MATHEMATICAL ANALYSIS AND NUMERICAL METHODS FOR INVERSE SCATTERING PROBLEMS

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ABSTRACT

Inverse scattering problems arise in diverse application areas, such as geophysical prospecting, submarine detection, near-field and nanooptical imaging, and medical imaging. For a given wave incident on a medium enclosed by a bounded domain, the scattering (direct) problem is to determine the scattered field or the energy distribution for the known scatterer. An inverse scattering problem is to determine the scatterer from the boundary measurements of the fields. Although significant recent progress has been made in solving the inverse problems, many challenging mathematical and computational issues remain unresolved. In particular, the severe ill-posedness has thus far limited the scope of inverse problem methods in practical applications. This paper is concerned with mathematical analysis and numerical methods for solving inverse scattering problems of broad interest. Based on multifrequency data, effective computational and mathematical approaches are presented for overcoming the ill-posedness of the inverse problems. A brief overview of these approaches and results is provided. Particular attention is paid to inverse medium, inverse obstacle, and inverse source scattering problems. Related topics and open problems are also discussed.

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1. INTRODUCTION

Research on scattering and inverse scattering plays a critical role in the advancement of exploration science, especially in medical imaging, stealth technology, oil and gas exploration, nondestructive testing, materials characterization, optical microscopy, and nanooptical imaging. Scattering involves studying the interaction of a medium, often inhomogeneous, with incident waves or particles, while inverse scattering deals with determining the medium, such as location, geometry, or material properties, by the wave field measured externally.

Over the last few decades, the ever-growing practical applications and scientific developments have driven the need for more sophisticated mathematical models and numerical algorithms to describe the scattering of complicated structures, to accurately compute scattered fields and thus to predict the performance of a given structure, as well as to carry out the optimal design of new structures. The rapid growth of computational capability and the development of fast algorithms have also made inverse scattering a viable option for solving many identification problems. Mathematically, inverse scattering has been an emerging and core field of modern mathematical physics. Significant progress has been made in the mathematical studies of uniqueness and stability, as well as the development of numerical methods for solving inverse scattering problems [66,74,78,88,94,102,103]. However, there are outstanding mathematical and computational challenges that remain to be resolved, especially the nonlinearity, ill-posedness, model uncertainty, and large-scale computation. In addition, in the area of nanotechnology and biology, optical measurement techniques are commonly used. Since the size of the measured structure is extremely small, how to overcome the diffraction limit to obtain superresolution imaging presents another key challenge.

This paper is not intended to cover all of the broad topics in inverse scattering theory for wave propagation. It is designed to be an introduction to the work of our research group to overcome the above challenges for solving the inverse scattering problems. Throughout, we are mainly concerned with multifrequency data for the following reasons. First, due to lack of stability, the inverse scattering problems are severely ill-posed at a fixed frequency, that is, small variations in the measured data can lead to large errors in the reconstructions. On the other hand, the problems become well-posed with Lipschitz-type stability estimates when all frequency data, corresponding to the time domain case, is available. Second, the nonlinearity of the inverse scattering problems at high wavenumber leads to many local minima for the associated optimization method. By properly designing a numerical method, such a highly nonlinear problem may be reduced to a set of linear problems at given frequencies. Physically, the approach based on multifrequency data is consistent with the Heisenberg uncertainty principle. According to the principle, one-half of the wavelength is the diffraction limit for resolving the sharpness of details that may be observed by optical microscopy [57,67,79]. The diffraction limit provides a limit on the accuracy of the reconstruction for a given wavelength. To improve the resolution, it is desirable to use an incident field with a shorter wavelength or a higher frequency to illuminate the scatterer.

The goal of this paper is two-fold. Concerning mathematical analysis, our recent stability results for the inverse scattering problems are discussed. Regarding numerical methods, we present the stable recursive linearization method (RLM) for solving quantitatively the inverse scattering problems with increased resolution.

The underlying physical model is usually a wave propagation system decided by means of measuring data. In this work, our primary focus is on acoustic and electromagnetic wave propagation governed by the Helmholtz equation and Maxwell's equations, respectively. Many of the approaches and methods may be extended to study inverse scattering problems in other wave propagation models, especially elastic waves. The inverse scattering problems for wave propagation can be broadly divided into three classes: the inverse medium problem (IMP), the inverse obstacle problem (IOP), and the inverse source problem (ISP), depending on the nature of reconstructions. To emphasize the significance of the spectral information for solving the inverse scattering problems, particular attention is paid to the frequency domain models or the time-harmonic cases. To further limit the scope, the numerical methods discussed here are nonlinear optimization-based iterative methods for solving inverse scattering problems. We refer the reader to [55,62,65,76,80,87] and references therein for noniterative, particularly direct imaging methods for solving inverse scattering problems.

