RESTRICTED PROBLEMS IN EXTREMAL COMBINATORICS

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ABSTRACT

Extremal combinatorics is a central research area in discrete mathematics. The field can be traced back to the work of Turán and it was established by Erdős through his fundamental contributions and his uncounted guiding questions. Since then it has grown into an important discipline with strong ties to other mathematical areas such as theoretical computer science, number theory, and ergodic theory.

We focus on extremal problems for *hypergraphs*, which were introduced by Turán. After solving the analogous question for graphs, Turán asked to determine the maximum cardinality of a set E of 3-element subsets of a given n-element set V such that for any 4 elements of V at least one triple is missing in E. This innocent looking problem is still open and, despite a great deal of effort over the last 80 years, our knowledge is still somewhat limited. We consider a variant of the problem by imposing additional restrictions on the distribution of the 3-element subsets in E. These additional assumptions yield a finer control over the corresponding extremal problem. In fact, this leads to many interesting and more manageable problems, some of which were already considered by Erdős and Sós in the 1980s. The additional assumptions on the distribution of the 3-element subsets are closely related to the theory of *quasirandom discrete structures*, which was pioneered by Szemerédi and became a central theme in the field. In fact, the hypergraph extensions by Gowers and by Rödl et al. of the *regularity lemma* provide essential tools for this line of research.

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1. INTRODUCTION

Extremal and probabilistic combinatorics is an important area of discrete mathematics with strong ties to Ramsey theory, random graph theory, number theory, theoretical computer science, and ergodic theory. This branch and those connections are central in discrete mathematics and have seen strong developments in the last few decades.

A prime example in that direction is Szemerédi's celebrated density theorem on arithmetic progressions [47]. Its connection to extremal problems for graphs and hypergraphs, provided by the *removal lemma*, was the source for some of the most important developments in the field and led to powerful techniques in extremal combinatorics, which include *Szemerédi's regularity lemma* for graphs [48], its extensions to hypergraphs due to Gowers [26] and Rödl et al. [33, 44], the systematic study of *quasirandom discrete structures* by Thomason [49, 50] and Chung, Graham, and Wilson [8], and the notion of *limits of sequences of graphs and hypergraphs* pioneered by Lovász and Szegedy [9, 30, 31]. Moreover, Szemerédi's theorem has interesting connections to *ergodic theory* (established by Furstenberg and his collaborators [20–22]), to *harmonic analysis* (see, e.g., the work of Roth [45, 46], Gowers [25], Bourgain [6], and others), and to *number theory* (see, e.g., the Green–Tao theorem [27]).

Pivotal in those works was the understanding of suitable notions of *quasirandomness* of discrete structures. A quasirandom structure resembles a truly random object by sharing significant properties with it. The systematic study of quasirandom graphs was initiated by Thomason [49, 50] and Chung, Graham, and Wilson [8]. Those authors considered sequences of deterministic (finite) graphs $G_n = (V_n, E_n)$ with the number of vertices $|V_n|$ tending to infinity with n and with *density* $|E_n|/\binom{|V_n|}{2}$ close to some constant $p \in [0, 1]$. Such a sequence of graphs is quasirandom if it shares some important properties with the binomial random graph $G(n, p)$ of the same density, i.e., G_n has some of the significant properties which hold for $G(n, p)$ with high probability. One of the key properties of $G(n, p)$ is its uniform edge distribution, and Thomason chose a quantitative version of it to define quasirandom graphs. The Chung–Graham–Wilson theorem established a deterministic equivalence between the uniform edge distribution and several other significant properties, including large spectral gap for the eigenvalues of the adjacency matrix, the number of cycles of length four appearing as a subgraph, and the expected number of copies of subgraphs of any fixed isomorphism type.

