ge q))_{\ell,\ell'\ge 1}$ is block-diagonal with entries of each block all equal to 1 and, thus, nonnegative definite. Using that $R_{\ell,\ell'} = \int_0^1 \mathrm{I}(R_{\ell,\ell'} \ge q)\,dq$, we see that the array $R$ is also nonnegative definite. Another way to express that (1.2) is an equivalence relation is to say that

$$R_{\ell_2,\ell_3} \ge \min(R_{\ell_1,\ell_2}, R_{\ell_1,\ell_3}) \tag{1.3}$$

for any three leaves $\ell_1$, $\ell_2$, and $\ell_3$. If property (b) holds then the array $R$ satisfying (1.3) is called an *ultrametric* array, because a subset $\{h_\ell : \ell \ge 1\}$ of the unit sphere in a Hilbert space such that $R_{\ell,\ell'} = h_\ell \cdot h_{\ell'}$ will form an ultrametric set satisfying

$$\|\sigma_{\ell_2} - \sigma_{\ell_3}\| \le \max(\|\sigma_{\ell_1} - \sigma_{\ell_2}\|, \|\sigma_{\ell_1} - \sigma_{\ell_3}\|).$$

One can also embed the entire tree isometrically into a unit ball of a Hilbert space, with the overlap equal to the scalar product in this embedding.

Property (c) is not obvious and requires some calculation, but it is not difficult and we will leave it as an exercise. The basic idea is that one can compute the probability of observing a finite tree in a particular configuration by "unwinding" how this tree was formed starting from clusters of closest leaves (those with the largest overlaps), and ignoring the order, because we will end up with the factor $1/n!$ no matter what the order was. So, we have this symmetry in distribution, although it does not appear immediately obvious from the construction. This also means that (1.1) also holds with indices permuted.

Notice that the properties (a), (b), and (c) do not explicitly refer to the tree structure in Figure 1. However, it turns out that these properties do imply that such a tree structure must be present, even if it is a priori not given.

**Theorem 1.1** ([102]). *If the array $(R_{\ell,\ell'})_{\ell,\ell'\ge 1}$ satisfies properties (a), (b), and (c) then (1.3) holds and the array can be generated as in Figure 1 with $\zeta = \mathcal{L}(R_{1,2})$.*

The distributional identities (1.1) in property (a) are called the *Ghirlanda–Guerra identities* [71] and, in all intended applications, properties (b) and (c) are, essentially, built into the construction. So, in words, Theorem 1.1 says that the Ghirlanda–Guerra identities (1.1) imply ultrametricity (1.3). As we will discuss below, the result is useful because the Ghirlanda–Guerra identities often appear rather naturally.

Below we will discuss the origin of this result in the setting of the so-called spin glass models from statistical physics. Before we begin, let us mention that the construction of the tree in Figure 1 is called the *Goldschmidt–Martin algorithm* [72] for generating a sample from the *Ruelle Probability Cascades* [115] corresponding to the overlap distribution $\zeta$ or, equivalently, for constructing the *Bolthausen–Sznitman coalescent* [34] (see also [98]). Nonnegative definite random arrays satisfying property (c) are called *Gram–de Finetti arrays*, and the analogue of de Finetti's representation for such arrays is called the *Dovbysh–Sudakov representation* [66] (see, e.g., [103, SECTION 1.5]).

## 2. SOME BACKGROUND

The name "spin glass" refers to certain dilute magnetic alloys (for example, dilute solutions of manganese in copper, or other magnetic atoms in nonmagnetic metals), and it seems to have been coined by Philip Anderson and Wai-Chao Kok (according to [7]). An entertaining account of a part of the history of spin glasses in physics up to 1990 can be found in [5–11]. Here I will only mention a few fundamental results related to the Sherrington–Kirkpatrick model [117]. In this model, given integer $N \geq 1$, one considers a Gaussian process $H_N(\sigma)$ indexed by $\sigma \in \Sigma_N := \{-1,+1\}^N$,

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j,$$

where the coefficients $g_{ij}$ are i.i.d. standard Gaussian random variables. This process is called the *Hamiltonian* of the model. It is sometimes viewed as the energy function of a random optimization problem of assigning students to two dorms, called dean's problem. Let $i, j$ be indices corresponding to $N$ students, $g_{ij}$ be an interaction parameter describing how much student $i$ likes or dislikes student $j$, and $\sigma_i$ be the label of one of two dorms $\{-1,+1\}$ that student $i$ is assigned to. If we write $\sigma_i\sigma_j = 2\mathrm{I}(\sigma_i = \sigma_j) - 1$, we can see that maximizing $H_N(\sigma)$ over all possible assignments $\sigma$ is equivalent to maximizing the so-called *comfort function* $\sum_{i\neq j} g_{ij}\mathrm{I}(\sigma_i = \sigma_j)$, which is the sum of interactions within the same dorms. It is not difficult to check that $\max_\sigma H_N(\sigma)$ is of order $\mathcal{O}(N)$, and so one may try to compute the exact limit of

$$\frac{1}{N} \max_{\sigma\in\Sigma_N} H_N(\sigma)$$

as $N \to \infty$. Related to this maximum is the *free energy*

$$F_N = F_N(\beta) := \frac{1}{N} \log \sum_{\sigma\in\Sigma_N} \exp \beta H_N(\sigma),$$

where $\beta > 0$ is called the *inverse temperature* parameter. Free energy can be viewed as a "smooth approximation" of the maximum, because

$$\frac{F_N(\beta)}{\beta} \le \frac{1}{N} \max_{\sigma\in\Sigma_N} H_N(\sigma) \le \frac{F_N(\beta)}{\beta} + \frac{\log 2}{\beta},$$

which can be seen by bounding the sum from below by the largest term, or replacing all terms by the largest one. This means that $F_N(\beta)/\beta$ is a good approximation of the maximum when $\beta$ is large, so one can try to compute the limit of the free energy for any fixed $\beta$ first. This turns out to be closely related to understanding the geometric and probabilistic structure of the *Gibbs measure* of the model,

$$G_N(\sigma) = \frac{\exp \beta H_N(\sigma)}{\sum_{\rho\in\Sigma_N} \exp \beta H_N(\rho)}.$$

