SCHUBERT CALCULUS AND QUIVER VARIETIES

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ABSTRACT

The Littlewood–Richardson rule (1934) is a combinatorial (and, in particular, manifestly positive) way to compute the structure constants of two a priori unrelated rings-with-basis: the representation ring of $\mathrm{GL}_k(\mathbb{C})$, and the cohomology ring of the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$. We recall a wealth of generalizations of the latter ring (changing the space, the cohomology theory, or the basis), all of which have non-manifestly-positive rules for computation, nowadays called their *Schubert calculus.* Until this century very few of these structure constants had combinatorial rules for their calculation, although many of the structure constants have been proven (ineffectively) to be nonnegative.

In recent years the formal similarity of one of these rules (the Knutson–Tao "puzzle" rule for equivariant cohomology) to quantum integrable systems has been traced to the geometry of *quiver varieties,* a class among which one finds the cotangent bundles to Grassmannians. This allowed for the discovery and proof of rules for many heretofore unsolved Schubert calculus problems, and new connections to representation theory.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 14M15; Secondary 16T25, 16G20

KEYWORDS

Schubert calculus, Knutson–Tao puzzles, quiver varieties, flag manifolds, R-matrices, stable envelopes, Chern–Schwartz–MacPherson classes

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 6, pp. 4582–4605
DOI 10.4171/ICM2022/118

Published by EMS Press

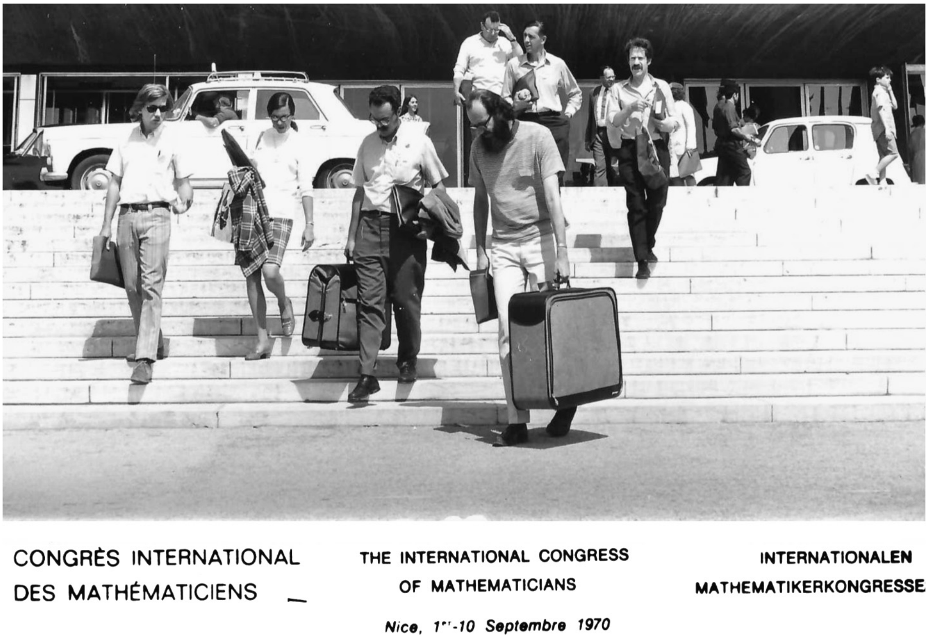

FIGURE 1
Donald Knutson, left, at the author's first ICM (see also Figure 2).

1. LITTLEWOOD–RICHARDSON COEFFICIENTS

1.1. From intersection theory on Grassmannians

Given a compact oriented manifold M (so, one enjoying Poincaré duality) and a Morse function, one obtains a decomposition of M into cells. With luck[1] the Morse function is "perfect," meaning that the cellular homology chain maps vanish, and the cells therefore give a basis of homology and (using the Poincaré duality) cohomology. The product $[X] \cdot [Y]$ in the cohomology ring[2] can be interpreted using the intersection $[X \cap Y]$, assuming that the cell closures X and Y have been moved to be transverse.

In this ring-with-basis, there is no reason to expect the structure constants to be non-negative. For example, if M is the blowup $\widetilde{\mathbb{CP}^2}$ of $\mathbb{CP}^2$ at a point, and E is the exceptional divisor, then $[E] \cdot [E]$ is *minus* the class of a point. (From this one can infer that the two-sphere E cannot be perturbed to some E' inside the real four-manifold $\widetilde{\mathbb{CP}^2}$ while *staying complex,* as the complex intersection $E \cap E'$ would then have the right orientation to be a positive number of points.)

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1 Of course, this is situation is very special—for example, it can only hold when the homology has no torsion.

2 It is worth noting that this homology/cohomology technology was invented exactly to answer the 15th question Hilbert proposed at the 1900 ICM **[29]**, about putting Schubert's calculus on a rigorous footing.

Sometimes there is a cheap source of such perturbations. If M is a *homogeneous space* for a complex Lie group G, then [34] shows that for $X, Y \subseteq M$ complex subvarieties and $g \in G$ a generically chosen group element, the subvarieties X and $g \cdot Y$ are transverse. From this and a certain duality property of the cohomology basis (to be defined below, in the case M compact), one finds that the structure constants are nonnegative.

We now focus on the first such M of real interest: the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$ of k-planes in n-space. One can overparametrize this manifold using the "row span" map taking a full-rank $k \times n$ matrix R to its row span, a k-plane. Since this map is invariant under row operations, we can use Gaussian elimination to restrict the domain to full-rank $k \times n$ matrices in reduced row-echelon form. There are now $\binom{n}{k}$ cases, according to where the k pivots occur, and each case gives a complex cell; in all this gives the *Bruhat decomposition* of $\mathrm{Gr}(k, \mathbb{C}^n)$ into complex cells. (There is also a Morse theory picture [1].)

The cells X°_λ are naturally indexed by partitions $(\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$ as follows: erase the pivot columns, and count the 0s in each row, from bottom to top. That is to say, the zeros form a "French partition" inside the smaller matrix:

$$\begin{bmatrix} 1 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \mapsto \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \mapsto \lambda = (2, 1, 1, 0). \qquad (*)$$

The closures $\{X_\lambda := \overline{X^\circ_\lambda}\}$ of these cells are the *Schubert varieties* in the Grassmannian, and we denote the Poincaré duals of their homology classes by $\{[X_\lambda]\}$, the *Schubert classes*.

Though we will not pursue the following viewpoint further here, it is worth recalling the reasons for the general interest in moduli spaces and especially in their cohomology, where the Grassmannian (the "moduli space of k-dimensional subspaces of $\mathbb{C}^n$") is the most basic example. Whenever one has a family $F \to X$ of some kind of mathematical object $\mathcal{O}$, one may hope to interpret it as the pullback of a *universal family* $\mathcal{F} \to M(\mathcal{O})$ along a "classifying map" $X \to M(\mathcal{O})$. What would be even better is if this recipe $\mathrm{Map}(X, M(\mathcal{O})) \to \{$isomorphism classes of $\mathcal{O}$-families over $X\}$ were bijective. Assuming both, and applying H^* to the classifying map gives us

$$\{\text{isomorphism classes of } \mathcal{O}\text{-families over } X\} \to \mathrm{Map}\big(H^*\big(M(\mathcal{O})\big) \to H^*(X)\big)$$

In the case $\mathcal{O} = \{k\text{-planes in } \mathbb{C}^\infty\}$ so $F \to X$ is a k-dimensional vector bundle, the cohomology ring $H^*(M(\mathcal{O}))$ is a polynomial ring in k generators $c_1, \ldots, c_k$, and their images in $H^*(X)$ are the Chern classes of F. The images of the Schubert classes arise as the classes of "degeneracy loci" of generic bundle maps from F to a (flagged) trivial bundle $\mathbb{C}^n \times X$. For more of this viewpoint on Schubert classes, see, e.g., [24].

1.2. From representation theory

The representation ring $\mathrm{Rep}(G)$ of a group G has a natural $\mathbb{Z}$-basis consisting of the finite-dimensional irreducible representations, and the multiplication in this basis has a positivity property: the expansion of (the semisimplification of) the tensor product of two irreps is an $\mathbb{N}$-combination of irreps. When G is a complex Lie group (or even Kac–Moody group), there has been a great deal of work on combinatorial interpretations of these structure constants, with a reasonably complete answer given by the work of Littelmann **[46]**.