We consider extremal problems for uniform hypergraphs. The classical extremal problem for hypergraphs, already posed by Turán [51] about 80 years ago, turned out to be notoriously hard and, despite a great deal of effort, our current knowledge is still somewhat limited. We investigate a variant of the classical problem by imposing additional restrictions on the distribution of the hyperedges. Roughly speaking, we shall consider *uniformly dense* hypergraphs, i.e., hypergraphs which induce on large sets of vertices at least a given edge density. This additional assumption yields a better control over the corresponding extremal problem. This leads to many interesting and sometimes more manageable subproblems, some of which were already considered by Erdős and Sós [12, 15]. In particular, those additional assumptions on the hyperedge distribution are closely related to the theory of quasirandom hypergraphs and make these problems amenable to the regularity method for hypergraphs. Extremal problems of this type were investigated in [4, 7, 23, 24, 32, 37–43].

2. EXTREMAL PROBLEMS FOR GRAPHS AND HYPERGRAPHS

Given a fixed graph F, a classical problem in extremal graph theory asks for the maximum number of edges that a (large) graph G on n vertices containing no copy of F can have. More formally, for a fixed graph F let the *extremal number* $\mathrm{ex}(n, F)$ be the number $|E|$ of edges of an F-free graph $G = (V, E)$ on $|V| = n$ vertices with the maximum number of edges. It is well known and not hard to observe that the sequence $\mathrm{ex}(n, F)/\binom{n}{2}$ is decreasing. Consequently, one may define the *Turán density*

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}},$$

which describes the maximum density of large F-free graphs. The systematic study of these extremal parameters was initiated by Turán [51], who determined $\mathrm{ex}(n, K_t)$ for complete graphs K_t. Recalling that the chromatic number $\chi(F)$ of a graph F is the minimum number of colors one can assign to the vertices of F in such a way that any two vertices connected by an edge receive distinct colors, it follows from a result of Erdős and Stone [16] that

$$\pi(F) = \frac{\chi(F) - 2}{\chi(F) - 1}, \tag{2.1}$$

while the connection with the chromatic number first appeared in the work of Erdős and Simonovits [14]. In particular, the value of $\pi(F)$ can be calculated in finite time. It also follows that the set $\Pi^{(2)} = \{\pi(F) : F \text{ is a graph}\}$ of all Turán densities of graphs is given by

$$\Pi^{(2)} = \left\{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{t-2}{t-1}, \dots\right\}.$$

Already in his original work [51], Turán asked for hypergraph extensions of these extremal problems. We mainly restrict ourselves here to 3-*uniform hypergraphs* $H = (V, E)$, where $V = V(H)$ is a finite set of *vertices* and the set of *hyperedges* $E = E(H) \subseteq V^{(3)}$, where $V^{(3)} = \{e \subseteq V : |e| = 3\}$ is a collection of 3-element sets of vertices. Despite considerable effort, even for 3-uniform hypergraphs F, no similar characterization (as in the graph case) is known. In fact, it is known that the corresponding set $\Pi^{(3)}$ of Turán densities for 3-uniform hypergraphs is much more complicated and, in particular as a subset of the reals, it is not well-ordered (see, e.g., [19] and [34]). Determining the value of $\pi(F)$ is a well known and hard problem even for "simple" hypergraphs like the complete 3-uniform hypergraph $K_4^{(3)}$ on four vertices and $K_4^{(3)-}$, the hypergraph with four vertices and three hyperedges. Currently the best known bounds for these Turán densities are

$$\frac{5}{9} \le \pi(K_4^{(3)}) \le 0.5615 \quad \text{and} \quad \frac{2}{7} \le \pi(K_4^{(3)-}) \le 0.2871,$$

where the lower bounds are given by what is believed to be optimal constructions due to Turán (see, e.g., [11]) and Frankl and Füredi [18]. The stated upper bounds are due to Razborov [36], Baber [1], and Baber and Talbot [2], and their proofs are based on the *flag algebra method* introduced by Razborov [35]. For a thorough discussion of Turán-type results and problems for hypergraphs we refer to the survey of Keevash [28].