The formula for the limit of $F_N(\beta)$ was proposed by David Sherrington and Scott Kirkpatrick in [117] based on the so-called replica formalism: using the formula $\log x = \lim_{n\downarrow 0} n^{-1}(x^n - 1)$, interchanging limits $N \to \infty$ and $n \to 0$, computing the $N \to \infty$ limit for integer $n \ge 0$, and hoping that the formula survives in the $n \to 0$ limit. They observed that the formula they obtained exhibited "unphysical behavior" at low temperature, which meant that it could only be correct for small enough values of $\beta$. Several years later, Giorgio Parisi [112,113] proposed another formula, now called the *Parisi formula*, which seemed to pass all the consistency checks. It stated that (almost surely)

$$\lim_{N\to\infty} F_N(\beta) = \inf_{\zeta}\left(\Phi(0,0) - \beta^2 \int_0^1 t\zeta(t)\,dt\right),$$

where the infimum is taken over all probability distributions $\zeta \in \Pr[0,1]$ on $[0,1]$ with $\zeta(t) := \zeta([0,t])$ being a cumulative distribution function, and $\Phi(t,x) : [0,1]\times\mathbb{R} \to \mathbb{R}$ the solution of

$$\Phi_t = -\beta^2\big(\Phi_{xx} + \zeta(t)(\Phi_x)^2\big), \quad \Phi(1,x) := \log 2\cosh(x).$$

Parisi's calculation started along the same lines of the replica formalism, but it required breaking from conventional wisdom in more than one way, as well as making some creative choices along the way. One of these choices was the *ultrametric parametrization* of the replica matrix that comes up in the calculation. Here this replica matrix appeared in a purely algebraic way, and it was only later given a physical meaning in another paper of Parisi [114], as the matrix of *overlaps*

$$R_{\ell,\ell'} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{\ell}\sigma_i^{\ell'} \tag{2.1}$$

of an i.i.d. sample $(\sigma^\ell)_{\ell\ge 1}$ from the Gibbs measure $G_N$. The overlap array $R$ discussed in the introduction and in Theorem 1.1 arises as a limit (in distribution) of the array (2.1) and, in the context of spin glass models, the purpose of Theorem 1.1 is to understand the distribution of this array in the thermodynamic limit $N \to \infty$. The reason why the overlaps (2.1) appear in the computation of the free energy is simple. The Hamiltonian is a Gaussian process, so its distribution is determined by its covariance, which in this case happens to be a function of the overlap,

$$\mathbb{E} H_N(\sigma^1) H_N(\sigma^2) = N(R_{1,2})^2. \tag{2.2}$$

Following the work of Parisi, there was a tremendous activity in physics using, extending, and analyzing these ideas. A classic summary of spin glasses at the end of the 1980s is the book of Mézard, Parisi, and Virasoro "Spin Glass Theory and Beyond" [91]. Some key developments appeared in a series of papers by Marc Mézard et al. [88–90, 92] in the mid-1980s, where, in particular, an algebraic choice of ultrametric parametrization in Parisi's replica calculation was expressed in terms of the familiar ultrametric geometry—in this case, the geometry of the support of the Gibbs measure in the thermodynamic limit $N \to \infty$. A very important role was also played by the study of toy models of spin glasses by Bernard Derrida et al.—the random energy model, REM, in [59, 60], and the generalized random energy model, GREM, in [61, 62]. It was shown in [58, 63, 90] that various statistics of the Gibbs sample in these toy models coincide with those in the SK model. When David Ruelle [115] gave an explicit description of the Gibbs measure in the GREM in terms of a certain family of Poisson processes, this meant that one now had an explicitly defined object conjecturally describing the Gibbs measure in the SK model. This object is now called the *Ruelle probability cascades* (RPC). For an explicit description, we will refer to Chapter 2 in [103], but Figure 1 above describes how to generate a sample from RPC corresponding to the parameter $\zeta \in \Pr[0, 1]$.

The fact that the limit of the free energy actually exists was proved by Guerra and Toninelli in [75]. The Parisi formula was proved by Michel Talagrand in a celebrated paper [126], following a discovery by Francesco Guerra [74] of an ingenious interpolation that showed that the Parisi formula is an upper bound on the limit of the free energy.

## 3. THE GHIRLANDA–GUERRA IDENTITIES

Talagrand's proof of the Parisi formula found a way around the ultrametricity (1.3) that played such an important role in the physics literature, but there is another approach based on the above Theorem 1.1 and the Aizenman–Sims–Starr scheme [4] (see [103, 104]). If we look at the array of overlaps (2.1), by definition, it satisfies properties (b) and (c) in Theorem 1.1, except for $R_{\ell,\ell'} \in [0, 1]$. The original SK Hamiltonian $H_N(\sigma)$ is symmetric under $\sigma \to -\sigma$, so the distribution of the overlaps is symmetric. However, there are various ways to break this symmetry in a way that enforces $R_{\ell,\ell'} \geq 0$ in the limit $N \to \infty$ without affecting the free energy much and, as a result, we can pretend that properties (b) and (c) always hold. This means that property (a)—the so-called Ghirlanda–Guerra identities—is really at the heart of Theorem 1.1. In fact, these identities also imply that $R_{\ell,\ell'} \geq 0$ (which is known as Talagrand's positivity principle), but, of course, their main role is to ensure that the ultrametricity of the overlap array in (1.3) holds.

