RECENT PROGRESS TOWARDS HADWIGER'S CONJECTURE

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ABSTRACT

In 1943 Hadwiger conjectured that every graph with no K_t minor is $(t-1)$-colorable for every $t \geq 1$. Hadwiger's conjecture generalizes the Four Color Theorem and is among most studied problems in graph theory.

In this paper we survey the ideas behind recent progress towards this conjecture, which, in particular, allowed for the first asymptotic improvement since 1980s on the number of colors sufficient to color every graph with no K_t minor.

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1. INTRODUCTION

In 1852 Francis Guthrie (see, e.g., [33]) conjectured that every planar graph is four colorable. This Four Color Conjecture was the central driving force behind many of the developments in graph theory for over a hundred years. Eventually, it was proved in 1976 by Appel and Haken [2,3] and became the Four Color Theorem. Appel and Haken's proof is one of the first and most well-known examples of computer assisted proofs. To date there are no known proofs of the Four Color theorem that can be reasonably considered to be human readable, and a deeper reason behind it remains elusive.

If true, the following famous conjecture made by Hadwiger [17] in 1943 points to such a reason. Its statement eliminates the topological component present in the Four Color Theorem's statement and instead involves graph minors.

Given graphs H and G, we say that G *has an* H *minor* or H *is a minor of* G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. We denote the complete graph on t vertices by K_t.

Conjecture 1.1 (Hadwiger's conjecture [17]). *For every integer* $t \geq 1$, *every graph with no* K_t *minor is* $(t-1)$*-colorable.*

We refer the reader to a comprehensive survey by Seymour [51] for the detailed history of the conjecture, and only present the background necessary to motivate the discussion of recent progress on two particular weakenings of the conjecture that we focus on here.

Hadwiger [17] and Dirac [10] independently showed that Conjecture 1.1 holds for $t \leq 4$. As the class of planar graphs is closed under taking minors and the complete graph K_5 is not planar, the case $t = 5$ of Hadwiger's conjecture implies the Four Color Theorem. In fact, Wagner already shown in 1937 that this case is equivalent to the Four Color Theorem. Robertson, Seymour, and Thomas [49] went one step further and proved Hadwiger's conjecture for $t = 6$, also by reducing it to the Four Color Theorem. Settling the conjecture exactly for $t \geq 7$ appears to be extremely challenging, in part due to the aforementioned absence of a transparent proof of the Four Color Theorem.

Another notable challenging case of Hadwiger's conjecture is the case of graphs with no independent set of size three. If G is such a graph on n vertices then properly coloring G requires at least $n/2$ colors, and so Hadwiger's conjecture implies that G has a $K_{\lceil n/2 \rceil}$ minor. This is still open. In fact, as mentioned in [51], it is not known whether there exists any $c > 1/3$ such that every graph G as above has a K_t minor for some $t \geq cn$.

The following natural weakening of Hadwiger's conjecture, which has been considered by several researchers, sidesteps the above challenges.

Conjecture 1.2 (Linear Hadwiger's conjecture [22,23,47]). *There exists* $C > 0$ *such that for every integer* $t \geq 1$, *every graph with no* K_t *minor is* Ct*-colorable.*

Until recently the best bound on the number of colors needed to color the graphs with no K_t minor was $O(t\sqrt{\log t})$, obtained independently by Kostochka [26,27] and Thomason [54] in the 1980s. The only improvement since then [24,55,59] and until the last two years since then has been in the constant factor. In the last two years, however, using in part the methods we survey here this bound has been improved, first, by Postle, Song, and I [40] to $O(t(\log t)^{\beta})$ for every $\beta > 1/4$, then by Postle [46] to $O(t(\log\log t)^6)$ and, very recently by Delcourt and Postle [9] to $O(t \log\log t)$. We sketch the proof of the first of these bounds here.

Investigation of another series of weakening of Hadwiger's conjecture has been proposed more recently by Seymour.

Conjecture 1.3 (H-Hadwiger's conjecture [50,51]). *For every graph H on t vertices, every graph with no H minor is $(t-1)$-colorable.*

Note that the bound on the number of colors in Conjecture 1.3 is tight for every graph H on t vertices, as K_{t-1} has no H minor and requires $t-1$ colors to properly color. Until recently, Conjecture 1.3 was only verified for a few very structured families of graphs H. As noted by Seymour [50], Conjecture 1.3 holds if H is a tree, and Kostochka [27] proved that Conjecture 1.3 holds for $H = K_{s,t}$ which is a sufficiently unbalanced complete bipartite graph, i.e., $t \geq C(s \log s)^3$ for some constant C. Using the methods surveyed in this paper, Turcotte and I [43] recently proved Conjecture 1.3 for a fairly large class of structurally sparse bipartite graphs H, and we present the sketch of our arguments in this survey.

