THE POSITIVE GRASSMANNIAN, THE AMPLITUHEDRON, AND CLUSTER ALGEBRAS

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ABSTRACT

The *positive Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$ is the subset of the real Grassmannian where all Plücker coordinates are nonnegative. It has a beautiful combinatorial structure as well as connections to statistical physics, integrable systems, and scattering amplitudes. The *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is the image of the positive Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ under a positive linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$. We will explain how ideas from oriented matroids, tropical geometry, and cluster algebras shed light on the structure of the positive Grassmannian and the amplituhedron.

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1. INTRODUCTION

The *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$ or (informally) the *positive Grassmannian* [40, 47] can be defined as the subset of the real Grassmannian $\mathrm{Gr}_{k,n}$ where all Plücker coordinates are nonnegative. It has a beautiful decomposition into *positroid cells* [47, 49, 52], where each cell is obtained by specifying that certain Plücker coordinates are strictly positive and the rest are zero. Since the work of Lusztig [40] and Postnikov [47, 48], there has been an extensive study of the positive Grassmannian, including approaches involving cluster algebras, tropical geometry, and matroids and the moment map.

Remarkably, the positive Grassmannian has several applications in theoretical physics. For example, the stationary distribution of the *asymmetric simple exclusion process* (a model for particles hopping on a one-dimensional lattice with open boundaries), can be described in terms of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ [13]. In a different direction, each point C of the real Grassmannian gives rise to a *soliton solution of the KP equation* (modeling interaction patterns of shallow water waves), whose asymptotics are determined by the matroid of C, and which is regular for all times t if and only if C lies in the positive Grassmannian [35, 36]. In yet a third direction, the positive Grassmannian encodes most of the physical properties of *scattering amplitudes in planar* $\mathcal{N} = 4$ *super Yang–Mills theory* [3, 4, 12] (which compute probabilities that certain particles are produced in a collision involving other particles). This insight combined with an idea of Hodges [29] led Arkani-Hamed and Trnka to introduce the *amplituhedron* [7], defined as the image of the positive Grassmannian under the *amplituhedron map*. In particular, any $n \times (k + m)$ matrix Z whose maximal minors are positive induces a map $\tilde{Z}$ from $\mathrm{Gr}_{k,n}^{\geq 0}$ to the Grassmannian $\mathrm{Gr}_{k,k+m}$, whose image (of full dimension km) is the *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ [7]. When $m = 4$, the *BCFW recurrence* [11] for computing scattering amplitudes can be used to produce collections of $4k$-dimensional cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ whose images conjecturally subdivide ("tile" or "triangulate") the amplituhedron.

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ generalizes the positive Grassmannian (obtained when $k + m = n$), cyclic polytopes (when $k = 1$) [7], and cyclic hyperplane arrangements (when $m = 1$) [32]. Moreover, the amplituhedron has intriguing and beautiful mathematical properties, many of them conjectural. For instance, we conjecture that for even m, the number of top-dimensional strata comprising a tiling of $\mathcal{A}_{n,k,m}(Z)$ is equal to the number of plane partitions contained in the $k \times (n - k - m) \times \frac{m}{2}$ box [33]. As another example, despite the fact that they have different dimensions and one of them is not a polytope, the hypersimplex $\Delta_{k+1,n}$ and the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ are closely related: for example, *T-duality* gives a bijection between positroid tilings of $\Delta_{k+1,n}$ and positroid tilings of $\mathcal{A}_{n,k,2}(Z)$ [39, 46].

In this article we explain how ideas from the theory of matroids, tropical geometry, and cluster algebras shed light on the structure of positive Grassmannians and amplituhedra. We start in Section 2 by introducing the matroid stratification of the Grassmannian and the positroid cell decomposition of the positive Grassmannian. Given a surjective map ϕ from a cell complex X onto another topological space Y, we also introduce the notion of *ϕ-induced tiling* of Y, which we will study in the case that X is the positive Grassmannian (and call a *positroid tiling*). In Section 3 we study positroid tilings when ϕ is the moment map, which are subdivisions of the hypersimplex into positroid polytopes, and are related to the positive tropical Grassmannian. In Section 4 we introduce the amplituhedron, giving two equivalent definitions, defining natural coordinates, characterizing its points when $m = 1$ and 2, and defining its *sign stratification*, which is an analogue of the matroid stratification. In Section 5 we then study positroid tilings when ϕ is the amplituhedron map. We give a conjectural link to plane partitions, and discuss the positroid cells on which the amplituhedron map is injective. In Section 6 we explain a mysterious notion called *T-duality*, which relates positroid tiles and tilings of the hypersimplex $\Delta_{k+1,n}$ to positroid tiles and tilings for the amplituhedron $\mathcal{A}_{n,k,2}(Z)$. One manifestation of this duality is the fact that the number of realizable sign strata of $\mathcal{A}_{n,k,2}(Z)$ equals the volume of $\Delta_{k+1,n}$ (an Eulerian number). Finally, in Section 7 we present several connections between the amplituhedron and cluster algebras, proved for $m = 2$ but conjectural in general.

A great many mathematicians and physicists have made tremendous contributions to the study of the positive Grassmannian and amplituhedron; it is impossible to give a complete account here. The results described below in which I played a role are joint with various collaborators including F. Ardila, S. Karp, T. Lukowski, M. Parisi, K. Rietsch, F. Rincón, M. Sherman-Bennett, D. Speyer, K. Talaska, E. Tsukerman, and Y. Zhang.

2. THE POSITIVE GRASSMANNIAN AND MATROID STRATIFICATION

2.1. The Grassmannian and the matroid stratification

The *Grassmannian* $\mathrm{Gr}_{k,n} = \mathrm{Gr}_{k,n}(\mathbb{K})$ is the space of k-dimensional subspaces of an n-dimensional vector space $\mathbb{K}^n$. Let $[n]$ denote $\{1, \dots, n\}$, and $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$. We can represent a point $V \in \mathrm{Gr}_{k,n}$ as the row-span of a full-rank $k \times n$ matrix C; then, for $I \in \binom{[n]}{k}$, we let $p_I(V)$ be the $k \times k$ minor of C occupying the columns in I. The $p_I(V)$ are called the *Plücker coordinates* of V, and are independent of the choice of matrix representative C (up to common rescaling). The map $V \mapsto \{p_I(V)\}_{I \in \binom{[n]}{k}}$ embeds $\mathrm{Gr}_{k,n}$ into projective space. We will occasionally *identify C with its row-span.*

Definition 2.1. A *matroid* $\mathcal{M}$ is a pair $(E, \mathcal{B})$, where E is a finite set and $\mathcal{B}$ a nonempty collection of subsets of E called *bases*, such that if B_1, B_2 are distinct bases and $b_1 \in B_1 \setminus B_2$, then there exists an element $b_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{b_1\}) \cup \{b_2\}$ is a basis.

Matroid theory originated in the 1930s as a combinatorial model that keeps track of, and abstracts, the dependence relations among a set of vectors.

Definition 2.2. Any full-rank $k \times n$ matrix C (with entries in a field $\mathbb{K}$), and consequently any point $C \in \mathrm{Gr}_{k,n}(\mathbb{K})$, gives rise to a matroid $\mathcal{M}(C) := ([n], \mathcal{B})$, where $\mathcal{B} = \{I \in \binom{[n]}{k} \mid p_I(C) \neq 0\}$. Such matroids are called *realizable* or *representable* over $\mathbb{K}$.

Example 2.3. Consider the full rank matrix

$$C = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \end{pmatrix} \quad \text{(or the corresponding point } C \in \mathrm{Gr}_{2,4}\text{).}$$

Here $p_{12}(C) = 1$, $p_{13}(C) = 2$, $p_{14}(C) = 4$, $p_{23}(C) = 1$, $p_{24}(C) = 2$, and $p_{34}(C) = 0$.

The corresponding matroid is $\mathcal{M}(C) = \{[4], \mathcal{B}\}$ where $\mathcal{B} = \{12, 13, 14, 23, 24\}$.

In what follows, we will be concerned with the *real* Grassmannian $\mathrm{Gr}_{k,n} = \mathrm{Gr}_{k,n}(\mathbb{R})$. While every full rank matrix gives rise to a matroid, there are many matroids which are *not* realizable (say over $\mathbb{R}$), that is, they cannot be realized by (real) matrices. The *non-Pappus matroid* is a matroid which is not realizable over any field.

The *matroid stratification* of the Grassmannian is the decomposition of $\mathrm{Gr}_{k,n}$ into strata consisting of all points with the same matroid. While this stratification has many beautiful properties [24], we also know that by Mnëv's universality theorem [43], a matroid stratum can have topology as bad as that of any algebraic variety!

One running theme in this article will be that matroids and the matroid stratification of the Grassmannian can exhibit pathological behavior, but when one adds the adjective "positive" to the picture, this bad behavior is replaced by the nicest possible statements.

2.2. The positive Grassmannian

Definition 2.4 ([40, 47]). We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if (up to a global change of sign) $p_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$. Similarly, V is *totally positive* if $p_I(V) > 0$ for all $I \in \binom{[n]}{k}$. We let $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k,n}^{>0}$ denote the set of totally nonnegative and totally positive elements of $\mathrm{Gr}_{k,n}$, respectively; $\mathrm{Gr}_{k,n}^{\geq 0}$ is called the *totally nonnegative Grassmannian*, or sometimes just the *positive Grassmannian*.

Note that the matrix C from Example 2.3 represents an element of $\mathrm{Gr}_{2,4}^{\geq 0}$.

The positive and nonnegative parts of a generalized partial flag variety G/P were first introduced by Lusztig [40], who gave a Lie-theoretic definition of $(G/P)_{>0}$ and $(G/P)_{\geq 0} := \overline{(G/P)_{>0}}$. Postnikov [47] subsequently defined $\mathrm{Gr}_{k,n}^{\geq 0}$ as in Definition 2.4. These definitions agree when $G/P = \mathrm{Gr}_{k,n}$ [51], [64, COROLLARY 1.2].

While the positive Grassmannian was introduced rather recently, the theory of totally positive matrices is much older. In fact, one can use results of Gantmakher and Krein [23] from 1950 to characterize $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k,n}^{>0}$ in terms of sign variation [31].

Definition 2.5. Given $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of sign changes of v, when v is viewed as a sequence of n numbers and zeros are ignored. We also define

$$\overline{\mathrm{var}}(v) := \max\{\mathrm{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } i \in [n] \text{ with } v_i \neq 0\},$$

i.e., $\overline{\mathrm{var}}(v)$ is the maximum number of sign changes after we choose a sign for each $v_i = 0$.

For example, if $v := (2, 0, 2, -1) \in \mathbb{R}^4$, then $\mathrm{var}(v) = 1$ and $\overline{\mathrm{var}}(v) = 3$.

The following result is based on [23, THEOREMS V.3, V.7, V.1, V.6].

Theorem 2.6 ([31, THEOREM 1.1]). *Let $V \in \mathrm{Gr}_{k,n}$ with orthogonal complement $V^{\perp} \in \mathrm{Gr}_{n-k,n}$.*

(i) $V \in \mathrm{Gr}_{k,n}^{\geq 0} \iff \mathrm{var}(v) \leq k-1 \quad \forall v \in V \iff \overline{\mathrm{var}}(w) \geq k \quad \forall w \in V^{\perp}$.