So where are the Ghirlanda–Guerra identities (1.1) coming from? If we denote by $\langle \cdot \rangle$ the average with respect to the Gibbs measure $G_N$ and by $\mathbb{E}$ the average with respect to the *Gaussian disorder* $(g_{ij})$ then, roughly speaking, the form (1.1) is simply another way to express the concentration of the Hamiltonian,

$$\mathbb{E}\Big\langle\Big|\frac{H_N(\sigma)}{N}-\mathbb{E}\Big\langle\frac{H_N(\sigma)}{N}\Big\rangle\Big|\Big\rangle \to 0, \tag{3.1}$$

by testing this concentration against a test function and then integrating by parts using the formula for covariance of $H_N$ in (2.2). Precise details are a bit more complicated, but the main question becomes: Where is the concentration (3.1) coming from? The particular statement (3.1) is not easy to prove (see Section 3.7 in [103]), but, for example, the same statement on average over the inverse temperature parameter $\beta$ is rather straightforward and follows readily from the convexity of $F_N(\beta)$ and its concentration around the expectation $\mathbb{E}F_N(\beta)$, as was demonstrated by Guerra in [73] and generalized by Ghirlanda and Guerra in [71]. Once the overall idea became clear, there was a lot of room to tweak this approach and make it applicable in a variety of situations.

For example, given some Hamiltonian $H_N(\sigma)$ with maximum of order $\mathcal{O}(N)$, one can add a smaller-order Gaussian perturbation term (with covariance given by a function of the overlap similarly to (2.2)) in such a way that the Ghirlanda–Guerra identities hold in the limit. The only requirement from the model is that the free energy satisfies some mild concentration assumptions. Hence, the Ghirlanda–Guerra identities become a property of the perturbation and not the model itself. Since the perturbation is of smaller order, its presence does not affect the limit of the free energy. Such perturbative approach even allows considering more general overlaps, leading to further applications as will be discussed in the next section.

Closely related to the Ghirlanda–Guerra identities is the so-called *Aizenman–Contucci stochastic stability* [3,56]. The first approach to ultrametricity using this stochastic stability was developed by Louis-Pierre Arguin and Michael Aizenman in [12], which inspired the line of research [99,101] that lead to Theorem 1.1 in [102]. The general idea of forcing nontrivial properties on a model using perturbations has emerged as one of the most important ideas on the mathematical side of spin glasses. As the physicists like to say, you can learn a lot about the system by observing how it reacts to small perturbations, and mathematicians like to think that in a small neighborhood of a system you might be able to find another one with better properties.

## 4. SYNCHRONIZATION MECHANISM

In this section, we will describe how Theorem 1.1 can be combined with the Ghirlanda–Guerra identities for more general overlaps to study various generalizations of the SK model. For illustration purposes, we will use the following two examples.

**Example 4.1** (Nonhomogeneous SK model). In the language of dean's problem, suppose that students are divided into two groups, Girls $=\{1,\ldots,N_1\}$ and Boys $=\{N_1+1,\ldots,N\}$, where $N_1/N\to\lambda\in(0,1)$, and suppose that the variance of the Gaussian interactions depends on the group membership,

$$\mathrm{Var}(g_{ij}) = \sigma^2_{S,S'}, \quad \text{if } i \in S, j \in S' \text{ for } S, S' \in \{\text{Girls}, \text{Boys}\}. \tag{4.1}$$

Otherwise, the Hamiltonian is the same as in the SK model. In this case, the computation of the free energy involves understanding the joints distribution of two types of overlaps,

$$R^S_{\ell,\ell'} = \frac{1}{N}\sum_{i\in S}\sigma_i^\ell\sigma_i^{\ell'} \quad \text{for } S \in \{\text{Girls}, \text{Boys}\},$$

over the entire array $\ell, \ell' \geq 1$.

**Example 4.2** (Potts SK model). Here we consider $K \geq 2$ dorms, so assignments $\sigma$ belong to $\{1, \ldots, K\}^N$, and suppose that the dorm sizes are fixed,

$$\lim_{N\to\infty}\frac{|\{i : \sigma_i = k\}|}{N} = p_k \in (0,1) \quad \text{for all } k \leq K,$$

where $\sum_{k\leq K} p_k = 1$. Now it is more natural to define the Hamiltonian as

$$H_N(\sigma) = \frac{1}{\sqrt{N}}\sum_{i,j=1}^N g_{ij}\mathrm{I}(\sigma_i = \sigma_j).$$

In this case, the computation of the free energy involves understanding the joints distribution over all $\ell, \ell' \geq 1$ of the matrix of overlaps

$$R(\sigma^\ell, \sigma^{\ell'}) = \big(R_{x,y}(\sigma^\ell, \sigma^{\ell'})\big)_{x,y\in\text{Dorms}}, \tag{4.2}$$

where, for $x, y \in \text{Dorms} := \{1, \ldots, K\}$,

$$R_{x,y}(\sigma^\ell, \sigma^{\ell'}) = \frac{1}{N}\sum_{i=1}^N \mathrm{I}(\sigma_i^\ell = x)\mathrm{I}(\sigma_i^{\ell'} = y),$$

which is the proportion of students assigned to dorm $x$ in the assignment $\sigma^\ell$ and dorm $y$ in another assignment $\sigma^{\ell'}$.

