GRAPHS OF LARGE CHROMATIC NUMBER

ALEX SCOTT

ABSTRACT

The chromatic number has been a fundamental topic of study in graph theory for more than 150 years. Graph coloring has a deep combinatorial theory and, as with many NP-hard problems, is of interest in both mathematics and computer science. An important challenge is to understand graphs with very large chromatic number. The chromatic number tells us something global about the structure of a graph: if G has small chromatic number then it can be partitioned into a few very simple pieces. But what if G has large chromatic number? Is there anything that we can say about its local structure? In particular, are there particular substructures that it must contain? In this paper, we will discuss recent progress and open problems in this area.

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1. INTRODUCTION

The chromatic number has been a fundamental topic of study in graph theory for more than 150 years. For example, the famous Four Color Conjecture, which states that every graph that can be embedded in the plane has chromatic number at most 4, was first raised in the 1850s by Francis Guthrie (a student of Augustus de Morgan), and was finally proved in the 1970s by Appel and Haken [4], in one of the first computer-assisted proofs. Attempts to solve the conjecture led to Birkhoff's [9] development in 1912 of the chromatic polynomial, which counts the number of k-colorings of a graph G. The chromatic polynomial was generalized by Tutte [85] to what is now known as the Tutte polynomial, which is closely connected to the Ising model and Potts model in statistical physics (see Fortuin and Kasteleyn [43], Sokal [83]), the random cluster model in probability (see Grimmett [44]) and the Jones polynomial in knot theory (see Jones [51]).

However, many fundamental questions about graph coloring remain. A particular challenge is to understand graphs with very large chromatic number. The chromatic number says something about the global structure of a graph: if G has small chromatic number then it can be partitioned into a few very simple pieces. But what if G has large chromatic number? Is there anything that we can say about its local structure? In particular, are there particular substructures that it must contain?

We will need a few definitions. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$ (all graphs in this paper are finite). A *complete graph* is a graph in which every pair of vertices is joined. A *stable set* (or *independent set*) in G is a set $S \subseteq V$ such that no two vertices of S are adjacent in G. The *clique number* $\omega(G)$ of G is the maximum number of vertices in a complete subgraph of G; and the *stability number* $\alpha(G)$ is the largest number of vertices in a stable set in G. A *k-coloring* of a graph is function from its vertices to $\{1, \ldots, k\}$ so that adjacent vertices have different colors. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that G has a k-coloring.

Graphs with chromatic number at most 2 are easily characterized: they are the graphs that do not contain an odd cycle. But for $k \geq 3$, there does not appear to be any simple structural characterization even of the minimal graphs with chromatic number more than k (see [10]). The algorithmic problem of k-colorability is well known to be NP-complete for $k \geq 3$, and was one of Karp's celebrated list [54] of 21 NP-complete problems; indeed, for $\varepsilon > 0$, it is NP-hard even to approximate the chromatic number within a factor $n^{1-\varepsilon}$, where n is the number of vertices. As with many NP-hard problems, graph coloring has a correspondingly deep combinatorial theory, and it has been the focus of extensive study in both mathematics and computer science, and understanding the connections between graph structure and chromatic number has been one of the fundamental goals of structural graph theory in the last 30 years.

Let us clarify the notion of *substructure*. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Thus H is obtained from G by deleting vertices and edges. We say that H is an *induced subgraph of* G if $V(H) \subset V(G)$, and $E(H)$ consists of the edges of G that are contained in $V(H)$ (and then H is the subgraph of G *induced*

by $V(H)$). For example, every graph is a subgraph of some complete graph; but if G is a complete graph then all of its induced subgraphs are complete graphs. In this paper we will be concerned primarily with *induced* subgraphs. We say that a graph G is H*-free* if G does not contain an induced subgraph that is isomorphic to H (more informally, if G does not contain an *induced copy* of H).

So, what can we say about the induced subgraphs of a graph with large chromatic number? One possibility is that G might itself be complete, in which case it only contains complete graphs as induced subgraphs. But what if G does not contain a large complete subgraph: are there particular structures that have to appear as induced subgraphs? In this paper we will be interested in statements of the form:

> *Every graph with sufficiently large chromatic number contains either a complete subgraph on k vertices or an induced ***.*

Equivalently, we will often say:

> *If G contains neither a complete subgraph on k vertices nor an induced *** then it has bounded chromatic number.*

The question is: what can we put in place of the asterisks?

The rest of this paper is organized as follows. In Section 2 we look at whether graphs with large chromatic number need to contain large complete subgraphs (they do not). In the next few sections, we investigate the relationship between chromatic number and clique number: after discussing perfect graphs in Section 3, we introduce χ-bounded classes in Section 4 and look at the effects of forbidding a single induced subgraph. In Section 5 we look at induced cycles in graphs of large chromatic number, and then Section 6 considers more complex induced subgraphs. Section 7 discusses the Erdős-Hajnal Conjecture, and Section 8 looks at its connection with polynomially χ-bounded classes. Finally, in Section 9, we compare the effects of excluding induced subgraphs with the effects of excluding graph minors.