(ii) $V \in \mathrm{Gr}_{k,n}^{>0} \iff \overline{\mathrm{var}}(v) \leq k-1 \quad \forall v \in V \setminus \{0\} \iff \mathrm{var}(w) \geq k \quad \forall w \in V^{\perp} \setminus \{0\}$.

2.3. The positroid cell decomposition

Despite the fact that the topology of matroid strata can be very bad, Postnikov realized that if one intersects these strata with the positive Grassmannian, one obtains a *cell decomposition* [47]. In fact, it is a regular CW decomposition [21, 49, 53, 67].

Theorem 2.7 ([47]). *For $\mathcal{M} \subseteq \binom{[n]}{k}$, let*

$$S_{\mathcal{M}} := \{V \in \mathrm{Gr}_{k,n}^{\geq 0} \mid p_I(V) > 0 \text{ if and only if } I \in \mathcal{M}\}.$$

Then $\mathrm{Gr}_{k,n}^{\geq 0} = \cup S_{\mathcal{M}}$ is a cell decomposition, i.e., each $S_{\mathcal{M}}$ is an open ball.

If $S_{\mathcal{M}} \neq \emptyset$, we call $\mathcal{M}$ a positroid *and $S_{\mathcal{M}}$ its* positroid cell.

More generally, Rietsch gave a cell decomposition of $(G/P)_{\geq 0}$ [52]. When $G/P = \mathrm{Gr}_{k,n}$, the two cell decompositions agree [64, COROLLARY 1.2].

As shown in [47] and explained below, the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ can be indexed by combinatorial objects such as *decorated permutations* π or move-equivalence classes of *plabic graphs* G, see, e.g., [19, CHAPTER 7]. We will correspondingly refer to such cells as S_π and S_G.

Definition 2.8. A *decorated permutation* on $[n]$ is a permutation $\pi \in S_n$ whose fixed points are each colored either black ("loop") or white ("coloop"). We denote a black fixed point i by $\pi(i) = \underline{i}$, and a white fixed point i by $\pi(i) = \overline{i}$. An *antiexcedance* of a decorated permutation π is an element $i \in [n]$ such that either $\pi^{-1}(i) > i$ or $\pi(i) = \overline{i}$. We say that a decorated permutation on $[n]$ is of *type* (k, n) if it has k antiexcedances.

For example, $\pi = (3, \underline{2}, 5, 1, 6, 8, \overline{7}, 4)$ has a loop in position 2 and a coloop in position 7. Its antiexcedances are 1, 4, and 7.

Definition 2.9. To a $k \times n$ matrix C with columns $(c^1, \dots, c^n)$ representing an element of $\mathrm{Gr}_{k,n}^{\geq 0}$, we associate a decorated permutation $\pi := \pi_C$ of type (k, n) as follows. We set $\pi(i) := j$ to be the label of the first column j such that $c^i \in \mathrm{span}\{c^{i+1}, c^{i+2}, \dots, c^j\}$, where the columns are listed in cyclic order (going from c^n to c^1 if $i + 1 > j$). If $c^i = \mathbf{0}$, then i is a *loop* of matroid $\mathcal{M}(C)$ and we set $\pi(i) = \underline{i}$, and if c^i is not in the span of the other column vectors, then i is a *coloop* of $\mathcal{M}(C)$ and we set $\pi(i) = \overline{i}$.

The construction above gives a well-defined map from $\mathrm{Gr}_{k,n}^{\geq 0}$ to decorated permutations of type (k, n). If C is the matrix from Example 2.3, then $\pi_C = (3, 1, 4, 2)$.

Proposition 2.10. *Let π be a decorated permutation of type (k, n), and let*

$$S_\pi = \{C \in \mathrm{Gr}_{k,n}^{\geq 0} \mid \pi_C = \pi\}.$$

Then S_π is a positroid cell, and all positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ have the form S_π for some decorated permutation π of type (k, n).

Definition 2.11. A *planar bicolored graph* (or *plabic graph*) is a planar graph G properly embedded into a closed disk with (uncolored) vertices lying on the boundary of the disk labeled $1, \dots, n$ in clockwise order for some positive n, such that: each boundary vertex is

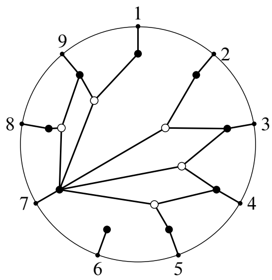

FIGURE 1
A plabic graph G with $\pi_G = (8, 5, 9, 2, 3, \underline{6}, 4, 1, 7)$. It has a black lollipop at 6.

incident to a single edge; each internal vertex is colored black or white; and each internal vertex is connected by a path to some boundary vertex. See Figure 1.

If a boundary vertex i is attached to an edge whose other endpoint is a leaf, we call this component a *lollipop*. *We will assume that* G *has no internal leaves except for lollipops*.

We next describe some local moves on plabic graphs, see Figure 2.

(M1) Square Move. If there is a square formed by four trivalent vertices whose colors alternate, then we can switch the colors of these four vertices.

(M2) Two adjacent internal vertices of the same color can be merged. Alternatively, we can split an internal vertex into two vertices of the same color joined by an edge.

(M3) We can remove/add degree 2 vertices, as shown.

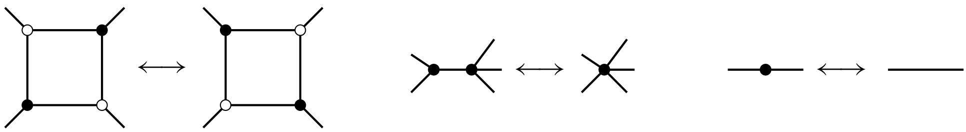

FIGURE 2
Moves (M1), (M2), (M3) on plabic graphs.

Definition 2.12. Two plabic graphs are *move-equivalent* if they can be obtained from each other by moves (M1)–(M3). A plabic graph is *reduced* if there is no graph move-equivalent to it in which two adjacent vertices u and v are connected by more than one edge.

Definition 2.13 and Proposition 2.16 give several ways to read a positroid off of a plabic graph. The positroid depends only on the move-equivalence class of the plabic graph.

Definition 2.13. Let G be a reduced plabic graph as above with boundary vertices $1, \ldots, n$. For each boundary vertex $i \in [n]$, we follow a path along the edges of G starting at i, turning (maximally) right at every internal black vertex, and (maximally) left at every internal white vertex. This path ends at some boundary vertex $\pi(i)$. By [47, SECTION 13], the fact that G is reduced implies that each fixed point of π is attached to a lollipop; we color each fixed point by the color of its lollipop. In this way we obtain the *(decorated) trip permutation* $\pi_G = \pi$ of G. We say that G is of *type* (k, n), where k is the number of antiexcedances of π_G.

In Figure 1 we have $\pi_G = (8, 5, 9, 2, 3, \underline{6}, 4, 1, 7)$, which has $k = 5$ antiexcedances.

Theorem 2.14 (Fundamental theorem of reduced plabic graphs, [47, THEOREM 13.4], see also [19, THEOREM 7.4.25]). *Let G and G' be reduced plabic graphs. Then G and G' are move-equivalent if and only if G and G' have the same decorated trip permutation.*

Definition 2.15. Let G be a bipartite plabic graph in which each boundary vertex is incident to a white vertex. An *almost perfect matching* of G is a subset M of edges such that each internal vertex is incident to exactly one edge in M (and each boundary vertex i is incident to either one or no edges in M). We let $\partial M = \{i \mid i \text{ is incident to an edge of } M\}$.

Given a plabic graph, we can use move (M3) to ensure that the resulting graph is bipartite and that each boundary vertex is incident to a white vertex. (Note that we can think of such a graph as a bipartite graph G in which all boundary vertices are colored black.)

Proposition 2.16 ([47, PROPOSITION 11.7, LEMMA 11.10]). *Let G be a bipartite plabic graph such that each boundary vertex is incident to a white vertex. Let*

$$\mathcal{M}(G) = \{\partial M \mid M \text{ an almost perfect matching of } G\}.$$

If $\mathcal{M}(G)$ is nonempty, then $\mathcal{M}(G)$ is the set of bases of a positroid on $[n]$. Moreover, all positroids arise from plabic graphs.

See Figure 3 for an example.

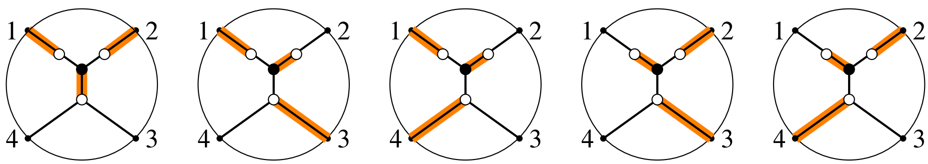

FIGURE 3
A bipartite plabic graph G_1 with $\pi_{G_1} = (3, 1, 4, 2)$ which has five almost-perfect matchings. The corresponding positroid is $([4], \mathcal{M}(G_1))$ where $\mathcal{M}(G_1) = \{12, 13, 14, 23, 24\}$.

Postnikov used plabic graphs to give parameterizations of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ [47]; this result can be recast in terms of *flows* [63] or as a variant of a theorem of Kasteleyn [56].

Theorem 2.17 ([47, 56, 63]). *Let G be a bipartite plabic graph with n boundary vertices, all of which are colored black. Suppose G has at least one almost perfect matching M_0, and let $k = |\partial M_0|$. Let $w : \mathrm{Edges}(G) \to \mathbb{R}_{>0}$ be any weight function, and for M an almost perfect* *matching, let* $w(M) := \prod_{e \in M} w_e$*, where* w_e *denotes the weight of edge* e*. Then there is a* $k \times n$ *matrix* $L = L(w)$ *representing a point of* $\mathrm{Gr}_{k,n}^{\geq 0}$ *such that*

$$p_I(L) = \sum_{M:\partial M = I} w(M) \quad \textit{for all } I \in \binom{[n]}{k}.$$

Moreover, if we let w *vary over weight functions, we obtain a positroid cell*

$$S_G := \{L(w) \mid w : \mathrm{Edges}(G) \to \mathbb{R}_{>0}\}.$$

If G is a tree, we call S_G a *tree positroid cell.*

Remark 2.18. If G is a plabic graph as in Theorem 2.17 which is reduced, with decorated permutation π_G and almost perfect matchings $\mathcal{M}(G)$, we have that $S_G = S_{\mathcal{M}(G)} = S_{\pi_G}$ [47]. So we can index positroid cells by plabic graphs, bases, or decorated permutations.

2.4. ϕ-induced subdivisions and positroid tilings

Given a surjective map $\phi : X \to Y$ from a cell complex X onto a topological space Y, it is natural to try to decompose Y using images of cells under ϕ.

Definition 2.19. Let $X = \bigsqcup_\pi S_\pi$ be a cell complex and let $\phi : X \to Y$ be a continuous surjective map onto Y, a d-dimensional cell complex or subset thereof. We define a *ϕ-induced dissection of* Y to be a collection $\{\overline{\phi(S_\pi)} \mid \pi \in \mathcal{C}\}$ of images of cells of X, indexed by the set $\mathcal{C}$, such that their union $\bigcup_{\pi \in \mathcal{C}} \overline{\phi(S_\pi)}$ equals Y, the interiors are pairwise disjoint, i.e., $\phi(S_\pi) \cap \phi(S_{\pi'}) = \emptyset$ for $\pi \neq \pi' \in \mathcal{C}$, and $\dim(\phi(S_\pi)) = d$ for all $\pi \in \mathcal{C}$;

We call a dissection $\{\overline{\phi(S_\pi)} \mid \pi \in \mathcal{C}\}$ a *ϕ-induced tiling* if additionally ϕ is injective on each S_π for $\pi \in \mathcal{C}$. And we call a dissection $\{\overline{\phi(S_\pi)} \mid \pi \in \mathcal{C}\}$ a *ϕ-induced subdivision* if whenever $\overline{\phi(S_\pi)} \cap \overline{\phi(S_{\pi'})} \neq \emptyset$, this intersection equals $\overline{\phi(S_{\pi''})}$, where $S_{\pi''}$ lies in $\overline{S_\pi} \cap \overline{S_{\pi'}}$.