3. HYPERGRAPHS UNIFORMLY DENSE ON SETS OF VERTICES

Erdős and Sós (see, e.g., [12,15]) suggested a variant, where one restricts to F-free hypergraphs H that are *uniformly dense* on large subsets of the vertices.

Definition 3.1. For reals $d \in [0,1]$ and $\eta > 0$, we say a 3-uniform hypergraph $H = (V, E)$ is $(d, \eta, \therefore)$-*dense* if all subsets $X, Y, Z \subseteq V$ induce at least

$$d|X||Y||Z| - \eta|V|^3$$

triples $(x, y, z) \in X \times Y \times Z$ such that $\{x, y, z\}$ is a hyperedge of H.

Restricting to $\therefore$-dense hypergraphs, the appropriate Turán density $\pi_{\therefore}(F)$ for a given hypergraph F can be defined as

$$\pi_{\therefore}(F) = \sup\big\{d \in [0,1]\colon \text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists}$$
$$\text{a 3-uniform, } F\text{-free, } (d, \eta, \therefore)\text{-dense hypergraph } H \text{ with } |V(H)| \geq n\big\},$$

and we obtain from the definitions that

$$\pi(F) \geq \pi_{\therefore}(F)$$

for every 3-uniform hypergraph F.

We first note that these Turán densities are nontrivial, i.e., there exist hypergraphs F such that $\pi_{\therefore}(F) > 0$, as the following examples show, which can be traced back to the work of Erdős and Hajnal [13].

Example 3.2. Consider a random tournament T_n on the vertex set $[n] = \{1, \ldots, n\}$, i.e., an orientation of all edges of the complete graph on the first n positive integers such that each of the two directions (i, j) or (j, i) of every pair of vertices $\{i, j\}$ is chosen independently with probability $1/2$. Given such a tournament T_n, we define the 3-uniform hypergraph $H(T_n)$ on the same vertex set, by including the triple $\{i, j, k\}$ in $E(H(T_n))$ if these three vertices span a cyclically oriented cycle of length three, i.e., $\{i, j, k\} \in E(H(T_n))$ if either (i, j), (j, k), and (k, i) are all in $E(T_n)$ or (i, k), (k, j), and (j, i) are all in $E(T_n)$. It is easy to check that for every $\eta > 0$ with probability tending to 1 as $n \to \infty$ the hypergraph $H(T_n)$ is $(1/4, \eta, \therefore)$-dense. Moreover, no hypergraph H obtained from a tournament in this way contains three hyperedges on four vertices, i.e., every such H is $K_4^{(3)-}$-free and this establishes $\pi_{\therefore}(K_4^{(3)-}) \geq 1/4$.

It was shown by Glebov, Král', and Volec [24] that indeed this construction is essentially optimal by providing a matching upper bound.

Theorem 3.3 (Glebov, Král', and Volec). *We have* $\pi_{\therefore}(K_4^{(3)-}) = 1/4$.

The proof in [24] is computer-assisted and based on flag-algebras. With Reiher and Rödl [41], we obtained an alternative proof, which relies on the regularity method for hypergraphs.

Note that $K_4^{(3)-}$ can be described as the hypergraph given by one vertex a having a triangle as its *link graph*, i.e., the graph consisting of the pairs of vertices that together with a form a hyperedge. From that point of view the following problem asks for a natural extension of Theorem 3.3.

Problem 3.4. *For $t \geq 3$, let S_t be the 3-uniform hypergraph on $t+1$ vertices $a, u_1, \dots, u_t$ such that $\{a, u_i, u_j\}$ is a hyperedge for all $1 \leq i < j \leq t$. Determine $\pi_{\therefore}(S_t)$ for $t \geq 4$.*

In [41] it is shown that

$$\frac{t^2 - 5t + 7}{(t-1)^2} \leq \pi_{\therefore}(S_t) \leq \left(\frac{t-2}{t-1}\right)^2,$$

which for $t = 3$ recovers Theorem 3.3 since $S_3 = K_4^{(3)-}$. For the first open case $t = 4$, we have $\frac{1}{3} \leq \pi_{\therefore}(S_4) \leq \frac{4}{9}$, and it would be interesting to close this gap and to find the extremal structures for this problem.