2. GIRTH AND CHROMATIC NUMBER

Suppose that a graph G has huge chromatic number. Are there induced subgraphs that it must contain? Perhaps the first question of this type to ask is: does every graph of large chromatic number contain a large complete subgraph? This question was answered in the negative in the 1940s by Tutte (writing as Blanche Descartes [33,34]), who showed that there are triangle-free graphs with arbitrarily large chromatic number. Many constructions are now known. For example, there is a simple construction of Mycielski from the 1950s [66]: given a graph G with vertices $\{v_1, \ldots, v_k\}$ we define a new graph $M(G)$ with vertices $\{x_1, \ldots, x_k, y_1, \ldots, y_k, z\}$; for each edge $v_i v_j$ of G, the graph $M(G)$ has edges $x_i x_j$, $y_i x_j$ and $x_i y_j$ (but not $y_i y_j$), and z is adjacent to all the y_i. It is straightforward to check that $\chi(M(G)) = \chi(G) + 1$, and if G is triangle-free then so is $M(G)$. Thus starting with $G_1 = K_1$

and inductively defining $G_{i+1} = M(G_i)$, we obtain a sequence of triangle-free graphs G_i with $\chi(G_i) = i$ for each i.

So graphs of large chromatic number need not have large complete subgraphs. But perhaps they must have short cycles? Or can they instead be "locally tree-like"? It turns out that the latter is the case. A *cycle of length* $k \geq 3$ is a graph with vertices $x_1, \ldots, x_k$ and edges $x_i x_{i+1}$ for each i (where indices are taken modulo k). The *girth* of a graph is the smallest k such that G contains a cycle of length k as a subgraph. In one of the earliest applications of probability in graph theory, Erdős [35] showed that there are graphs with arbitrarily large girth and chromatic number. Indeed, consider a random graph on n vertices in which every edge is present with probability $(\log n)/n$. A simple first moment argument shows that, with positive probability, only $o(n)$ vertices are contained in short cycles, while any stable set has size at most $o(n)$ (so the chromatic number is large, as a coloring is a partition into stable sets). Deleting all vertices in short cycles gives the desired graph.

Constructing *explicit* examples of graphs with large girth and chromatic number is rather harder. There is a pretty example of a graph with large chromatic number and no short *odd* cycles: the *Kneser graph* $K(n,k)$ has as its vertex set all k-sets contained in $\{1, \ldots, n\}$, with A and B adjacent if and only if they are disjoint [57]. It is easy to check that if $n = 2k + t$ then there are no odd cycles of length less than about n/t. It is rather harder to show that Kneser graphs can have large chromatic number: in fact, it turns out that for $k > n/2$, the Kneser graph $K(n,k)$ has chromatic number exactly $n - 2k + 2$. This was proved in a celebrated paper of Lovász [63], which developed the connection between chromatic number and the topology of the neighborhood complex of a graph; shortly afterwards, Bárány [6] found a second beautiful (and surprisingly simple) topological proof.

There are now a number of explicit constructions of graphs with large chromatic number and *no* short cycles. These include a construction of Lovász [62]; the Ramanujan graphs of Lubotzky, Phillips, and Sarnak [64]; a construction of Nešestřil and Rödl [67] using the "amalgamation method"; and an ingenious recent construction of Alon, Kostochka, Reiniger, West, and Zhu [2] based on careful augmentation of trees.

Even here, though, there are basic questions that remain. For example, the following question of Erdős and Hajnal [36] concerning (not necessarily induced) subgraphs has been open for 50 years.

Conjecture 2.1. *For every pair of positive integers k, t, every graph of sufficiently large chromatic number contains a subgraph with chromatic number more than t and girth more than k.*

The best current result is due to Rödl [72], who proved the conjecture for $k = 3$.

3. PERFECT GRAPHS

Recall that the *clique number* $\omega(G)$ of a graph G is the maximum number of vertices in a complete subgraph in G. Every graph has chromatic number at least as large as its clique

number, as the vertices in any clique must all have different colors in a proper coloring. But when is the chromatic number larger than the clique number?

Here are two examples where this happens:

- Let C be an cycle of odd length. Then $\chi(C) = 3$ and (unless C is a triangle) $\omega(C) = 2$.
- Let $\overline{C}$ be the complement of a cycle of odd length. Then it can be checked that (unless C is a triangle) $\chi(\overline{C}) > \omega(\overline{C})$.

Let us say that an induced subgraph of a graph G is a *hole (in G)* if it is a cycle of length at least four, and an *antihole* if it is the complement of a cycle of length at least four (or, equivalently, if it corresponds to a hole in the complement of G). A hole or antihole is *odd* if it has an odd number of vertices. In the 1960s, Claude Berge [7] conjectured that the minimal graphs with chromatic number larger than clique number are precisely the odd holes and odd antiholes. This became known as the Strong Perfect Graph Conjecture, and was a central problem in structural graph theory for many years.