In the specific case $G = \mathrm{GL}_k(\mathbb{C})$, the basis is naturally indexed by dominant weights $\{\lambda \in \mathbb{Z}^k : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k\}$, and many authors restrict to the subcase $\lambda_k \geq 0$ of "polynomial representations" V_λ when considering the **Littlewood–Richardson coefficients**

$$c^\nu_{\lambda\mu} := \dim \mathrm{Hom}_{\mathrm{GL}_k(\mathbb{C})}(V_\nu, V_\lambda \otimes V_\mu).$$

Effectively, this subcase is the representation theory of the Lie *monoid* $(\mathrm{End}(\mathbb{C}^k), \bullet)$ rather than of the group $\mathrm{GL}_k(\mathbb{C})$ sitting densely within. If one is willing to stray this far from groups, it is natural to consider the representation theory of the entire category **Vec** (of finite-dimensional complex vector spaces), i.e., functors **Vec** $\to$ **Vec**, where the irreps are the "Schur functors" such as $V \mapsto \mathrm{Alt}^m V$. (For technical reasons, one usually restricts to representations that are finite direct sums of Schur functors.)

This Rep(**Vec**) picture lets one observe a nice stability: for $\lambda, \mu, \nu \in \mathbb{N}^k$ and $k' \geq k$, if we construct $\lambda', \mu', \nu' \in \mathbb{N}^{k'}$ by concatenating $k' - k$ zeros at the end, then $c^\nu_{\lambda\mu} = c^{\nu'}_{\lambda'\mu'}$.

Theorem 1. *The linear map* $\mathrm{Rep}(\mathrm{End}(\mathbb{C}^k)) \twoheadrightarrow H^*(\mathrm{Gr}(k, \mathbb{C}^n))$ *taking*

$$[V_\lambda] \mapsto \begin{cases} [X_\lambda] & \text{if } \lambda_1 \leq n-k, \\ 0 & \text{otherwise} \end{cases}$$

is a ring homorphism. In particular, the structure constants in the Schubert basis of the Grassmannian are again Littlewood–Richardson coefficients.

The original proof **[45]** of Theorem 1 is rather indirect—essentially, one checks that the "Pieri rule" (which governs multiplication in the case $\mu_1 = 1$) holds in both cases. These Pieri classes generate the ring, and associativity takes care of the rest.

Since then, there have been a number of more satisfying linkages drawn between the two rings-with-bases. In **[41, §8]** Kostant, developing further a proof by Horrocks **[30]** (see also **[14]**), approaches $H^*(\mathrm{Gr}(k, \mathbb{C}^n)) \cong H^*(\mathrm{GL}_n(\mathbb{C})/P_{k,n-k})$ using de Rham cohomology and $U(n)$-invariant forms. The resulting Lie algebra cohomology differential vanishes because the radical $\mathrm{Rad}(P_{k,n-k})$ of the parabolic subgroup $P_{k,n-k}$ is abelian, and the representation theory points very directly to the Schubert basis. Compact homogeneous spaces G/P with $\mathrm{Rad}(P)$ abelian are called **cominuscule** and will appear again in Section 2.5.

In the cunningly titled paper "The connection between Schubert calculus and representation theory" **[58]**, a natural map $\mathrm{Rep}(\mathrm{Vec}) \to H^*(\mathrm{Gr}(k, \mathbb{C}^n))$ is constructed, applying a Schur functor to the tautological bundle over the Grassmannian and (à la Chern–Weil theory) using the $U(n)$-invariant Hermitian connection on that bundle to build a cohomology class. Unfortunately, while the *map* is natural and visibly multiplicative, the proof that it takes basis elements to basis elements (or zero) again amounts to observing that both rings enjoy the Pieri rule.

A tighter connection appears in [5] (see also [6]), in which Belkale uses a point in the transverse triple intersection $X_\lambda \cap (g \cdot X_\mu) \cap (g' \cdot X_\nu)$ to define a vector in

$$(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{SL}(\mathbb{C}^k)},$$

and shows that the resulting vectors are linearly independent. (This proves only an inequality between the intersection-theoretical vs. the representation-theoretical numbers. A related approach in [49] establishes the equality.) This is perhaps the most satisfying (or "most categorical") in that it works directly with the vector space rather than just its dimension. The same is true in a more general statement about quivers in [20].

1.3. From group theory

Given a partition λ and a prime p, one can construct a finite abelian p-group $\Gamma_\lambda := \prod_i (\mathbb{Z}/p^{\lambda_i})$. The number of short exact sequences $0 \to \Gamma_\lambda \to \Gamma_\nu \to \Gamma_\mu \to 0$ turns out to be a polynomial in p, with leading coefficient $c_{\lambda\mu}^{\nu}$. We refer the reader to [22] for more on this source of LR coefficients.

1.4. Combinatorial approaches

Before going further, we draw a distinction here between the computation of Schubert *classes* vs. their products, the Schubert *calculus*. Every one of the rings-with-bases we will consider here and in Section 2 has a known presentation with generators and relations, and (with greater difficulty in some cases than others) a known system of polynomial representatives for the desired basis elements. These polynomials themselves often have interesting positivity properties, giving statements such as "The Schubert polynomials of Lascoux–Schützenberger have nonnegative coefficients." However, such positivity results (or even combinatorial formulæ) do not directly give positivity results about the multiplication. As such we will not focus further here on the (very interesting) questions of constructing these representatives.

We cannot emphasize strongly enough that the name of the game is to give *manifestly nonnegative formulæ for the (known to be nonnegative) structure constants.* We mention three reasons to seek such formulæ, even where nonpositive formulæ are readily available.

(1) For applications (including real-world engineering applications), it is more important to know that some structure constant c is *positive*, than it is to know its actual value. This is much more easily studied with a noncancelative formula. The same is true for another problem of frequent interest, determining when $c = 1$.

(2) Alternating sum formulæ tend to be *much* less efficient computationally.

(3) When positive integers appear, they suggest that there may be a possibility for categorification, in which each coefficient is promoted to a vector space of that dimension. A combinatorial rule for the coefficient then suggests an indexing of a basis for the vector space.

There are several rules of the form "$c^{\nu}_{\lambda\mu}$ is the number of Young tableaux of a certain shape and content satisfying several conditions [semistandard/ballot/Yamanouchi/reverse lattice word]" all of which go by the name "Littlewood–Richardson rules." The history of these rules is somewhat convoluted—in particular, the original proof had an error not corrected for decades—and we refer the interested reader to [22, 62] for it. The most concise modern proofs seem to be [56] (based on a sign-reversing involution) and [12] (which uses the associativity argument of [45]).

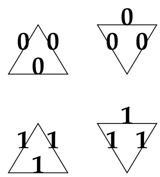
For these same numbers there are a wealth of other rules counting combinatorial objects, e.g., the "pictures" of [65] (which allows for a generalization involving skew Schur functions), the "cartons" of [59] which manifest an S_3-symmetry in the problem, the "Mondrian tableaux" of [17]—but we focus now on the *puzzles* that we introduced in [36, 37] which have so far admitted the most generalizations.

We will need to index Schubert classes on $\mathrm{Gr}(k, \mathbb{C}^n)$ not by partitions as in (∗), but by binary strings where the 0s indicate the pivot columns. In the example from (∗), the string would be 0100101, and more generally has content $0^k 1^{n-k}$. One typical cohomology calculation is $S_{101}^2 = S_{110} \in H^*(\mathrm{Gr}(1, 3))$, which says that two lines in the projective plane intersect in a point.

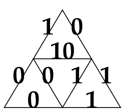
A $c^{\nu}_{\lambda\mu}$ **puzzle** will be an equilateral triangle of edge-length n, with the northwest, northeast, and south sides of this Δ labeled by λ, μ, ν all written left-to-right.