Another intriguing problem concerns $K_4^{(3)}$, the so-called *tetrahedron*. The following random construction of Rödl [43] shows that $\pi_{\therefore}(K_4^{(3)}) \geq 1/2$, and Erdős [12] suggested that this might be best possible.

Example 3.5. Given any map $\varphi\colon [n]^{(2)} \to \{\text{red}, \text{green}\}$, we define the 3-uniform hypergraph H_φ with vertex set $[n]$ by putting a triple $\{i, j, k\}$ with $i < j < k$ into $E(H_\varphi)$ if and only if the colors of the two pairs $\{i, j\}$ and $\{i, k\}$ differ. Irrespective of the choice of the coloring φ, the hypergraph H_φ contains no tetrahedra: for if a, b, c, and d are any four distinct vertices, say with $a = \min(a, b, c, d)$, then it is impossible for all three of the pairs $\{a, b\}$, $\{a, c\}$, and $\{a, d\}$ to have distinct colors, whence not all three of the triples $\{a, b, c\}$, $\{a, b, d\}$, and $\{a, c, d\}$ can be hyperedges of H_φ. Moreover, it was noticed in [43] that if the coloring φ is chosen uniformly at random, then for any $\eta > 0$ the hypergraph H_φ is with high probability $(1/2, \eta, \therefore)$-dense as n tends to infinity. This is easily checked using standard tail estimates for binomial distributions. In other words, this examples show that $\pi_{\therefore}(K_4^{(3)}) \geq \frac{1}{2}$ holds.

It is believed that this construction is optimal, which leads to the following beautiful problem suggested by Erdős [12].

Problem 3.6. *Show that $\pi_{\therefore}(K_4^{(3)}) = \frac{1}{2}$.*

There is some evidence in support of that conjecture. Recently, Balogh, Clemen, and Lidický [3] showed that

$$\pi_{\therefore}(K_4^{(3)}) \leq 0.529$$

and, hence, $\pi_{\therefore}(K_4^{(3)})$ is strictly smaller than the Turán density $\pi(K_4^{(3)})$. In joint work with Reiher and Rödl, we were able to resolve Problem 3.6 affirmatively for a slightly stronger notion of uniform edge distribution [38], which is also satisfied by the hypergraphs from Examples 3.2 and 3.5 (see Theorem 4.2 below). The construction in Example 3.5 can be extended for arbitrary cliques and this leads to the following general problem.

Problem 3.7. *For every fixed integer $t \geq 4$, show that $\pi_{\therefore}(K_t^{(3)}) = \frac{t-3}{t-2}$.*

This problem seems to be one of the main problems in the area. However, for $t = 6$ a second different lower bound constructions is known (see [41, CONCLUDING REMARKS]), which may indicate that the general problem might be more challenging.

Example 3.8. Similarly as in Example 3.5, we consider a random 2-coloring φ of $[n]^{(2)}$. However, this time we include all triples as hyperedges in H_φ if the three underlying pairs are not all of the same color. Again it is easy to check that for every $\eta > 0$ the hypergraph H_φ is with high probability $(3/4, \eta, \therefore)$-dense and, due to the first nontrivial instance of Ramsey's theorem, it is also $K_6^{(3)}$-free.

Very recently, Bucić, Cooper, Král', Mohr, and Munhá Correia [7] could determine the $\pi_{\therefore}(C_\ell)$ for hypergraph cycles. Here a hypergraph cycle C_ℓ for $\ell \geq 4$ is defined by

$$V(C_\ell) = \mathbb{Z}/\ell\mathbb{Z} \quad \text{and} \quad E(C_\ell) = \big\{\{i, i+1, i+2\} : i \in \mathbb{Z}/\ell\mathbb{Z}\big\}.$$