The outline of this paper is as follows. In Section 2, the IMP is introduced. Stability results for the multiple frequency models are presented. Section 3 is devoted to the ISP. Stability for the multifrequency ISP of Maxwell's equations is discussed. The recent development of stochastic inverse source problems is provided. The IOP is addressed in Section 4. Of particular interest is the inverse diffraction problem. The paper is concluded with some general remarks and discussions on related problems. Some significant open problems are also presented in Section 5.

2. INVERSE MEDIUM PROBLEM

In this section, we consider the IMP, which is to reconstruct the inhomogeneous medium from boundary measurements of the scattered field surrounding the medium. The main difficulties are the ill-posedness, especially lack of stability, and the nonlinearity. In the static case (zero frequency), the problem is related to the celebrated Calderón problem [56], which is known to be severely unstable, in general [2,101]. In fact, such severe ill-posedness carries over to the inverse medium problems for acoustic and electromagnetic waves at a fixed frequency [4,81]. Our remedy to overcome the difficulties is to consider multifrequency boundary data. For the Maxwell equations model, we present a stable reconstruction method based on recursive linearization. The stability of the model IMP is also investigated.

2.1. Model problem

Consider the time harmonic Maxwell equation in three dimensions, namely

$$\nabla \times (\nabla \times E^t) - \kappa^2(1+q)E^t = 0 \quad \text{in } \mathbb{R}^3, \tag{2.1}$$

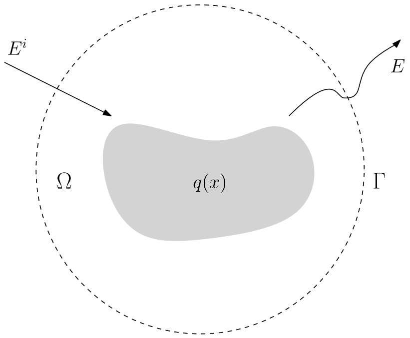

FIGURE 1
The inverse medium problem geometry. A plane wave E^i is incident on the scatterer q with a compact support contained in Ω.

where E^t is the total electric field, $\kappa > 0$ is the wavenumber or frequency, and q is a real function known as the scatterer representing the inhomogeneous medium. The scatterer is assumed to have a compact support contained in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ, and satisfies $-1 < q \leq q_{\max} < \infty$ where $q_{\max}$ is a positive constant. The problem geometry is shown in Figure 1.

The scatterer is illuminated by a plane wave

$$E^i(x) = \vec{p}e^{\mathrm{i}\kappa x\cdot\vec{n}},$$

where $\vec{n} \in \mathbb{S}^2$ is the propagating direction and $\vec{p} \in \mathbb{S}^2$ is the polarization vector satisfying $\vec{p}\cdot\vec{n} = 0$. Evidently, the incident wave satisfies the homogeneous Maxwell equation

$$\nabla \times (\nabla \times E^i) - \kappa^2 E^i = 0 \quad \text{in } \mathbb{R}^3. \tag{2.2}$$

Since the total field E^t consists of the incident field E^i and the scattered field E, it follows from (2.1)–(2.2) that the scattered field satisfies

$$\nabla \times (\nabla \times E) - \kappa^2(1+q)E = \kappa^2 q E^i \quad \text{in } \mathbb{R}^3. \tag{2.3}$$

In addition, the scattered field is required to satisfy the Silver–Müller radiation condition

$$\lim_{r\to\infty}\big((\nabla \times E)\times x - \mathrm{i}\kappa r E\big) = 0, \qquad r = |x|.$$

Denote by ν the unit outward normal to Γ. Computationally, it is convenient to reduce the problem to a bounded domain by imposing a suitable (artificial) boundary condition on Γ. For simplicity, we employ the first-order absorbing boundary condition

$$\nu \times (\nabla \times E) + \mathrm{i}\kappa\nu \times (\nu \times E) = 0 \quad \text{on } \Gamma. \tag{2.4}$$

Given the incident field E^i, the direct problem is to determine the scattered field E for the known scatterer q. This work is devoted to the solution of the IMP, i.e., determining the scatterer q from the tangential trace of the electric field, $\nu \times E|_\Gamma$ at multiple frequencies.