The conjecture was finally resolved by Chudnovsky, Robertson, Seymour, and Thomas in 2006 [19]:

Theorem 3.1. *If the chromatic number of G is larger than its clique number, then G contains an odd hole or an odd antihole.*

The proof of the Strong Perfect Graph Theorem is a tour de force of structural techniques. The details are rather complicated, but the strategy of the proof is to show that if G has no odd holes and no odd antiholes, then either it belongs to one of a small number of well-understood "basic classes" of graphs, or it has a "nice" decomposition into smaller graphs.

This type of approach is frequently used in structural graph theory and has been remarkably successful in understanding a wide variety of graph classes, but it is only effective when the classes being examined have some sort of nice structure. In the rest of this paper, we will mostly be interested in larger classes, where it is unlikely that there are nice decomposition results, and so very different techniques need to be used.

4. χ-BOUNDED CLASSES AND THE GYÁRFÁS–SUMNER CONJECTURE

The Strong Perfect Graph Theorem characterizes when the chromatic number $\chi(G)$ is larger than the clique number $\omega(G)$, but what induced subgraphs can we get when the chromatic number is *much* larger than the clique number? In the 1980s, András Gyárfás wrote an influential paper, *Problems from the world surrounding perfect graphs*, in which he initiated the systematic investigation of this question, using the language of χ-bounded classes. Gyárfás laid out a research programme for the study of χ-bounded classes and made a sequence of important conjectures, many of which have been resolved only in the last few years.

We will always be concerned with *hereditary* classes, namely those that are closed under taking induced subgraphs.

Definition 4.1. A hereditary class $\mathcal{G}$ of graphs is *χ-bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$ (see [46,79]).

The class of all graphs is not χ-bounded, as there are triangle-free graphs with arbitrarily large chromatic number (so we cannot even define $f(3)$). So any χ-bounded class must exclude at least one induced subgraph. In this section, we look at the question: when is it enough to forbid a single induced subgraph H?

Let us fix a graph H. If H contains a cycle C then the class of H-free graphs is not χ-bounded: we know from Section 2 that there are graphs with arbitrarily large girth and chromatic number (if their girth is more than the length of C, they do not contain a copy of H). So the interesting case is when H is acyclic, i.e., a *forest*. This is the subject of the well-known *Gyárfás–Sumner Conjecture* [45,84]:

Conjecture 4.2. *For every forest H the class of H-free graphs is χ-bounded.*

The Gyárfás–Sumner Conjecture can equivalently be stated as follows:

Conjecture 4.3. *For every forest H and every $k \geq 1$, every graph with sufficiently large chromatic number contains either a complete graph on k vertices or an induced copy of H.*

The conjecture has proved extremely resistant. It is not hard to show that it suffices to consider the case when H is a tree (for a forest F, the class of F-free graphs is χ-bounded if and only if the class of H-free graph is χ-bounded for every component H of F). But the conjecture is only known for a few quite special trees, for example:

- If H is a star, then it follows easily from Ramsey's theorem.
- If H is a path then there is a simple and elegant argument due to Gyárfás [45]. It is also known for the broom [45] and the double broom [22].
- It is true if H is a tree of radius 2: the triangle-free case was proved by Gyárfás, Szemerédi, and Tuza [48], and the general case by Kierstead and Penrice [55].
- It is known for some special trees of radius three [56,78].

In most cases, the proofs are quite intricate. However, the argument when H is a path is simple and elegant, so let us sketch it. Suppose we are looking for a path P with t vertices, and G is a graph with huge chromatic number that does not contain an induced copy of P or a complete subgraph on k vertices. By induction, we may assume that for every vertex v in G, its neighborhood $N(v)$ has small chromatic number (as it does not contain a complete subgraph on $k-1$ vertices). We may also assume that G is connected, by just considering the component with largest chromatic number. Now choose any vertex x_1. If we delete x_1 and its neighbors from G, then the remaining graph falls into components $C_1, \ldots, C_r$ for some r, and as the neighborhood of x_1 has small chromatic number and G has large chromatic number, one of the these components (say C_1) must also have large chromatic number. Since G is connected, there must be some vertex x_2 that is both adjacent to x_1 and has a neighbor in C_1. We focus on x_2 and C_1 and repeat the argument, deleting

neighbors of x_2 from C_1, choosing a component C of the remainder with large chromatic number, and choosing a vertex x_3 that is adjacent to x_2 and has neighbors in C. Continuing in this way, we walk into the graph, always heading towards a region with large chromatic number, and build an induced path along the way. This type of argument crops up repeatedly, and has become known as the *Gyárfás path argument*.

Another (rather more complicated) technique is the *method of templates*. Building on the work of Gyárfás, Szemerédi, and Tuza [48], this was developed by Kierstead and Penrice [55]. The idea is to look for complete multipartite subgraphs with (large) constant size. Thus we look for a large complete bipartite graph, or a (slightly less) large complete tripartite graph, and so on. The process terminates as we are assuming that there is no complete graph on k vertices. A *template* T consists of one of these subgraphs, say K, together with all the vertices that are moderately dense to K (in some appropriate sense). We define a sequence $T_1, T_2, \ldots$ of templates by repeatedly choosing one that is maximal (by some measure), deleting it from the graph, and then looking at templates in the graph that remains. When we are finished, we are left with a graph containing no templates, and show that it has small chromatic number. The key now is to show that edges between the templates we removed are rather restricted: there is usually quite a complex argument to partition and "clean up" the templates into progressively more simply structured pieces, until all the pieces have small chromatic number (and so we are done, as we have partitioned the whole graph into a bounded number of pieces with small chromatic number).