Note that for $\ell = 4$ we have $C_4 = K_4^{(3)}$ and the best known lower bound $\pi_{\therefore}(C_4) \geq 1/2$ is given by Example 3.5. For $\ell = 5$, Reiher [37] gave an example which shows $\pi_{\therefore}(C_5) \geq 4/27$. On the other hand, for ℓ divisible by 3, the hypergraph cycle C_ℓ is tripartite, and it follows from the definition and the work of Erdős [10] that

$$\pi_{\therefore}(C_{3k}) \leq \pi(C_{3k}) = 0$$

for every $k \geq 2$. Bucić et al. [7] showed that the construction of Reiher is optimal and established the same bound for all $\ell \geq 5$ that are not divisible by 3.

Theorem 3.9 (Bucić et al.). *For every $\ell \geq 5$ with $\ell \not\equiv 0 \pmod 3$, we have $\pi_{\therefore}(C_\ell) = 4/27$.*

Besides determining $\pi_{\therefore}(\cdot)$ for particular hypergraphs, as in the problems and results above, it would be interesting to study the set $\Pi_{\therefore}^{(3)} = \{\pi_{\therefore}(F) : F \text{ is a 3-uniform hypergraph}\}$ of all such Turán densities. In that direction as a first problem one may consider the smallest nonzero value. In [10] Erdős showed that $\pi(F) = 0$ if and only if F is tripartite and from this characterization it follows that the smallest nonzero classical Turán density is at least $2/9$. It was proved in [5, 17] that it is in fact $2/9$. For $\pi_{\therefore}(\cdot)$, we showed in [39], similarly as Erdős for $\pi(\cdot)$, a characterization of the hypergraphs F with $\pi_{\therefore}(F) = 0$.

Theorem 3.10. *For a 3-uniform hypergraph F, the following are equivalent:*

(a) $\pi_{\therefore}(F) = 0$.

(b) *There is an enumeration of the vertex set $V(F) = \{v_1, \dots, v_f\}$ and there is a three-coloring $\varphi: \partial F \to \{red, blue, green\}$ of the pairs of vertices ∂F covered by hyperedges of F such that every hyperedge $\{v_i, v_j, v_k\} \in E(F)$ with $i < j < k$*

satisfies

$$\varphi(v_i, v_j) = red, \quad \varphi(v_i, v_k) = blue, \quad and \quad \varphi(v_j, v_k) = green.$$

This characterization implies that the smallest nonzero Turán density in this context is at least $1/27$.

Corollary 3.11. *If a hypergraph F satisfies $\pi_{\therefore}(F) > 0$, then $\pi_{\therefore}(F) \geq \frac{1}{27}$.*

Proof. Given a positive integer n, consider a three-coloring $\varphi\colon [n]^{(2)} \to \{\text{red, blue, green}\}$ of the pairs of the first n positive integers. We define a hypergraph H_φ with vertex set $[n]$ by regarding a triple $\{i, j, k\}$ with $1 \leq i < j < k \leq n$ as being a hyperedge if and only if $\varphi(i, j) = \text{red}$, $\varphi(i, k) = \text{blue}$, and $\varphi(j, k) = \text{green}$. Standard probabilistic arguments show that when φ is chosen uniformly at random, then for any fixed $\eta > 0$ the probability that H_φ is $(1/27, \eta, \therefore)$-dense tends to 1 as n tends to infinity. On the other hand, as F does not satisfy condition (b) from Theorem 3.10, it is in a deterministic sense the case that F is never a subgraph of H_φ no matter how large n becomes. Thus we have indeed $\pi_{\therefore}(F) \geq \frac{1}{27}$. ∎

Recently, Garbe, Král', and Lamaison [23] complemented Corollary 3.11 and established a matching upper bound for the smallest nonzero value of $\pi_{\therefore}(\cdot)$.

Theorem 3.12 (Garbe, Král', and Lamaison). *There is a hypergraph F with $\pi_{\therefore}(F) = 1/27$.*

It seems plausible that $\Pi^{(3)}_{\therefore}$ is structurally similar to $\Pi^{(2)}$. For example, the construction in Example 3.5 in some sense transfers the extremal example for triangle-free graphs into our context here. In contrast to $\Pi^{(3)}$ we put forward the following problem.