Although this is a classical problem in inverse scattering theory, progress has been difficult to make on reconstruction methods, due to the nonlinearity and ill-posedness associated with the inverse scattering problem. We refer to [5,58,64,71,85,86,93,109] for related results on the IMP.

To overcome the difficulties, an RLM was proposed in [59–61] for solving the IMP of the two-dimensional Helmholtz equation. Based on the Riccati equations for the scattering matrices, the method requires full aperture data and needs to solve a sensitivity matrix equation at each iteration. Due to the high computational cost, it is numerically difficult to extend the method to three-dimensional problems. Recently, new and more efficient RLMs have been developed for solving the two-dimensional Helmholtz equation and the three-dimensional Maxwell equations for both full and limited aperture data by directly using the differential equation formulations [13,20,21,23,24,28,29,31,32,37,51]. In the case of a fixed frequency, a novel RLM has also been developed by making use of the evanescent waves [22,25]. Direct imaging techniques have been explored to replace the weak scattering for generating the initial guesses in [19,33]. More recently, the RLM has been extended to solve the inverse medium scattering problem in elasticity [44].

Next, we present an RLM that solves the IMP of Maxwell's equations in three dimensions, which first appeared in our work [20,25]. The algorithm requires multifrequency scattering data, and the recursive linearization is obtained by a continuation method on the wavenumber. The algorithm first solves a linear equation under the Born approximation at the lowest wavenumber. Updates are made by using the data at higher wavenumbers sequentially. Following the idea of the Kaczmarz method, we use partial data and solve an underdetermined minimal norm solution at each step. For each iteration, one forward and one adjoint state of the Maxwell equations are solved, which may be implemented by using the symmetric second-order edge elements.

2.2. Born approximation

Rewrite (2.3) as

$$\nabla \times (\nabla \times E) - \kappa^2 E = \kappa^2 q(E^i + E), \tag{2.5}$$

where the incident wave is taken as a plane wave $E^i = \vec{p}_1 e^{\mathrm{i}\kappa x\cdot\vec{n}_1}$. Consider a test function $F = \vec{p}_2 e^{\mathrm{i}\kappa x\cdot\vec{n}_2}$, where $\vec{p}_2, \vec{n}_2 \in \mathbb{S}^2$ satisfy $\vec{p}_2 \cdot \vec{n}_2 = 0$. Clearly, the plane wave F satisfies (2.2).

Multiplying equation (2.5) by F and integrating over Ω on both sides, we have, by integrating by parts and noting (2.2) for F, that

$$\int_\Gamma \big[E \times (\nabla \times F) - F \times (\nabla \times E)\big] \cdot \nu \mathrm{d}s = \kappa^2 \int_\Omega qF \cdot (E^{\mathrm{i}} + E)\mathrm{d}x.$$

A simple calculation yields

$$\begin{aligned}\int_\Omega q(x)(\vec{p}_1 \cdot \vec{p}_2) e^{\mathrm{i}\kappa x\cdot(\vec{n}_1+\vec{n}_2)}\mathrm{d}x = {}& \frac{\mathrm{i}}{\kappa}\int_\Gamma (\nu \times E)\cdot\big((\vec{n}_2 + \nu) \times \vec{p}_2\big) e^{\mathrm{i}\kappa x\cdot\vec{n}_2}\mathrm{d}s \\ & - \int_\Omega q(x)(\vec{p}_2 \cdot E) e^{\mathrm{i}\kappa x\cdot\vec{n}_2}\mathrm{d}x.\end{aligned}$$

For the weak scattering, either the wavenumber κ is small, or the domain of support Ω is small, or $\|q\|_{L^\infty(\Omega)}$ is small, we may drop the second (nonlinear) term on the right-hand side of the above equation to obtain the linear integral equation

$$\int_\Omega q(x)e^{\mathrm{i}\kappa x\cdot(\vec{n}_1+\vec{n}_2)}\mathrm{d}x = \frac{\mathrm{i}}{(\vec{p}_1\cdot\vec{p}_2)\kappa}\int_\Gamma (\nu\times E)\cdot\big((\vec{n}_2+\nu)\times\vec{p}_2\big)e^{\mathrm{i}\kappa x\cdot\vec{n}_2}\mathrm{d}s,$$

which is the Born approximation.