To study joint distributions of more than one type of overlap, a *synchronization mechanism* was developed in [106,110,111], which we will now describe. Let $\Sigma_N$ be the space of assignments, for example, $\Sigma_N = \{-1, +1\}^N$ in the SK model and nonhomogeneous SK model, and $\Sigma_N = \{1, \ldots, K\}^N$ in the Potts model. Let $H$ be a Hilbert space and

$$\Phi_N : \Sigma_N \to H$$

be such that $\|\Phi_N(\sigma)\|_H = \text{const}$, where for simplicity we will assume that this constant in independent of $N$. We will call

$$R_{\ell,\ell'} = \Phi_N(\sigma^\ell)\cdot\Phi_N(\sigma^{\ell'})$$

a *generalized overlap*. Since we can define a Gaussian process with the covariance equal to this generalized overlap (or its powers), the perturbation approach we described in the previous section allows us to force any such generalized overlap to satisfy the Ghirlanda–Guerra identities and, by way of Theorem 1.1, satisfy ultrametricity in the limit $N \to \infty$.

Moreover, if we consider two generalized overlaps $R_{\ell,\ell'}$ and $Q_{\ell,\ell'}$ then $R^n_{\ell,\ell'}Q^m_{\ell,\ell'}$ will also be a generalized overlap for any integer $n, m \geq 0$, and the perturbative approach allows us to simultaneously force all of them be ultrametric in the limit. The ultrametricity puts strong constraints on how this can happen and, in fact, implies that $R_{\ell,\ell'}$ and $Q_{\ell,\ell'}$ have to be synchronized in the following sense (Theorem 4 in [106]):

$$R_{\ell,\ell'} = f(R_{\ell,\ell'} + Q_{\ell,\ell'}), \quad Q_{\ell,\ell'} = g(R_{\ell,\ell'} + Q_{\ell,\ell'}), \tag{4.3}$$

for some deterministic 1-Lipschitz functions $f$ and $g$, which depend only on the distribution of the array $(R, Q)$.

In the case of nonhomogeneous SK model, this means that both overlaps $R^S_{\ell,\ell'}$ for $S \in \{\text{Girls, Boys}\}$ are determined by their sum, which is just the usual overlap $R_{\ell,\ell'}$. When the matrix $\Sigma = (\sigma^2_{S,S'})_{S,S' \in \{\text{Girls,Boys}\}}$ of variances in (4.1) is nonnegative definite, this allows computing the Parisi-type formula for the free energy [28,106]. When $\Sigma$ is not nonnegative definite, the upper bound via Guerra's interpolation [74] is missing and the problem is still open, but the synchronization mechanism plays a crucial role in another promising approach to this problem developed in a series of papers [2,29,94–97].

In the case of the Potts SK model, the quadratic form

$$\sum_{x,y \in \text{Dorms}} R_{x,y}(\sigma^\ell, \sigma^{\ell'})^m \lambda_x \lambda_y$$

is a generalized overlap for any $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}^K$, and, again, one can force all of them to be synchronized in the sense of equation (4.3). This yields even more surprising constraints on the overlap matrix (4.2) in the limit $N \to \infty$. Namely, in this case, one can show that

$$R(\sigma^\ell, \sigma^{\ell'}) = \Phi\big(\text{tr}\big(R(\sigma^\ell, \sigma^{\ell'})\big)\big)$$

for some deterministic function $\Phi : \mathbb{R} \to \text{SPD}$ (where SPD is the set of symmetric nonnegative definite matrices) that depends only on the distribution of the array $R$. Moreover, $\Phi$ is Lipschitz elementwise, and nondecreasing in SPD, i.e., $\Phi(a) - \Phi(b) \in \text{SPD}$ for any $a \geq b$. The reason such constraints are surprising is that a priori the matrix $R(\sigma^\ell, \sigma^{\ell'})$ in (4.2) is not even symmetric. However, synchronization of the generalized overlaps above enforces such strong symmetries and, in particular, all the overlaps are determined by the trace $\text{tr}(R(\sigma^\ell, \sigma^{\ell'}))$. This yields a Parisi-type formula for the free energy [110,111].

Besides the above two examples, some more general results include showing that the upper bounds of Talagrand [128,129] for multiple systems with overlap constraints are sharp. These results and the synchronization mechanism itself were used in a variety of applications (see, e.g., [1,31,44,57,65,77,83]).

## 5. OTHER APPLICATIONS OF ULTRAMETRICITY

Theorem 1.1 or the ideas in its proof were used in a number of other places. We will not describe them in detail, but will at least mention briefly. One application is to show the so-called *chaos in temperature* for generic mixed $p$-spin models [108]. The phenomenon of chaos in temperature in spin glass models was first studied in the physics literature by Fisher and Huse [70] and Bray and Moore [36], and its states that, if we change the inverse temperature parameter $\beta$ even a little, configurations sampled from the Gibbs measure will become uncorrelated with those sampled at the original temperature. In other words, the Gibbs measure is chaotic under small changes in temperature.

Thouless–Anderson–Palmer approach [130] to computing the free energy in the SK model is a landmark work in the physics literature and its ideas play an important role also in connection to the Parisi solution [91]. In [50,51], the so-called *generalized TAP free energy* was studied, extending the ideas of Eliran Subag from the setting of spherical models [121] to the original SK and related models. Theorem 1.1 played a role there in computing the TAP correction, similarly to the approach in [104].

The Ghirlanda–Guerra identities and Theorem 1.1 also played a key role in the series of papers [105,107,109] that studied the so-called Mézard–Parisi ansatz [87] in the setting of diluted spin glass models. The main problem here (the so-called reproducibility hypothesis) is still open, but quite general special case was proved in [109].