Theorem 2. [36, 37] *There exists a finite set of "puzzle pieces" (unit triangles with edge labels, oriented either as Δ or ∇) such that the number of $c^{\nu}_{\lambda\mu}$ puzzles assembled from them is the Littlewood–Richardson coefficient, for all k, n and λ, μ, ν.*

Once we grant this oracular statement, it is extremely easy to reverse-engineer the pieces (which is why I did not include them in the statement above). On the point $\mathrm{Gr}(1, \mathbb{C}^1)$ there is a unique Schubert class $S_0 = 1$, hence $S_0^2 = S_0$, so there should exist a unique puzzle with boundary 0 on all three sides. We have found our first piece, the 0–0–0 Δ-piece. The same argument on the point $\mathrm{Gr}(0, \mathbb{C}^1)$ lets us also discover a Δ-piece with 1 on each side. The similar formula $S_{00}^2 = S_{00}$ in $H^*(\mathrm{Gr}(2, \mathbb{C}^2))$ is almost as easy; the 0–0–0 Δ-piece fits nicely into the three corners, forcing us to invent a 0–0–0 ∇-piece to go in the middle. Again, the corresponding calculation on $H^*(\mathrm{Gr}(0, \mathbb{C}^2))$ suggests we admit a 1–1–1 ∇-piece as well.



A new phenomenon enters when we consider $H^*(\mathrm{Gr}(1, 2))$, where we need to find a puzzle computing $S_{01}^2 = S_{01}$. In the southwest and southeast corners we can place 0–0–0 and 1–1–1 Δ-pieces. If we try to put either the label 0 or 1 on the remaining edge, we run into problems (overcounting $c^{\nu}_{\lambda\mu}$ somewhere down the line), so we invent a new label "10" to go on this edge. The two rotations of this puzzle give the $S_{01}S_{10} = S_{10}S_{01} = S_{10}$ computations, so all six rotations of the 1–0–10 piece should be admitted.



It is then an easily checked experimental fact (for $n \leq 10$, say) that no more labels or pieces are needed: these three (up to rotation) pieces are already giving the right count for every $c_{\lambda\mu}^{\nu}$! Of course, this is not a proof, and the first proof of Theorem 2 was a bit unsatisfying—just a reduction to another, known, rule for LR coefficients. A much more concrete link between the Grassmannian geometry and combinatorics was first laid out in [61], and connected to puzzles in [35,40].

These puzzles enjoy six symmetries: rotation by multiples of 120°, left–right reflection composed with the label swap $0 \leftrightarrow 1$, and composites thereof. In fact, the LR coefficients have these symmetries and more, once one observes

$$c_{\lambda\mu}^{\nu} = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} S_\lambda S_\mu S_{\nu \text{ reversed}} = \int_{\mathrm{Gr}(n-k,\mathbb{C}^n)} S_{\lambda^*} S_{\mu^*} S_{\nu^* \text{ reversed}}$$

where λ^* means "reverse and swap $0 \leftrightarrow 1$." The first equality comes from the dual-basis statement $\int_{\mathrm{Gr}(k,\mathbb{C}^n)} S_\lambda S_{\nu \text{ reversed}} = \delta_{\lambda\nu}$, the second from Grassmannian duality. It is rather hard to directly see that the puzzle rule defines a commutative product, which is manifest in the carton rule from [60].

2. SEVERAL INDEPENDENT AXES OF GENERALIZATION

In Sections 2.1–2.6 below we present various mutually compatible axes of generalization "**KTFQGC**" of the basic problem (that being Schubert calculus in $H^*(\mathrm{Gr}(k,\mathbb{C}^n))$), and comment afterward on the combinations thereof. While we have endeavored to report the state-of-the-art (as concerns combinatorial rules), for reasons of space we have not included a complete timeline of earlier results.

2.1. K-theory [K]

The K-theory of the Grassmannian is again a free $\mathbb{Z}$-module of dimension $\binom{n}{k}$, but there are two big differences between it and the cohomology: it is naturally filtered rather than graded (with $H^*(\mathrm{Gr}(k,\mathbb{C}^n))$ as the associated graded ring), and there are *two* natural bases for it, dual to one another under a natural pairing. The more commonly studied basis $\{G_\lambda\}$ consists of (Klasses of the) structure sheaves of the Schubert varieties. The other basis $\{G_\lambda^*\}$ consists of the "ideal sheaves", functions on a Schubert variety vanishing on all smaller Schubert varieties.

The first proof that the multiplicative structure constants of the $\{G_\lambda\}$ basis are positive (up to a sign convention) came hand in hand with a formula for their computation, in terms of Buch's "set-valued semistandard Young tableaux" [8]. A (more widely applicable, but noneffective) geometric argument for this positivity appeared afterward in [7].

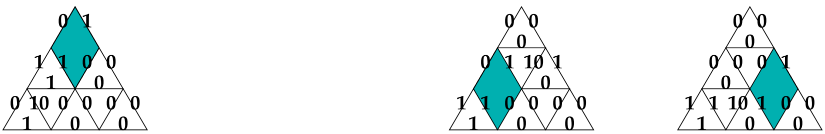
It is easy to guess puzzle pieces for these two products, from the computations $(G_{01}^*)^2 = G_{01}^* - G_{10}^*$ and $G_{0101}^2 = G_{0110} + G_{1001} - G_{1010}$. For the $\{G_\lambda\}$ multiplication, one introduces a Δ-piece with labels 10–10–10 (announced in [61]). For the $\{G_\lambda^*\}$ multiplication, one introduces the 10–10–10 ∇-piece [64]. In both cases, the sign convention requires that each 10–10–10 piece contribute a factor of -1, but this does not lead to any cancelation.

2.2. T-equivariant cohomology [T]

The invertible diagonal matrices $T \leq \mathrm{GL}_n(\mathbb{C})$ act on $\mathrm{Gr}(k, \mathbb{C}^n)$ preserving each of the Schubert varieties X_λ. Hence, each X_λ defines an element of the T-equivariant cohomology ring $H^*_T(\mathrm{Gr}(k, \mathbb{C}^n))$. These classes (again called $\{S_\lambda\}$) form a basis not over $\mathbb{Z} = H^*(pt)$, but over the base ring $H^*_T(pt) \cong \mathbb{Z}[y_1, \dots, y_n]$, and hence the structure constants live in this polynomial ring. The coefficients were shown (again, geometrically and more generally but ineffectively) in **[27]** to lie in $\mathbb{N}[y_1 - y_2, \dots, y_{n-1} - y_n]$.

The first two computations $S_{10}^2 = (y_1 - y_2)S_{10}$ and $S_{010}^2 = S_{100} + (y_2 - y_3)S_{010}$ suggest a vertically[3] rhomboidal **equivariant piece** whose **fugacity** $\mathrm{fug}(\diamond) = y_i - y_j$ depends on the piece's location in the puzzle (namely, the $\diamond$ is in the ith SW/NE diagonal and the jth NW/SE diagonal). See the examples below.

Theorem 3 (**[36]**). *Define an **equivariant puzzle** P to be one in which this additional piece is allowed, and define its **fugacity** $\mathrm{fug}(P)$ to be the product of the fugacities of the equivariant pieces. Then the equivariant structure constant $c^\nu_{\lambda\mu}$ is the sum of the fugacities of the equivariant puzzles with boundary λ, μ, ν as in Theorem 2.*



$$S_{010}\, S_{100} = (y_1 - y_3)S_{100} \quad = \quad S_{100}\, S_{010} = \quad (y_1 - y_2)S_{100} \quad + \quad (y_2 - y_3)S_{100}.$$

(In particular, $i < j$ for any equivariant piece, so this formula verifies the Grassmannian case of Graham's positivity theorem **[27]**.) The proof proceeds from the "most equivariant case" $c^\lambda_{\lambda\lambda}$, with an induction based on a recursive formula for the $\{c^\nu_{\lambda\mu}\}$. The recursion involves a denominator that vanishes if one passes to the nonequivariant case $y_i \equiv 0$, so does not allow for a directly nonequivariant proof of Theorem 2.

We take a moment to foreshadow the framework that will come in Sections 3–4.