Since the scatterer q has a compact support, we use the notation

$$\hat{q}(\xi) = \int_\Omega q(x)e^{\mathrm{i}\kappa x\cdot(\vec{n}_1+\vec{n}_2)}\mathrm{d}x,$$

where $\hat{q}(\xi)$ is the Fourier transform of $q(x)$ with $\xi = \kappa(\vec{n}_1+\vec{n}_2)$. Choose

$$\vec{n}_i = (\sin\theta_i\cos\phi_i, \sin\theta_i\sin\phi_i, \cos\theta_i), \quad i = 1,2,$$

where θ_i, ϕ_i are the latitudinal and longitudinal angles, respectively. It is clear to note that the domain $[0,\pi]\times[0,2\pi]$ of (θ_i,ϕ_i), $i=1,2$, corresponds to the ball $B_{2\kappa} = \{\xi\in\mathbb{R}^3 : |\xi|\le 2\kappa\}$. Thus, the Fourier modes of $\hat{q}$ in the ball $B_{2\kappa}$ can be determined. The scattering data with higher wavenumber κ must be used in order to recover more modes of the scatterer q.

2.3. Recursive linearization

As discussed in the previous subsection, when the wavenumber κ is small, the Born approximation allows the reconstruction of those Fourier modes less than or equal to 2κ for the function $q(x)$. We now describe a procedure that recursively determines q_κ, an approximation of $q(x)$ at $\kappa = \kappa_j$ for $j = 1,2,\ldots,$ with the increasing wavenumber.

Suppose now that the scatterer $q_{\tilde{\kappa}}$ has been recovered at some $\tilde{\kappa}$, and that κ is slightly larger than $\tilde{\kappa}$. We wish to determine q_κ or to determine equivalently the perturbation

$$\delta q = q_\kappa - q_{\tilde{\kappa}}.$$

Let E and $\tilde{E}$ be solutions of the scattering problem (2.3)–(2.4) corresponding to q_κ and $q_{\tilde{\kappa}}$, respectively. Taking the difference of the scattering problem (2.3)–(2.4) corresponding to q_κ and $q_{\tilde{\kappa}}$, omitting the second-order smallness in δq and in $\delta E = E - \tilde{E}$, we obtain

$$\begin{cases} \nabla\times(\nabla\times\delta E) - \kappa^2(1+q_{\tilde{\kappa}})\delta E = \kappa^2\delta q(E^{\mathrm{i}}+\tilde{E}) & \text{in } \Omega, \\ \nu\times(\nabla\times\delta E) + \mathrm{i}\kappa\nu\times(\nu\times\delta E) = 0 & \text{on } \Gamma. \end{cases} \tag{2.6}$$

For the scatterer q_κ and the incident wave E^{i}, we define the scattering map

$$M(q_\kappa, E^{\mathrm{i}}) = \nu\times E|_\Gamma,$$

where E is the solution of (2.3)–(2.4) with the scatterer q_κ. For simplicity, we denote $M(q_\kappa, E^{\mathrm{i}})$ by $M(q_\kappa)$ since the scattering map $M(q_\kappa, E^{\mathrm{i}})$ is linear with respect to E^{i}.

Next, we examine the boundary data $\nu\times E(x;\theta_1,\phi_1;\kappa)$. Here, the variable x is the observation point which has two degrees of freedom on the artificial boundary Γ, θ_1 and ϕ_1 are latitudinal and longitudinal angles of the incident wave E^{i}, respectively. At each frequency, we have four degrees of freedom, and thus data redundancy, which may be addressed by fixing one of the incident angles, say θ_1.