We overview the main tools behind the above mentioned recent progress towards Conjectures 1.2 and 1.3, which mainly relies on the interplay between the very basic parameters of the graph G with no H minor, namely

- $\mathsf{v}(G)$—the number of vertices of G,
- $\mathsf{e}(G)$—the number of edges of G,
- $\mathsf{d}(G) = \mathsf{e}(G)/\mathsf{v}(G)$—the *density* of G,
- $\chi(G)$—the *chromatic number* of G, that is the minimum positive integer r such that G is r-colorable,
- $\kappa(G)$—the *connectivity* of G, the maximum positive integer $k < \mathsf{v}(G)$ such that G remains connected after deleting any set of fewer than k vertices.

The rest of the paper is structured as follows. In Section 2 we present the basic tools relating connectivity, density, and chromatic number of graphs with no K_t minor (and more generally, for graphs in classes closed under taking minors). In Section 3 we survey known bounds on density of graphs with no H minors. In Section 4, we present the crucial tool behind the recent progress—the density increment theorem, which is used to locate small dense subgraphs in large graphs without a dense minor. Using this theorem, one can build the minors of the graph under consideration by combining smaller pieces found in the dense subgraphs. This procedure is described in Section 5. Finally, in Section 6 we sketch how the presented tools are combined to obtain the results of [40] and [43] mentioned above.

2. BASIC DEPENDENCIES

For both Conjectures 1.2 and 1.3, it is enough to investigate *contraction critical* graphs G, that is, graphs G such that $\chi(H) < \chi(G)$ for every minor H of G unless H is isomorphic to G.

The basic relationship between density and the chromatic number of contraction critical graphs is given by the following standard *degeneracy* argument. Let G be a contraction critical graph, and let $t = \chi(G)$. If $\deg(v) \leq t-2$ for some $v \in V(G)$, then we have $\chi(G \setminus v) \leq t-1$ and any $(t-1)$-coloring of $G \setminus v$ can be extended to v, a contradiction. Thus every vertex of G has degree at least $t-1$ and so

$$\mathsf{d}(G) \leq \frac{\chi(G)-1}{2}, \tag{2.1}$$

by averaging.

The following harder theorem of Kawarabayshi [22] guarantees that the connectivity of every contraction critical graph is also linear in the chromatic number.

Theorem 2.1 ([22]). *If G is a contraction critical graph, then*

$$\kappa(G) \geq \frac{2}{27}\chi(G).$$

In the more technical arguments, one works with subgraphs of contraction critical graphs, which are not by themselves contraction critical. To be useful for building the minors, we need these subgraphs to be highly-connected. The following classical result of Mader allows us to gain connectivity without losing to much density.

Theorem 2.2 ([36]). *Every graph G contains a subgraph G' such that $\kappa(G') \geq \mathsf{d}(G)/2$.*

We frequently want to additionally guarantee that by passing to the highly-connected subgraph or minor, we do not reduce the chromatic number excessively. This is possible due to a recent theorem of Girao and Narayanan [16].

Theorem 2.3 ([16]). *For every positive integer k, every graph G with $\chi(G) \geq 7k$ contains a subgraph G' such that $\kappa(G') \geq k$ and $\chi(G') \geq \chi(G) - 3k$.*

Finally, for small graphs G, we have another tool, the following classical bound due to Duchet and Meyniel [12], on the independence number of graphs with no K_t minor.

The set $X \subseteq V(G)$ is *independent* in G if no pair of vertices of X are adjacent. The *independence number* $\alpha(G)$ of a graph G is the maximum size of an independent set in G.

Theorem 2.4 ([12]). *For every $t \geq 2$, every graph G with no K_t minor has an independent set of size at least $\frac{\mathsf{v}(G)}{2(t-1)}$.*

Theorem 2.4 implies that every graph with no K_t minor contains a t-colorable subgraph on a constant proportion of vertices. Woodall [60] proved the following stronger result, which as observed by Seymour [51] also follows from the proof of Theorem 2.4 in [12].