The matching requirement for adjacent puzzle labels can be interpreted nicely in terms of matrix multiplication (where the formula $(AB)_{iq} = \sum_{j=p} A_{ij} B_{pq}$ requires a similar matching). Following **[66]**, we introduce a 9×9 matrix $R(a, b)$ whose 3^2 columns are labeled by the possible rhombus tops, and rows by the 3^2 possible rhombus bottoms. A matrix entry is 1 if the resulting labeled rhombus can be filled by two triangular puzzle pieces; is

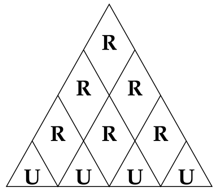
3 In fact, the search was also directed by the need to break the $\mathbb{Z}/3$-symmetry, whose derivation depended on the dual-basis equation $\int_{\mathrm{Gr}(k,\mathbb{C}^n)} S_\lambda S_\nu = \delta_{\lambda,\nu \text{ backwards}}$. This equation makes sense, but does not hold, in equivariant cohomology. The $\mathbb{Z}/3$-symmetry of the K-theory puzzles comes from a different source, $G^*_\lambda = G_\lambda[\mathcal{O}(1)]$ for all λ, a property of "minuscule" flag manifolds (which includes Grassmannians); see Section 2.5.

$a-b$ in the one case that it can be filled by an equivariant piece; and is 0 otherwise:

	$1\wedge 1$	$1\wedge 0$	$1\wedge 10$	$0\wedge 1$	$0\wedge 0$	$0\wedge 10$	$10\wedge 1$	$10\wedge 0$	$10\wedge 10$
$1\vee 1$	1	0	0	0	0	1	0	0	0
$1\vee 0$	0	0	0	$a-b$	0	0	0	0	0
$1\vee 10$	0	0	0	0	1	0	1	0	0
$0\vee 1$	0	1	0	0	0	0	0	0	0
$0\vee 0$	0	0	0	0	1	0	1	0	0
$0\vee 10$	0	0	0	0	0	0	0	0	0
$10\vee 1$	0	0	0	0	0	0	0	0	0
$10\vee 0$	1	0	0	0	0	1	0	0	0
$10\vee 10$	0	0	0	0	0	0	0	0	0

There is a corresponding 3×9 matrix U, with rows labeled $\underline{1}, \underline{0}, \underline{10}$, controlled by the triangular pieces taken alone. With R and U, one reinterprets the sum in Theorem 3 as a *matrix entry* inside the $3^n \times (3^n \cdot 3^n)$ matrix

$$U^{\otimes n}\left(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\left(I_{3^{2j-i-2}} \otimes R(y_{j-i}, y_j) \otimes I_{3^{2n-2j+i}}\right)\right).$$



The key observation in [66] is that this R satisfies the "rational Yang–Baxter equation"

$$\big(R(a,b)\otimes I_3\big)\big(I_3\otimes R(a,c)\big)\big(R(b,c)\otimes I_3\big) = \big(I_3\otimes R(b,c)\big)\big(R(a,c)\otimes I_3\big)\big(I_3\otimes R(a,b)\big)$$

whose relevance will be explained in Section 3.

There is also a "trigonometric Yang–Baxter equation" whose R-matrix entries depend on a/b instead of $a-b$. This becomes relevant in creating puzzle formulæ in equivariant K-theory, as in [38, 39, 64], foreshadowed in [54].

2.3. Larger flag manifolds [F]

Define the d**-step flag manifold** $\mathrm{Fl}(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$ to be the space of chains $(V_1 \le V_2 \le \cdots \le V_d : \dim(V_i) = n_i)$, so Grassmannians are 1-step flag manifolds. (Of course, one can reduce to the case (n_i) strictly increasing, in which case $d = n$ is the maximal situation, but it will be mildly convenient not to.) The row-span and Gaussian elimination technique from Section 1.1 lead again to a cell decomposition with strata now labeled by strings with content $0^{n_1}1^{n_2-n_1}\cdots d^{n-n_d}$.

In 1999 I followed the oracle of Theorem 2 to $d > 1$, and came up with a set of edge labels and puzzle pieces that looked promising for general d. But it was already wrong for $d = 3$ and $n = 5$, so I (prematurely) abandoned it, without even pursuing $d = 2$. Buch observed experimentally that my incorrect rule was consistently *undercounting* at $d = 3$, and he suggested some additional puzzle labels, though $d \ge 4$ remained seemingly out of reach. My 2-step puzzle conjecture was proven in [10], again by an associativity check, and in [38] we proved Buch's modified 3-step conjecture (plus a 151-piece extension to K-theory) by techniques to be recalled in Section 4. Another combinatorial rule for 2-step appears in [17], where it is suggested that the techniques involved should extend to higher d (see the survey [19] for a $d = 3$ example).

The "Schur times Schubert" subproblem, in which one of the two classes is pulled back from the Grassmannian, is easily shown to be equivalent to the problem of expanding the class of a "positroid variety" into Schubert classes. Positroid varieties have become of much interest in the physical theory of scattering amplitudes [4].

2.4. Quantum cohomology [Q]

On a compact oriented manifold, where cohomology classes can be thought of homologically, the structure constants of multiplication can be computed as the finite number of points $p \in X \cap Y \cap Z$ in a transverse triple intersection. Physicists introduced in the 1990s the "quantum cohomology" of an almost complex manifold M, in which one instead counts almost complex maps $\gamma : \mathbb{CP}^1 \to M$ such that $\gamma(0) \in X$, $\gamma(1) \in Y$, and $\gamma(\infty) \in Z$. To define the (small) quantum product for most honest complex manifolds, one must deform the complex structure to generic almost complex (the 1990s solution), or involve an "obstruction bundle" over the moduli space of maps (the 21st-century solution), but for Grassmannians these niceties are unnecessary [23].

It turns out that for generic enough $\gamma : \mathbb{CP}^1 \to \mathrm{Gr}(k, \mathbb{C}^n)$ (and as proved in [11], the γs we want to count *will* be generic enough), the two nested subspaces $\bigcap_{z \in \mathbb{CP}^1} \gamma(z)$, $\sum_{z \in \mathbb{CP}^1} \gamma(z)$ will have dimensions $k - \deg(\gamma), k + \deg(\gamma)$, respectively (where $\deg(\gamma)$ is defined by $\gamma_*([\mathbb{CP}^1]) = \deg(\gamma)[X_{1\ldots1010\ldots0}]$). With a little more work, it is shown in [11] that the degree m structure constants in $QH(\mathrm{Gr}(k, \mathbb{C}^n))$ can be calculated as structure constants in *ordinary* $H^*(\mathrm{Fl}(k - m, k + m; \mathbb{C}^n))$. (This sparked particular interest in the 2-step case, causing Buch–Kresch–Purbhoo–Tamvakis to revisit and eventually prove my 2-step conjecture from Section 2.3.)

2.5. Other Lie groups [G]

Grassmannians $\mathrm{Gr}(k, \mathbb{C}^n)$ are minimal homogeneous spaces for $\mathrm{GL}_n(\mathbb{C})$, with the property that their point stabilizers have abelian unipotent radical. Such **cominuscule flag manifolds** are very rare—each connected simply-connected group G has only $|Z(G)| - 1$ of them up to conjugacy (and even fewer up to isomorphism, such as $\mathrm{Gr}(k, \mathbb{C}^n) \cong \mathrm{Gr}(n - k, \mathbb{C}^n)$). For example, there are two E_6 cominuscule flag manifolds, but they are isomorphic.

In [60] is given a uniform rule for Schubert calculus in the ordinary cohomology of cominuscule flag manifolds. The proof of its validity, however, is case-by-case.

There is a Langlands dual notion, of "minuscule" flag manifold, which is one whose minimal equivariant embedding $G/P \hookrightarrow \mathbb{P}V$ is into a G-representation V with only extremal weights. This tight control on the homogeneous coordinate ring is felicitous for K-theory considerations (see, e.g., [13]).