The method of templates has been a powerful approach for handling small-radius trees, but at present there seem to be significant technical obstacles to extending it to trees of radius more than 3. It is worth noting that the base case (finding complete bipartite graphs) is not straightforward, but can usually be handled with the following result (proved by Rödl but not published):

Theorem 4.4. *For every k and t, and every tree T, every graph with sufficiently large chromatic number contains either K_k, $K_{t,t}$ or T as an induced subgraph.*

Here K_k denotes the complete graph on k vertices, and $K_{t,t}$ denotes the complete bipartite graph with t vertices in each class. Kierstead and Penrice [55] strengthened Theorem 4.4, showing that such graphs have bounded degeneracy.[1] See Theorem 8.5 below for a further strengthening.

Perhaps the most general result related to the Gyárfás–Sumner conjecture concerns *induced subdivisions* of forests. We say that a graph F is a *subdivision* of a graph H (or is *homeomorphic to* H) if F can be obtained from H by adding vertices along the edges, or equivalently by replacing some subset of the edges by paths. For example, every cycle is a subdivision of a triangle. The following weakening of the Gyárfás–Sumner conjecture was obtained in [73].

1 The *degeneracy* of a graph G is the maximum integer r such that every subgraph of G has a vertex with degree at most r. The chromatic number of a graph is at most one more than its degeneracy.

Theorem 4.5. *Let H be a forest. The class of graphs that do not contain an induced copy of any subdivision of H is χ-bounded.*

A special case of this implies the Gyárfás–Sumner conjecture when H is a subdivision of a star (as any subdivision of H contains an induced copy of H).

In fact, something slightly stronger than Theorem 4.5 was shown in [73]:

Theorem 4.6. *For every forest H there is a finite list $H_1, \ldots, H_t$ of subdivisions of H such that the class of graphs that do not contain an induced copy of any H_i is χ-bounded.*

5. HOLES IN GRAPHS OF LARGE CHROMATIC NUMBER

What other structures must appear in graphs of large chromatic number? If we do not forbid a forest, then the existence of graphs with large girth and chromatic number implies that it is not enough to forbid a single induced subgraph, or indeed any finite list of induced subgraphs. Perhaps the simplest example of this is where we forbid a collection of induced holes (i.e., induced cycles of length at least four). Gyárfás made several important conjectures concerning holes, and we will focus on these in this section.

The Strong Perfect Graph Theorem tells us that the class of graphs with no odd holes and no odd antiholes is χ-bounded, and furthermore with the best possible function $f(\omega) = \omega$. Long before this theorem was proved, Gyárfás conjectured that for χ-boundedness it would suffice to exclude only odd holes.

Conjecture 5.1. *The class of graphs with no odd holes is χ-bounded.*

He also conjectured that it would be enough to exclude only long holes; and more adventurously that it would be enough to exclude *long* odd holes:

Conjecture 5.2. *For every integer t, the class of graphs with no holes of length more than t is χ-bounded.*

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For some time, there was no progress on these conjectures. As noted earlier, the structural techniques used to prove the Strong Perfect Graph theorem rely on the fact that perfect graphs have nice structural features, and a minimum counterexample to the theorem can be decomposed in some nice way. The larger classes considered by Gyárfás have much wilder structure, and do not appear to be amenable to decomposition techniques. So a different approach is required.

For a long time, all three conjectures appeared intractable. However, the three conjectures have now been proved: the first was proved in a paper with Seymour [74], giving the following bound.

Theorem 5.4. *For $k \geq 1$, every graph with chromatic number at least $2^{2^{k+2}}$ contains either a complete subgraph on k vertices or an odd hole.*

The doubly exponential bound is probably far from best possible. Indeed, there is only one obstacle in the proof that causes the bound to jump from single to double exponential, and it seems likely that this could be circumvented with new ideas. One approach would be to prove the Hoàng–McDiarmid conjecture [50], which says:

Conjecture 5.5. *Let G be a graph with no odd hole and at least one edge. Then the vertices of G can be partitioned into two sets such that every maximum clique in G intersects both sets.*

Conjecture 5.5 would imply immediately that graphs without odd holes satisfy $\chi(G) \leq 2^{\omega(G)}$.

The class of graphs without odd holes has also been of significant algorithmic interest. Following the proof of the Strong Perfect Graph Theorem, Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [18] showed that there is a polynomial-time algorithm to recognize perfect graphs (i.e., graphs with no odd hole and no odd antihole). However, it was only recently shown that it was shown that there is a polynomial-time algorithm to test for the presence of an odd hole [26]; indeed, the problem had been open since the 1980s (and there was reason to expect that the problem might be hard, as Bienstock showed that testing for the presence of an odd hole containing a specific vertex is NP-complete [8]). In subsequent work, it has been shown that finding a shortest odd hole [23] and an odd hole of at least a fixed length [25] can also be solved in polynomial time.