Theorem 2.5 ([60]). *Let G be a graph with no K_t minor. Then there exists $X \subseteq V(G)$ with $|X| \geq \frac{\mathsf{v}(G)}{2}$ such that $\chi(G[X]) \leq t-1$.*

Theorem 2.5 straightforwardly implies the following bound on the chromatic number of graphs with no K_t minor.

Corollary 2.6. *Let G be a graph with no K_t minor. Then*

$$\chi(G) \le \left(\log_2\left(\frac{\mathsf{v}(G)}{t}\right) + 2\right)t.$$

3. DENSITY

Until recently the best bounds on the chromatic number of graph with no K_t minor for large t relied exclusively on the degeneracy bound (2.1). To determine the optimum bounds that can be obtained in this manner towards Conjecture 1.3, we investigate the maximum density of graphs G with no H minor for a fixed H.

More formally, following [38], for a graph H with $\mathsf{v}(H) \ge 2$, we define the *extremal function* $c(H)$ of H as the supremum of $\mathsf{d}(G)$ taken over all nonnull graphs G not containing H as a minor.

Mader [34] proved that $c(H)$ is finite for every graph H. The exact value has been determined for various small graphs H. For example, if K_t is the complete graph on $t \le 9$ vertices, then $c(K_t) = t - 2$ (see [11,21,35,52]); and if P is the Petersen graph, then $c(P) = 5$ (see [20]). We primarily focus on asymptotic results for classes of graphs H.

The asymptotic behavior of $c(K_t)$ was studied in [26,27,54], and was determined precisely by Thomason [55], who showed that

$$c(K_t) = \big(\lambda + o(1)\big)t\sqrt{\log t}, \tag{3.1}$$

where

$$\lambda = \max_{\alpha>0} \frac{1-e^{-\alpha}}{2\sqrt{\alpha}} = 0.319\ldots$$

Improving on results of [38,48], Thomason and Wales [56] recently extended the upper bound from (3.1) to general graphs, by showing that for every graph H,

$$c(H) \le \big(\lambda + o_{\mathsf{d}(H)}(1)\big)\,\mathsf{v}(H)\sqrt{\log \mathsf{d}(H)}. \tag{3.2}$$

The inequality (3.2) is tight in many regimes. Myers and Thomason [38] showed that it is tight (up to the choice of the error term) for almost all graphs with n vertices and $n^{1+\varepsilon}$ edges for every fixed $\varepsilon > 0$, and for all regular graphs with these parameters. They also gave an explicit asymptotic formula for $c(H)$ for all such polynomially dense graphs.

Reed, Thomason, Wood, and I [41] recently showed that (3.2) is also tight for almost all regular graphs of constant density, that is, for almost all d-regular graphs H,

$$c(H) \ge \big(\lambda - o_d(1)\big)\,\mathsf{v}(H)\sqrt{\log d}. \tag{3.3}$$

However, for several concrete sparse families, the extremal function behaves qualitatively differently:

- Chudnovsky, Reed, and Seymour [7] proved that $c(K_{2,t}) = \frac{t+1}{2}$ for all $t \geq 2$;
- Kostochka and Prince [28] proved that $c(K_{3,t}) = \frac{t+3}{2}$ for all $t \geq 6300$;
- More generally, Myers [37] considered the asymptotic behavior of $c(K_{s,t})$ for fixed s and t and conjectured that $c(K_{s,t}) \leq c_s t$ for some constant independent on t. Kühn and Osthus [32] and Kostochka and Prince [25] independently proved this conjecture by showing that $c(K_{s,t}) = (\frac{1}{2} + o_s(1))t$.
- Csóka et al. [8] proved that if H is a disjoint union of cycles, then

$$c(H) \leq \frac{\mathsf{v}(H) + \mathrm{comp}(H)}{2} - 1,$$

which is tight whenever every component of H is an odd cycle.

All of the above families are structurally sparse and the extremal function is linear in the number of vertices. (In fact, $c(H) < (1 + o(1))\,\mathsf{v}(H)$ for all these graphs.)