2.6. Cotangent Schubert calculus [C]

Although a complex manifold M and its cotangent bundle T^*M are homotopic, hence bear the same cohomology ring, they may have different natural bases. One source for the latter is the *characteristic cycle* $\mathrm{cc}(\mathcal{F})$ of a $\mathcal{D}_M$-module $\mathcal{F}$. This is a Lagrangian cycle inside T^*M, and is invariant under the $\mathbb{C}^\times$-action dil dilating the cotangent fibers, hence defines a class

$$\left[\mathrm{cc}(\mathcal{F})\right] \in H^*_{\mathrm{dil}}(T^*M) \cong H^*_{\mathrm{dil}}(M) \cong H^*_{\mathrm{dil}}(pt) \otimes H^*(M) \cong H^*(M)[\hbar]$$

(where $\hbar$ is the standard generator of $H^*_{\mathrm{dil}}(pt)$). When $\iota : A \hookrightarrow M$ is the inclusion of a locally closed submanifold and $\mathcal{F} = \iota_*(\mathcal{O}_A)$ is the sheaf of distributions supported on A, this class $[\mathrm{cc}(\mathcal{F})]$ is essentially the **Chern–Schwarz–MacPherson class** of the submanifold [26], up to a sign.

If we invert $\hbar$ (or set it to -1, as is conventional for CSM classes) then the classes associated to the ($\mathcal{D}$-modules of distributions on the) Bruhat cells are again a basis, now of $H^*(\mathrm{Gr}(k, \mathbb{C}^n))[\hbar^\pm]$. There is some work on their structure constants of multiplication [16], but it has been more fruitful to consider multiplying the **Segre–Schwarz–MacPherson classes** $\{\mathrm{SSM}_\lambda := [\mathrm{cc}(\iota_*(\mathcal{O}_{X^\circ_\lambda}))] \,/\, e(T^*\,\mathrm{Gr}(k, \mathbb{C}^n))\}$. The necessity of introducing this denominator is hinted at in Section 4.

Theorem 4 ([39]). *The product of SSM classes on* $\mathrm{Gr}(k, \mathbb{C}^n)$ *can be computed using puzzles as before, within which one now allows* both *the* Δ *and* ∇ *10–10–10 pieces.*

As a consequence, the Euler characteristic of a transverse triple intersection $X^\circ_\lambda \cap (g \cdot X^\circ_\mu) \cap (h \cdot X^\circ_\nu)$ *is* $(-1)^{\text{its dimension}}$ *times the number of such puzzles, where* λ, μ, ν *are all written clockwise on the puzzle boundary. That dimension also predicts the number of 10–10–10 pieces in every puzzle.*

"Cotangent" is the newest adjective in the subject and is getting a lot of attention, e.g., [2, 57]. The involvement of $\mathcal{D}_{G/P}$-modules is especially exciting because of the Beilinson–Bernstein localization theorem, which relates them to representations of $U\mathfrak{g}$. The representations that are relevant here are the parabolic Verma modules of central character 0.

In addition, it appears that a proper understanding of Schubert calculus in *elliptic* cohomology (the next step beyond K-theory, in a sense) requires passage to the cotangent bundle [42].

2.7. Mixing and matching

The theorems in Sections 2.1–2.6 may make the subject sound closed, but each of the 2^6 combinations of **KTFQGC** is its own problem, and most are unsolved. By time of writing, the maximal positively solved problems (in the sense of having a manifestly noncancelative combinatorial rule for all products) are

- **KG** for minuscule flag manifolds [13],
- **KTFC** for $d = 2$ and **KFC** for $d = 3$ [39],

- **QT** via the connection to 2-step [9–11],
- **KQT** for projective space [15].

There are many partial results concerning multiplication by special classes, as well as non-effective positivity results such as [3, 48].

2.8. A few other generalizations

The cohomology of a space M bears a ring structure exactly because M has a canonical map $M \to M \times M$, the diagonal inclusion; the multiplication then comes from that pullback. If more generally we have a map $F/P \to G/Q$ of generalized flag manifolds, we can consider the map $H^*(G/Q) \to H^*(F/P)$ in the bases of Schubert classes.

Theorem 5 ([28]). *Let* $\iota : \mathrm{SpGr}(k, \mathbb{C}^{2n}) \hookrightarrow \mathrm{Gr}(k, \mathbb{C}^{2n})$ *be the inclusion of the Grassmannian of isotropic* k*-planes with respect to a symplectic form. Then the pullback in* T^n*-equivariant cohomology can be computed using puzzles with* 10*-labels allowed on the bottom, and that are invariant under flipping left–right while exchanging* $0 \leftrightarrow 1$. *(Another, nonequivariant, rule appears in* [18].*)*

The *affine Grassmannian* Gr_G is a homogeneous space for the affine Lie group, so the study of its cohomology is covered by case **G** above. But since Gr_G is homotopic to a group, its *homology* also bears a ring structure. Fascinatingly, this ring is tightly connected to the quantum cohomology of the corresponding finite-dimensional *full* flag manifold [33, 43, 44] (itself very far from having a positive rule).

There is also a ring structure on the K-homology $\bigoplus_{a,b} K_\bullet(\mathrm{Gr}(a, \mathbb{C}^{a+b}))$ induced by the "direct sum" map, computed with new puzzle pieces in [55]. Finally, the "separated descents" pullback along the inclusion $\mathrm{Fl}(\mathbb{C}^n) \hookrightarrow \mathrm{Fl}(1, \ldots, k;\ \mathbb{C}^n) \times \mathrm{Fl}(k, \ldots, n;\ \mathbb{C}^n)$ is computed in cohomology in [31] and will be given a **KTC** puzzle rule elsewhere.

3. QUIVER VARIETIES, STABLE ENVELOPES, AND STABLE BASES

We switch gears to define a very different family of varieties, following [25,47,50–52].

We comment briefly on the 20-year journey we took from puzzles to these *quiver varieties*. P. Zinn-Justin [66] reproved the equivariant puzzle rule from [36], along the way showing that one could build an "R-matrix" (meaning, a solution to the "Yang–Baxter equation") from the equivariant pieces; see Section 2.2. Through this he was able to replace much of the bespoke combinatorial arguments we had used in [36] with standard tricks from the theory of quantum integrable systems. Further puzzle results were obtained by this algebraic technique in [38, 64]—in particular, *discovering* and proving the rule for K(3-step), which requires 151 new puzzle pieces—but the deeper relation of this algebra to flag manifolds was not clear.

Solutions to the Yang–Baxter equation typically come from commutors of representations of quantized affine algebras (see, e.g., [21, §13] and [32]). Nakajima constructed some such representations on the K-theory of quiver schemes (theorem 7 below), and Maulik–

Okounkov interpreted the R-matrices directly on the quiver varieties [47,53]. In the remainder of this paper we recall the constructions from [39, §7], in which we geometrically reinterpret the algebraic results of [38] and extend to cotangent Schubert calculus in the bargain.

3.1. Quiver varieties

Consider a directed graph (Γ_0, Γ_1) with some of the vertices Γ_0 called "gauged" and the others called "framed", which we will generally indicate by $\boxed{\text{framing}}$ them. To simplify notation (while not ruling out any of the cases of most interest here), we assume that there is at most one edge connecting any two vertices.

To each labeling $d : \Gamma_0 \to \mathbb{N}$ of the vertices, called a **dimension vector**, we construct a **quiver variety** $\mathcal{M}(\Gamma, d)$ in four steps:

(1) Consider the vector space $\prod_{(t\to h)\in\Gamma_1}(\mathrm{Hom}(\mathbb{C}^{d(t)}, \mathbb{C}^{d(h)}) \times \mathrm{Hom}(\mathbb{C}^{d(h)}, \mathbb{C}^{d(t)}))$, where a typical element is a tuple $(M_{ab} \in \mathrm{Hom}(\mathbb{C}^{d(a)}, \mathbb{C}^{d(b)}))$.

(2) Impose the closed "complex moment map" condition that at each gauged vertex v, $\sum_{(v\to w)\in\Gamma_1} M_{vw}M_{wv}$ equals $\sum_{(w\to v)\in\Gamma_1} M_{wv}M_{vw}$ plus a scalar.

(3) Impose the open "stability" condition that for any $\vec{w} \neq \vec{0}$ at a gauged vertex v_0, there exists an undirected path $v_0 \leftrightarrow v_1 \leftrightarrow \cdots \leftrightarrow v_m$ with v_m framed, such that $M_{v_{m-1},v_m} \cdots M_{v_1,v_0}\vec{w} \neq \vec{0}$. (There are other stability conditions one might use but we will not need them.)