When $\phi : X \to Y$ is an affine projection of convex polytopes, the above notion of ϕ-induced subdivision recovers Billera–Sturmfels' notion of ϕ-induced polyhedral subdivision [9]. The subdivisions which are also ϕ-induced tilings are their *tight* ϕ-induced subdivisions. The relation to polyhedral subdivisions suggests a number of questions. What can one say about the *Baues poset* of ϕ-induced subdivisions, partially ordered by refinement? What can one say about the *flip graph*, the restriction of the Hasse diagram of $\omega(\phi : X \to Y)$ to elements of rank 0 and 1? When is it connected? Is there an analogue of *fiber polytopes* [9] (perhaps along the lines of [41]), which control the *coherent* π-induced subdivisions?

Definition 2.20. In Definition 2.19, let X be the positive Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ with its positroid cell decomposition. For S_π a d-dimensional positroid cell, we say that $\overline{\phi(S_\pi)}$ is a *positroid tile* (for ϕ) if ϕ is injective on S_π. We will refer to ϕ-induced dissections, tilings, and subdivisions, as *positroid dissections, tilings*, and *subdivisions*. Our notion of positroid subdivision is closely related to the *good dissections* studied in [39].

In this article we will take X to be the positive Grassmannian, and consider the case where ϕ is the *moment map* (Section 3) or the *amplituhedron map* (Section 5).

3. THE MOMENT MAP AND POSITROID TILINGS

The foundational 1987 paper of Gelfand–Goresky–MacPherson–Serganova [24] initiated the study of the Grassmannian and its matroid stratification via the moment map. Here we will consider the restriction of the moment map to the positive Grassmannian.

Given a subset $I \subset [n]$ and a point $x \in \mathbb{R}^n$, we use the notation $x_I := \sum_{i \in I} x_i$. We also let $e_I := \sum_{i \in I} e_i \in \mathbb{R}^n$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

The torus $T = (\mathbb{C}^*)^n$ acts on $\mathrm{Gr}_{k,n}$ by scaling the columns of a matrix representative C. This torus action gives rise to a *moment map* $\mu : \mathrm{Gr}_{k,n} \to \mathbb{R}^n$.

Definition 3.1. Let $C \in \mathrm{Gr}_{k,n}$. The *moment map* $\mu : \mathrm{Gr}_{k,n} \to \mathbb{R}^n$ is defined by

$$\mu(C) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(C)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(C)|^2}.$$

Let TC denote the orbit of C under the action of T, and $\overline{TC}$ its closure. It follows from [26] that the image $\mu(\overline{TC})$ is a convex polytope, whose vertices are the images of the torus-fixed points. This polytope is the *matroid polytope* $\Gamma_{\mathcal{M}(C)}$ [24], as defined below.

Definition 3.2. Given a matroid $\mathcal{M} = ([n], \mathcal{B})$, the (basis) *matroid polytope* $\Gamma_{\mathcal{M}}$ is the convex hull of the indicator vectors of the bases of $\mathcal{M}$,

$$\Gamma_{\mathcal{M}} := \mathrm{conv}\{e_B \mid B \in \mathcal{B}\} \subset \mathbb{R}^n.$$

A special case of a matroid polytope is the *hypersimplex* $\Delta_{k,n}$, the convex hull of all points e_I for $I \in \binom{[n]}{k}$. We have that $\mu(\mathrm{Gr}_{k,n}) = \Delta_{k,n}$.

3.1. Classification of positroid tiles for the moment map

Now let us consider the positive analogues of some of the above objects. If $\mathcal{M}$ is a *positroid*, the matroid polytope $\Gamma_{\mathcal{M}}$ is called a *positroid polytope*. If one restricts the moment map to $\mathrm{Gr}_{k,n}^{\geq 0}$, one can show that the moment map image $\mu(\overline{S_{\mathcal{M}}}) = \overline{\mu(S_{\mathcal{M}})}$ of $S_{\mathcal{M}}$ is precisely the corresponding positroid polytope $\Gamma_{\mathcal{M}}$ [65, PROPOSITION 7.10]. In particular, the moment map image of $\mathrm{Gr}_{k,n}^{\geq 0}$ is again the hypersimplex $\Delta_{k,n}$. If $\mathcal{M}$ is the positroid associated to a cell S_π or S_G, we also use the notation Γ_π and Γ_G to refer to $\Gamma_{\mathcal{M}}$. See Figure 4.

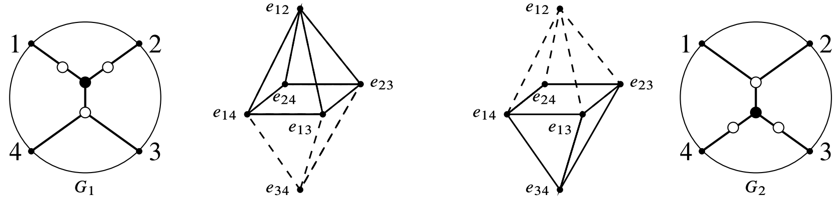

FIGURE 4

Positroid polytopes Γ_{G_1} and Γ_{G_2} associated to graphs G_1 and G_2, cf. Figure 3.

Applying Definition 2.20 to the moment map $\mu : \mathrm{Gr}_{k,n}^{\geq 0} \to \Delta_{k,n}$ onto the hypersimplex, a polytope of dimension $n-1$, we see that a positroid tile is the (closure of the) image of an $(n-1)$-dimensional positroid cell on which the moment map is injective.

Proposition 3.3 ([39, PROPOSITION 3.16], based on [1, 65]). *The positroid tiles for the moment map are exactly the positroid polytopes Γ_G associated to the tree positroid cells S_G. Two positroid tiles Γ_G and $\Gamma_{G'}$ are the same if and only if G and G' are related by move (M2).*

3.2. The positive tropical Grassmannian and positroid subdivisions

How can one produce positroid tilings and, more generally, positroid subdivisions of the hypersimplex $\Delta_{k,n}$? One way is to use the *positive tropical Grassmannian*.

The *tropical Grassmannian* $\mathrm{Trop}\,\mathrm{Gr}_{k,n}$ [27, 34, 57] is the space of *realizable tropical linear spaces*, obtained by applying the valuation map to elements of the Grassmannian $\mathrm{Gr}_{k,n}(K)$ over the field $K = \mathbb{C}\{\{t\}\}$ of *Puiseux-series*. Meanwhile the *Dressian* $\mathrm{Dr}_{k,n}$ is the space of *tropical Plücker vectors* $P = \{P_I\}_{I \in \binom{[n]}{k}}$, also known as *valuated matroids*. Thinking of each $P \in \mathrm{Dr}_{k,n}$ as a *height function* on the vertices of the hypersimplex $\Delta_{k,n}$, one can show that the Dressian parameterizes regular matroid subdivisions $\mathcal{D}_P$ of $\Delta_{k,n}$ [30, 56], which in turn are dual to *abstract tropical linear spaces* [56].

There are positive notions of both of the above spaces. The *positive tropical Grassmannian* [58] is the space of *realizable positive tropical linear spaces*, obtained by applying the valuation map to Puiseux-series valued elements of the positive Grassmannian. The *positive Dressian* is the space of *positive tropical Plücker vectors*.

Definition 3.4. We say $P = \{P_I\}_{I \in \binom{[n]}{k}} \in \mathbb{R}^{\binom{[n]}{k}}$ is a *positive tropical Plücker vector* if for each three-term Plücker relation, it lies in the positive part of the associated tropical hypersurface: for $1 \leq a < b < c < d \leq n$ and $S \in \binom{[n]}{k-2}$ disjoint from $\{a, b, c, d\}$, either

$$P_{Sac} + P_{Sbd} = P_{Sab} + P_{Scd} \leq P_{Sad} + P_{Sbc} \quad \text{or}$$
$$P_{Sac} + P_{Sbd} = P_{Sad} + P_{Sbc} \leq P_{Sab} + P_{Scd}.$$

The *positive Dressian* $\mathrm{Dr}_{k,n}^{+}$ is the set of positive tropical Plücker vectors.

In general, the Dressian $\mathrm{Dr}_{k,n}$ contains the tropical Grassmannian $\mathrm{Gr}_{k,n}$ but is much larger [28]. However, the situation for their positive parts is different, see [59] and [5].

Theorem 3.5 ([59, THEOREM 3.9]). *The positive tropical Grassmannian* $\mathrm{Trop}^{+}\,\mathrm{Gr}_{k,n}$ *equals the positive Dressian* $\mathrm{Dr}_{k,n}^{+}$. *That is, all abstract positive tropical linear spaces are realizable.*

There are two natural fan structures on the Dressian—the Plücker fan and the secondary fan—which coincide [45]. The cones of $\mathrm{Trop}^{+}\,\mathrm{Gr}_{k,n}$ control the *regular subdivisions* of $\Delta_{k,n}$ into positroid polytopes, with maximal cones giving rise to positroid tilings.

Consider a point $P = \{P_I\}_{I \in \binom{[n]}{k}} \in \mathbb{R}^{\binom{[n]}{k}}$, which we also think of as a real-valued function $\{e_I\} \mapsto P_I$ on the vertices of $\Delta_{k,n}$. We define a polyhedral subdivision $\mathcal{D}_P$ of $\Delta_{k,n}$ as follows: consider the points $(e_I, P_I) \in \Delta_{k,n} \times \mathbb{R}$ and take their convex hull. Take the lower faces (those whose outwards normal vector have last component negative) and project them down to $\Delta_{k,n}$. This gives us the *regular subdivision* $\mathcal{D}_P$ of $\Delta_{k,n}$. Note that $\mathcal{D}_P$ is a *polytopal subdivision* and that its facets (top-dimensional faces) comprise a positroid subdivision for the moment map (cf. Definition 2.20).

Theorem 3.6 ([39, THEOREM 9.12], see also [5]). *The point $P = \{P_I\}_{I \in \binom{[n]}{k}}$ is a positive tropical Plücker vector if and only if every face of $\mathcal{D}_P$ is a positroid polytope.*

Example 3.7. Consider a positive tropical Plücker vector $P = \{P_I\}_{I \in \binom{[4]}{2}} \in \mathbb{R}^{\binom{[4]}{2}}$. If we have $P_{13} + P_{24} = P_{14} + P_{23} < P_{12} + P_{34}$, then P induces the subdivision $\mathcal{D}_P$ of $\Delta_{2,4}$ into the two square pyramids shown in Figure 4. If $P_{13} + P_{24} = P_{12} + P_{34} < P_{14} + P_{23}$, we get the subdivision of $\Delta_{2,4}$ into two square pyramids separated by the square with vertices $e_{12}, e_{13}, e_{24}, e_{34}$, see Figure 7. These are both positroid tilings of $\Delta_{2,4}$.