This property generalizes to the large and well-studied class of sparse graph families defined as follows. A graph family is *monotone* if it is closed under taking subgraphs. A *separation* of a graph G is a pair (A_1, A_2) of subsets of $V(G)$ such that $G = G[A_1] \cup G[A_2]$ and $A_1 \setminus A_2 \neq \emptyset$ and $A_2 \setminus A_1 \neq \emptyset$. A separation (A_1, A_2) has *order* $|A_1 \cap A_2|$. A separation (A_1, A_2) is *balanced* if $|A_1|, |A_2| \geq \frac{\mathsf{v}(G)}{3}$. A graph family $\mathcal{F}$ admits *strongly sublinear separators* (written $\mathcal{F}$ is *s.s.s.*, for brevity) if $\mathcal{F}$ is monotone, and there exist $\beta < 1$ and $c > 0$ such that every graph $G \in \mathcal{F}$ has a balanced separation of order at most $c\,\mathsf{v}(G)^{\beta}$. For example, every *proper minor-closed family* (a family that is closed under isomorphisms and taking minors, and does not include all graphs) is s.s.s., as proved by [1] with $\beta = \frac{1}{2}$. More generally, every family with polynomial expansion is s.s.s. [13].

Before formally stating the general asymptotic bound on the extremal function of graphs in s.s.s. graph families, we describe two natural lower bounds on $c(H)$. First, since H is not a minor of $K_{\mathsf{v}(H)-1}$,

$$c(H) \geq \mathsf{d}(K_{\mathsf{v}(H)-1}) = \frac{\mathsf{v}(H)}{2} - 1. \tag{3.4}$$

A *vertex cover* of H is a set $S \subseteq V(H)$ such that $H - S$ has no edges. Let $\tau(H)$ be the minimum size of a vertex cover of H. For the second bound, observe that $\tau(H) \leq \tau(G)$ whenever H is a minor of G. It follows that H is not a minor of the complete bipartite graph $K_{\tau(H)-1,n}$ for any n and

$$c(H) \geq \lim_{n \to \infty} \mathsf{d}(K_{\tau(H)-1,n}) = \tau(H) - 1. \tag{3.5}$$

Hendrey, Wood, and I [19] have recently shown that the lower bounds (3.4) and (3.5) are asymptotically tight for 4-colorable graphs in s.s.s. families, strengthening the result of Haslegrave, Kim, and Liu [18] for bipartite graphs. The resulting density theorem below is one of the main tools used in the recent progress towards Conjecture 1.3 discussed below.

Theorem 3.1. *For every s.s.s. family $\mathcal{F}$ and for every $H \in \mathcal{F}$ with $\chi(H) \le 4$,*

$$c(H) = \big(1 + o_{\mathcal{F}}(1)\big) \cdot \max\left(\frac{\mathsf{v}(H)}{2}, \tau(H)\right), \tag{3.6}$$

where the error term $o_{\mathcal{F}}(1)$ depends on $\mathcal{F}$ and satisfies $o_{\mathcal{F}}(1) \to 0$ as $\mathsf{v}(H) \to \infty$.

Finally, we mention the following tight bound on density of unbalanced bipartite graph without a K_t minor, due to Postle and I. It is used to supplement the density increment arguments presented in the next section

Theorem 3.2 ([39]). *There exists $C > 0$ such that, for every $t \ge 3$ and every bipartite graph G with bipartition (A, B) and no K_t minor, we have*

$$\mathsf{e}(G) \le Ct\sqrt{\log t}\sqrt{|A||B|} + (t-2)\mathsf{v}(G). \tag{3.7}$$

4. DENSITY INCREMENT

Perhaps the most important new ingredient in the recent progress towards Conjectures 1.2 and 1.3 is a density increment argument, which informally says that every graph either contains a substantially denser minor, or a small subgraph with density not much smaller than that of the whole graph.

Let us proceed by giving a more detailed motivation. By (3.1) there exists $D = O(t\sqrt{\log t})$ such that every graph G with density $\mathsf{d}(G) \ge D$ has a K_t minor. For a graph G with smaller density, one might still hope to guarantee a K_t minor by finding a minor H of G with $\mathsf{d}(H) \ge D$. Thus we are interested, for given d and D, in properties of graphs G of density $\mathsf{d}(G) = d$ and no minor of density D.

One family of obstructions are graphs G which simply do not have enough edges. As every graph of density D has at least D^2 edges, if G has a minor of density D, we must have $D^2 \le \mathsf{e}(G) = d \cdot \mathsf{v}(G)$. It follows that all the graphs G with $\mathsf{v}(G) < D^2/d$ are among the obstructions to our approach. One can obtain further obstructions by taking disjoint union of such graphs, and, more generally, by gluing smaller obstructions along small sets in a "tree-like fashion." However, the graphs obtained in this way contain a subgraph with at most D^2/d vertices and density close to d.