Let $(\phi_1)_j = 2\pi(j-1)/m$, $j = 1, \dots, m$, and define the residual operator

$$R_j(q_{\tilde{\kappa}}) = \nu \times E\big(x; \theta_1, (\phi_1)_j; \kappa\big)\big|_\Gamma - \nu \times \tilde{E}\big(x; \theta_1, (\phi_1)_j; \kappa\big)\big|_\Gamma,$$

where $\tilde{E}(x; \theta_1, (\phi_1)_j; \kappa)$ is the solution of (2.3)–(2.4) with the incident longitudinal angle $(\phi_1)_j$ and the scatterer $q_{\tilde{\kappa}}$. For each j, the linearized problem (2.6) can be written as the operator equation

$$DM_j(q_{\tilde{\kappa}})\delta q_j = R_j(q_{\tilde{\kappa}}), \tag{2.7}$$

where $DM_j(q_{\tilde{\kappa}})$ is the Fréchet derivative of the scattering map $M_j(q_\kappa)$ corresponding to the incident angle $(\phi_1)_j$. Applying the Landweber–Kaczmarz iteration [90] to (2.7) yields

$$\delta q_j = \beta_\kappa DM_j^*(q_{\tilde{\kappa}})R_j(q_{\tilde{\kappa}}),$$

where $\beta_\kappa > 0$ is a relaxation parameter and $DM_j^*(q_{\tilde{\kappa}})$ is the adjoint operator of $DM_j(q_{\tilde{\kappa}})$.

An adjoint state method is adopted to compute the correction δq_j efficiently [25]. For each incident wave with the longitudinal angle $(\phi_1)_j$, it is necessary to solve one direct and one adjoint problem for Maxwell's equations. Since the adjoint problem takes a similar variational form to the direct problem, we need to compute essentially two direct problems at each step. Once δq_j is determined, $q_{\tilde{\kappa}}$ is updated by $q_{\tilde{\kappa}} + \delta q_j$. After the mth sweep is completed, we get the reconstructed scatterer q_κ at the wavenumber κ. Assume that the scattering data is for $\kappa \in [\kappa_{\min}, \kappa_{\max}]$ and let $\kappa_{\min} = \kappa_0 < \kappa_1 < \cdots < \kappa_n = \kappa_{\max}$. The algorithm of the RLM can be illustrated in Table 1.

Start with the Born approximation q_{k_0}.

Do the outer loop on the wavenumber k_i, $i = 1, 2, \dots, n$.

Let $q_{k_i}^0 = q_{k_{i-1}}$.

Do the inner loop on the incident direction ϕ_j, $j = 1, 2, \dots, m$,

$$\delta q_j = \beta_k DM_j^*(q_{k_i}^{j-1})R_j(q_{k_i}^{j-1}),$$

$$q_{k_i}^j = q_{k_i}^{j-1} + \delta q_j.$$

End

End

TABLE 1
The algorithm beyond Born approximation

2.4. Numerical experiments

We present an example to illustrate the performance of the method. Let $\tilde{q}(x_1, x_2, x_3) = 2x_1^2 e^{-(x_1^2+x_2^2+x_3^2)}$ and reconstruct the scatterer defined by

$$q(x_1, x_2, x_3) = \tilde{q}(3x_1, 3.5x_2, 3x_3).$$

Figure 2 shows the surface plot of the true scatterer at slices $x_1 = 0.3$, $x_2 = 0$, and $x_3 = 0$, respectively. Six equally-spaced wavenumbers are used in the construction, starting from the lowest wavenumber $\kappa_{\min} = 0.5\pi$ and ending at the highest wavenumber $\kappa_{\max} = 2.5\pi$. The incident fields are taken at 20 randomly chosen directions, which accounts for 20 Landweber iterations at each wavenumber. The relaxation parameter is 0.01 and the noise level of the data is 5%. Figure 3 shows the reconstructed scatterer at the Born approximation with the wavenumber $\kappa = 0.5\pi$. Figures 4–5 illustrate the reconstructed scatterers at the different wavenumbers. It can be observed from the numerical results that the Born approximation generates a poor reconstruction, but the result can be improved as the wavenumber increases.

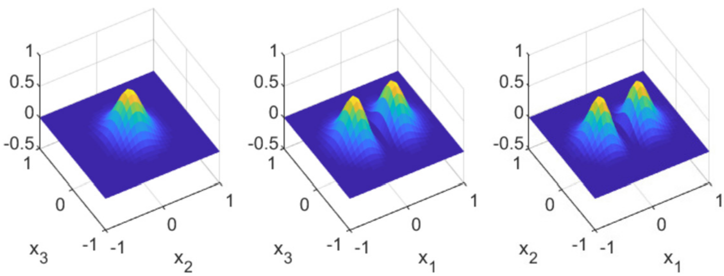

FIGURE 2
The true scatterer: (left) the slice $x_1 = 0.3$; (middle) the slice $x_2 = 0$; (right) the slice $x_3 = 0$.