(4) Divide by the action of the group of basis transformations at the gauge vertices.

It is more traditional to fix the scalars used in the moment map condition, especially to 0, but for convenience of exposition we work with this enlarged point of view. While quiver varieties (as defined here) are naturally Poisson varieties, with symplectic leaves given by fixing those scalars, we will focus attention on a certain circle action that does *not* preserve the Poisson structure. It is induced from the scaling action on half the original variables, $\prod_{(t\to h)\in\Gamma_1}\mathrm{Hom}(\mathbb{C}^{d(t)}, \mathbb{C}^{d(h)})$, and we call it the **dilation** action dil.

The following is a folklore observation, which we include in part to recall the Grothendieck–Springer deformation of $T^*\,\mathrm{Fl}(n_1, \ldots, n_d; \mathbb{C}^n)$:

Theorem 6 (see, e.g., [50]).

$$\mathcal{M}\begin{pmatrix} \boxed{n} & & \\ \uparrow & & \\ n_d \leftarrow & \cdots & \leftarrow n_1 \end{pmatrix}$$

is isomorphic to the Grothendieck–Springer deformation of the cotangent bundle $T^*\,\mathrm{Fl}(n_1, \ldots, n_d; \mathbb{C}^n)$*, where the deformation parameters are the d scalars used at the gauge vertices. The dilation action on the cotangent bundle is given by scaling the cotangent vector.*

Proof sketch (the morphism in one direction). For convenience, index the vertices by their dimensions. The gauge-invariant functions we will use are the endomorphism

$M_{n_d,n}M_{n,n_d} \circlearrowright \mathbb{C}^n$ and the nested subspaces $V_i := im(M_{n,n_d} M_{n_{d-1},n_d} \cdots M_{n_{i+1},n_i}) \leq \mathbb{C}^n$. The stability condition ensures that $\dim(V_i) = n_i$, giving us the flag, and the moment map condition implies that $(X - \varepsilon_i)V_i \leq V_{i+1}$ where ε_i is the scalar at vertex i. This resulting space $\{(X, V_\bullet \in \mathrm{Fl}(n_1, \ldots, n_d; \mathbb{C}^n), \vec{\varepsilon}) : (X - \varepsilon_i)V_i \leq V_{i-1}\}$ is the **Grothendieck–Springer family**, whose central fiber $\vec{\varepsilon} = \vec{0}$ is the Springer resolution of the closure of a nilpotent orbit. ■

It will also be convenient to fix only the dimension vector $\boxed{d}$ on the framed vertices, and define the **quiver scheme** $\mathcal{M}(\Gamma, \boxed{d})$ as the disjoint union of all $\mathcal{M}(\Gamma, e)$ where e agrees with $\boxed{d}$ on the framed vertices. (That may seem like a lot of components, but because of the stability condition these $\mathcal{M}(\Gamma, e)$ are frequently empty, such as when the (n_i) in the example of Theorem 6 are not weakly increasing.) One way these enter geometric representation theory is as follows:

Theorem 7 ([51]). *Assume Γ's gauge vertices form an ADE quiver, corresponding to a simple Lie algebra $\mathfrak{g}$. Then there is a family of natural actions of the quantized loop algebra $U_q(g[z^\pm])$ on the finite-dimensional vector space $K(\mathcal{M}(\Gamma, \boxed{d}))$. The decomposition $K(\mathcal{M}(\Gamma, \boxed{d})) = \bigoplus_e K(\mathcal{M}(\Gamma, e))$ is into the weight spaces.*

Moreover, if $d = d_1 + d_2$, then $K(\mathcal{M}(\Gamma, \boxed{d})) \cong K(\mathcal{M}(\Gamma, \boxed{d_1})) \otimes K(\mathcal{M}(\Gamma, \boxed{d_2}))$ generically in this family.

The parameters on the family can be interpreted geometrically as follows. Pick a basis of each $\boxed{\text{framed}}$ space, and let T be the torus that acts by scaling these basis elements. Then $U_q(g[z^\pm])$ acts on $K_T(\mathcal{M}(\Gamma, \boxed{d}))$, and the base of the family above is the space $T \cong$ Spec $K_T(pt)$ of equivariant parameters.

Applying this to the A_d example in Theorem 6 when $\boxed{n} = 1$, the nonempty quiver varieties are the $d+1$ points $T^* \mathrm{Fl}(0, \ldots, 0, 1, \ldots, 1; \mathbb{C})$. Their total K-theory gives the standard representation $\mathbb{C}^{d+1}$ of $U_q(\mathfrak{sl}_{d+1}(\mathbb{C}[z^\pm]))$. For general $\boxed{n}$, we get the rep $(\mathbb{C}^{d+1})^{\otimes n}$, with a basis consisting of length n strings in $0, 1, \ldots, d$, compatible with the indexing from Section 2.3.

One can degenerate the algebra to a Yangian (essentially) and act instead on $H(\mathcal{M}(\Gamma, \boxed{d}))$ [63], with the benefit that one can extract $H_{\mathrm{top}}(\mathcal{M}(\Gamma, \boxed{d}))$; this latter space bears a less-natural *irreducible* action of $U\mathfrak{g}$ itself [50] that was found first. In the example above, we recover the irrep $\mathrm{Sym}^n(\mathbb{C}^{d+1})$ of $U\mathfrak{sl}_{d+1}$, whose highest weight is $n\omega_1$; this reflects the fact that we attached the $\boxed{n}$ to the first vertex of A_d.

3.2. Stable envelopes and bases

Fix Γ and d. A circle action S on the framed vector spaces induces a circle action on $\mathcal{M}(\Gamma, d)$. Loosely following [47, THEOREM 3.7.4], we define the **stable envelope**

$$\mathrm{env}(S) := \overline{\left\{\left(p \in \mathcal{M}(\Gamma, d)^S, q \in \mathcal{M}(\Gamma, d)\right) : \lim_{z \to 0} S(z) \cdot q \text{ exists and is } p\right\}} \subseteq \mathcal{M}(\Gamma, d)^S \times \mathcal{M}(\Gamma, d)$$

which we regard as providing a *correspondence,* not an actual map, $\mathcal{M}(\Gamma, d)^S \to \mathcal{M}(\Gamma, d)$. Sidestepping some compactness issues (discussed, in e.g., [39, §7]), the envelope induces an isomorphism $\tilde{H}^*_{\text{dil}}(\mathcal{M}(\Gamma, d)^S) \to \tilde{H}^*_{\text{dil}}(\mathcal{M}(\Gamma, d))$ where dil is the dilation action, and the tilde indicates that (as in Section 2.6) we have inverted the $\hbar$ in $H^*_{\text{dil}}(pt) = \mathbb{Z}[\hbar]$. This is of particular interest when $\mathcal{M}(\Gamma, d)^S$ is a finite set, in which case we can use the isomorphism to carry the obvious basis of $\tilde{H}^*_{\text{dil}}(\mathcal{M}(\Gamma, d)^S)$ to a **stable basis** of the target. This basis depends on S, though the action $S'(z) \cdot m := S(z^N) \cdot m$ for fixed $N > 0$ leads to the same envelope and basis.

Lemma 1 (Special case and restatement of [47, LEMMA 3.6.1]). *Fix Γ, d and let A_1, A_2 be two commuting circle actions on the framed spaces. Then for $N \gg 1$, the triangle*

$$\begin{array}{ccc} \mathcal{M}(\Gamma, d)^{A_1, A_2} & \xrightarrow{\text{env}(A_{1+})} & \mathcal{M}(\Gamma, d) \\ \text{env}(A_2) \searrow & & \nearrow \text{env}(A_1) \\ & \mathcal{M}(\Gamma, d)^{A_1} & \end{array}$$

commutes (in the sense of convolutions of correspondences), where $A_{1+}(z) = A_1(z^N) A_2(z)$.

In particular, if $\mathcal{M}(\Gamma, d)^{A_1, A_2}$ is finite, this implies that the correspondence $\text{env}(A_1)$ takes stable basis elements to stable basis elements.