A series of density increment results culminating in the following result by Wang [57] shows that a similar subgraph can be found in every obstruction.

Theorem 4.1 ([57]). *There exists $C > 0$ satisfying the following. Let $D > 0$ be real, G be a graph with $\mathsf{d}(G) \ge C$, and let $s = D/\mathsf{d}(G)$. Then G contains at least one of the following:*

(i) *a minor J with $\mathsf{d}(J) \ge D$, or*

(ii) *a subgraph H with $\mathsf{v}(H) \le g(s)D^2/\mathsf{d}(G)$ and $\mathsf{d}(H) \ge \mathsf{d}(G)/g(s)$,*

where $g(s) = C(1 + \log s)^5$.

The first variant of Theorem 4.1 was proved by Song and I [42] with $g(s) = Cs^\alpha$ for a particular constant α. The magnitude of $g(s)$ was subsequently improved by Postle, first in [44] to $g(s) = o(s^\delta)$ for every $\delta > 0$, then in [45] in $g(s) = C(1 + \log s)^6$, and, finally, by Wang [57] to the bound stated in Theorem 4.1.

A similar theorem with narrower scope of application, but stronger bounds, was very recently obtained, using Theorem 3.2, by Delcourt and Postle [9].

Theorem 4.2 ([9]). *There exists $C > 0$ satisfying the following. Let $t \geq 1$ be an integer and let G be a graph with $\mathsf{d}(G) \geq Ct$. Then G contains at least one of the following:*

(i) *a K_t minor, or*

(ii) *a subgraph H with $\mathsf{v}(H) \leq Ct \log^3 t$ and $\mathsf{d}(H) \geq Ct$,*

We finish this section with an example of application of Theorem 4.1 due to Postle, Song, and I [40].

For a pair of graphs G and H, we say that G is *H-free* if no subgraph of G is isomorphic to H. The next theorem due to Kühn and Osthus [31] shows that H-free graphs have exceptionally dense minors for every complete bipartite graph H.

Theorem 4.3 ([31]). *For every integer $s \geq 2$, every $K_{s,s}$-free graph G has a minor J with*

$$\mathsf{d}(J) \geq \big(\mathsf{d}(G)\big)^{1+\frac{1}{2(s-1)}-o_{\mathsf{d}(G)}(1)}. \tag{4.1}$$

Krivelevich and Sudakov [29] tightened (4.1) to $\mathsf{d}(J) \geq c_s(\mathsf{d}(G))^{1+\frac{1}{s-1}}$ for some $c_s > 0$ independent of $\mathsf{d}(G)$. They also proved the following, strengthening a result of Kühn and Osthus [30].

Theorem 4.4 ([29]). *For every integer $k \geq 2$, there exists $c_k > 0$ such that every C_{2k}-free G has a minor J with*

$$\mathsf{d}(J) \geq c_k\big(\mathsf{d}(G)\big)^{\frac{k+1}{2}}.$$

The exponents appearing in Theorems 4.3 and 4.4 cannot be improved, subject to well known conjectures on the Turán numbers of $K_{s,s}$ and C_{2k}, which we mention below.

In this section we use Theorem 4.1 to extend Theorems 4.3 and 4.4 to general bipartite graphs. Stating our result requires a couple of definitions. The *Turán number* $\mathrm{ex}(n, H)$ of a graph H with $\mathsf{e}(H) \neq 0$ is the maximum number of edges in an H-free graph G with $\mathsf{v}(G) = n$. The *Turán exponent* $\gamma(H)$ of a graph H with $\mathsf{e}(H) \geq 2$ is defined as

$$\gamma(H) := \limsup_{n\to\infty} \frac{\log \mathrm{ex}(n, H)}{\log n}.$$

Many fundamental questions about Turán exponents of bipartite graphs remain open. In particular, a famous conjecture of Erdős and Simonovits (see [15, CONJECTURE 1.6]) states that $\gamma(H)$ is rational for every graph H, and that $\lim_{n\to\infty} \mathrm{ex}(n, H)/n^{\gamma(H)}$ exists and is positive. We refer the reader to a comprehensive survey by Füredi and Simonovits [15] for further background.