Conjectures 5.2 and 5.3 have also now been proved. The second conjecture was proved in a paper with Chudnovsky and Seymour [21], and the third in a paper with Chudnovsky, Seymour, and Spirkl [27] (both with significantly larger bounds). But this raises a natural further question: why ask only for *odd* holes? In the light of the (then) Strong Perfect Graph Conjecture, it was very natural for Gyárfás and others to think about holes of odd or even parity. However, motivated by topological considerations, Kalai and Meshulam [52,53] also conjectured that the class of graphs with no triangle and no hole of length divisible by 3 does not contain graphs of arbitrarily large chromatic number. This was proved in a breakthrough paper of Bonamy, Charbit, and Thomassé [12].

It turns out that much stronger results hold. The current state-of-the-art is the following, which was proved in a paper with Seymour [76].

Theorem 5.6. *For all integers $k \geq 0$ and $\ell \geq 1$, the class of graphs with no hole of length k modulo ℓ is χ-bounded.*

As an application, this resolves two further conjectures of Kalai and Meshulam, connecting the chromatic number of a graph with the homology of its independence complex.

It seems likely that even stronger results are true. Indeed, perhaps we can break away from parity conditions altogether and just use some sort of density condition:

Conjecture 5.7. *Let $A \subset \mathbb{N}$ be an infinite set with bounded gaps. Then the class of graphs that do not contain a hole of any length in A is χ-bounded.*

This has been proved in the special case of triangle-free graphs (in another paper with Seymour [75]). The proof is long and complicated, and extending it to the general case will require significant new ideas.

It would also be very interesting to answer the following question:

Conjecture 5.8. *Is there a set $A \subset \mathbb{N}$ with upper density 0 such that the class of graphs that do not contain any hole with length in A is χ-bounded?*

What can be said about the techniques? Proving these results has required a substantially different toolbox from the decomposition techniques used to study perfect graphs. The methods use a mixture of structural and extremal techniques, and can perhaps be thought of as a "rougher structural" approach.

A useful framework is provided by using the *local chromatic number*. For an integer $r \geq 0$ and a graph G, we define the *r-local chromatic number $\chi^{(r)}(G)$ of G* to be the maximum of $\chi(B)$ over all subgraphs B induced by r-balls in G (using the shortest path metric). The relationship between $\chi^{(r)}(G)$ and $\chi(G)$ is interesting: roughly speaking, it is interesting to distinguish between graphs in which some small ball has large chromatic number, and graphs where the chromatic number is not visible locally (for instance, if the graph is locally treelike) so that it is somehow "spread out" in the graph. More precisely, given any graph G with very large chromatic number, it is possible to drop to an induced subgraph G' with one of the two following properties:

- G' has large chromatic number and small r-local chromatic number;
- G' has large chromatic number, and every induced subgraph of G' with large chromatic number contains an r-ball with large chromatic number.

This framework was introduced in [73], and has been the starting point for many subsequent proofs. The "local" and "spread-out" cases have very different structural behaviors, and usually require very different methods.

6. INDUCED SUBDIVISIONS AND GEOMETRIC CONSTRUCTIONS

So far, we have discussed forests and cycles. But it is natural to ask whether we can ask for more complicated local structures. In 1997, it was conjectured in [73] that if we allow subdivisions, then any structure can be found:

Conjecture 6.1. *For every graph H, the class of graphs with no induced subdivision of H is χ-bounded.*

Equivalently, the conjecture claims that any graph with large chromatic number contains either a large clique or an induced subdivision of H. When H is a forest, this is true by Theorem 4.5 [73]; and when H is a cycle, this follows from the truth of Conjecture 5.2 [21,46]. Motivated by Conjecture 6.1, Kühn and Osthus [58] also proved the following beau-

tiful result, showing that if we forbid a complete *bipartite* graph then large minimum degree is already enough.

Theorem 6.2. *For every graph H and positive integer k, every graph with sufficiently large minimum degree contains either a complete graph K_k, a complete bipartite graph $K_{k,k}$ or a subdivision of H as an induced subgraph.*

As we will see below, Conjecture 6.1 ultimately turned out to be incorrect, but it remained open for more than 15 years. Part of the difficulty in finding a counterexample lies in the fact that we do not have many ways to generate structured examples of graphs with large chromatic number. For, example, random graphs provide a simple way to create graphs with large chromatic number; but typically they also have good expansion and connectivity properties, and contain subdivisions of any fixed graph H. And while there are many ways to construct examples of graphs with large chromatic number and (for example) no triangles, it is similarly hard to constrain their larger-scale structure.

One fruitful line of construction has come from considering geometric graphs. It is not enough to consider graphs embeddable on a fixed surface, as these have bounded chromatic number (this can be deduced easily from Euler's formula, which implies that graphs embeddable on a fixed surface have bounded degeneracy). But it is rather more interesting to consider *intersection graphs*: these have vertex set consisting of a family $\mathcal{C}$ of sets, with $A, B \in \mathcal{C}$ adjacent if $A \cap B$ is nonempty.