3.3. Comparison of stable bases

There is another important application of Lemma 1. Assume that A, A' are commuting **regular** circle actions on $M := \mathcal{M}(\Gamma, d)$ in the sense that they each have isolated fixed points—necessarily the *same* set of fixed points, as each of A, A' acts (trivially) on the fixed points of the other. How can we compute the change-of-basis matrix between the two stable bases?

Let $\langle A, A' \rangle$ be the 2-torus they generate, and $\Lambda := \text{Hom}(\mathbb{C}^\times, \langle A, A' \rangle) \cong \mathbb{Z}^2$ be its coweight lattice. Within this plane Λ, the subset $\{S \in \Lambda \ : M^S \textit{ not } \text{finite}\}$ is easily shown to be the union of a finite number of lines through the origin. These lines cut the plane into sectors, and within each sector the associated stable basis is constant.

Draw a path from A to A' inside this coweight lattice. It may pass through many sectors, giving us many stable bases along the way, and give thereby a factorization of the change-of-basis matrix as a product. Wherever the path crosses a wall $C_+ \cap C_-$ between two chambers C_+, C_-, we can pick coweights S_1, S_2 where S_1 lies in the interior of the wall, and for all $N \gg 0$ we have $S_1^N S_2 \in C_+$, $S_1^N S_2^{-1} \in C_-$. Then we get a diagram of correspondences

$$\begin{array}{ccccc} M^{\langle A, A' \rangle} & \xrightarrow{\text{env}(S_2)} & M^{S_1} & \xleftarrow{\text{env}(S_2^{-1})} & M^{\langle A, A' \rangle} \\ \text{env}(S_1^N S_2) \searrow & & \downarrow \text{env}(S_1) & & \swarrow \text{env}(S_1^N S_2^{-1}) \\ & & M & & \end{array}$$

whose triangles commute (by Lemma 1), and whose diagonal arrows induce the stable bases from chambers C_+, C_-.

The change-of-basis matrix R across the wall $C_+ \cap C_-$ amounts to following the induced map on cohomology from the northwest $M^{\langle A,A'\rangle}$, down to the M, followed by the inverse of the map from the northeast $M^{\langle A,A'\rangle}$. Because the triangles commute we can instead work across the top line, from the northwest $M^{\langle A,A'\rangle}$, to M^{S_1}, followed by the inverse of the map from the northeast $M^{\langle A,A'\rangle}$. Typically, M^{S_1} is highly disconnected, from which we infer that the change-of-basis matrix R is very sparse.

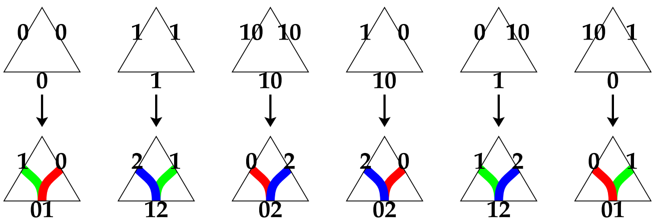
In the example of the next section, each component of M^{S_1} is either a point or $T^*\mathbb{CP}^1$, so (up to reordering of rows and columns) R is a direct sum of blocks of size 1 or 2.

4. A FACTORIZATION OF CORRESPONDENCES GIVES THE PUZZLE RULE

This material is ahistorically drawn from [38, 39]. In this section, we specialize the deformation parameters in our quiver varieties to 0, and work with the action of $U_{\exp(\hbar)}(\mathfrak{g}[z])$ on cohomology rather than $U_q(\mathfrak{g}[z^{\pm}])$ on K-theory.

We have a big clue: in [66] the puzzle pieces for $H^*_T(\mathrm{Gr}(k, \mathbb{C}^n))$ are utilized to construct an R-matrix (see Section 2.2), which by the work of Drinfel'd and Jimbo referenced above suggests we seek a quantum group representation on $\mathbb{C}^3$ (where $3 = \#\{0, 1, 10\}$ is the number of edge labels). There is an obvious guess: $\mathfrak{sl}_3 \circlearrowright \mathbb{C}^3$ or, more precisely, $U_{\exp(\hbar)}(\mathfrak{sl}_3[z]) \circlearrowright \mathbb{C}^3(y)$ where y is the "evaluation parameter" of the representation (arising in Theorem 7 as an equivariant parameter).

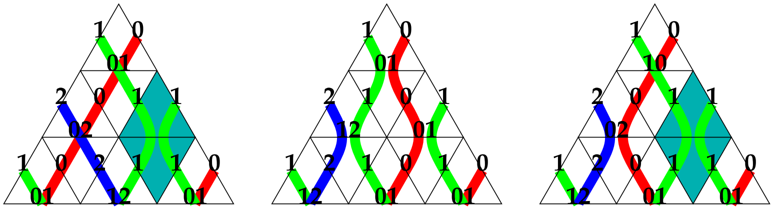
The R-matrix[4] $\mathbb{C}^3(a) \otimes \mathbb{C}^3(b) \to \mathbb{C}^3(b) \otimes \mathbb{C}^3(a)$ gives[5] the rhombi, but we will also need triangular pieces $U : \mathbb{C}^3(a) \otimes \mathbb{C}^3(b) \to V(c)$, where V is again 3-dimensional. The $\mathfrak{sl}_3$-equivariance allows for only one possibility: V must be $\mathrm{Alt}^2\,\mathbb{C}^3$. This asymmetry suggests a reformulation of the puzzle pieces:



4 Technically, in the quantum integrable systems literature this is the "$\check{R}$-matrix."

5 Actually the R-matrix from Section 2.2 is only a degenerate limit of the one provided by the representation theory—for example, in the nondegenerate R-matrix there are nonzero entries corresponding to fillings that use 10–10–10 pieces. One of the purposes of [39] was to provide a cohomological question that would be answered by the richer puzzles constructed from the nondegenerate R-matrix. This turned out to be the **C**otangent story.

The $\mathfrak{t}$-equivariance, or weight conservation, of the map U becomes the statement that the $0, 1, 2$-pipes propagate all the way to the boundary of a puzzle. See these:



Under this new labeling, the puzzles have content $0^0 1^k 2^{n-k}$ on the northwest side, $0^k 1^{n-k} 2^0$ on the northeast side, and $(01)^k (02)^0 (12)^{n-k}$ on the south side. Those contents pick out certain weight spaces of $(\mathbb{C}^3)^{\otimes n}$, $(\mathbb{C}^3)^{\otimes n}$, and $(\mathrm{Alt}^2\, \mathbb{C}^3)^{\otimes n}$, suggesting that we look at the following A_2 quiver varieties,

$$\begin{array}{ccccccc} / & & \backslash & & \wedge & & \text{—} \\ \mathcal{M}\begin{pmatrix} \boxed{n} & \\ k & 0 \end{pmatrix} & \times & \mathcal{M}\begin{pmatrix} \boxed{n} & \\ n & k \end{pmatrix} & \to & \mathcal{M}\begin{pmatrix} \boxed{2n} & \\ n+k & k \end{pmatrix} & \to & \mathcal{M}\begin{pmatrix} & \boxed{n} \\ k & k \end{pmatrix} \\ T^* \mathrm{Fl}(0, k; \mathbb{C}^n) & \times & T^* \mathrm{Fl}(k, n; \mathbb{C}^n) & & T^* \mathrm{Fl}(k, n+k; \mathbb{C}^{2n}) & & T^* \mathrm{Fl}(k, k; \mathbb{C}^n) \end{array}$$

(The middle dimension vector is the sum of the two on the left. The two arrows are yet to be discussed.)

This is looking good: the first space is $T^*(\mathrm{Gr}(k, \mathbb{C}^n)^2)$, the last is $T^*(\mathrm{Gr}(k, \mathbb{C}^n))$, and multiplication (our goal) is pullback along the diagonal inclusion $\mathrm{Gr}(k, \mathbb{C}^n) \hookrightarrow \mathrm{Gr}(k, \mathbb{C}^n)^2$. There is no natural *map* between their cotangent bundles, but there is a natural correspondence[6]—the conormal bundle to the graph of the diagonal inclusion. (It does not *quite* induce the multiplication map on $H^*(\mathrm{Gr}(k, \mathbb{C}^n))$; as explained in [39, LEMMA 11], this will be why we need introduce the denominator in our SSM classes.)