Positroid tilings are particularly nice. The following result refines [59, THEOREM 6.6].

Theorem 3.8 ([59, THEOREM 6.6]). *Let $P \in \mathrm{Trop}^+ \mathrm{Gr}_{k,n}$ and consider the positroid subdivision $\mathcal{D}_P$. The following statements are equivalent:*

- *The facets of $\mathcal{D}_P$ comprise a positroid tiling.*
- *The facets of $\mathcal{D}_P$ comprise a finest positroid subdivision.*
- *Every face of $\mathcal{D}_P$ is the matroid polytope of a series-parallel matroid.*
- *Every octahedron in $\mathcal{D}_P$ is subdivided.*

Combining Theorem 3.8 with Speyer's f-vector theorem [55] gives the following.

Corollary 3.9. *Let $\mathcal{D}_P$ be a positroid tiling of $\Delta_{k,n}$ as in Theorem 3.8. Then this polyhedral subdivision contains precisely $\frac{(n-c-1)!}{(k-c)!(n-k-c)!(c-1)!}$ interior faces of dimension $n - c$. In particular, $\mathcal{D}_P$ consists of precisely $\binom{n-2}{k-1}$ positroid tiles.*

3.3. Realizability of positively oriented matroids

Positroid polytopes and the positive tropical Grassmannian have applications to realizability questions. One of the early goals of matroid theory was to find axioms that characterized the realizable matroids, i.e., those that arise from full rank matrices as in Definition 2.2. However, this problem (over a field of characteristic 0) is now considered to be intractible [42, 66]: in Vámos's words, "the missing axiom of matroid theory is lost forever."

There is a notion of *oriented matroids*, in which bases have signs: an oriented matroid is a matroid $([n], \mathcal{B})$ together with a *chirotope*, a function $\chi : [n]^k \to \{0, +, -\}$ obeying certain axioms which roughly encode the three-term Plücker relations [10]. If M is a full rank $k \times n$ real matrix, the function taking $(i_1, i_2, \ldots, i_k)$ to the sign of the minor using columns $(i_1, i_2, \ldots, i_k)$ is a chirotope, and the realizable oriented matroids are precisely the chirotopes occurring in this way. Thus, if M represents a point of the positive Grassmannian, then M gives a chirotope χ with $\chi(i_1, i_2, \ldots, i_k) \in \{0, +\}$ for $1 \le i_1 < i_2 < \cdots < i_k \le n$.

A *positively oriented matroid* is a chirotope χ such that $\chi(i_1, i_2, \ldots, i_k) \in \{0, +\}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The following was conjectured by da Silva in 1987 [14].

Theorem 3.10 ([2,59]). *Every positively oriented matroid is realizable. In other words, every positively oriented matroid is a positroid.*

The first proof used the geometry of positroid polytopes [2]. The second used the fact that if $P \in \operatorname{Trop}\operatorname{Gr}_{k,n}$, every face of $\mathcal{D}_P$ corresponds to a realizable matroid [59].

4. THE AMPLITUHEDRON AND THE SIGN STRATIFICATION

Building on [3], Arkani-Hamed and Trnka [7] introduced the *(tree) amplituhedron*, which is the image of the positive Grassmannian under a positive linear map. Let $\operatorname{Mat}_{n,p}^{>0}$ denote the set of $n \times p$ matrices whose maximal minors are positive. Throughout this section we will make the convention that k, n, m are positive integers such that $k + m \leq n$.

Definition 4.1. Let $Z \in \operatorname{Mat}_{n,k+m}^{>0}$. The *amplituhedron map* $\tilde{Z} : \operatorname{Gr}_{k,n}^{\geq 0} \to \operatorname{Gr}_{k,k+m}$ is defined by $\tilde{Z}(C) := CZ$, where C is a $k \times n$ matrix representing an element of $\operatorname{Gr}_{k,n}^{\geq 0}$, and CZ is a $k \times (k+m)$ matrix representing an element of $\operatorname{Gr}_{k,k+m}$. The *amplituhedron* $\mathcal{A}_{n,k,m}(Z) \subset \operatorname{Gr}_{k,k+m}$ is the image $\tilde{Z}(\operatorname{Gr}_{k,n}^{\geq 0})$.

The condition $Z \in \operatorname{Mat}_{n,k+m}^{>0}$ ensures that $\operatorname{rank}(CZ) = k$ and hence $\tilde{Z}$ is well defined [7]. This condition can be relaxed: the map $\tilde{Z}$ is well defined if and only if $\operatorname{var}(v) \geq k$ for all nonzero $v \in \ker(Z)$ [31, THEOREM 4.2].

If $k + m = n$, $\mathcal{A}_{n,k,m}(Z)$ is isomorphic to $\operatorname{Gr}_{k,k+m}^{\geq 0}$. The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ has full dimension km inside $\operatorname{Gr}_{k,k+m}$, but it does not lie inside $\operatorname{Gr}_{k,k+m}^{\geq 0}$ in general.

Example 4.2. If $k = 1$ and $m = 2$, $\mathcal{A}_{n,1,2}(Z)$ is a polygon in projective space $\mathbb{P}^2$. To see this, let $Z_1, \ldots, Z_n$ denote the rows of $Z \in \operatorname{Mat}_{n,3}^{>0}$; we can think of each Z_i as a point in $\mathbb{P}^2$. Since $Z \in \operatorname{Mat}_{n,3}^{>0}$, the points $Z_1, \ldots, Z_n$ are arranged in convex position like the vertices of a polygon, see Figure 8. Elements $C \in \operatorname{Gr}_{1,n}^{\geq 0}$ are nonzero vectors with coordinates in $\mathbb{R}_{\geq 0}$. If $C = e_i$ then $CZ = Z_i$, so all points Z_i lie in $\mathcal{A}_{n,1,2}$. As C varies over $\operatorname{Gr}_{1,n}^{\geq 0}$, the image CZ varies over all convex combinations of the Z_i's, and hence $\mathcal{A}_{n,1,2}(Z)$ is an n-gon. More generally, it follows from [61] that $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* in $\mathbb{P}^m$.

Physicists are most interested in the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. Since $\mathcal{A}_{n,k,m}(Z) \subset \operatorname{Gr}_{k,k+m}$, when m is small it is sometimes more convenient to take orthogonal complements and work with $\operatorname{Gr}_{m,k+m}$ instead of $\operatorname{Gr}_{k,k+m}$. This motivates the following definition of the *B-amplituhedron*, which is homeomorphic to the "A-amplituhedron."

Definition 4.3 ([32, DEFINITION 3.8]). Choose $Z \in \operatorname{Mat}_{n,k+m}^{>0}$, and let $W \in \operatorname{Gr}_{k+m,n}^{>0}$ be the column span of Z. We define the *B-amplituhedron* to be

$$\mathcal{B}_{n,k,m}(W) := \{V^{\perp} \cap W \mid V \in \operatorname{Gr}_{k,n}^{\geq 0}\} \subset \operatorname{Gr}_m(W),$$

where $\operatorname{Gr}_m(W)$ denotes the Grassmannian of m-planes in W.

The idea behind the identification $\mathcal{B}_{n,k,m}(W) \cong \mathcal{A}_{n,k,m}(Z)$ is that we obtain $\mathcal{B}_{n,k,m}(W) \subset \mathrm{Gr}_m(W) \subset \mathrm{Gr}_{m,n}$ from $\mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}_{k,k+m}$ by taking orthogonal complements in $\mathbb{R}^{k+m}$, then applying an isomorphism between $\mathbb{R}^{k+m}$ and W so that our subspaces lie in W, not $\mathbb{R}^{k+m}$.

Proposition 4.4 ([32, LEMMA 3.10 AND PROPOSITION 3.12]). *Choose $Z \in \mathrm{Mat}_{n,k+m}^{>0}$, and let $W \in \mathrm{Gr}_{k+m,n}^{>0}$ be the column span of Z. We define a map $f_Z : \mathrm{Gr}_m(W) \to \mathrm{Gr}_{k,k+m}$ by*

$$f_Z(X) := Z(X^\perp) = \{Z(x) : x \in X^\perp\}.$$

Then f_Z restricts to a homeomorphism from $\mathcal{B}_{n,k,m}(W)$ onto $\mathcal{A}_{n,k,m}(Z)$, sending $V^\perp \cap W$ to $\tilde{Z}(V)$ for all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$.

We next discuss coordinates on $\mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}_{k,k+m}$ and $\mathcal{B}_{n,k,m}(W) \subset \mathrm{Gr}_{m,n}$.

Definition 4.5. Choose $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ with rows $Z_1, \dots, Z_n \in \mathbb{R}^{k+m}$. Given a matrix Y with rows $y_1, \dots, y_k$ representing an element of $\mathrm{Gr}_{k,k+m}$, and $\{i_1, \dots, i_m\} \subset [n]$, we define the *twistor coordinate*, denoted

$$\langle Y Z_{i_1} Z_{i_2} \cdots Z_{i_m} \rangle \quad \text{or} \quad \langle y_1, \dots, y_k, Z_{i_1}, \dots, Z_{i_m} \rangle,$$

to be the determinant of the matrix with rows $y_1, \dots, y_k, Z_{i_1}, \dots, Z_{i_m}$.

If $I = \{i_1 < \cdots < i_m\}$, we also use $\langle Y Z_I \rangle$ to denote $\langle Y Z_{i_1} Z_{i_2} \cdots Z_{i_m} \rangle$.

Lemma 4.6 shows that the twistor coordinates of the amplituhedron $\mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}_{k,k+m}$ are equal to the Plücker coordinates of the B-amplituhedron $\mathcal{B}_{n,k,m}(W) \subset \mathrm{Gr}_{m,n}$.

Lemma 4.6 ([32, (3.11)]). *If we let $Y := f_Z(X)$ in Proposition 4.4, we have*

$$p_I(X) = \langle Y Z_I \rangle \quad \textit{for all } I \in \binom{[n]}{m}. \tag{4.7}$$

By Definition 4.3 and Theorem 2.6, if $X \in \mathcal{B}_{n,k,m}(W)$ and $w \in X \setminus \{0\}$ then

$$k \leq \overline{\mathrm{var}}(w) \leq k + m - 1.$$

When $m = 1$ this leads to the following sign variation description of the amplituhedron.

Theorem 4.8 ([32, COROLLARY 3.19] and Lemma 4.6). *We have*

$$\mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) \mid \overline{\mathrm{var}}(w) = k\} \subset \mathbb{P}(W) \quad \textit{or, equivalently,} \tag{4.9}$$

$$\mathcal{A}_{n,k,1}(Z) = \{Y \in \mathrm{Gr}_{k,k+1} \mid \overline{\mathrm{var}}(\langle Y Z_1 \rangle, \langle Y Z_2 \rangle, \dots, \langle Y Z_n \rangle) = k\}. \tag{4.10}$$

Moreover, $\mathcal{A}_{n,k,1}(Z) \cong \mathcal{B}_{n,k,1}(W)$ can be identified with the complex of bounded faces of a cyclic hyperplane arrangement.

