# KKL'S INFLUENCE ON ME


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*Dedicated to my dear friends and mentors K., K., and L.*


### ABSTRACT

In 1988 Kahn, Kalai, and Linial published their landmark paper in which they proved a lower bound on the maximal influence of variables on a Boolean function. Their use of Fourier analysis to solve the question, and especially their introduction of a hypercontractive inequality (due to Bonami, Beckner, and Gross), has shaped the field of study of Boolean functions and has had a great influence on combinatorics and theoretical computer science.

In this paper I survey how my own work has been influenced by their approach, via a collection of various problems that I have approached throughout the years.



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## 1. INTRODUCTION

In 1988 Kahn, Kalai, and Linial published their landmark paper [34] in which they proved that if $f : \{0,1\}^n \to \{0,1\}$ is a function with expectation $\alpha$ then there exists a coordinate $i$ for which the (discrete) derivative in the $i$th direction, $f_i$, is substantial: $\|f_i\|_2^2 \geq c\alpha(1-\alpha)\log(n)/n$, for some global constant $c$, which does not depend on $n$. Stated in this language this result might well be categorized by the reader as a theorem in analysis. However, the range of the function $f$, the set $\{0,1\}$, is indicative of the fact that the motivation for this problem comes from a combinatorial and computer-science-theoretical angle. Indeed, what made this paper so influential is the use of discrete Fourier analysis, and especially the introduction of the hypercontractive inequality of Beckner [4], Bonami [6], and Gross [32] (henceforth the BBG inequality), to the end of proving a conjecture that has a strong combinatorial flavor, and arose in the context of theoretical computer science. In the third of a century since, this approach has had a huge impact on the study of Boolean functions, and their applications to combinatorics. Seeing that a substantial part of combinatorics deals with sets and their subsets, it is no surprise that the Boolean functions that indicate these subsets are such a useful language for dealing with these problems, and that the analytical tools applied to these functions are so fruitful.

When I approached Gil Kalai in 1992, and requested a research project, with the intention of it potentially growing to be a PhD thesis, he offered me several papers to read: a paper on the diameter of polytopes, a paper dealing with Cohen–Macaulay rings, and the duo of papers [34] and its sequel where Bourgain and Katznelson joined to produce [7]. I was intrigued by the latter two papers, and have been working on related problems ever since, to this very day. In this paper I want to present some samples of my related work. By doing this, I hope to give a glimpse into some of the interesting problems, notions, and techniques that I have encountered over the years during my work in the field. Of course, this sample is in no way representative of the progress of the field of Boolean functions, and the rich collection of results produced in it during the past decades; it is heavily skewed by focusing on my own work. For a nice reference for many of the foundational and important results in the field (at least up to 2014), I recommend Ryan O'Donnell's book on Boolean functions [39].

Following this introduction, this paper contains two main sections: in the first, I present a set of problems, and in the second, I present the tools and ideas used to solve each of them.

### 1.1. Basic terminology and definitions

Any function $f : \{0,1\}^n \to \mathbb{R}$ has a unique Fourier expansion

$$f = \sum_{S \subset \{1,2,\dots,n\}} \hat{f}(S)\chi_S$$

where the functions $\chi_S$ are the characters of $\mathbb{Z}_2^n$,

$$\chi_S(x) = (-1)^{\sum_{i \in S} x_i}.$$

The *degree* of $f$ is the maximal $|S|$ for which $\hat{f}(S)$ is nonzero.

We often consider the product measure $\mu_p$ on $\{0,1\}^n$, where

$$\mu_p(A) = \sum_{x \in A} p^{\sum x_i}(1-p)^{\sum(1-x_i)}.$$

When $p = 1/2$, this is the uniform measure. For other $p$, the measure induces the inner product

$$\langle f, g\rangle = E_x\big[f(x)g(x)\big],$$

and we expand functions according to an orthogonal basis with respect to this inner product. We refer to the elements of this basis also as characters (although this is a misnomer, as there is no group involved). So now $f = \sum_{S \subset \{1,2,\ldots,n\}} \hat{f}(S)\chi_S$ where $\chi_S$ is defined by

$$\chi_S(x) = \big(-\sqrt{(1-p)/p}\big)^{\sum_{i\in S} x_i}\big(\sqrt{p/(1-p)}\big)^{\sum_{i\in S}(1-x_i)}.$$

When considering the distribution of the Fourier coefficients of a function $f$, we refer to the *Fourier weight* of $f$ on a set $A \subset \{0,1\}^n$, meaning $\sum_{S\in A} \hat{f}^2(S)$.