## 6. SOME RELATED WORK

Finally, we will briefly summarize some related work in spin glasses. It is not really feasible to give a detailed overview, so we will only list some important results. Talagrand first discovered in [124] that the Ghirlanda–Guerra identities hold in the setting of the simplest Ruelle probability cascades (Poisson–Dirichlet processes) and, following a similar idea, the Ghirlanda–Guerra identities in the setting of the general Ruelle probability cascades were proved by Bovier and Kurkova in [35], where the Gibbs measure of the GREM was studied rigorously. The Ghirlanda–Guerra identities in a strong sense (not on average over $\beta$) were derived in [18, 38, 100]. The ultrametricity of the overlap array in the thermodynamic limit can be translated to a similar description for finite-size systems in some approximate sense, which was done in [41,76]. The properties of the Parisi formula were studied in [15,16,19,23,78, 80,127,131], and the Parisi formula at zero temperature (for the maximum of the Hamiltonian) was obtained and studied in [17,45,52,79]. Various results related to chaos in temperature and chaos in disorder appeared in [32, 37, 39, 40, 43, 47, 48, 69]. A tiny sample of results related to diluted spin glass models is [54,55,64]. Spherical analogues of the SK and related models were studied in [24–27, 42, 52, 79, 81–86, 119, 122, 125]. The complexity of critical points in spherical models was analyzed in great detail in [13, 14, 20, 118, 123]. Various results related to the TAP approach and optimization can be found in [21, 22, 30, 33, 46, 49, 53, 67, 68, 93, 116, 120].

## FUNDING

This work was partially supported by NSERC and Simons Fellowship.

## REFERENCES


[1] A. Adhikari and C. Brennecke, Free energy of the quantum Sherrington–Kirkpatrick spin-glass model with transverse field. *J. Math. Phys.* **61** (2020), 083302.

[2] E. Agliari, A. Barra, R. Burioni, and A. Di Biasio, Notes on the $p$-spin glass studied via Hamilton–Jacobi and smooth-cavity techniques. *J. Math. Phys.* **53** (2012), no. 6, 063304.

[3] M. Aizenman and P. Contucci, On the stability of the quenched state in mean-field spin-glass models. *J. Stat. Phys.* **92** (1998), no. 5–6, 765–783.

[4] M. Aizenman, R. Sims, and S. L. Starr, An extended variational principle for the SK spin-glass model. *Phys. Rev. B* **68** (2003), 214403.

[5] P. W. Anderson, Spin glass I: a scaling law rescued. *Phys. Today* **41** (1988), no. 1, 9–11.

[6] P. W. Anderson, Spin glass II: is there a phase transition? *Phys. Today* **41** (1988), no. 3, 9–11.

[7] P. W. Anderson, Spin glass III: theory raises its head. *Phys. Today* **41** (1988), no. 6, 9–11.

[8] P. W. Anderson, Spin glass IV: glimmerings of trouble. *Phys. Today* **41** (1988), no. 9, 9–11.

[9] P. W. Anderson, Spin glass V: real power brought to bear. *Phys. Today* **42** (1989), no. 7, 9–11.

[10] P. W. Anderson, Spin glass VI: spin glass as cornucopia. *Phys. Today* **42** (1989), no. 9, 9–11.

[11] P. W. Anderson, Spin glass VII: spin glass as paradigm. *Phys. Today* **43** (1990), no. 3, 9–11.

[12] L.-P. Arguin and M. Aizenman, On the structure of quasi-stationary competing particles systems. *Ann. Probab.* **37** (2009), no. 3, 1080–1113.

[13] A. Auffinger and G. Ben, Arous, Complexity of random smooth functions on the high dimensional sphere. *Ann. Probab.* **41** (2013), no. 6, 4214–4247.

[14] A. Auffinger, G. Ben Arous, and J. Cerný, Random matrices and complexity of spin glasses. *Comm. Pure Appl. Math.* **66** (2013), no. 2, 165–201.

[15] A. Auffinger and W.-K. Chen, On properties of Parisi measures. *Probab. Theory Related Fields* **161** (2015), no. 3, 817–850.

[16] A. Auffinger and W.-K. Chen, The Parisi formula has a unique minimizer. *Comm. Math. Phys.* **335** (2015), no. 3, 1429–1444.

[17] A. Auffinger and W.-K. Chen, Parisi formula for the ground state energy in the mixed $p$-spin model. *Ann. Probab.* **45** (2017), 4617–4631.

[18] A. Auffinger and W.-K. Chen, On concentration properties of disordered Hamiltonians. *Proc. Amer. Math. Soc.* **146** (2018), 1807–1815.

[19] A. Auffinger, W.-K. Chen, and Q. Zeng, The SK model is infinite step replica symmetry breaking at zero temperature. *Comm. Pure Appl. Math.* **73** (2020), no. 5, 921–943.

[20] A. Auffinger and J. Gold, The number of saddles of the spherical $p$-spin model. 2020, arXiv:2007.09269.

[21] A. Auffinger and A. Jagannath, On spin distributions for generic $p$-spin models. *J. Stat. Phys.* **174** (2019), 316–332.

[22] A. Auffinger and A. Jagannath, Thouless–Anderson–Palmer equations for generic $p$-spin glasses. *Ann. Probab.* **47** (2019), no. 4, 2230–2256.

[23] A. Auffinger and Q. Zeng, Existence of two-step replica symmetry breaking for the spherical mixed spin glass at zero temperature. *Comm. Math. Phys.* **370** (2019), no. 1, 377–402.

[24] J. Baik, E. Collins-Woodfin, P. Le Doussal, and H. Wu, Spherical spin glass model with external field. *J. Stat. Phys.* **183** (2021), no. 2.

[25] J. Baik and J. O. Lee, Fluctuations of the free energy of the spherical Sherrington–Kirkpatrick model. *J. Stat. Phys.* **165** (2016), no. 2, 185–224.

[26] J. Baik and J. O. Lee, Free energy of bipartite spherical Sherrington–Kirkpatrick model. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020), no. 4, 2897–2934.

[27] J. Baik, J. O. Lee, and H. Wu, Ferromagnetic to paramagnetic transition in spherical spin glass. *J. Stat. Phys.* **173** (2018), no. 5, 1484–1522.