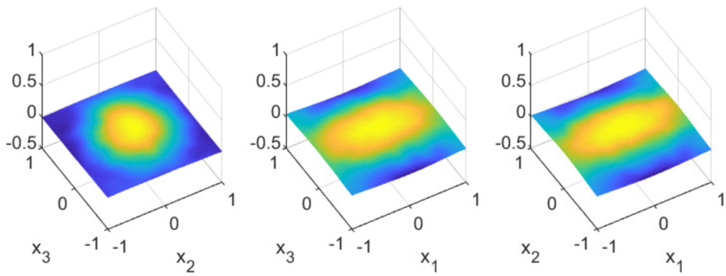

FIGURE 3
The Born approximation at the wavenumber $\kappa = 0.5\pi$: (left) the slice $x_1 = 0.3$; (middle) the slice $x_2 = 0$; (right) the slice $x_3 = 0$.

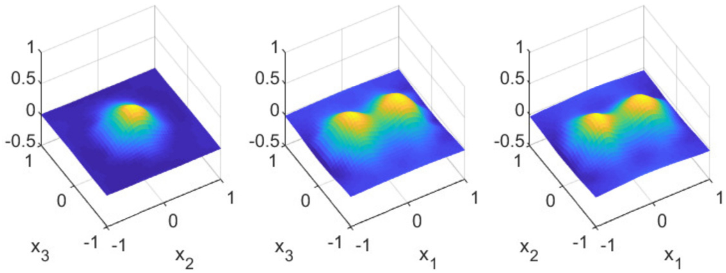

FIGURE 4

The reconstructed scatterer at the wavenumber $\kappa = 1.3\pi$: (left) the slice $x_1 = 0.3$; (middle) the slice $x_2 = 0$; (right) the slice $x_3 = 0$.

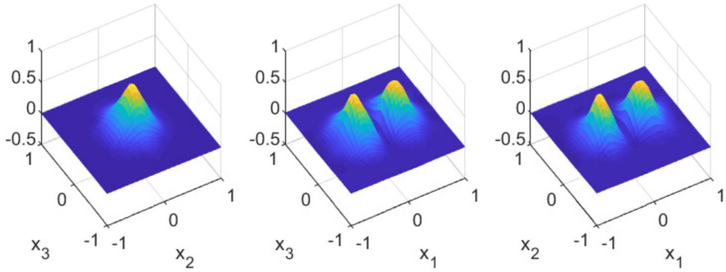

FIGURE 5

The reconstructed scatterer at the wavenumber $\kappa = 2.5\pi$: (left) the slice $x_1 = 0.3$; (middle) the slice $x_2 = 0$; (right) the slice $x_3 = 0$.

2.5. Stability analysis

It is well known that when the data is given for all frequencies and under certain geometrical assumptions, the IMP is well-posed with Lipschitz type stability estimates [45, 46, 96, 108]. However, in practice, the boundary measurements are often taken only at a finite number of frequencies. Our numerical method based on recursive linearization takes advantage of the regularity of the problem at high frequencies without being undermined by local minima. Numerical tests have shown that the method is very stable with data driven accuracy. Some preliminary convergence results of the RLM for solving the IMP with multifrequency are available in [27, 41].

Next, we present stability estimates for the multifrequency IMP in one-dimension. Stability in several dimensions is still open due to the difficulties of strong nonlinearity for high frequencies and trapped rays of the frequency-dependent scattering relation.

Consider the one-dimensional Helmholtz equation

$$\phi''(x,\kappa) + \kappa^2\big(1 + q(x)\big)\phi(x,\kappa) = 0, \quad x \in \mathbb{R},$$

where the scatterer q is assumed to be supported in $(0,1)$. Denote by ψ_+ and ψ_- the scattering waves corresponding to the left and right excitation $e^{\pm i\kappa x}$ which satisfy

$$\phi_\pm(x,k) = \psi_\pm + e^{\pm i\kappa x}.$$

Assume $q(x) \in C_0^{m+1}([0,1])$ and define the reflection coefficients by

$$\psi_+(x,\kappa) = \mu_+(\kappa)e^{-\mathrm{i}\kappa x}, \quad \psi_-(x,\kappa) = \mu_-(\kappa)e^{\mathrm{i}\kappa x}$$

and their associated measurements

$$d_\pm(\kappa) = \frac{1-\mu_\pm(\kappa)}{1+\mu_\pm(\kappa)}.$$

Given the measurement data $d_\pm(\kappa), \kappa \in (0,\kappa_0)$, the IMP is to reconstruct the scatterer $q(x)$. In the following, we present two stability results obtained recently in [42].