Finally, we need the important notion of *influence* (appearing also in the title of this paper): given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, the influence of the $i$th coordinate on $f$ is the probability that $f(x) \neq f(y)$, where $x$ is chosen uniformly at random, and $y$ is obtained by flipping the $i$th coordinate of $x$.

## 2. PROBLEMS

In this section I describe the parting point of several papers, i.e., the problems that they set out to solve.

### 2.1. Juntas rule

For a subset $S$ of the discrete cube $\{0,1\}^n$, the *edge boundary* of $S$, denoted by $\partial_e(S)$ is the set of all pairs $(x, y)$ with $x \in S$, $y \notin S$, and such that $x$ and $y$ differ in a single coordinate (they form an edge in the Hamming graph on $\{0,1\}^n$). Finding the minimum size of $\partial_e(S)$ given the size of $S$ is a classical isoperimetric problem, first solved by Harper [33]. It is quite easy to prove that if $|S| = 2^{n-1}$ the unique minimizers of $\partial_e(S)$ are the subcubes of codimension 1, for which $|\partial_e(S)| = 2^{n-1}$. These are sometimes known as *dictators*, as their characteristic functions are dictated by a single coordinate. The question is what can be said if the edge boundary is within a multiplicative constant of the minimum:

**Question 1.** Given $S \subset \{0,1\}^n$, with, say, $|S| = 2^{n-1}$ and $|\partial_e(S)| = c \cdot 2^{n-1}$, is it true that $S$ may be approximated by *a junta*, a set whose characteristic function is determined by a small set of coordinates, whose size depends only on the constant $c$ and on the precision of the approximation?

### 2.2. Almost-dictator functions

A Boolean function on $\{0,1\}^n$ of the form $f(x) = x_i$ is called a dictator; $f(x) = 1 - x_i$ is called an antidictator. It is easy to show that dictators, antidictators, and the constant functions 0 and 1 are the only Boolean functions of degree 1. A question that arose quite naturally in the work of Kalai on social choice was whether this is robust:

**Question 2.** Is it true that a Boolean function which has almost all of its Fourier weight on the first two levels (empty set and singletons) is close to a dictator, antidictator, or constant? In other words, if $f : \{0,1\}^n \to \{0,1\}$ and

$$\sum_{|S|>1} \hat{f}^2(S) < \varepsilon,$$

is there a Boolean function $g$ of degree 1 such that $\|f - g\|_2^2$ is small as a function of $\varepsilon$?

### 2.3. Thresholds for every graph property

The very first problem I worked on during my PhD studies was the following question, proposed by Gil Kalai and Nati Linial.

**Question 3.** Given a property of graphs on $n$ vertices, how wide can the threshold interval be for the appearance of the property in $G(n,p)$?

For example, for a given $n$, if $k$ is the most likely size of the maximal clique in $G(n,1/2)$, then for $\varepsilon = O(1/\log^2(n))$ there is an interval of length $\varepsilon$ in $(0,1)$, such that for all $p$ in that interval the probability that the largest clique in $G(n,p)$ is of size at least $k$ is between 0.01 and 0.99. Fixing these parameters (0.01 and 0.99), are there other graph properties for which the threshold interval for $p$ is much larger?

### 2.4. Sharp/coarse thresholds – global/local properties

The previous question is interesting mainly when the threshold interval is bounded away from 0 and 1. However, for many interesting graph properties, the critical probability for the appearance of the property in the random graph $G(n,p)$ is $p$ which is a function of $n$, tending to zero as $n$ grows. If $p^*$ is such that a given monotone graph property appears in $G(n,p^*)$ with probability 1/2, then a result of Bollobás and Thomason [5] tells us that the threshold length is at most of order $p^*$. The following is a natural, if ambitious, question.