An important example is given by the intersection graph of a collection of axis-aligned boxes in $\mathbb{R}^d$. When $d = 1$, we obtain the family of *interval graphs*. These are well-known to be perfect [49]. When $d = 2$, we are considering intersections of rectangles in the plane: Asplund and Grünbaum [5] showed that these satisfy $\chi(G) = O(\omega(G)^2)$ (recently improved to $O(\omega \log \omega)$ by Chalermsook and Walczak [15]). However, in three dimensions more happens: Burling constructed triangle-free intersection graphs of boxes in three dimensions with arbitrarily large chromatic number ([14]; see also [69]). It follows that intersection graphs of families of boxes in d-dimensions are χ-bounded for $d = 1, 2$, but not for $d \geq 3$.

A larger class of two-dimensional intersection graphs is provided by the family of *string graphs*, namely intersection graphs of curves in the plane (see, for example, [65]). Many special families of string graphs have been of interest. For example, the intersection graphs of straight line segments in the plane: Erdős asked in the 1970s whether this family is χ-bounded. It was a surprise when, in 2014, Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [68] came up with a way to represent Burling's graphs in two dimensions. Their beautiful construction shows the following.

Theorem 6.3. *There are triangle-free intersection graphs of line segments in the plane that have arbitrarily large chromatic number.*

As a corollary, Conjecture 6.1 does not hold (for example, for any graph H that is obtained by subdividing every edge of a nonplanar graph). However, it remains an interesting problem to determine when the conjecture does hold. Chalopin, Esperet, Li, and Ossona de Mendez [16] analyzed the construction from [68] in detail, further limiting the graphs that

could satisfy Conjecture 6.1. In the case of string graphs, the problem was completely solved in [24], which also proved the following result:

Theorem 6.4. *Every string graph with large chromatic number contains a 2-ball with large chromatic number.*

Perhaps this is a necessary feature in any family of counterexamples to Conjecture 6.1? The following resuscitation of that conjecture is proposed in [77]. Informally:

Conjecture 6.5. *For every graph H, every graph with large chromatic number contains either a 2-ball with large chromatic number or an induced subdivision of H.*

7. THE ERDŐS–HAJNAL CONJECTURE

In this section, we look at the largest complete subgraph or independent set in a graph. Frank Ramsey [70] showed in 1930 that every infinite graph contains an infinite complete subgraph or stable set. The finite version of this result is the following:

Theorem 7.1. *For every $k \geq 1$ there is an integer $R(k)$ such that every graph with at least $R(k)$ vertices contains a complete subgraph or stable set of size k.*

So "large" graphs contain "large" homogeneous structures. But how large is large? Ramsey gave an explicit bound on $R(k)$, but a nice quantitative version of Ramsey's Theorem was proved by Erdős and Szekeres [40] in the 1930s:

Theorem 7.2. *Every graph with at least $\binom{s+t-2}{s-1}$ vertices contains either a complete subgraph of size s or a stable set of size t.*

By taking $s = t$, it follows that every graph on n vertices contains a complete subgraph or stable set of size at least $c_1 \log n$. On the other hand, by considering random graphs, it is not hard to show that most graphs on n vertices do not contain a complete subgraph or stable set of size more than $c_2 \log n$.

How does the picture change if we know something about the local structure of a graph? Erdős and Hajnal speculated in the 1980s [37,38] that H-free graphs exhibit a very different behavior:

Conjecture 7.3. *For every graph H, there is a constant $c = c(H) > 0$ such that the following holds: every H-free graph with n vertices has a complete subgraph or stable set with at least n^c vertices.*

In other words, if we exclude any induced subgraph then the largest stable set or complete subgraph that must occur jumps in size from logarithmic to polynomial. The Erdős–Hajnal Conjecture has become one of the central conjectures in graph theory.

Despite considerable work, Conjecture 7.3 is only known for a small family of graphs. There are a few small examples: complete graphs (this follows from the quantitative form of Ramsey's Theorem 7.2), the four-vertex path P_4 (P_4-free graphs are perfect),

and the bull (Chudnovsky and Safra [20]). The class of graphs H for which the Erdős–Hajnal Conjecture holds also satisfies two closure properties:

- It follows immediately from the statement of the conjecture that if it holds for H then it also holds for $\overline{H}$.
- Alon, Pach, and Solymosi [3] proved that the class of graphs H for which Erdős–Hajnal holds is closed under substitution (the operation of *substituting* a graph F for a vertex x of H deletes x and replaces it with a copy of F; every new vertex is joined to every vertex that was adjacent to x).

Recently, a new graph was added to the list [29]:

Theorem 7.4. *The Erdős–Hajnal Conjecture holds when H is a cycle of length 5.*

This was of particular interest, as it had been highlighted as an important case both by Erdős and Hajnal [38] and Gyárfás [47], and was part of the original motivation for the conjecture. However, the conjecture remains open even for the five vertex path and the best bound known for general graphs is due to Erdős and Hajnal [38], who showed that the conjecture holds with $e^{c\sqrt{\log n}}$ in place of n^c.