Problem 3.13. *Show that $\Pi^{(3)}_{\therefore} = \{\pi_{\therefore}(F)\colon F \text{ is a 3-uniform hypergraph}\}$ is well-ordered as a subset of the reals.*

Another intriguing open problem from [39] concerns the comparison of $\pi_{\therefore}(F)$ with $\pi(F)$.

Problem 3.14. *Is $\pi_{\therefore}(F) < \pi(F)$ for every 3-uniform hypergraph F with $\pi(F) > 0$?*

Roughly speaking, this questions has an affirmative answer, if no 3-uniform hypergraph F with positive Turán density has an extremal hypergraph H that is uniformly dense with respect to large vertex sets $U \subseteq V(H)$ (see also [15, PROBLEM 7] for a related assertion). Problem 3.14 is motivated by the fact that currently all known extremal constructions for such 3-uniform hypergraphs F are obtained from blow-ups or iterated blow-ups of smaller hypergraphs, which fail to be $(d, \eta, \therefore)$-dense for all $d > 0$ and sufficiently small $\eta > 0$, which may suggest that the answer to Problem 3.14 is affirmative.

The work in [7, 39, 41] may indicate that the regularity method for hypergraphs provides a suitable approach for the problems stated in this section. Moreover, those proofs require Ramsey-type arguments and new results from extremal graph theory which are of independent interest.

4. HYPERGRAPHS UNIFORMLY DENSE ON VERTICES AND PAIRS

In Definition 3.1 we defined uniform hyperedge distribution with respect to vertex sets, and Examples 3.2, 3.5, and 3.8 showed that this notion alone with density bounded away from 0 does not suffice to embed arbitrary 3-uniform hypergraphs. In contrast, it follows that if we define uniform hyperedge density with respect to sets of pairs, then such a notion would allow the embedding of arbitrary fixed 3-uniform hypergraphs (see, e.g., [29]) and, in fact, these considerations led to the more involved concepts in the hypergraph regularity projects of Gowers and Rödl et al.

Moreover, there seem to be at least two intermediate variants of uniformly dense hypergraphs. In connection with extremal problems, those notions were already partly investigated in [4, 38, 42] and they lead to several interesting problems. The first strengthening is the following stronger concept of uniformly dense hypergraphs, where we "replace" the two sets Y and Z from Definition 3.1 by an arbitrary set of pairs P.

Definition 4.1. For reals $d \in [0,1]$ and $\eta > 0$, we say a 3-uniform hypergraph $H = (V, E)$ is $(d, \eta, \underline{\therefore})$-dense if for every subset $X \subseteq V$ of vertices and every subset of pairs of vertices $P \subseteq V \times V$ the number $e_{\underline{\therefore}}(X, P)$ of pairs $(x, (y, z)) \in X \times P$ with $\{x, y, z\} \in E$ satisfies

$$e_{\underline{\therefore}}(X, P) \geq d|X||P|| - \eta|V|^3.$$

Since for any hypergraph $H = (V, E)$ and sets $X, Y, Z \subseteq V$ we may apply the definition for X and $P = Y \times Z$, it follows from these definitions that $(d, \eta, \underline{\therefore})$-dense hypergraphs are also $(d, \eta, \therefore)$-dense. Moreover, we can introduce the corresponding Turán density $\pi_{\underline{\therefore}}(F)$ for a given hypergraph F by

$$\pi_{\underline{\therefore}}(F) = \sup\big\{d \in [0,1]\colon \text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists}$$
$$\text{a 3-uniform, } F\text{-free, } (d, \eta, \underline{\therefore})\text{-dense hypergraph } H \text{ with } |V(H)| \geq n\big\}$$