**Question 4.** Which monotone graph properties have a *sharp threshold*? That is, for which properties is the length of the threshold interval $o(p^*)$?

For example, the appearance of a triangle in $G(n,p)$ typically occurs for $p = \Theta(1/n)$, but for any constant $c$ the probability that $G(n,c/n)$ contains a triangle is bounded away from 0 and 1. In contrast, the critical probability for connectivity is $p = \log(n)/n$, whereas the threshold is of width only $\Theta(1/n)$.

### 2.5. Maximizing copies of one (hyper)graph in another

**Question 5.** Given a fixed graph (or uniform hypergraph) $H$, and an integer $m$, what is the maximal number of copies of $H$ one can have as subgraphs of a graph with $m$ edges? (And what in the world does this possibly have to do with hypercontractive inequalities?)

It turns out that the answer is of the form $\Theta(m^{\pi^*(H)})$. The main challenge is to understand the function $\pi^*$ (although finding the precise constants is also a challenging question.) In his MSc thesis, Noga Alon [2] provided an algorithm that computes $\pi^*(H)$. It was later recognized that this is the fractional covering number of $H$. In a joint paper with Jeff Kahn [26], we proved this for all uniform hypergraphs, using an entropy approach. Prior to this, a chance encounter with Noga led me to consider whether this result is related to the Bonami–Beckner–Gross hypercontractive inequality. I was trying to use a special case of the graph-theoretic inequality in order to prove a special case of the BBG inequality, and Noga, upon hearing this, first introduced me to his old(est) result. Studying this led me to the realization that the opposite implication is also a possibility, and led me to wonder whether one can prove the graph theorem using hypercontractivity.

### 2.6. The traffic light problem

Consider the following combinatorial problem:

**Question 6.** Assume that at a given road junction there are $n$ three-position switches that control the red–amber–green position of the traffic light. You are told that, whenever you change the position of all the switches, the color of the light changes. Is it true that the light is, in fact, controlled by a single switch?

This problem was solved by Greenwell and Lovász [31] in 1974, using straightforward combinatorial arguments. In 2003, with Alon, Dinur, and Sudakov [3] we tackled this problem and generalizations thereof using spectral analysis, and also used the existing techniques from the study of Boolean functions to prove robustness results, i.e., what can be said if the hypothesis holds only for, say, 99.99% of the switch-configurations?

### 2.7. Subsets of independent sets in product graphs

The problem in the previous section gives rise to characterizing the maximal independent sets in the graph $(K_3)^{\otimes n}$, and showing that they are, in fact, cylinders of codimension 1, i.e., dictators. These sets have measure $1/3$ (according to the uniform measure on the graph). But what about independent sets whose size differs by a fixed multiplicative constant from the maximum?

**Question 7.** What can be said about independent sets in $K_3^{\otimes n}$ whose measure is, say, $1/100$? Are such sets essentially described by a small (constant) number of coordinates?

The hope to completely describe, or even approximate such sets, using a bounded number of coordinates is, of course, too far-reaching, as a random subset of a maximal independent set is also independent, and cannot be characterized by a small number of coordinates. Could it be, however, that any large independent set is (essentially) *contained* in an independent set determined by few coordinates (a junta)?

### 2.8. $t$-intersecting subsets of the cube

The Erdős–Ko–Rado theorem [18] is a (or perhaps the) fundamental theorem in extremal combinatorics. The ground set it considers is all subsets of size $k$ of $\{1, 2, \ldots, n\}$ for some $k \leq n/2$. It then bounds the size of a maximal intersecting family of such subsets. The "complete intersection theorem" of Ahlswede and Khachatarian [1] expands this to families of subsets, where every two of them have an intersection of size at least $t$, for some integer $t > 1$.