Theorem 4.1 implies an essentially tight analogue of Theorems 4.3 and 4.4 for H-free graphs G for general bipartite H.

Theorem 4.5 ([40]). *For every bipartite graph H with $\gamma(H) > 1$, every H-free graph G has a minor J with*

$$\mathsf{d}(J) \geq \big(\mathsf{d}(G)\big)^{\frac{\gamma(H)}{2(\gamma(H)-1)} - o_{\mathsf{d}(G)}(1)}.$$

5. BUILDING THE MINORS

As mentioned earlier, we build the required minors from pieces. Describing how the minors combine together is more convenient in the language of models. A *model μ of a graph H in a graph G* assigns to every vertex of $v \in V(H)$ a set $\mu(v)$ of vertices of G such that

- $\mu(u) \cap \mu(v) = \emptyset$ for every pair of distinct $u, v \in V(H)$,
- the subgraph $G[\mu(v)]$ of G induced by $\mu(v)$ is connected for every $v \in V(H)$, and
- for every edge $uv \in E(H)$ there exist $u' \in \mu(u)$ and $v' \in \mu(v)$ such that $u'v' \in E(G)$.

It is well known and not hard to see that G has an H minor if and only if there exists a model of H in G.

Given an injection $\phi : V(H) \to V(G)$, we say that a model μ of H in G is *ϕ-rooted* if $\phi(v) \in \mu(v)$ for every $v \in V(H)$. Finally, we say that G is *H-linked* if $\mathsf{v}(G) \geq \mathsf{v}(H)$ and for every injection $\phi : V(H) \to V(G)$ there exists a ϕ-rooted model of H in G. Thus every H-linked graph has an H minor, but the converse does not hold.

The case when H is a matching of size k, i.e., $H = kK_2$ is of particular interest. Note that a graph G is kK_2-linked, if and only if $\mathsf{v}(G) \geq 2k$ and, for every collection of distinct $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k \in V(G)$, there exist pairwise vertex disjoint paths $P_1, \ldots, P_k$ such that P_i has ends s_i and t_i for every $i \in [k]$. We will write *k-linked* instead of kK_2-linked for brevity. (Our definition coincides with the standard definition of k-linked graphs.)

The following theorem of Thomas and Wollan [53], improving an earlier result of Bollobás and Thomason [4], ensures that connectivity linear in k is sufficient to guarantee that the graph is k-linked.

Theorem 5.1 ([53]). *For every integer $k \geq 1$, every graph G with $\kappa(G) \geq 10k$ is k-linked.*

Connectivity linear in t is certainly insufficient to guarantee that a graph is K_t-linked, but interestingly, as observed by Delcourt and Postle [9], connectivity linear in t together with a slightly larger complete minor is sufficient.

Lemma 5.2 ([9]). *For every integer $t \geq 1$, every graph G with $\kappa(G) \geq t$ that has a $K_{\lceil 5t/2 \rceil}$ minor is K_t-linked.*

We are further interested in a more general setting, where we need to find a rooted model of a disjoint union H' of a given graph H and a matching of size k. If a graph G is

H'-linked for such an H' then we write that G is $(H+k)$*-linked* for brevity. The following theorem can be easily derived from the results of Wollan [58].

Theorem 5.3 ([58]). *There exists $C > 0$ satisfying the following. Let H and G be graphs and let $k \geq 0$ be an integer. If*

$$\kappa(G) \geq C\big(c(H)+k\big)$$

then G is $(H+k)$-linked.

Our final tool describes the conditions under which we can glue a larger minor from small pieces in a highly connected graph. Let $H_1, H_2, \ldots, H_s$ be graphs and let $H = H_1 \cup H_2 \cup \cdots \cup H_s$. Then we say that $\{H_i\}_{i\in[s]}$ is *a decomposition* of H with *excess* $(\sum_{i\in[s]} \mathsf{v}(H_i)) - \mathsf{v}(H)$.

Theorem 5.4 ([43]). *There exists $C > 1$ satisfying the following. Let $\{H_i\}_{i\in[s]}$ is a decomposition of a graph H with excess k, let G be a graph and let $G_1, \ldots, G_s$ be pairwise vertex disjoint subgraphs of G. If*

- *G_i is $(H_i + k)$-linked for every $i \in [s]$, and*
- *G is k-linked,*

then G has an H minor.