and obtain from the definitions that

$$\pi(F) \geq \pi_{\therefore}(F) \geq \pi_{\underline{\therefore}}(F)$$

for every 3-uniform hypergraph F. One can check that the random constructions in Examples 3.2 and 3.5 also give lower bounds for $\pi_{\underline{\therefore}}(K_4^{(3)-})$ and $\pi_{\underline{\therefore}}(K_4^{(3)})$, as the constructed hypergraphs in these examples are also $\underline{\therefore}$-dense. In particular, we have $\pi_{\underline{\therefore}}(K_4^{(3)}) \geq 1/2$ and a matching upper bound was proved in joint work with Reiher and Rödl [38], which can be viewed as some evidence towards an affirmative answer for Problem 3.6.

Theorem 4.2. *We have* $\pi_{\underline{\therefore}}(K_4^{(3)}) = 1/2$.

Moreover, these considerations naturally suggest that a possible first step towards Problems 3.7, 3.4, and 3.13 is to consider these problems for $\pi_{\underline{\therefore}}(\cdot)$.

Problem 4.3.

(i) *Show that* $\pi_{\underline{\therefore}}(K_t^{(3)}) = \frac{t-3}{t-2}$ *for every* $t > 4$.

(ii) *Determine* $\pi_{\underline{\therefore}}(S_t)$ *for* $t \geq 4$.

(iii) *Show that* $\Pi^{(3)}_{\dot{\smile}} = \{\pi_{\dot{\smile}}(F) \colon F$ *is a* 3*-uniform hypergraph*$\}$ *is well-ordered as a subset of the reals.*

Finding the smallest nonzero value of $\pi_{\dot{\smile}}(\cdot)$ would also be of high interest. However, the situation here is less clear and maybe as a first step it would be useful to establish a meaningful characterization of the hypergraphs F with $\pi_{\dot{\smile}}(F) = 0$. By definition, the set of those hypergraphs must contain all hypergraphs F with $\pi_{\therefore}(F) = 0$, but finding a useful characterization appears to be an interesting problem on its own.

Problem 4.4. *Find a useful characterization of the* 3*-uniform hypergraphs* F *with* $\pi_{\dot{\smile}}(F) = 0$.

5. HYPERGRAPHS UNIFORMLY DENSE ON PAIRS OF SETS OF PAIRS

The following further strengthening of the notion of $\dot{\smile}$-dense hypergraphs is in some sense the strongest nontrivial uniform density condition for extremal problems in 3-uniform hypergraphs.

Definition 5.1. For reals $d \in [0,1]$ and $\eta > 0$, we say a 3-uniform hypergraph $H = (V, E)$ is (d, η, Λ)-dense if for any two subsets of pairs $P, Q \subseteq V \times V$ the number $e_{\Lambda}(P, Q)$ of pairs of pairs $((x,y),(x,z)) \in P \times Q$ with $\{x,y,z\} \in E$ satisfies

$$e_{\Lambda}(P,Q) \geq d|\mathcal{K}_{\Lambda}(P,Q)| - \eta|V|^3,$$

where $\mathcal{K}_{\Lambda}(P,Q)$ denotes the set of pairs in $P \times Q$ of the form $((x,y),(x,z))$.

The corresponding Turán density $\pi_{\Lambda}(F)$ can be defined similarly as above by

$$\pi_{\Lambda}(F) = \sup\big\{d \in [0,1] \colon \text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists}$$
$$\text{a 3-uniform, } F\text{-free, } (d,\eta,\Lambda)\text{-dense hypergraph } H \text{ with } |V(H)| \geq n\big\}$$

and again the definition ensures that

$$\pi(F) \geq \pi_{\therefore}(F) \geq \pi_{\dot{\smile}}(F) \geq \pi_{\Lambda}(F).$$

With respect to cliques $K^{(3)}_t$, parameter $\pi_{\Lambda}(\cdot)$ behaves differently and grows much more slowly. In [42], together with Reiher and Rödl, we could show the following upper bound.