To make the latter identification, we choose a basis $w^{(0)}, w^{(1)}, \dots, w^{(k)}$ for $W \subset \mathbb{R}^n$ such that $\mathrm{span}(w^{(1)}, \dots, w^{(k)}) \in \mathrm{Gr}_{k,n}^{>0}$ and $w^{(0)}$ is *positively oriented* as in [32, DEFINITION 6.10]. Then we let $\mathcal{H}^W$ be the hyperplane arrangement in $\mathbb{R}^k$ with hyperplanes

$$H_i := \{x \in \mathbb{R}^k \mid w_i^{(1)} x_1 + \cdots + w_i^{(k)} x_k + w_i^{(0)} = 0\} \quad \text{for } i \in [n].$$

By [32, THEOREM 6.16], the map

$$\Psi_{\mathcal{H}^W} : \mathbb{R}^k \to \mathbb{P}(W), \quad x \mapsto \langle x_1 w^{(1)} + \cdots + x_k w^{(k)} + w^{(0)} \rangle$$

is a homeomorphism from the bounded complex $B(\mathcal{H}^W)$ of $\mathcal{H}^W$ to $\mathcal{B}_{n,k,1}(W)$. Moreover, if we partition the elements $w = (w_1, \ldots, w_n) \in \mathcal{B}_{n,k,1}(W)$ based on the signs of the w_i, we obtain the bounded faces of $B(\mathcal{H}^W)$. See Figure 5.

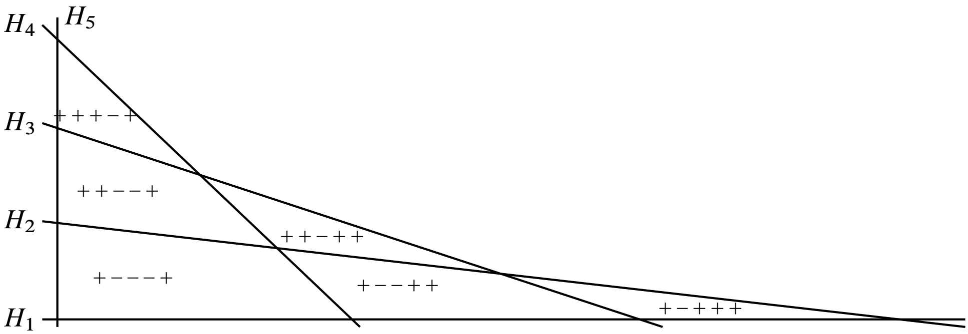

FIGURE 5
The hyperplane arrangement $\mathcal{H}^W$ from $\mathcal{B}_{5,2,1}(W) \cong B(\mathcal{H}^W)$, where $w^{(0)} = (0, -9, -6, -3, 0)$, $w^{(1)} = (0, 1, 1, 1, 1)$, and $w^{(2)} = (1, 9, 3, 1, 0)$. Its bounded faces are labeled by sign vectors. Here we have labeled only the maximal faces.

Before turning to the case $m = 2$, we need to introduce some notation.

Remark 4.11. Let $Z \in \mathrm{Mat}_{n,p}^{\geq 0}$ with $n \geq p$ have rows $Z_1, Z_2, \ldots, Z_n$, and let $\hat{Z}_i$ denote $(-1)^{p-1} Z_i$. Then the matrix with rows $Z_2, \ldots, Z_n, \hat{Z}_1$ also lies in $\mathrm{Mat}_{n,p}^{>0}$. Thus matrices with maximal minors nonnegative (and elements of $\mathrm{Gr}_{k,n}^{\geq 0}$) exhibit a *twisted cyclic symmetry*.

The following sign variation description of $\mathcal{A}_{n,k,2}(Z)$ was conjectured in [6].

Theorem 4.12 ([46, THEOREM 5.1]). *Fix $k < n$ and $Z \in \mathrm{Mat}_{n,k+2}^{>0}$. Let*

$$\mathcal{F}_{n,k,2}^{\circ}(Z) := \{Y \in \mathrm{Gr}_{k,k+2} \mid \langle Y Z_i Z_{i+1} \rangle > 0 \text{ for } 1 \leq i \leq n-1, \text{ and } \langle Y Z_n \hat{Z}_1 \rangle > 0, \\ \text{and } \mathrm{var}(\langle Y Z_1 Z_2 \rangle, \langle Y Z_1 Z_3 \rangle, \ldots \langle Y Z_1 Z_n \rangle) = k.\}$$

Then $\mathcal{A}_{n,k,2}(Z) = \overline{\mathcal{F}_{n,k,2}^{\circ}(Z)}$.

To generalize (4.10) and Theorem 4.12 for $m > 2$, we first observe the following [6].

Lemma 4.13. *Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ and $Y = CZ$ for $C \in \mathrm{Gr}_{k,n}^{>0}$. Let $r = \lfloor \frac{m}{2} \rfloor$.*

$$\text{If } m \text{ is even, } \langle Y Z_I \rangle > 0 \quad \forall I := \{i_1 < i_1 + 1 < i_2 < i_2 + 1 < \cdots < i_r < i_r + 1\} \in \binom{[n]}{m}$$

$$\text{and } \langle Y Z_I Z_n \hat{Z}_1 \rangle > 0 \quad \forall I := \{i_1 < i_1 + 1 < \cdots < i_{r-1} < i_{r-1} + 1\} \in \binom{[2, n-1]}{m-2}.$$

If m is odd, $(-1)^k\langle YZ_I\rangle > 0 \quad \forall I := \{1 = i_0 < i_1 < i_1+1 < \cdots < i_r < i_r+1\} \in \binom{[n]}{m}$

and $\langle YZ_I\rangle > 0 \quad \forall I := \{i_1 < i_1+1 < \cdots < i_r < i_r+1 < i_{r+1} = n\} \in \binom{[n]}{m}$.

Proof. The lemma follows from the fact that $Z \in \mathrm{Mat}^{>0}_{n,k+m}$ and

$$\langle CZ, Z_{i_1}, \dots, Z_{i_m}\rangle = \sum_{J=\{j_1<\cdots<j_k\}\in\binom{[n]}{k}} p_J(C)\langle Z_{j_1}, \dots, Z_{j_k}, Z_{i_1}, \dots, Z_{i_m}\rangle, \qquad (4.14)$$

by considering how many swaps are necessary to put $\{j_1, \dots, j_k, i_1, \dots, i_m\}$ in order. ∎

The following conjecture is (up to taking the closure) the statement [6, (5.14)].

Conjecture 4.15 ([6, (5.14)]). *Fix* $Z \in \mathrm{Mat}^{>0}_{n,k+m}$*, and define*

$$\mathcal{F}^\circ_{n,k,m}(Z) := \big\{Y \in \mathrm{Gr}_{k,k+m} \mid \textit{the conclusion of Lemma 4.13 holds, and}$$
$$\mathrm{var}\big(\langle YZ_1 \dots Z_{m-1}Z_m\rangle, \langle YZ_1 \dots Z_{m-1}Z_{m+1}\rangle, \dots, \langle YZ_1 \dots Z_{m-1}Z_n\rangle\big) = k\big\}.$$

Then $\mathcal{A}_{n,k,m}(Z) = \overline{\mathcal{F}^\circ_{n,k,m}(Z)}$.

To see that $\mathcal{A}_{n,k,m}(Z) \subseteq \overline{\mathcal{F}^\circ_{n,k,m}(Z)}$, see [6, SECTION 5.4] or [32, COROLLARY 3.21].

Given the evident importance of signs of coordinates, and taking inspiration from the $m = 1$ example, we define the *sign stratification* for the amplituhedron; this stratification is closely related to the *oriented matroid stratification* of the Grassmannian.

Definition 4.16. Let $\sigma = (\sigma_I) \in \{0, +, -\}^{\binom{n}{m}}$ be a nonzero sign vector with coordinates indexed by subsets $I \in \binom{[n]}{m}$. We consider σ *modulo multiplication by* ± 1 (since Plücker and twistor coordinates are coordinates in projective space). Set

$$\mathcal{A}^\sigma_{n,k,m}(Z) := \big\{Y \in \mathcal{A}_{n,k,m}(Z) \mid \mathrm{sign}\langle YZ_I\rangle = \sigma_I \text{ for all } I\big\} \quad \text{and}$$
$$\mathcal{B}^\sigma_{n,k,m}(W) := \big\{X \in \mathcal{B}_{n,k,m}(W) \mid \mathrm{sign}\big(p_I(X)\big) = \sigma_I \text{ for all } I\big\}.$$

We call $\mathcal{A}^\sigma_{n,k,m}(Z)$ (respectively, $\mathcal{B}^\sigma_{n,k,m}(W)$) an *(amplituhedron) sign stratum.* Clearly,

$$\mathcal{A}_{n,k,m}(Z) = \bigsqcup_\sigma \mathcal{A}^\sigma_{n,k,m}(Z) \quad \text{and} \quad \mathcal{B}_{n,k,m}(W) = \bigsqcup_\sigma \mathcal{B}^\sigma_{n,k,m}(W).$$

If $\sigma \in \{+, -\}^{\binom{n}{m}}$, we call $\mathcal{A}^\sigma_{n,k,m}(Z)$ and $\mathcal{B}^\sigma_{n,k,m}(W)$ open *(amplituhedron) chambers*.

For arbitrary σ, $\mathcal{A}^\sigma_{n,k,m}(Z)$ (or $\mathcal{B}^\sigma_{n,k,m}(W)$) may be empty. We call $\mathcal{A}^\sigma_{n,k,m}$ *realizable* if there is some Z for which $\mathcal{A}^\sigma_{n,k,m}(Z)$ is nonempty. It is an open problem to classify the realizable chambers/strata of the amplituhedron for $m > 2$. When $m = 1$, the realizable strata are precisely the $\mathcal{A}^\sigma_{n,k,1}(Z)$ with $\overline{\mathrm{var}}(\sigma) = k$ (cf. (4.10) and [32]). When $m = 2$, we will show in Section 6 that the realizable chambers of $\mathcal{A}_{n,k,2}$ are counted by the *Eulerian numbers*. This is related to the fact that the volume of $\Delta_{k+1,n}$ is the Eulerian number.

5. THE AMPLITUHEDRON MAP AND POSITROID TILINGS

In this section we begin our discussion of positroid tilings of the amplituhedron, cf. Definition 2.20 with $\phi = \tilde{Z}$. These have also beeen called *(positroid) triangulations*.

Definition 5.1. Choose $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. Given a positroid cell S_π of $\mathrm{Gr}_{k,n}^{\geq 0}$, we let $Z_\pi^\circ := \tilde{Z}(S_\pi)$ and $Z_\pi := \overline{\tilde{Z}(S_\pi)} = \tilde{Z}(\overline{S_\pi})$, and we refer to Z_π and Z_π° as *Grasstopes* and *open Grasstopes*, respectively. As in Definition 2.20, we call Z_π and Z_π° a *(positroid) tile* and an *open (positroid) tile* for $\mathcal{A}_{n,k,m}(Z)$ if $\dim(S_\pi) = km$ and $\tilde{Z}$ is injective on S_π. Since positroid cells are also indexed by plabic graphs, we will also use Z_G and Z_G° to denote the Grasstopes associated to the positroid cell S_G.

Images of positroid cells under the map $\tilde{Z}$ have been studied since [7], where the authors conjectured that the images of various *BCFW* collections of $4k$-dimensional cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ give a "triangulation" (positroid tiling) of the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. (For the "canonical" BCFW tiling studied in [33], this conjecture has been recently proved in a beautiful paper of Even-Zohar–Lakrec–Tessler [17].)