6. BRINGING IT ALL TOGETHER

Having introduced the necessary toolkit in the preceding section, let us describe how combining them we can progress forward.

We start by sketching a proof of the following theorem by Postle and I [39].

Theorem 6.1 ([39]). *For every $\beta > \frac{1}{4}$, if G is a graph with $\kappa(G) = \Omega(t(\log t)^{\beta})$ no K_t minor then $\mathsf{v}(G) = O(t(\log t)^{7/4})$.*

Note that using Theorem 2.1 and Corollary 2.6, we immediately obtain from Theorem 6.1 the following bound on the chromatic number of graphs with no K_t minors, originally proved in [40].

Theorem 6.2 ([40]). *For every $\beta > \frac{1}{4}$, if G is a graph with no K_t minor then $\chi(G) = O(t(\log t)^{\beta})$.*

Proof sketch of Theorem 6.1. Note that there exists a decomposition of K_t into $O((\log t)^{1/4})$ complete subgraphs $H_1, H_2, \ldots, H_s$ of excess $k = O(t(\log t)^{1/4})$ such that $s = O((\log t)^{1/2})$ and $\mathsf{v}(H_i) = O(t/(\log t)^{1/4})$ for every $i \in [s]$.

Assume that $\mathsf{v}(G) = \Omega(t(\log t)^{7/4})$. By Theorem 5.4, it suffices to find vertex disjoint subgraphs $G_1, \ldots, G_s$ of G such that G_i is $(H_i + k)$-linked for every $i \in [s]$, as G is k-linked by Theorem 5.1 and the lower bound on $\kappa(G)$. By (3.1), we have $c(H_i) = O(t(\log t)^{1/4})$ and so, by Theorem 5.3, it suffices to guarantee that

$\kappa(G_i) \geq Ct(\log t)^{1/4})$ for some C' independent on t. By Theorem 2.3, we can further relax this condition to $\mathsf{d}(G_i) \geq Ct(\log t)^{1/4})$ (possibly changing C).

Finally, to find the required G_i's, select the maximum collection $G_1, \dots, G_{s'}$ of vertex disjoint subgraphs of G such that $\mathsf{d}(G_i) \geq Ct(\log t)^{1/4}$ and $\mathsf{v}(G_i) \leq t(\log t)^{3/4}$. Assume for a contradiction that $s' < s$. Let $A = \bigcup_{i \in [s']}(V(G_i))$, then $|A| = Ct(\log t)^{5/4}$. Let $B = V(G) - A$. Then $|B| = (1 - o(1))\mathsf{v}(G)$. By Theorem 3.2, there are $O(t^2(\log t)^{9/8}\sqrt{\mathsf{v}(G)})$ edges joining A and B, and so as the minimum degree of G is $\Omega(t(\log t)^{\beta})$, and

$$t^2(\log t)^{9/8}\sqrt{\mathsf{v}(G)} = o\big(t(\log t)^{\beta}\mathsf{v}(G)\big),$$

the average degree of the subgraph $G[B]$ of G induced by B is still $\Omega(t(\log t)^{\beta})$. As $G[B]$ has no K_t minor, applying Theorem 4.1 with $D = c(K_t)$ to $G[B]$, we conclude that G contains a subgraph $G_{s'+1}$ with $\mathsf{v}(G_{s'+1}) = O(t(\log t)^{1-\beta}(\log\log t)^5)$ and $\mathsf{d}(G_{s'+1}) = \Omega(t(\log t)^{\beta}/(\log\log t)^5)$, contradicting the choice of s'. ■

Secondly, let us outline the proof of the following recent theorem due to Turcotte and I **[43]**.

Theorem 6.3 (**[43]**). *For every s.s.s. graph family $\mathcal{F}$ and every positive integer Δ, there exists N such that for every bipartite graph $H \in \mathcal{F}$ with $\Delta(H) \leq \Delta$ and $\mathsf{v}(H) \geq N$, every graph G with $\chi(G) \geq \mathsf{v}(H)$ has an H minor. (That is, Conjecture 1.3 holds for H.)*

To prepare for the proof of Theorem 6.3, we need to introduce a couple of final tools from the literature. The first is the well-known "bandwidth theorem" of Böttcher, Schachts, and Taraz **[6]**, which using the results of **[5]** can be adapted to our setting, to imply the following.