It is quite natural to consider a "smoothed" version of this question. For $p \in (0, 1/2]$, consider the product measure $\mu = \mu_p$ on the discrete cube $\{0,1\}^n$, and ask what the maximal measure of a family of vectors in the cube is if every two of them have nondisjoint support, or a common support of size at least $t$. The fact that if $A \subseteq \{0,1\}^n$ is intersecting, then $\mu_p(A) \leq p$ was proven in many papers, e.g., [20] and [22] to mention just a few.

**Question 8.** Does this fact have a Fourier-theoretic proof? Does it have a robust version? What about $t$-intersecting families?

### 2.9. Triangle-intersecting families of graphs

There are many beautiful questions that generalize the original question of Erdős–Ko–Rado regarding intersecting families. One of my favorites was posed by Miklos Simonovits and Vera Sós, and I was intrigued by it from the moment I heard Vera introduce it in an open problems session in Oberwolfach. The question arises if we impose some structure on the ground set from which the intersecting subsets are taken. Their question is the following:

**Question 9.** Given a family of subgraphs of the complete graph on $n$ vertices, if the intersection of every two members of the family contains a triangle, how large can the family be? Is it true that the maximum is attained by taking all graphs containing a fixed triangle?

### 2.10. Intersecting families of permutations

Another nice generalization of EKR was considered by Deza and Frankl in [11]. A family of permutations in $S_n$ is called intersecting if for every two permutations in the family, $\sigma, \tau$, there exists $i$ such that $\sigma(i) = \tau(i)$. The family is $t$-intersecting for an integer $t \geq 1$ if every two permutations in the family agree on $t$ points. Deza and Frankl made the observation that an intersecting family has size at most $(n-1)!$, and conjectured that this is achieved only by dictatorships, namely families of the form $\{\tau \in S_n : \tau(i) = j\}$ for some $i$ and $j$. This conjecture, made in 1977, was finally proved in 2003 by Cameron and Ku in [9].

The more general conjecture made by Deza and Frankl was that $t$-intersecting families are of size at most $(n-t)!$, with the unique maximizers being families of the form

$$\{\tau \in S_n : \tau(i_r) = j_r, r = 1, \ldots, t\}$$

for some $i_1, \ldots, i_t$ and $j_1, \ldots, j_t$. Here even the conjectured upper bound of $(n-t)!$, which was very simple in the case $t = 1$, turned out to be difficult for $t > 1$.

**Question 10.** Prove that the families mentioned above are the unique maximal size $t$-intersecting families of permutations.

## 3. APPROACHES AND TECHNIQUES

In this section I give a bird's-eye view of the ways in which the questions presented in the previous section were resolved.

### 3.1. Juntas rule

In [23] I prove that, indeed, if the edge boundary of $S \subset \{0,1\}^n$ is small, then $S$ is close to a set that is determined by a small number of coordinates. Here are the main steps and ideas of the proof:

(1) Let $f$ be the characteristic function of $S$. The size of the edge boundary of $S$ is equal (after normalizing) to the sum of the influences of the variables on $f$. Thus the sum of the influences in this case is small (bounded).

(2) Partition the variables according to their influence into two sets, $J$ – those with large influence, and those with small influence, where the cutoff is appropriately chosen. The variables in $J$ will form the junta. Since the sum of influences is bounded there cannot be too many variables in $J$.

(3) Let $f^{(1)} = \sum_{|S|<K} \hat{f}(S)\chi_S$ be a truncated version of $f$, where the cutoff point $K$ is appropriately chosen. It is easy to show that $f^{(1)}$ is close to $f$ (the fact that the sum of influences is small implies that there is not much Fourier weight on large sets.)

(4) Next, discard all coefficients of characters $\chi_S$ where $S$ is a set not contained in the junta $J$, i.e.,

$$f^{(2)} = \sum_{|S|<K, S\subseteq J} \hat{f}(S)\chi_S.$$

Clearly, $f^{(2)}$ depends only on the junta variables.