There has been substantial recent progress in looking at analogues of the Erdős–Hajnal Conjecture with more than one excluded graph. A hereditary class $\mathcal{G}$ of graphs has the *Erdős–Hajnal property* if there is $c > 0$ such that every $G \in \mathcal{G}$ has a stable set or complete subgraph with at least $|G|^c$ vertices. Thus the Erdős–Hajnal Conjecture says that the class of H-free graphs has the Erdős–Hajnal property.

One approach to proving that graph classes satisfy the Erdős–Hajnal property has been through looking at large *bipartite* structures. Disjoint sets A, B of vertices in a graph G are *complete* if G contains all edges between A and B and *anticomplete* if G contains no edges between A and B. There is a substantial body of work on finding this type of structure in various graph classes (see [32,39] and the sequence of papers starting with [28]). It is particularly helpful when it is possible to find *linear* complete or anticomplete pairs. A hereditary class $\mathcal{G}$ of graphs has the *strong Erdős–Hajnal property* if there is $\delta > 0$ such that every $G \in \mathcal{G}$ has disjoint sets A, B of at least δn vertices such that the pair (A, B) is either complete or anticomplete.

The strong Erdős–Hajnal property is useful for the following reason:

Lemma 7.5. *The strong Erdős–Hajnal property implies the Erdős–Hajnal property.*

So when does the strong Erdős–Hajnal property hold for the class of H-free graphs? By considering sparse random graphs (for instance, with $p \sim \log n/n$), it can be seen that H must be a forest; on the other hand, by considering complements of sparse random graphs, it follows that the complement of H must also be a forest. But if both H and its complement are forests, then H has at most four vertices, and the conjecture is already known for these cases. So it seems that the strategy gives us nothing. But here is an interesting result of Bousquet, Lagoutte, and Thomassé [13]:

Theorem 7.6. *For every positive integer t, the class of graphs G such that neither G nor its complement contains a t-vertex path as an induced subgraph satisfies the strong Erdős–Hajnal property.*

Thus the strong Erdős–Hajnal property holds if we exclude *two* graphs: one sparse (a path on t vertices) and one dense (the complement of a path on t vertices). Theorem 7.6 was extended by Choromanski, Falik, Liebenau, Patel, and Pilipczuk [17], and then further by Liebenau, Pilipczuk, Seymour, and Spirkl [60]. An optimal result was given in [28]:

Theorem 7.7. *Let T be a forest. Then the class of graphs G such that neither G nor its complement contains an induced copy of T satisfies the strong Erdős–Hajnal property.*

Since we must exclude both a forest and the complement of a forest to obtain the strong Erdős–Hajnal property, the result characterizes all hereditary classes that are defined by a finite set of excluded subgraphs and satisfy the strong Erdős–Hajnal property. (See [30] for an analogous result where we forbid all induced subdivisions of a single graph H in both G and its complement.)

We end the section by noting that there is a natural connection between the Erdős–Hajnal Conjecture and problems about χ-boundedness such as the Gyárfás–Sumner Conjecture: a graph with small chromatic number must contain large stable sets (as a coloring is a partition into stable sets); and the Erdős–Hajnal Conjecture tells us that H-free graphs have "large" cliques or stable sets. However, there is no immediate implication. For example, the class of triangle-free graphs has the Erdős–Hajnal Property, but contains graphs of arbitrarily large chromatic number. And Theorem 5.4 shows that the class of graphs with no odd holes is χ-bounded, but the bounds do not imply anything like polynomial behavior of cliques or stable sets. However, under some conditions it is possible to deduce the Erdős–Hajnal Property from χ-boundedness: we will discuss this in the next section.

8. POLYNOMIAL BOUNDS AND ESPERET'S CONJECTURE

So far, we have discussed classes in which the chromatic number is bounded as a function of the clique number, without considering what type of function provides the bound. In most of the results we have mentioned, the proofs give multiply exponential functions, either because there are repeated applications of Ramsey-type results, or because there is some blowup at the inductive step. In this section, we will be concerned with *polynomially χ-bounded* classes, namely classes $\mathcal{G}$ for which there is a *polynomial* function f such that $\chi(G) \leq f(\mathrm{cl}(G))$ for every $G \in \mathcal{G}$. Polynomially χ-bounded classes are of particularly interest because of their connection to the Erdős–Hajnal Conjecture: it follows immediately that any polynomially χ-bounded class has the Erdős–Hajnal property.