This suggests we attach correspondences to the two arrows in the diagram above, so as to make their composite the conormal bundle to the graph of the diagonal inclusion (or, more correctly, to its transpose). We have a good choice for the first arrow: a certain component of the stable envelope for the circle $S(z) := \mathrm{diag}(z^n, 1^n) \in \mathrm{GL}_{2n}(\mathbb{C})$. As an effect of working with the "leftmost" component in some sense—the gauge dimensions $k, 0$ in the left factor are smallest possible—the closure step in the definition of stable envelope may be skipped. Identifying each cotangent bundle with its Springer description (as in Theorem 6), this component of the envelope is the correspondence

$$\left\{ \begin{array}{c} (((A, W), (D, V)), (X, (W', V'))) : \\ X = \begin{bmatrix} A & 0 \\ * & D \end{bmatrix}, \begin{array}{l} W' = W \oplus 0 \\ V' = \mathbb{C}^n \oplus V \end{array} \end{array} \right\} \subseteq T^* \mathrm{Gr}(k, \mathbb{C}^n)^2 \times T^* \mathrm{Fl}(k, n+k; \mathbb{C}^n).$$

6 This is an example of a *Lagrangian* correspondence and, more specially, is a "conical Lagrangian correspondence" as it is invariant under the dilation action on the cotangent bundle. Although it is intriguing, we will not make any real use of this additional geometry.

The second correspondence is subtler in that it must break symmetry—the middle variety has a $\mathrm{GL}_{2n}(\mathbb{C})$-action, whereas the third has only a $\mathrm{GL}_n(\mathbb{C})$-action. We now draw inspiration from the $U_{\exp(\hbar)}(\mathfrak{sl}_3[z])$-equivariance of the maps $\mathbb{C}^3(y_i) \otimes \mathbb{C}^3(z_i) \to \mathrm{Alt}^2\, \mathbb{C}^3(c)$: according to the representation theory, these maps can only exist if $y_i = \hbar/2 + c = \hbar + z_i$ (otherwise the tensor product is either irreducible, or has only $\mathrm{Sym}^2(\mathbb{C}^3)$ as a quotient). Specializing the evaluation parameters is equivalent (thanks to Theorem 7) to specializing the equivariant parameters, which is equivalent to shrinking the group action. That clue helped suggest a correspondence that does the job:

$$\left\{ \begin{array}{l} (X, (W', V')), (((A', W''))) : \\ X = \begin{bmatrix} A & * \\ I_n & D \end{bmatrix}, \begin{array}{c} W'' = W'/(0 \oplus \mathbb{C}^n) \\ W'' = V' \cap (0 \oplus \mathbb{C}^n) \\ A' = A + D \end{array} \end{array} \right\} \subseteq T^* \mathrm{Fl}(k, n+k; \mathbb{C}^n) \times T^* \mathrm{Gr}(k, \mathbb{C}^n).$$

(Because of the I_n, the space of such X is only invariant under $\{[\begin{smallmatrix} M & 0 \\ 0 & M \end{smallmatrix}] : M \in \mathrm{GL}_n(\mathbb{C})\}$, which is how the symmetry is broken.)

Theorem 8 ([39, §7.6.2]). *The composite of these two correspondences is the conormal bundle to the transpose of the graph of the diagonal inclusion.*

The standard stable basis on $T^* \mathrm{Fl}(k, n+k; \mathbb{C}^n)$ does not interact well with the second correspondence; as will be explained in a moment, we need to change to the stable basis based on the Weyl chamber corresponding to the riffle–shuffle permutation $1\ 3\ 5\ \ldots\ 2n-1\ 2\ 4\ \ldots\ 2n$, of length $\binom{n}{2}$. Changing from one basis to another, as explained in Section 3.3, involves passing through $\binom{n}{2}$ walls and thereby composing $\binom{n}{2}$ many very sparse change-of-basis matrices. Puzzlewise this amounts to filling in the $\binom{n}{2}$ vertical rhombi.

The benefit of working in this second stable basis on $T^* \mathrm{Fl}(k, n+k; \mathbb{C}^n)$ is that under the map induced by the second arrow, each basis element maps either to 0 on $T^* \mathrm{Gr}(k, \mathbb{C}^n)$, or to a (fixed, rational-function) multiple of a basis element. The corresponding puzzle calculation amounts to filling in the n triangles at the bottom (if possible). Together, modulo an $\hbar \to \infty$ limit to be discussed in a moment, this is the calculation we did at the end of Section 2.2.

Theorem 9 ([39]). *The structure constants for multiplying equivariant SSM classes on $T^* \mathrm{Gr}(k, \mathbb{C}^n)$ can be computed with the puzzle pieces from Theorem 4, plus two new equivariant pieces, where the fugacities are derived from the entries of the R-matrix for $\mathbb{C}^3 \otimes \mathbb{C}^3$.*

The Schubert classes arise as a limit of the SSM classes, $S_\lambda = \lim_{\hbar \to \infty} \hbar^{\ell(\lambda)}\, \mathrm{SSM}_\lambda$. If one distributes powers of $\hbar$ carefully among the fugacities (essentially, conjugating the R-matrix with a diagonal matrix), one can then derive Theorem 3 as a limit of Theorem 4.

There is an analogous basis in K-theory, the "motivic Segre classes," and a similar theorem holds but we do not yet have a fully geometric proof. One should be available through upgrading the correspondences, from cycles to instead sheaves supported on those cycles.

4.1. $d = 2, 3, 4, \geq 5$

There are 8 labels in the $d = 2$ puzzle rule, so the R-matrix technology suggests we look for a group G with 8-dimensional representations $V_{r,g,b}$ and an intertwiner $V_r \otimes V_g \to V_b$. This (plus some extra weight-conservation considerations, spelled out in [38, §2.3–§2.6]) suggests $G = \mathrm{Spin}(8)$ acting on its three minuscule representations. The 3-fold symmetry of puzzles is then based on D_4's triality! It is worth noting that unlike at $d = 1$, the intermediate quiver variety is not a cotangent bundle (proof: its middle homology is not 1-dimensional). We are properly in quiver variety territory here.

A new phenomenon arises at $d = 3$ (based on the 27-dim reps of E_6): there is no way to distribute the powers of $\hbar$ so as to regularize the limit $\hbar \to \infty$ of the puzzle piece fugacities. One can just barely sidestep this, but only through giving up T-equivariance.

Another new phenomenon arises at $d = 4$: the representations involved (each the $(e_8 \oplus \mathbb{C})$-rep of $U_q(E_8[z^{\pm}])$) have a weight space of dimension > 1. Without a canonical choice of basis and dual basis, we can no longer guarantee that the dot products and fugacities all come out simultaneously positive. So there *is* a puzzle rule, but it is not positive.

At $d \geq 5$ there is still a natural choice of Cartan matrix and representation [38, §2], but both the Lie algebra and its representations are infinite-dimensional, and the rule will undoubtedly suffer the same lack of positivity as at $d = 4$.

The appearance of the Dynkin diagrams A_2, D_4, E_6, E_8 at $d = 1, 2, 3, 4$ is very suggestive of a connection to cluster algebras, as these are the types of the finite-type cluster varieties $\mathrm{Gr}(3, n)$ for $n = 5, 6, 7, 8$. (For $n \geq 9$ they, like our Lie algebras, are infinite type.)

5. FUTURE DIRECTIONS

Obviously there is a great deal of work left to do on the 2^6 problems from Section 2. In my personal estimation, the problems likely to have the most impact on/interaction with other fields are those around

- **C**otangent, for the connection to representation theory,
- **Q**uantum, for the connection to 2-d mirror symmetry,
- elliptic cohomology, for the connection to 3-d mirror symmetry, and
- H_* (affine Grassmannian), for the connection to the geometric Satake correspondence.

That admitted, my heart tells me to continue pursuing **F**lag manifolds.

FIGURE 2
The author (center) en route to the Nice ICM (see Figure 1) .

FUNDING

This work was partially supported by NSF grant 1953948.

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