(5) The following step is the heart of the argument – one wishes to show that $\|f - f^{(2)}\|_2^2$ is small, i.e., that the Fourier weight on sets containing variables with small influence is small. Note that this is not immediate since, although they each have small influence individually, it is not clear that their collective influence on $f$ is small. To this end, the hypercontractive estimate of Bonami–Beckner–Gross is applied à-la KKL. By comparing different norms of the derivatives $f_i$, where $i$ is a coordinate with small influence, one can show that the total Fourier weight on sets involving these variables is small.

(6) Using a rounding procedure, replace $f^{(2)}$ by a Boolean function that also depends only on the junta variables, and is a good approximation of $f$.

### 3.2. Almost-dictator functions

In [28], together with Kalai and Naor, we proved that, indeed, Boolean functions with almost all their Fourier weight concentrated on the first two levels are close to either a constant function, or a dictator, or an antidictator. In the paper we supply two different proofs. One proof uses a Berry–Eseen-type theorem of König, Schütt, and Tomczak-Jaegermann, [37], to show that there is at most one singleton $\{i\}$ such that $\hat{f}(\{i\})$ is large. The other proof uses the Bonami–Beckner–Gross hypercontractive inequality to show that once one truncates all Fourier coefficients above the first level, one is still left with a very well-behaved function, a fact that can only be explained by it being close to a degree-1 Boolean function. In particular, if $f^{(1)} = \sum_{|S|\leq 1} \hat{f}(S)\chi_S$, we measure the "Booleanity" of $f^{(1)}$ by bounding the variance of $(f^{(1)})^2 - f^{(1)}$. Ultimately, this enables us to conclude that if for a Boolean function $f$ one has $\sum_{|S|>1} \hat{f}^2(S) < \varepsilon$ then there exists a degree-1 Boolean function $g$ such that $\|f - g\|_2^2 < O(\varepsilon)$.

### 3.3. Thresholds for every graph property

In [27] Kalai and I proved that for every monotone graph property the threshold length is of order at most $O(1/\log(n))$. The proof is embarrassingly simple: we simply observe that this is true for every weakly symmetric function – a function where there is a transitive group action on the variables of the function. By KKL and symmetry, if there are $m$ coordinates, then all $m$ influences are of order at least $\log(m)/m$, and their sum is of order at least $\log(m)$. Then, the Russo–Margulis lemma relates the sum of the influences to the slope of $\mu(p)$, the probability that $G(n,p)$ possesses the property in question, as a function of $p$. This slope determines the length of the threshold interval.

The paper [27], published in 1996, contains a conjecture that is still open, and, if true (as is widely believed), would have many interesting consequences. It is known as the *Entropy-Influence Conjecture*.

**Conjecture 1.** *There exists a constant $c$ such that for every Boolean function $f : \{0,1\}^n \to \{0,1\}$ it holds that*

$$\sum \hat{f}^2(S)\log\bigl(1/\bigl|\hat{f}(S)\bigr|\bigr) \leq c \sum \hat{f}^2(S)|S|.$$

Another interesting question raised in [27] is that of the dependence of the maximal possible threshold interval length on the structure of the orbits of the symmetry group of the property. In particular, we suspected that the maximal interval length for graph properties should be $O(1/\log^2(n))$, and not $1/\log(n)$. This was finally (almost) proved by Bourgain and Kalai in [8] where they gave a bound of $O(1/\log^{2-\varepsilon}(n))$, for every $\varepsilon$. Recently Kelman, Kindler, Lifshitz, Minzer, and Safra [36] improved this to $O(\frac{\log\log(n)}{\log^2(n)})$.

### 3.4. Sharp/coarse thresholds – global/local properties

In my PhD thesis [24], I settled this problem by proving that any monotone graph property which does not have a sharp threshold for appearance in $G(n,p)$ must be close to a "local" property: one of containing a graph from a fixed list of graphs (e.g., the property of containing a triangle or a cycle of length 4). The main steps of the proof are:

(1) Given a graph property with a coarse threshold, consider $f$, the characteristic function of the graphs containing this property, and its "skew-Fourier" expansion.

(2) Using the information that the threshold interval is large, prove that there exists a list of graphs $S$ such that almost all the Fourier weight of $f$ is concentrated on basis-functions indexed by graphs from this list. This is the main difficulty in the proof.