Theorem 5.2. *For every* $t \geq 2$,

$$\pi_{\Lambda}(K^{(3)}_{2^t}) \leq \frac{t-2}{t-1},$$

which is tight for $t = 2, 3,$ *and* 4.

Maybe somewhat surprisingly, [42] establishes the precise value of $\pi_{\Lambda}(K^{(3)}_s)$ for $s \in \{4,\ 6,7,8,\ 11,12,\dots,16\}$, but the cases $s = 5$, 9, and 10, were left open. Very recently, in joint work with Berger, Piga, Reiher, and Rödl [4], we could resolve the case $s = 5$ and showed

$$\pi_{\Lambda}(K^{(3)}_5) = \frac{1}{3}.$$

Comparing Theorem 5.2 with the known (or believed to be optimal) lower bounds for $\pi_{\underline{\therefore}}(K_{t+1}^{(3)})$, we have

$$\pi_{\wedge}(K_{2^t}^{(3)}) \leq \frac{t-2}{t-1} \leq \pi_{\underline{\therefore}}(K_{t+1}^{(3)}) \leq \pi_{\therefore}(K_{t+1}^{(3)}).$$

So in particular, the Turán densities for $\wedge$-dense hypergraphs for $K_t^{(3)}$ grow much slower compared to $\underline{\therefore}$-dense or $\therefore$-dense hypergraphs.

Maybe the most urgent questions related to $\pi_{\wedge}(\cdot)$ are an appropriate version of Problem 4.4 and determining $\pi_{\wedge}(K_s^{(3)})$ for the missing small values of $s = 9$ and 10 mentioned above.

Problem 5.3. (i) *Find a useful characterization of the* 3*-uniform hypergraphs* F *with* $\pi_{\wedge}(F) = 0$.

(ii) *Determine* $\pi_{\wedge}(K_s^{(3)})$ *for* $s = 9$ *and* 10.

It seems plausible that by combining the main ideas from **[42]** and **[4]** one can derive the improved upper bound

$$\pi_{\wedge}(K_{10}^{(3)}) \leq \frac{3}{5}.$$

More generally, one may show that if $\pi_{\wedge}(K_s^{(3)}) \leq \alpha$, then $\pi_{\wedge}(K_{2s}^{(3)}) \leq \frac{1}{2-\alpha}$, which was suggested by Reiher in **[37]**.

Determining the value $\pi_{\wedge}(K_s^{(3)})$ for large values of s might be a challenging problem, and one may first focus on the asymptotic behavior. For every $s \geq 3$, Theorem 5.2 tells us

$$\pi_{\wedge}(K_s^{(3)}) \leq 1 - \frac{1}{\log_2(s)}. \tag{5.1}$$

For a lower bound, we consider the following well-known random construction.

Example 5.4. For $r \geq 2$, we consider random hypergraphs $H_\varphi = (V, E_\varphi)$ with the edge set defined by the nonmonochromatic triangles of a random r-coloring $\varphi\colon V^{(2)} \to [r]$ for a sufficiently large vertex set V. It is easy to check that for any fixed $\eta > 0$ with high probability such hypergraphs H_φ are $(\frac{r-1}{r}, \eta, \wedge)$-dense. On the other hand, if s is at least as large as $R(3;r)$, the r-color Ramsey number for graph triangles, then every such H_φ is $K_s^{(3)}$-free.

Consequently, Example 5.4 yields

$$\pi_{\wedge}(K_s^{(3)}) \geq 1 - \frac{1}{r}, \qquad \text{whenever } s \geq R(3;r)$$

and, using the simple upper bound $R(3;r) \leq 3r!$, we arrive at

$$\pi_{\wedge}(K_s^{(3)}) \geq 1 - \frac{\log_2 \log_2(s)}{\log_2(s)} \tag{5.2}$$

for sufficiently large s. Comparing the bounds in (5.1) and (5.2) leads to the following problem.

Problem 5.5. *Determine the asymptotic behavior of* $1 - \pi_{\wedge}(K_s^{(3)})$.

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