[28] A. Barra, P. Contucci, E. Mingione, and D. Tantari, Multi-species mean-field spin-glasses. Rigorous results. *Ann. Henri Poincaré* **16** (2015), 691–708.

[29] A. Barra, A. Di Biasio, and F. Guerra, Replica symmetry breaking in mean-field spin glasses through the Hamilton–Jacobi technique. *J. Stat. Mech. Theory Exp.* (2010), P09006.

[30] D. Belius and N. Kistler, The TAP–Plefka variational principle for the spherical SK model. *Comm. Math. Phys.* **367** (2019), no. 3, 991–1017.

[31] G. Ben Arous and A. Jagannath, Spectral gap estimates in mean field spin glasses. *Comm. Math. Phys.* **361** (2018), no. 1, 1–52.

[32] G. Ben Arous, E. Subag, and O. Zeitouni, Geometry and temperature chaos in mixed spherical spin glasses at low temperature – the perturbative regime. 2018, arXiv:1804.10573.

[33] E. Bolthausen, An iterative construction of solutions of the TAP equations for the Sherrington–Kirkpatrick model. *Comm. Math. Phys.* **325** (2014), no. 1, 333–366.

[34] E. Bolthausen and A.-S. Sznitman, On Ruelle's probability cascades and an abstract cavity method. *Comm. Math. Phys.* **197** (1998), no. 2, 247–276.

[35] A. Bovier and I. Kurkova, Derrida's generalized random energy models. I. Models with finitely many hierarchies. *Ann. Inst. Henri Poincaré Probab. Stat.* **40** (2004), no. 4, 439–480.

[36] A. J. Bray and M. A. Moore, Chaotic nature of the spin-glass phase. *Phys. Rev. Lett.* **58** (1987), no. 1, 5760.

[37] S. Chatterjee, Disorder, chaos, and multiple valleys in spin glasses. 2008, arXiv:0907.3381.

[38] S. Chatterjee, The Ghirlanda–Guerra identities without averaging. 2009, arXiv:0911.4520.

[39] S. Chatterjee, *Superconcentration and related topics*. Springer Monogr. Math., Springer, Berlin–Heidelberg, 2014.

[40] S. Chatterjee, Chaos, concentration, and multiple valleys. 2018, arXiv:0810.4221.

[41] S. Chatterjee and L. Sloman, Average Gromov hyperbolicity and the Parisi ansatz. *Adv. Math.* **376** (2021), 107417.

[42] W.-K. Chen, The Aizenman–Sims–Starr scheme and Parisi formula for mixed $p$-spin spherical models. *Electron. J. Probab.* **18** (2013), no. 94, 1–14.

[43] W.-K. Chen, Chaos in the mixed even-spin models. *Comm. Math. Phys.* **328** (2014), no. 3, 867–901.

[44] W.-K. Chen, Phase transition in the spiked random tensor with Rademacher prior. *Ann. Statist.* **47** (2019), no. 5, 2734–2756.

[45] W.-K. Chen, M. Handschy, and G. Lerman, On the energy landscape of the mixed even $p$-spin model. *Probab. Theory Related Fields* **171** (2018), no. 1–2, 53–95.

[46] W.-K. Chen and W.-K. Lam, Universality of approximate message passing algorithms. *Electron. J. Probab.* **26** (2021), no. 36, 1–44.

[47] W.-K. Chen and D. Panchenko, Temperature chaos in some spherical mixed $p$-spin models. *J. Stat. Phys.* **166** (2017), no. 5, 1151–1162.

[48] W.-K. Chen and D. Panchenko, Disorder chaos in some diluted spin glass models. *Ann. Appl. Probab.* **28** (2018), no. 3, 1356–1378.

[49] W.-K. Chen and D. Panchenko, On the TAP free energy in the mixed $p$-spin models. *Comm. Math. Phys.* **362** (2018), no. 1, 219–252.

[50] W.-K. Chen, D. Panchenko, and E. Subag, The generalized TAP free energy. 2018, arXiv:1812.05066.

[51] W.-K. Chen, D. Panchenko, and E. Subag, The generalized TAP free energy II. *Comm. Math. Phys.* **381** (2021), no. 1, 257–291.

[52] W.-K. Chen and A. Sen, Parisi formula, disorder chaos and fluctuation for the ground state energy in the spherical mixed $p$-spin models. *Comm. Math. Phys.* **350** (2017), no. 1, 129–173.

[53] W.-K. Chen and S. Tang, On convergence of the cavity and Bolthausen's TAP iterations to the local magnetization. 2020, arXiv:2011.00495.

[54] A. Coja-Oghlan and K. Panagiotou, Going after the $k$-SAT threshold. In *Proc. 45th STOC*, pp. 705–714, ACM, 2013.

[55] A. Coja-Oghlan and K. Panagiotou, The asymptotic $k$-SAT threshold. *Adv. Math.* **288** (2016), 985–1068.

[56] P. Contucci and C. Giardinà, Spin-glass stochastic stability: a rigorous proof. *Ann. Henri Poincaré* **6** (2005), no. 5, 915–923.

[57] P. Contucci and E. Mingione, A multi-scale spin-glass mean-field model. *Comm. Math. Phys.* **268** (2018), no. 3, 1323–1344.

[58] C. de Dominicis and H. Hilhorst, Random (free) energies in spin glasses. *J. Phys. Lett.* **46** (1985), L909–L914.

[59] B. Derrida, Random-energy model: limit of a family of disordered models. *Phys. Rev. Lett.* **45** (1980), no. 2, 79–82.

[60] B. Derrida, Random-energy model: an exactly solvable model of disordered systems. *Phys. Rev. B (3)* **24** (1981), no. 5, 2613–2626.