The positroid tiles for $\mathcal{A}_{n,k,m}(Z)$ were classified for $m = 1$ in [32, THEOREM 8.10]. For $m = 2$ and $k = 1$ (cf. Example 4.2), the amplituhedron $\mathcal{A}_{n,1,2}(Z)$ is a convex n-gon in $\mathbb{P}^2$. The positroid tiles are exactly the triangles on vertices $Z_1, \ldots, Z_n$ of the polygon, and positroid tilings of $\mathcal{A}_{n,1,2}(Z)$ are just ordinary triangulations of a polygon. More generally, the positroid tiles have been classified for $m = 2$ in [46, THEOREM 4.25], as we now describe.

5.1. Classification of positroid tiles for the amplituhedron map when $m = 2$

When $m = 2$, positroid tiles are in bijection with *bicolored subdivisions* of an n-gon.

Definition 5.2. Let $\mathbf{P}_n$ be a convex n-gon with boundary vertices labeled from 1 to n in clockwise order. A *bicolored triangulation* $\mathcal{T}$ of $\mathbf{P}_n$ is a triangulation whose edges connect vertices of $\mathbf{P}_n$ and whose triangles are colored black or white. If $\mathcal{T}$ has exactly k black triangles, we say it has *type* (k, n). Two bicolored triangulations $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent* if the union of the black triangles is the same for both of them. If we erase the diagonals separating pairs of like-colored triangles in $\mathcal{T}$ sharing an edge, we obtain a *bicolored subdivision* $\overline{\mathcal{T}}$ of $\mathbf{P}_n$ into black and white polygons, which represents the equivalence class of $\mathcal{T}$.

Given $\mathcal{T}$ as above, we build a labeled bipartite graph $\hat{G}(\mathcal{T})$ by placing black boundary vertices at the vertices of the n-gon, and placing a trivalent white vertex in the middle of each black triangle, connecting it to the three vertices of the triangle. See Figure 6. We can think of $\hat{G}(\mathcal{T})$ as a *plabic graph* if we enclose it in a slightly larger disk and add n edges connecting each black vertex to the boundary of the disk. See Figure 6.

Note that if $\mathcal{T}$ and $\mathcal{T}'$ are equivalent, $\hat{G}(\mathcal{T})$ and $\hat{G}(\mathcal{T}')$ are move-equivalent. Bicolored triangulations and subdivisions are special cases of *plabic tilings* [44].

The following statement refines a conjecture from [38].

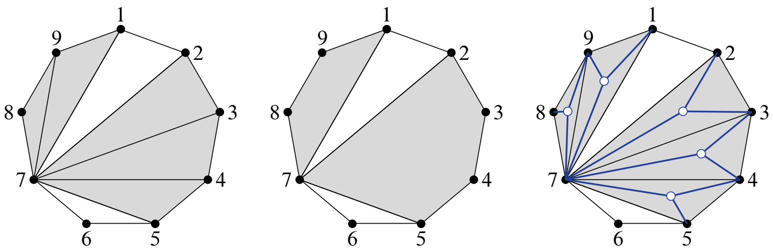

FIGURE 6
A bicolored triangulation $\mathcal{T}$; the corresponding bicolored subdivision $\overline{\mathcal{T}}$; the graph $\hat{G}(\mathcal{T})$.

Theorem 5.3 ([46, THEOREM 4.25]). *Fix $k < n$ and $Z \in \mathrm{Mat}_{n,k+2}^{>0}$. Then $\tilde{Z}$ is injective on the $2k$-dimensional cell $S_{\mathcal{M}}$ if and only if $S_{\mathcal{M}} = S_{\hat{G}(\mathcal{T})}$ for some bicolored triangulation $\mathcal{T}$ of type (k,n). That is, the positroid tiles for $\mathcal{A}_{n,k,2}$ are exactly the Grasstopes $Z_{\hat{G}(\mathcal{T})}$.*

It is an open problem to classify positroid tiles of $\mathcal{A}_{n,k,m}(Z)$ for $m > 2$.

5.2. Numerology of positroid tilings of the amplituhedron

When $m = 4$, each (conjectural) BCFW tiling of $\mathcal{A}_{n,k,4}(Z)$ has cardinality equal to the *Narayana number* $N_{n-3,k+1} = \frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$. What should be the cardinality for $m \neq 4$? Table 1 gives data about special cases studied thus far.

Special case	Cardinality of tiling of $\mathcal{A}_{n,k,m}(Z)$	Explanation
$m = 0$ or $k = 0$ or $k + m = n$	1	$\mathcal{A}$ is a point or $\mathcal{A} \cong \mathrm{Gr}_{k,n}^{\geq 0}$
$m = 1$	$\binom{n-1}{k}$	[32]
$m = 2$	$\binom{n-2}{k}$	[6, 8, 46]
$m = 4$	$\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$	[7, 17]
$k = 1$, m even	$\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$	$\mathcal{A} \cong$ cyclic polytope $C(n,m)$

TABLE 1
Cardinalities of tilings of $\mathcal{A}_{n,k,m}(Z)$.

As we will see later, the appearance of the number $\binom{n-2}{k}$ in the $m = 2$ row of the table is related to the appearance of the number $\binom{n-2}{k}$ in Corollary 3.9.

The special cases in the table led us to make the following intriguing conjecture.

Conjecture 5.4 ([33, CONJECTURE 8.1]). *If m is even, the cardinality of a positroid tiling of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is $M(k, n-k-m, \frac{m}{2})$, where*

$$M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$$

is the number of plane partitions *contained in an $a \times b \times c$ box.*

For odd m, we believe the *maximum* cardinality achieved by a positroid tiling of $\mathcal{A}_{n,k,m}(Z)$ is $M(k, n-k-m, \lceil \frac{m}{2} \rceil)$. This is consistent with results for $m = 1$ [32] and the fact that for odd m, the number of top-dimensional simplices in a triangulation of the cyclic polytope $C(n, m)$ can lie anywhere between $\binom{n-1-\frac{m+1}{2}}{\frac{m-1}{2}}$ and $\binom{n-\frac{m+1}{2}}{\frac{m+1}{2}}$ [50, COROLLARY 1.2(II)].

Remark 5.5. Clearly, $M(a, b, c)$ is symmetric in a, b and c. Conjecture 5.4 thus suggests that for even m, there is a symmetry of (the positroid tilings of) the amplituhedron $\mathcal{A}_{n,k,m}$ that allows one to exchange k, $n-k-m$, and $\frac{m}{2}$. The symmetry between k and $n-k-m$ is called *parity duality* and was subsequently verified in [22].

Remark 5.6. The numbers $M(a, b, c)$ also count collections of c noncrossing lattice paths inside an $a \times b$ rectangle; rhombic tilings of a hexagon with side lengths a, b, c, a, b, c; and perfect matchings of a certain honeycomb lattice. See [33, SECTION 8.1].

6. T-DUALITY AND POSITROID TILINGS OF $\Delta_{k+1,n}$ AND $\mathcal{A}_{n,k,2}$

In this section we will fix $1 \leq k \leq n-2$, and explore a mysterious duality between the hypersimplex $\Delta_{k+1,n}$—an $(n-1)$-dimensional polytope in $\mathbb{R}^n$—and the amplituhedron $\mathcal{A}_{n,k,2}(Z)$—a $2k$-dimensional (non-polytopal) subset of $\mathrm{Gr}_{k,k+2}$ [39, 46]. This duality was first discovered in [39], after we observed that for small k and n, the f-vector of the positive tropical Grassmannian $\mathrm{Trop}^+ \mathrm{Gr}_{k+1,n}$ [58] agrees with the numbers of positroid subdivisions of $\mathcal{A}_{n,k,2}(Z)$ [39, SECTION 11]. By Theorem 3.6, the cones of $\mathrm{Trop}^+ \mathrm{Gr}_{k+1,n}$ parameterize the regular positroid subdivisions of $\Delta_{k+1,n}$, so this leads to the idea that positroid subdivisions of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ must be related [39].

Example 6.1. Continuing Example 3.7, there are two maximal cones of $\mathrm{Trop}^+ \mathrm{Gr}_{2,4}$, defined by the inequalities $P_{13} + P_{24} = P_{14} + P_{23} < P_{12} + P_{34}$ and $P_{13} + P_{24} = P_{12} + P_{34} < P_{14} + P_{23}$. These two cones give rise to the two subdivisions of $\Delta_{2,4}$ shown in Figure 7. These subdivisions are both positroid tilings for the moment map (and there are no others). Meanwhile, the amplituhedron $\mathcal{A}_{4,1,2}(Z)$ is a quadrilateral, which has precisely two positroid tilings for the amplituhedron map, as shown in Figure 8.

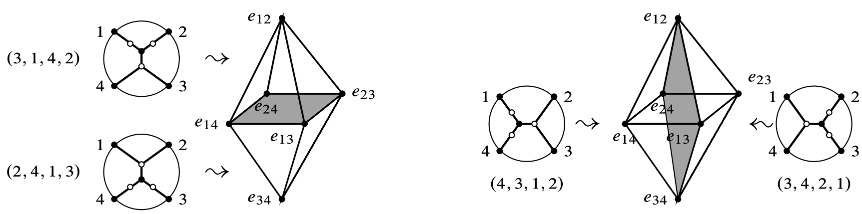

FIGURE 7
The two positroid tilings of $\Delta_{2,4}$ for the moment map. The plabic graphs specify the positroid cells whose images are the positroid tiles (positroid polytopes).

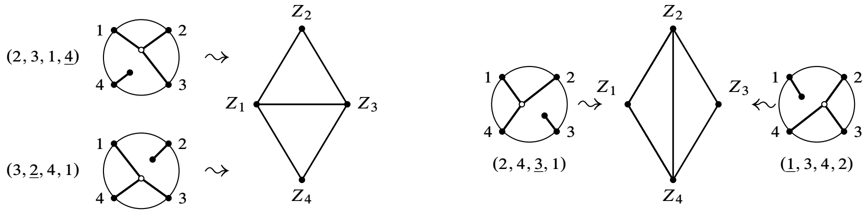

FIGURE 8
The two positroid tilings of $\mathcal{A}_{4,1,2}(Z)$ for $\tilde{Z}$. The plabic graphs specify the positroid cells whose images are the positroid tiles (Grasstopes).

Definition 6.2. Let $\pi = (a_1, a_2, \ldots, a_n)$ be a loopless decorated permutation (written in one-line notation). Its *T-dual* decorated permutation is $\hat{\pi} : i \mapsto \pi(i-1)$, so that $\hat{\pi} = (a_n, a_1, a_2, \ldots, a_{n-1})$. Any fixed points in $\hat{\pi}$ are declared to be loops.[1]

For example, the four permutations $(3,1,4,2)$, $(2,4,1,3)$, $(4,3,1,2)$, $(3,4,2,1)$ labeling the positroid tilings of $\Delta_{2,4}$ in Figure 7 are loopless. Their T-dual images are $(2,3,1,\underline{4})$, $(3,\underline{2},4,1)$, $(2,4,\underline{3},1)$, and $(\underline{1},3,4,2)$—precisely the permutations labeling the positroid tilings of $\mathcal{A}_{4,1,2}(Z)$ in Figure 8!

The T-duality map appears in [33, 39], and is a version of an $m = 4$ map from [3].