Esperet [41] made the remarkable (and provocative) conjecture that all χ-bounded classes are polynomially χ-bounded:

Conjecture 8.1. *If a hereditary class $\mathcal{G}$ is χ-bounded then it is polynomially χ-bounded.*

If Esperet's conjecture is true, then the Gyárfás–Sumner could be strengthened to the following:

Conjecture 8.2. *For every forest H, the class of H-free graphs is polynomially χ-bounded.*

Since the Gyárfás–Sumner Conjecture is only known for some small families of trees, the polynomial Gyárfás–Sumner Conjecture looks very challenging (and may well turn out to be incorrect). However, there has been some progress, and it is known for a few very small trees [81]:

Theorem 8.3. *The polynomial Gyárfás–Sumner Conjecture holds for every tree of diameter at most 3.*

Paths form a particularly interesting case. Let P_k be the path on k vertices. Graphs that are P_3- or P_4-free are well known to be perfect, so the polynomial Gyárfás–Sumner Conjecture follows immediately. However, in general, the best bounds are exponential, even when excluding paths. The current borderline case is the five vertex path, where until recently the best bound was exponential [42]. This was improved in [82]:

Theorem 8.4. *Every graph with chromatic number at least $k^{\log_2 k}$ contains either a clique on k vertices or an induced path on five vertices.*

This is just slightly superpolynomial, but it is not yet small enough to prove the Erdős–Hajnal Conjecture for P_5.

Polynomial bounds are also known when a tree and a complete bipartite graph are excluded. The following result [80] strengthens Theorem 4.4 and answers a question of Bonamy, Bousquet, Pilipczuk, Rzazewski, Thomassé, and Walczak [11]:

Theorem 8.5. *For every tree T, there is a polynomial $p(t)$ such that, for every $t \geq 1$, every graph with minimum degree at least $p(t)$ contains either an induced copy of T or a (not necessarily induced) copy of $K_{t,t}$.*

It seems likely that even more could hold. Indeed, Paul Seymour and I conjecture the following strengthening of Theorem 6.2:

Conjecture 8.6. *For every graph H there is a polynomial $p(t)$ such that, for every $t \geq 1$, every graph with minimum degree at least $p(t)$ contains either an induced subdivision of H or a (not necessarily induced) copy of $K_{t,t}$.*

9. GRAPH MINORS AND INDUCED SUBGRAPHS

Throughout this paper, we have been looking at the large-scale structural consequences of forbidding one or more induced subgraphs. In this final section, we compare this with the effects of excluding graph minors. A graph H is a *minor* of a graph G if H can be obtained from G by contracting edges and deleting edges and vertices (a *contraction* of an edge xy replaces the vertices x and y by a single vertex z adjacent to all other vertices

that were previously adjacent to x or to y; for simplicity, we will ignore loops and parallel edges). A class $\mathcal{G}$ of graphs is *minor-closed* if, whenever $G \in \mathcal{G}$ then all its minors are in $\mathcal{G}$.

Minor-closed classes arise in many contexts: for example, the class of all graphs embeddable on a fixed surface is minor-closed. For the plane, Wagner [86] proved the following result (which also follows from work of Kuratowski [59]):

Theorem 9.1. *A graph is planar if and only if it does not contain a minor of K_5 or $K_{3,3}$.*

The theory of graph minors was developed in a major series of papers by Robertson and Seymour. A celebrated result in this theory is the following [71]:

Theorem 9.2. *Let $G_1, G_2, \ldots$ be an infinite sequence of graphs. Then there are $i < j$ such that G_i is a minor of G_j.*

In other words, finite graphs are well-quasiordered under the excluded minor relation. A corollary of this is a vast extension of Theorem 9.1: for any class $\mathcal{G}$ of graphs that is closed under minors, there is a finite set $\mathcal{M}$ of graphs such that a graph G is in $\mathcal{G}$ if and only if it does not contain any graph in $\mathcal{M}$ as a minor. In other words, any minor-closed class has a finite set of minimal excluded minors.

A central result in the theory of graph minors is the *Graph Minor Structure Theorem*, which states (very roughly) that for every fixed graph H, any H-minor-free graph can be obtained by gluing together (in a treelike way) a sequence of graphs that can (almost) be embedded in surfaces of bounded genus. This is not a structural description, but can be thought of as an approximate structure theorem: the class $\mathcal{G}$ of H-minor-free graphs is contained in a class $\mathcal{G}'$ in which the graphs can all be built in a certain way, and which does not contain graphs that are much more "complex" than H.

So can anything similar be said for induced subgraphs? The class of finite graphs is not well-quasiordered by the induced subgraph relation: consider, for example, the class of cycles. So no theorem directly analogous to Theorem 9.2 can hold (see, for example, [61] for further discussion). On the positive side, there are good structural descriptions of H-free graphs for some very small H, although precise structural descriptions look intractable for larger H. For arbitrary H, it is known that every H-free graph can be partitioned into a bounded number of pieces that are either dense or sparse [31]; and there is a great deal known about the structure of *typical* H-free graphs (see, for instance, [1]). But what is really missing is an analogue for induced subgraphs of the Graph Minor Structure Theorem.

At the moment, it is not yet clear what such a theorem would say: what would the "basic" graph classes be? How would they be glued together? And would the theory describe the whole graph, or just some suitably well-structured "core"? But such a theorem could draw together a large body of work, and would have widespread applications. An essential part of this theory will be understanding the relationship between chromatic number and induced subgraphs; the size of cliques and independent sets will also be crucial. The Gyárfás–Sumner and Erdős–Hajnal Conjectures are major challenges in our understanding of induced subgraphs, and resolving either of them would be a substantial milestone in the development of a more general theory.

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ALEX SCOTT

Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom,
scott@maths.ox.ac.uk