[61] B. Derrida, A generalization of the random energy model that includes correlations between the energies. *J. Phys. Lett.* **46** (1985), 401–407.

[62] B. Derrida and E. Gardner, Solution of the generalised random energy model. *J. Phys. C* **19** (1986), 2253–2274.

[63] B. Derrida and G. Toulouse, Sample to sample fluctuations in the random energy model. *J. Phys. Lett.* **46** (1985), L223–L228.

[64] J. Ding, A. Sly, and N. Sun, Proof of the satisfiability conjecture for large $k$. In *STOC'15*, pp. 59–68, ACM, 2015.

[65] T. Dominguez, The $\ell^p$-Gaussian–Grothendieck problem with vector spins. Preprint, 2021.

[66] L. N. Dovbysh and V. N. Sudakov, Gram–de Finetti matrices. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **119** (1982), 77–86.

[67] A. El Alaoui and A. Montanari, Algorithmic thresholds in mean field spin glasses. 2020, arXiv:2009.11481.

[68] A. El Alaoui, A. Montanari, and M. Sellke, Optimization of mean-field spin glasses. 2020, arXiv:2001.00904.

[69] R. Eldan, A simple approach to chaos for $p$-spin models. *J. Stat. Phys.* **181** (2020), 1266–1276.

[70] D. S. Fisher and D. A. Huse, Ordered phase of short-range Ising spin glasses. *Phys. Rev. Lett.* **56** (1986), no. 15, 16011604.

[71] S. Ghirlanda and F. Guerra, General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity. *J. Phys. A* **31** (1998), no. 46, 9149–9155.

[72] C. Goldschmidt and J. B. Martin, Random recursive trees and the Bolthausen–Sznitman coalescent. *Electron. J. Probab.* **10** (2005), no. 21, 718–745.

[73] F. Guerra, About the overlap distribution in mean field spin glass models. *Internat. J. Modern Phys. B* **10** (1996), no. 13–14, 1675–1684.

[74] F. Guerra, Broken replica symmetry bounds in the mean field spin glass model. *Comm. Math. Phys.* **233** (2003), no. 1, 1–12.

[75] F. Guerra and F. L. Toninelli, The thermodynamic limit in mean field spin glass models. *Comm. Math. Phys.* **230** (2002), no. 1, 71–79.

[76] A. Jagannath, Approximate ultrametricity for random measures and applications to spin glasses. *Comm. Pure Appl. Math.* **70** (2017), no. 4, 611–664.

[77] A. Jagannath, J. Ko, and S. Sen, A connection between MAX $\kappa$-CUT and the inhomogeneous Potts spin glass in the large degree limit. *Ann. Appl. Probab.* **28** (2018), no. 3, 1536–1572.

[78] A. Jagannath and I. Tobasco, A dynamic programming approach to the Parisi functional. *Proc. Amer. Math. Soc.* **144** (2016), 3135–3150.

[79] A. Jagannath and I. Tobasco, Low temperature asymptotics of spherical mean field spin glasses. *Comm. Math. Phys.* **352** (2017), no. 3, 979–1017.

[80] A. Jagannath and I. Tobasco, Some properties of the phase diagram for mixed $p$-spin glasses. *Probab. Theory Related Fields* **167** (2017), no. 3–4, 615–672.
[81] P. Kivimae, Critical fluctuations for the spherical Sherrington–Kirkpatrick model in an external field. 2019, arXiv:1908.07512.
[82] J. Ko, The Crisanti–Sommers formula for spherical spin glasses with vector spins. 2019, arXiv:1911.04355.
[83] J. Ko, Free energy of multiple systems of spherical spin glasses with constrained overlaps. *Electron. J. Probab.* **25** (2020), no. 28, 1–34.
[84] B. Landon, Free energy fluctuations of the 2-spin spherical SK model at critical temperature. 2020, arXiv:2010.06691.
[85] B. Landon and P. Sosoe, Fluctuations of the overlap at low temperature in the 2-spin spherical SK model. 2019, arXiv:1905.03317.
[86] B. Landon and P. Sosoe, Fluctuations of the 2-spin SSK model with magnetic field. 2020, arXiv:2009.12514.
[87] M. Mézard and G. Parisi, The Bethe lattice spin glass revisited. *Eur. Phys. J. B* **20** (2001), no. 2, 217–233.
[88] M. Mézard, G. Parisi, N. Sourlas, G. Toulouse, and M. A. Virasoro, On the nature of the spin-glass phase. *Phys. Rev. Lett.* **52** (1984), 1156.
[89] M. Mézard, G. Parisi, N. Sourlas, G. Toulouse, and M. A. Virasoro, Replica symmetry breaking and the nature of the spin-glass phase. *J. Phys.* **45** (1984), 843.
[90] M. Mézard, G. Parisi, and M. A. Virasoro, Random free energies in spin glasses. *J. Phys. Lett.* **46** (1985), L217–L222.
[91] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin glass theory and beyond*. Lecture Notes in Phys. 9, World Scientific, Teaneck, NJ, 1987.
[92] M. Mézard and M. A. Virasoro, The microstructure of ultrametricity. *J. Phys.* **46** (1985), 1293–1307.
[93] A. Montanari, Optimization of the Sherrington–Kirkpatrick Hamiltonian. *SIAM J. Comput.* **FOCS19-38** (2021).
[94] J.-C. Mourrat, Parisi's formula is a Hamilton–Jacobi equation in Wasserstein space. 2019, arXiv:1906.08471.
[95] J.-C. Mourrat, Free energy upper bound for mean-field vector spin glasses. 2020, arXiv:2010.09114.
[96] J.-C. Mourrat, Nonconvex interactions in mean-field spin glasses. *Probab. Math. Phys.* **2** (2021), no. 2, 281–339.
[97] J.-C. Mourrat and D. Panchenko, Extending the Parisi formula along a Hamilton–Jacobi equation. *Electron. J. Probab.* **25** (2020), no. 23, 1–17.
[98] J. Neveu, *A continuous-state branching process in relation with the GREM model of spin glass theory*. Rapport interne no. 267 Ecole Polytechnique, 1992.
[99] D. Panchenko, A connection between Ghirlanda–Guerra identities and ultrametricity. *Ann. Probab.* **38** (2010), no. 1, 327–347.
[100] D. Panchenko, The Ghirlanda–Guerra identities for mixed $p$-spin model. *C. R. Acad. Sci. Paris, Ser. I* **348** (2010), 189–192.