Proposition 6.3 ([39, LEMMA 5.2] and [46, PROPOSITION 8.1]). *T-duality is a bijection between loopless cells of* $\mathrm{Gr}_{k+1,n}^{\geq 0}$ *and coloopless cells of* $\mathrm{Gr}_{k,n}^{\geq 0}$. *Moreover, it is a poset isomorphism: we have* $S_\mu \subset \overline{S_\pi}$ *if and only if* $S_{\hat{\mu}} \subset \overline{S_{\hat{\pi}}}$.

One can also describe T-duality as a map on reduced plabic graphs G; we say G is *black-trivalent* (*white-trivalent*) if all of its interior black (white) vertices are trivalent.

Definition 6.4 ([46, DEFINITION 8.6]). Let G be a reduced black-trivalent plabic graph. The *T-dual* of G, denoted $\hat{G}$, is the graph obtained as follows (see Figure 9).

- In each face f of G, place a black vertex $\hat{b}(f)$.
- "On top of" each black vertex b of G, place a white vertex $\hat{w}(b)$;
- For each black vertex b of G incident to face f, add edge $(\hat{w}(b), \hat{b}(f))$;
- Put $\hat{i}$ on the boundary of G between vertices $i-1$ and i and draw an edge from $\hat{i}$ to $\hat{b}(f)$, where f is the adjacent boundary face.

Note that the graphs in Figures 8 and 7 are related by T-duality.

1 Our use of the "hat" notation here is unrelated to that from Remark 4.11.

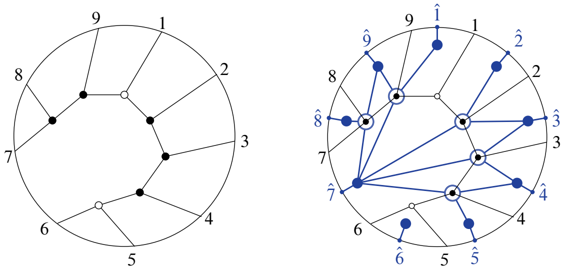

FIGURE 9
(Left) A plabic graph G with trip permutation $\pi_G = (5, 9, 2, 3, 6, 4, 1, 7, 8)$; (Right) G with the T-dual graph $\hat{G}$ superimposed. We have $\pi_{\hat{G}} = (8, 5, 9, 2, 3, \underline{6}, 4, 1, 7)$.

Proposition 6.5 ([46, PROPOSITION 8.8]). *Let G be a reduced black-trivalent plabic graph with trip permutation $\pi_G = \pi$. Then $\hat{G}$ is a reduced white-trivalent plabic graph with $\pi_{\hat{G}} = \hat{\pi}$.*

T-duality provides a link between positroid tilings of $\mathcal{A}_{n,k,2}(Z)$ and $\Delta_{k+1,n}$. The first result is that T-duality gives a bijection between positroid tiles of $\Delta_{k+1,n}$ (see Proposition 3.3) and positroid tiles of $\mathcal{A}_{n,k,2}$ (see Theorem 5.3).

Given a bicolored triangulation $\mathcal{T}$, we define $\mathrm{area}(a \to b)$ to be the number of black triangles to the left of $a \to b$ in any triangulation of $\overline{\mathcal{T}}$ compatible with $a \to b$.

Theorem 6.6 ([46, SECTION 8]). *Given a bicolored triangulation $\mathcal{T}$ of type (k, n), we can read off T-dual graphs G and $\hat{G}$ giving positroid tiles Γ_G and $Z_{\hat{G}}$ as follows:*

- *$G := G(\mathcal{T})$ is the dual graph of $\mathcal{T}$, as shown at the left of Figure 10.*
- *$\hat{G} := \hat{G}(\mathcal{T})$ is the graph from Definition 5.2, as shown at the right of Figure 10.*

This correspondence gives a bijection between positroid tiles of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$, both of which depend only on $\overline{\mathcal{T}}$.

Moreover, if we let $h \to j$ (with $h < j$) range over arcs of $\mathcal{T}$, the inequalities

(1) $\mathrm{area}(h \to j) + 1 \geq x_h + x_{h+1} + \cdots + x_{j-1} \geq \mathrm{area}(h \to j)$ *for* $x \in \Gamma_{G(\mathcal{T})}$;

(2) $(-1)^{\mathrm{area}(h \to j)} \langle Y Z_h Z_j \rangle \geq 0$ *for* $Y \in Z_{\hat{G}(\mathcal{T})}$

cut out the positroid tiles $\Gamma_{G(\mathcal{T})}$ and $Z_{\hat{G}(\mathcal{T})}$.

We now explain how Eulerian numbers enter the story.

Definition 6.7. Let $w \in S_n$. We call a letter $i \geq 2$ in w a *left descent* if $w^{-1}(i) < w^{-1}(i-1)$. And we say that $i \in [n]$ in w is a *cyclic left descent* if either $i \geq 2$ is a left descent of w or if $i = 1$ and $w^{-1}(1) < w^{-1}(n)$. Let $\mathrm{cDes_L}(w)$ denote the set of cyclic left descents of w.

Let $D_{k+1,n}$ be the set of permutations $w \in S_n$ with $k+1$ cyclic descents and $w_n = n$. Note that $|D_{k+1,n}|$ equals the *Eulerian number* $E_{k,n-1} := \sum_{\ell=0}^{k+1} (-1)^\ell \binom{n}{\ell} (k+1-\ell)^{n-1}$.

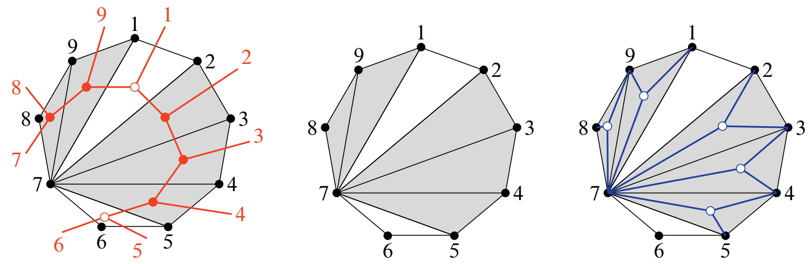

FIGURE 10
In the middle: a bicolored triangulation $\mathcal{T}$, with the dual graph $G(\mathcal{T})$ to its left, and the T-dual graph $\hat{G}(\mathcal{T})$ to its right.

Definition 6.8. For $w \in D_{k+1,n}$, let $w^{(a)}$ denote the cyclic rotation of w ending at a. Let $I_1 = I_1(w) := \mathrm{cDes}_{\mathrm{L}}(w)$ and for $2 \le r \le n$, let $I_r = I_r(w) := \mathrm{cDes}_{\mathrm{L}}(w^{(r-1)})$. We then define the *$w$-simplex* Δ_w to be the convex hull $\Delta_w := \mathrm{conv}(e_{I_1}, \dots, e_{I_n}) \subseteq \Delta_{k+1,n}$.

Example 6.9. If $w = (1,3,2,4)$, then we have $I_1 = \{1,3\}$, $I_2 = \{2,3\}$, $I_3 = \{3,4\}$, and $I_4 = \{2,4\}$, so $\Delta_w = \mathrm{conv}(e_{13}, e_{23}, e_{34}, e_{24})$. See Figure 7.

Stanley gave the first triangulation of the hypersimplex **[60]**, see also **[62]** and **[37]**.

Proposition 6.10 (**[60]**). *We have*

$$\Delta_{k+1,n} = \bigcup_{w \in D_{k+1,n}} \Delta_w.$$

Example 6.11. For example, $\Delta_{24} = \Delta_{1324} \cup \Delta_{2134} \cup \Delta_{2314} \cup \Delta_{3124}$. This decomposition refines the two positroid tilings shown in Figure 7 (and this holds in general **[37]**).

Definition 6.12. Let $w \in D_{k+1,n}$ and let $I_1, \dots, I_n$ be as in Definition 6.8. We define $\hat{\Delta}^{\circ}_w(Z)$ to be the open amplituhedron chamber consisting of $Y \in \mathcal{A}_{n,k,2}(Z)$ such that for $a = 1, \dots, n$, the sign flips of the sequence

$$\big(\langle Y Z_a \hat{Z}_1\rangle, \langle Y Z_a \hat{Z}_2\rangle, \dots, \langle Y Z_a \hat{Z}_{a-1}\rangle, \langle Y Z_a Z_a\rangle, \langle Y Z_a Z_{a+1}\rangle, \dots, \langle Y Z_a Z_n\rangle\big)$$

occur precisely in positions $I_a \setminus \{a\}$, where we say a sign flip occurs in position j if $\langle Y Z_a Z_j\rangle$ and $\langle Y Z_a Z_{j+1}\rangle$ are nonzero and have different signs (if $j = n$ we consider $j+1 = 1$).

We refer to $\hat{\Delta}^{\circ}_w(Z)$ and $\hat{\Delta}_w(Z) := \overline{\hat{\Delta}^{\circ}_w(Z)}$ as open and closed *w-chambers*.

Example 6.13. If $w = (1,3,2,4)$, then $I_1 = \{1,3\}$, $I_2 = \{2,3\}$, $I_3 = \{3,4\}$, and $I_4 = \{2,4\}$, so $\hat{\Delta}^{\circ}_w(Z)$ consists of $Y \in \mathcal{A}_{n,1,2}(Z)$ such that $\langle Y Z_1 Z_4\rangle < 0$, $\langle Y Z_2 Z_4\rangle < 0$, and the other four $\langle Y Z_i Z_j\rangle$ with $i < j$ are positive. In Figure 8, this corresponds to the triangle with vertices Z_3, Z_4, and the point where the two diagonals of the quadrilateral intersect.

For some choices of Z, $\hat{\Delta}^{\circ}_w(Z)$ can be empty. However, for each $w \in D_{k+1,n}$ one can find explicit matrices $Z \in \mathrm{Mat}^{>0}_{n,k+2}$ such that $\hat{\Delta}^{\circ}_w(Z)$ is nonempty **[46, SECTION 11]**. Moreover, the w-chambers are the only amplituhedron chambers which are realizable.

Theorem 6.14 ([46, THEOREM 10.10]). *For any* $Z \in \mathrm{Mat}^{>0}_{n,k+2}$*, we have*

$$\mathcal{A}_{n,k,2}(Z) = \bigcup_{w \in D_{k+1,n}} \hat{\Delta}_w(Z).$$

By Corollary 6.6, each positroid tile $Z_{\hat{G}}$ is a union of closed w-chambers.

Example 6.15. For example, $\mathcal{A}_{4,1,2}(Z) = \hat{\Delta}_{1324}(Z) \cup \hat{\Delta}_{2134}(Z) \cup \hat{\Delta}_{2314}(Z) \cup \hat{\Delta}_{3124}(Z)$. This decomposition refines the two positroid tilings shown in Figure 8.

Given that positroid tiles $\Gamma_G \subset \Delta_{k+1,n}$ are unions of w-simplices, and positroid tiles $Z_{\hat{G}} \subset \mathcal{A}_{n,k,2}(Z)$ are unions of w-chambers, the following is the key to proving that positroid tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ are in bijection.

Proposition 6.16 ([46, PROPOSITION 11.1]). *Let* $Z \in \mathrm{Mat}^{>0}_{n,k+2}$*. Suppose* $w \in D_{k+1,n}$ *and that* $\hat{\Delta}_w(Z) \neq \emptyset$*. For any positroid tile* Γ_π*,* $\Delta_w \subset \Gamma_\pi$ *if and only if* $\hat{\Delta}_w(Z) \subset Z_{\hat{\pi}}$*.*

Figures 7 and 8 illustrate the fact that the two positroid tilings of $\mathcal{A}_{4,1,2}(Z)$ are related to the two positroid tilings of $\Delta_{2,4}$ by T-duality. The following result, first conjectured in [39, CONJECTURE 6.9], generalizes this example to arbitrary k and n.