(3) For a function $f$, with Fourier expansion as above, partition the space of graphs according to the subgraph count of graphs from the list $S$, and show that on each part $f$ is close to being constant.

(4) Show that any monotone function that behaves as described on the different parts of the space is a "local" function, i.e., its behavior on the different parts of the space is consistent.

### 3.5. Maximizing copies of one (hyper)graph in another

In [26] Jeff Kahn and I proved that $\pi^*(H)$ is equal to the fractional covering number of $H$ for any hypergraph $H$. The fractional covering number of a graph is the solution to a certain linear programming problem, and we used the strong duality theorem of linear programming to prove the (equal) upper and lower bounds using the problem and its dual. The challenging part was the upper bound, for which we provided two different proofs. The first used an information-theoretic approach, via Shearer's entropy lemma. The other proof, which actually quite surprised us, was by relating the subgraph count to two different norms of a certain low degree polynomial function, and then using the BBG hypercontractive inequality to bound one norm in terms of the other, deducing the required inequality.

### 3.6. The traffic light problem

In [3] we proved for a large family of graphs (which includes complete graphs, and in particular triangles, which are the basis for the traffic light problem) that all maximal independent sets in tensor powers of these graphs are cylinders of codimension 1. Furthermore, we proved a stability result, showing that independent sets of size close to the maximal must also be close in structure to one of the extremal cylinders. The idea of the proof was to take the proof of the Hoffman bound on the size of independent sets in the base graph, and to "tensor" it. We built an orthonormal basis for the space of functions on the powers of the base graph, that consisted of tensor powers of the eigenvectors of the base graph. We then proceeded to use spectral analysis in a manner that is completely analogous to the Fourier analysis on $\{0,1\}^n$, including proving robustness results by using a variant of [28].

### 3.7. Subsets of independent sets in product graphs

In [12] Dinur, Regev, and I prove that, indeed, any large independent set in $K_3^{\otimes n}$ (and in many other product graphs, and also in Kneser graphs) is essentially contained in a set determined by few coordinates, a junta. Later, in [29] a paper with Regev, we complement this by proving that the junta itself is also an independent set (whereas in [12] we only proved that it is sparse.) The main tool in [12] is *noise*. The noise operator is the object which is the focus of the BBG hypercontractive inequality. Recall that the inequality studies a noisy version of a function $f$, i.e., $Tf(x) = E[f(y)]$, where $y$ is a randomly perturbed version of $x$. By applying noise to an independent set contained in a junta, one is able to recover the junta, and the hypercontractivity serves to control this procedure.

In [29] we complement this result by showing that the junta we have recovered (which we proved in [12] to span very few edges) can be made independent by removing a small set of vertices. To this end, we prove an edge removal lemma, and prove it in the spirit of Fox's proof of the graph removal lemma [21]. We show that for the graphs in question (e.g., $K_3^{\otimes n}$ and also Kneser graphs) any sparse set can be made independent by removing a small set of vertices.

### 3.8. $t$-intersecting subsets of the cube

In [25] I manage to apply a Fourier approach to prove the claim that for $p \leq 1/2$ and an intersecting family $I \subseteq \{0,1\}^n$ we have $\mu_p(I) \leq p$. The idea of the proof comes from encoding the problem via a weighted graph, and applying (the proof of) Hoffman's bound to this graph. It turns out that the $t$-intersecting subsets' problem is more intricate. First of all, the natural extremal candidate of a $t$-umvirate, the AND of $t$ bits, is extremal only for $p \leq 1/(t+1)$. For larger values of $p$, other examples take over (such as "three out of four" for 2-intersecting families). Still, for $p \leq 1/(t+1)$ it is possible to give a Fourier proof, and a robustness theorem.