Theorem 6.4 (**[5,6]**). *For every s.s.s. graph family $\mathcal{F}$, every positive integer Δ, every $\gamma > 0$, and for every bipartite graph $H \in \mathcal{F}$ with $\Delta(H) \leq \Delta$ and $\mathsf{v}(H) \geq N$, if G is a graph such that $\mathsf{v}(G) \geq \mathsf{v}(H)$ and $\deg(v) \geq (1+\gamma)\frac{\mathsf{v}(G)}{2}$ for every $v \in V(G)$ then G contains a subgraph isomorphic to H.*

The second is a fairly straightforward lemma, present, in particular, in **[14]**.

Lemma 6.5 (**[14]**). *Let $\mathcal{F}$ be s.s.s. graph family. Then for every $\varepsilon > 0$ there exists C such that every graph $G \in \mathcal{F}$ admit a decomposition into subgraphs on at most C vertices with excess at most $\varepsilon\mathsf{v}(G)$.*

We are now ready to sketch the proof of Theorem 6.3 using our toolkit.

Proof sketch of Theorem 6.3. By Theorem 2.1, we may assume that $\kappa(G) \geq 2/27 \cdot \mathsf{v}(H)$. By the argument in Section 2, we may further assume $\deg(v) \geq \mathsf{v}(H) - 1$ for every $v \in V(G)$. In particular, $\mathsf{d}(G) \geq (\mathsf{v}(H) - 1)/2$.

If $\mathsf{v}(G) \leq 3/2 \cdot \mathsf{v}(H)$ then, assuming N is chosen to be appropriately large, G contains a subgraph isomorphic to H by Theorem 6.4.

Assume next that $\mathsf{v}(G) \leq K\dot{\mathsf{v}}(H)$ for some sufficiently large K dependent only on the constant in the preceding theorems. By Theorem 3.1, $c(H) = (1+o(1))\mathsf{v}(H)/2$. Thus by Theorem 4.1 applied with $D = c(H)$, G contains a subgraph G_1 with $\mathsf{v}(G_1) \leq C\mathsf{v}(H)$ and $\mathsf{d}(G_1) \geq \mathsf{v}(H)/C$ for some constant C. In fact, if K is sufficiently large compared to k and C, we can find vertex disjoint subgraphs $G_1, \ldots, G_k$ of G with the same properties, using a variant of the argument used for a similar purpose in the proof sketch of Theorem 6.1 above. By Lemma 6.5, for any $\varepsilon > 0$ there exists a decomposition of H with excess $k' \leq \varepsilon\mathsf{v}(H)$ into subgraphs $H_1, \ldots, H_k$ such that $(1-\varepsilon)\mathsf{v}(H)/k \geq \mathsf{v}(H_i) \leq (1+\varepsilon)\mathsf{v}(H)/k$, as long as $\mathsf{v}(H)$ is large enough as a function of k and ε. Applying Theorem 3.1 to H_i this time, we have $c(H_i) \leq \mathsf{v}(H)/k$, and so G_i is $(H_i + k')$-linked for every $i \in [k]$ by Theorem 5.3, as long as ε is sufficiently small and k sufficiently large compared to C. By Theorem 5.4, it now follows that H is a minor of G as desired.

It remains to consider the case $3/2 \cdot \mathsf{v}(H) \leq \mathsf{v}(G) \leq K \cdot \mathsf{v}(H)$. This regime is somewhat complicated and we do not present all the details. We consider a decomposition $H_1, \ldots, H_k$ of H with excess at most $\varepsilon\mathsf{v}(H)$ such that $\mathsf{v}(H_i) \leq C$ for every $i \in [k]$ and excess at most $\varepsilon\mathsf{v}(H)$, where k is no longer constant, but C is. We reserve a randomly chosen $Z \subseteq V(G)$ with $\varepsilon\mathsf{v}(H) \ll |Z| \ll \mathsf{v}(G)$ for future use. Next, we choose l maximum such that the disjoint union of graphs $H_1, H_2, \ldots, H_l$ is isomorphic to a subgraph G' of $G \setminus Z$. If $l < k$, by further choosing G' such that $G[V(G')]$ is as sparse as possible, we guarantee that $G \setminus V(G') \setminus Z$ is dense enough to contain a subgraph isomorphic to H_{l+1} by Theorem 4.5, which is a contradiction, implying $l = k$. The subgraphs $H_1, H_2, \ldots, H_k$ can now be linked together to obtain the H minor by using Z. ∎

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