Theorem 6.17 ([46]). *A collection* $\{\Gamma_\pi\}$ *of positroid polytopes in* $\Delta_{k+1,n}$ *gives a positroid tiling of* $\Delta_{k+1,n}$ *if and only if for all* $Z \in \mathrm{Mat}^{>0}_{n,k+2}$*, the collection* $\{Z_{\hat{\pi}}\}$ *of Grasstopes gives a positroid tiling of* $\mathcal{A}_{n,k,2}(Z)$*.*

In light of the fact that $\Delta_{k+1,n}$ is an $(n-1)$-dimensional polytope, and $\mathcal{A}_{n,k,2}(Z)$ is a $2k$-dimensional nonpolytopal subset of $\mathrm{Gr}_{k,k+2}$, we find Theorem 6.17 very surprising!

We believe that more generally, Theorem 6.17 extends to give a bijection between positroid dissections (respectively, subdivisions) of $\Delta_{k+1,n}$ and positroid dissections (respectively, subdivisions) of $\mathcal{A}_{n,k,2}(Z)$, see [39, CONJECTURES 6.9 AND 8.8].

Given that $\mathrm{Trop}^+ \mathrm{Gr}_{k+1,n}$ controls the regular positroid subdivisions of $\Delta_{k+1,n}$, which are (conjecturally) in bijection with positroid subdivisions of $\mathcal{A}_{n,k,2}(Z)$, it is natural to ask: can we make a direct connection between $\mathrm{Trop}^+ \mathrm{Gr}_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$? Is there a way to think of points of $\mathrm{Trop}^+ \mathrm{Gr}_{k+1,n}$ as giving "height functions" for $\mathcal{A}_{n,k,2}(Z)$?

7. THE AMPLITUHEDRON AND CLUSTER ALGEBRAS

Cluster algebras are a remarkable class of commutative rings introduced by Fomin and Zelevinsky [18,20], see also [19]. Many homogeneous coordinate rings of "nice" algebraic varieties have a cluster algebra structure, including the Grassmannian [54]. Starting in 2013, various authors connected scattering amplitudes of planar $\mathcal{N} = 4$ super-Yang–Mills theory to cluster algebras [16,25,38,46]. In this section we explain several connections between the amplituhedron $\mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}_{k,k+m}$ and the cluster algebra structure on $\mathrm{Gr}_{m,n}$.

7.1. Cluster adjacency for facets of positroid tiles

Facets of positroid tiles in $\mathcal{A}_{n,k,m}(Z)$ are related to the cluster structure on $\mathrm{Gr}_{m,n}$.

Definition 7.1. Let Z_π be a Grasstope of $\mathcal{A}_{n,k,m}(Z)$. We say that $Z_{\pi'}$ is a *facet* of Z_π if it is maximal by inclusion among the Grasstopes satisfying the following three properties: $Z_{\pi'} \subset \partial Z_\pi$; the cell $S_{\pi'}$ is contained in $\overline{S_\pi}$; and $Z_{\pi'}$ has codimension 1 in Z_π.

Recall from [19, 20] that the cluster variables for $\mathrm{Gr}_{2,n}$ are the Plücker coordinates p_{ij}, and a collection of Plücker coordinates is *compatible* if the corresponding diagonals in an n-gon are noncrossing. When $m = 2$, we have the following theorem (whose first paragraph is the *cluster adjacency conjecture* from [38]).

Theorem 7.2 ([46, THEOREM 9.12]). *Let $Z_{\hat{G}(\mathcal{T})}$ be a positroid tile of $\mathcal{A}_{n,k,2}(Z)$. Each facet lies on a hypersurface $\langle YZ_iZ_j\rangle = 0$, and the collection of Plücker coordinates $\{p_{ij}\}_{\hat{G}(\mathcal{T})}$ corresponding to facets is a collection of compatible cluster variables for* $\mathrm{Gr}_{2,n}$.

If p_{hl} is compatible with $\{p_{ij}\}_{\hat{G}(\mathcal{T})}$, then $\langle YZ_hZ_l\rangle$ has a fixed sign on $Z^\circ_{\hat{G}(\mathcal{T})}$.

For $m > 2$ the Grassmannian $\mathrm{Gr}_{m,n}$ has infinitely many cluster variables, each of which can be written as a polynomial $Q(p_I)$ in the $\binom{n}{m}$ Plücker coordinates. Meanwhile, each facet of a positroid tile of the amplituhedon $\mathcal{A}_{n,k,m}(Z)$ lies on a hypersurface defined by the vanishing of some polynomial $Q(\langle YZ_I\rangle)$ in the $\binom{n}{m}$ twistor coordinates $\langle YZ_I\rangle_{I\in\binom{[n]}{m}}$.

Conjecture 7.3 ([46, CONJECTURE 6.2]). *Let Z_π be a positroid tile of $\mathcal{A}_{n,k,m}(Z)$ and let*

$$\mathrm{Facet}(Z_\pi) := \big\{Q(p_I) \mid \textit{a facet of } Z_\pi \textit{ lies on the hypersurface } Q(\langle YZ_I\rangle) = 0\big\},$$

where Q is a polynomial in the $\binom{n}{m}$ Plücker coordinates. Then each $Q \in \mathrm{Facet}(Z_\pi)$ is a cluster variable for $\mathrm{Gr}_{m,n}$, and $\mathrm{Facet}(Z_\pi)$ consists of compatible cluster variables. Moreover, if $\tilde{Q}$ is a cluster variable compatible with $\mathrm{Facet}(Z_\pi)$, the polynomial $\tilde{Q}(\langle YZ_I\rangle)$ in twistor coordinates has a fixed sign on Z°_π.

7.2. Positroid tiles as totally positive parts of cluster varieties

Following [46, SECTION 6.2], we now build a cluster variety $\mathcal{V}_{\overline{\mathcal{T}}}$ in $\mathrm{Gr}_{k,k+2}(\mathbb{C})$ for each positroid tile $Z_{\hat{G}(\overline{\mathcal{T}})}$ of $\mathcal{A}_{n,k,2}(Z)$. The positroid tile $Z^\circ_{\hat{G}(\overline{\mathcal{T}})}$ is exactly the *totally positive part* of $\mathcal{V}_{\overline{\mathcal{T}}}$ (in the sense of [18]).

Fix a bicolored subdivision $\overline{\mathcal{T}}$ of type (k, n), with black polygons $P_1, \dots, P_r$. Let $\mathcal{S}(\overline{\mathcal{T}})$ denote the set of all bicolored triangulations represented by $\overline{\mathcal{T}}$. For each black polygon P_i, fix a *distinguished boundary arc* $h_i \to j_i$ with $h_i < j_i$ in the boundary of P_i. We will build $\mathcal{V}_{\overline{\mathcal{T}}}$ by defining seeds in the field of rational functions on $\mathrm{Gr}_{k,k+2}(\mathbb{C})$.

Definition 7.4 (Cluster variables). Let $a \to b$ with $a < b$ be an arc which is contained in a black polygon P_i and is not the distinguished boundary arc $h_i \to j_i$. We define

$$x_{ab} := \frac{(-1)^{\mathrm{area}(a\to b)}\langle YZ_aZ_b\rangle}{(-1)^{\mathrm{area}(h_i\to j_i)}\langle YZ_{h_i}Z_{j_i}\rangle},$$

which is a rational function on $\mathrm{Gr}_{k,k+2}(\mathbb{C})$.

Definition 7.5 (Seeds). Let $\mathcal{T} \in \mathcal{S}(\overline{\mathcal{T}})$. We define the quiver $Q_{\mathcal{T}}$ as follows:

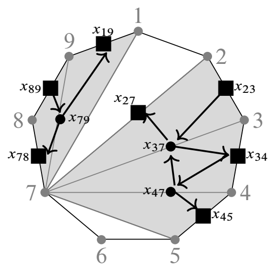

FIGURE 11
In gray, a bicolored triangulation $\mathcal{T}$. In black, the seed $\Sigma_{\mathcal{T}}$. The distinguished boundary arcs are $1 \to 7$ and $5 \to 7$.

- Place a frozen vertex on each non-distinguished boundary arc of $P_1, \ldots, P_r$ and a mutable vertex on every other arc (a "black arc") bounding a triangle of $\mathcal{T}$.
- If arcs $a \to b$, $b \to c$, $c \to a$ form a triangle, we put arrows in Q between the corresponding vertices, going clockwise around the triangle.

We label the vertex of $Q_{\mathcal{T}}$ on arc $a \to b$ of $\mathcal{T}$ with the function x_{ab}. The collection of vertex labels is the *(extended) cluster* $\mathbf{x}_{\mathcal{T}}$. The pair $(Q_{\mathcal{T}}, \mathbf{x}_{\mathcal{T}})$ is the *seed* $\Sigma_{\mathcal{T}}$.

See Figure 11 for an example. Note that the cluster $\mathbf{x}_{\mathcal{T}}$ has size $2k$.

Theorem 7.6 ([46, SECTION 6.2]). *Let $\mathcal{T} \in \mathcal{S}(\overline{\mathcal{T}})$. Then*

$$\mathcal{V}_{\mathcal{T}} := \left\{ Y \in \mathrm{Gr}_{k,k+2}(\mathbb{C}) : \prod_{a \to b \text{ black arc of } \mathcal{T}} \langle Y Z_a Z_b \rangle \neq 0 \right\}$$

is birational to an algebraic torus of dimension $2k$, and its field of rational functions is the field $\mathbb{C}(\mathbf{x}_{\mathcal{T}})$ of rational functions in the cluster $\mathbf{x}_{\mathcal{T}}$.

The set $\mathcal{V}_{\overline{\mathcal{T}}} := \bigcup_{\mathcal{T} \in \mathcal{S}(\overline{\mathcal{T}})} \mathcal{V}_{\mathcal{T}}$ is a cluster variety in $\mathrm{Gr}_{k,k+2}(\mathbb{C})$. In particular, if $\mathcal{T}$ and $\mathcal{T}'$ are related by flipping arc $a \to b$, seeds $\Sigma_{\mathcal{T}}$ and $\Sigma_{\mathcal{T}'}$ are related by mutation at x_{ab}.

$$\text{The positive part } \mathcal{V}_{\overline{\mathcal{T}}}^{\geq 0} := \left\{ Y \in \mathcal{V}_{\overline{\mathcal{T}}} : x_{ab}(Y) > 0 \text{ for all cluster variables } x_{ab} \right\}$$

of the cluster variety $\mathcal{V}_{\overline{\mathcal{T}}}$ is equal to the positroid tile $Z^{\circ}_{\hat{G}(\overline{\mathcal{T}})}$.

We conjecture that for even m, each positroid tile of $\mathcal{A}_{n,k,m}(Z)$ can be realized as the totally positive part of a cluster variety in $\mathrm{Gr}_{k,k+m}(\mathbb{C})$.

8. FUTURE DIRECTIONS

There are other geometric objects related to the amplituhedron, including the *loop amplituhedron* [7], and the *momentum amplituhedron* (defined for $m = 4$ in [15] and for even m in [39]). It would be interesting to explore these objects systematically as above.

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