Since the heart of the proof for the 1-intersecting case relies on the product structure of the cube, it uses the deep mathematical fact that if $x, y \in \{0,1\}$ then $x \cdot y = 1$ if and only if $x = y = 1$. This suffices to "penalize" two sets that intersect. However, when we want to penalize sets that have intersection of size at least, say, 2, we do not want to disqualify two sets that have one joint element, only "warn" them. To turn this into a scheme that is amenable to products, we need an element $X \neq 0$ such that $X^2 = 0$. The solution is to work over a ring of polynomials in the formal variable $X$, modulo the relation $X^2 = 0$. This adds various complications, as the eigenvalues of the resulting matrices are no longer real numbers, rather they are ring elements, hence it no longer makes sense to speak of the minimal eigenvalue. These difficulties can be handled, and eventually one can produce a set of $t$ linear inequalities involving the Fourier distribution of the characteristic function of the $t$-intersecting family. Taking an appropriate linear combination of the inequalities, one may prove that the Fourier transform of the function sits on the first $t+1$ levels, and from there the road to proving the uniqueness and stability results is not too rough.

I would like to mention that Ryan O'Donnell [38] also discovered a simpler, more elementary way to deduce the inequalities regarding the Fourier coefficients of the functions that are used in this proof.

### 3.9. Triangle-intersecting families of graphs

In [14] we proved that, indeed, the largest triangle-intersecting families of subgraphs of $K_n$ are those consisting of all graphs containing a given, fixed, triangle.

The difference between the 1-intersecting and $t$-intersecting for $t > 1$ in the previous subsection is that in the former case one gets an inequality regarding the Fourier coefficients of the characteristic function of the family in question, and in the latter case one gets a system of $t$ inequalities. These inequalities are generated by the various constraints on the intersections of the family elements. In the problem now at hand, the constraints we used were of the form that for any bipartite graph $B$, and any two graphs $G$ and $H$ in the family, the intersection of $G$ and $H$ is not contained in $B$ (as it contains a triangle). The potentially unbounded set of inequalities arising from these constraints is formidable, yet it offers a wealth of possibilities. By taking various appropriate linear combinations of these inequalities, we were able to prove that the characteristic function of the family in question has its Fourier coefficients concentrated precisely on the characters we expect.

### 3.10. Intersecting families of permutations

This problem was settled in [17], written with David Ellis and Haran Pilpel. The idea of using weighted graphs in order to prove a Hoffman-type bound is a recurring theme in solving the problems we have seen so far. This usually leads to expanding the functions in question in terms of a nice orthonormal basis, often the characters of a group, or the "skew" version, especially if the graph in question is a Cayley graph. In the case of the permutation version of EKR, this approach leads us to study the representation theory of $S_n$, and the (non-Abelian) Fourier expansion of functions. It turns out that, after carefully choosing the weights on the edges of the graph that represents the problem, one can deduce that the Fourier transform of a $t$-intersecting family is concentrated on representations indexed by partitions where the first part is of length at most $n - t$. This enables us to prove that the natural examples of extremal $t$-intersecting families, cosets of stabilizers of $t$ points, are indeed maximal and unique. In later papers with Ellis and Filmus ([15] and [16]), we proved results that imply stability versions of [17], that $t$-intersecting families in $S_n$ whose size is close to the maximum must be close in structure to the true maximizers. The results of [15] and [16] are analogs of known results regarding the Fourier transform of Boolean functions on $\{0, 1\}^n$. In [15] we prove an analog of the FKN theorem: Boolean functions on $S_n$ whose Fourier transform is mainly concentrated on the trivial representation and the representation indexed by $(n - 1, 1)$ must be close to a dictator. In [16] we study functions on $S_n$ whose Fourier transform is concentrated mostly on the representations indexed by by partitions of $n$ with first part of size at least $n - t$, and proving that they are "junta-like." This is an analog of several similar theorems regarding Boolean functions on $\{0, 1\}^n$ whose Fourier transform is mostly concentrated on low levels (almost-low-degree functions), see, e.g., [35] and its references for the latest results in this vein.

It turns out, however, that there was a gap in the proof in [17], one of the final lemmas in the proof was false, thus, for $t > 1$ our proof of uniqueness of the extremal examples is incorrect. This was noticed by Filmus [19]. Luckily, this has been circumvented, both in a 2011 paper by Ellis [13] who used a clever bootstrapping approach, and in a recent paper [10].

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