[101] D. Panchenko, Ghirlanda–Guerra identities and ultrametricity: an elementary proof in the discrete case. *C. R. Acad. Sci. Paris, Ser. I* **349** (2011), no. 13–14, 813–816.

[102] D. Panchenko, The Parisi ultrametricity conjecture. *Ann. of Math. (2)* **177** (2013), no. 1, 383–393.

[103] D. Panchenko, *The Sherrington–Kirkpatrick model*. Springer Monogr. Math., Springer, New York, 2013.

[104] D. Panchenko, The Parisi formula for mixed $p$-spin models. *Ann. Probab.* **42** (2014), no. 3, 946–958.

[105] D. Panchenko, Structure of 1-RSB asymptotic Gibbs measures in the diluted $p$-spin models. *J. Stat. Phys.* **155** (2014), no. 1, 1–22.

[106] D. Panchenko, Free energy in the multi-species Sherrington–Kirkpatrick model. *Ann. Probab.* **43** (2015), no. 6, 3494–3513.

[107] D. Panchenko, Hierarchical exchangeability of pure states in mean field spin glass models. *Probab. Theory Related Fields* **161** (2015), no. 3, 619–650.

[108] D. Panchenko, Chaos in temperature in generic $2p$-spin models. *Comm. Math. Phys.* **346** (2016), no. 2, 703–739.

[109] D. Panchenko, Structure of finite-RSB asymptotic Gibbs measures in the diluted spin glass models. *J. Stat. Phys.* **162** (2016), no. 1, 1–42.

[110] D. Panchenko, Free energy in the mixed $p$-spin models with vector spins. *Ann. Probab.* **46** (2018), no. 2, 865–896.

[111] D. Panchenko, Free energy in the Potts spin glass. *Ann. Probab.* **46** (2018), no. 2, 829–864.

[112] G. Parisi, Infinite number of order parameters for spin-glasses. *Phys. Rev. Lett.* **43** (1979), 1754–1756.

[113] G. Parisi, A sequence of approximate solutions to the S-K model for spin glasses. *J. Phys. A* **13** (1980), L-115.

[114] G. Parisi, Order parameter for spin glasses. *Phys. Rev. Lett.* **50** (1983), 1946.

[115] D. Ruelle, A mathematical reformulation of Derrida's REM and GREM. *Comm. Math. Phys.* **108** (1987), no. 2, 225–239.

[116] M. Sellke, Optimizing mean field spin glasses with external field. 2021, arXiv:2105.03506.

[117] D. Sherrington and S. Kirkpatrick, Solvable model of a spin glass. *Phys. Rev. Lett.* **35** (1975), 1792–1796.

[118] E. Subag, The complexity of spherical $p$-spin models – a second moment approach. *Ann. Probab.* **45** (2017), no. 5, 3385–3450.

[119] E. Subag, The geometry of the Gibbs measure of pure spherical spin glasses. *Invent. Math.* **210** (2017), no. 1, 135–209.

[120] E. Subag, Following the ground-states of full-RSB spherical spin glasses. 2018, arXiv:1812.04588.

[121] E. Subag, Free energy landscapes in spherical spin glasses. 2018, arXiv:1804.10576.

[122] E. Subag, The free energy of spherical pure $p$-spin models – computation from the TAP approach. 2021, arXiv:2101.04352.
[123] E. Subag and O. Zeitouni, The extremal process of critical points of the pure $p$-spin spherical spin glass model. *Probab. Theory Related Fields* **168** (2017), no. 3–4, 773–820.
[124] M. Talagrand, *Spin glasses: a challenge for mathematicians*. Ergeb. Math. Grenzgeb. (3)/Ser. Mod. Surv. Math. 43, Springer, 2003.
[125] M. Talagrand, Free energy of the spherical mean field model. *Probab. Theory Related Fields* **134** (2006), no. 3, 339–382.
[126] M. Talagrand, The Parisi formula. *Ann. of Math. (2)* **163** (2006), no. 1, 221–263.
[127] M. Talagrand, Parisi measures. *J. Funct. Anal.* **231** (2006), no. 2, 269–286.
[128] M. Talagrand, Large deviations, Guerra's and A.S.S. schemes, and the Parisi hypothesis. *J. Stat. Phys.* **126** (2007), no. 4–5, 837–894.
[129] M. Talagrand, *Mean-field models for spin glasses. Volumes I and II*. Ergeb. Math. Grenzgeb. (3)/Ser. Mod. Surv. Math., Springer, 2011.
[130] D. J. Thouless, P. W. Anderson, and R. G. Palmer, Solution of 'solvable model of a spin glass'. *Phys. Mag.* **35** (1977), no. 3, 593–601.
[131] F. L. Toninelli, About the Almeida–Thouless transition line in the Sherrington–Kirkpatrick mean-field spin glass model. *Europhys. Lett.* **60** (2002), no. 5, 764.

**DMITRY PANCHENKO**
Department of Mathematics, University of Toronto, Toronto, Canada,
panchenk@math.toronto.edu