Theorem 2.1. *Assume that $q, \tilde{q}$ are two scatterer functions. Let $d_\pm, \tilde{d}_\pm$ be their boundary measurements in $(0,\kappa_0)$. Then there exist a positive constant C and a function η such that the following estimate holds:*

$$\|q - \tilde{q}\|_{L^\infty(\mathbb{R})} \le C\|d_\pm - \tilde{d}_\pm\|_{L^\infty(0,\kappa_0)}^{\eta(\kappa_0)}.$$

Remark 2.2. We refer to [42] for the complete statement. The proof is based on a combination of the trace formula, Hitrik's pole-free strip for the Schrödinger operator, the meromorphic extension, and the Two Constant Theorem. The Hölder exponent $\eta \in (0,1)$ in the estimate is an explicit increasing function of κ_0. It tends to zero when κ_0 tends to zero which shows as expected that the ill-posedness of the inversion increases when the band of frequency shrinks. We conclude from the stability estimate that the reconstruction of the scatterer function is accurate when the band of frequency is large enough and deteriorates when this later shrinks toward zero. These theoretical results confirm the numerical observations and the physical expectations for the increasing stability phenomena by taking multifrequency data.

By taking into account the uncertainty principle, it is reasonable to consider the observable part of the scatterer. In the one-dimensional setting, the observable part of the scatterer q over the frequency band $(0,\kappa_0)$ may be well-defined by using the truncated trace formula [42].

The next theorem gives the stability estimate on the observable part of the scatterer, which shows that the reconstruction of the observable part of the scatterer is stable for κ_0 sufficiently large.

Theorem 2.3. *Assume that $q, \tilde{q}$ are two scatterer functions and $q_{\kappa_0}, \tilde{q}_{\kappa_0}$ are their corresponding observable parts. Let $d_\pm, \tilde{d}_\pm$ be the boundary measurements in $(0,\kappa_0)$. There exist two constants ρ_Q and κ_Q such that the following estimate holds for all $\kappa_0 \ge \kappa_Q$:*

$$\|q_{\kappa_0} - \tilde{q}_{\kappa_0}\|_{L^\infty(\mathbb{R})} \le \rho_Q \left\| d(k) - \tilde{d}(k) \right\|_{L^1(0,\kappa_0)}.$$

3. INVERSE SOURCE PROBLEM

In this section, we consider the ISP that determines the unknown current density function from boundary measurements of the radiated fields at multiple wavenumbers. The ISP has many significant applications in biomedical engineering and antenna synthesis [7, 88].

In medical applications, it is often desirable to use the measurement of the radiated electromagnetic field on the surface of the human brain to infer abnormalities inside the brain [68].

3.1. Model problem

Consider the time-harmonic Maxwell equation in a homogeneous medium

$$\nabla \times (\nabla \times E) - \kappa^2 E = \mathrm{i}\kappa J \quad \text{in } \mathbb{R}^3, \tag{3.1}$$

where $\kappa > 0$ is the wavenumber, E is the electric field, J is the electric current density which is assumed to have a compact support Ω. The Silver–Müller radiation condition is required to ensure the well-posedness of the direct problem

$$\lim_{r\to\infty} \big((\nabla \times E) \times x - \mathrm{i}\kappa r E\big) = 0, \quad r = |x|. \tag{3.2}$$

Given $J \in L^2(\Omega)^3$, it is known that the scattering problem (3.1)–(3.2) has a unique solution

$$E(x,\kappa) = \int_\Omega G(x, y; \kappa) \cdot J(y)\mathrm{d}y,$$

where $G(x, y; \kappa)$ is Green's tensor for the Maxwell system (3.1). Explicitly, we have

$$G(x, y; \kappa) = \mathrm{i}\kappa g(x, y; \kappa) I_3 + \frac{\mathrm{i}}{\kappa} \nabla_x \nabla_x^\top g(x, y; \kappa),$$

where g is the fundamental solution of the three-dimensional Helmholtz equation and I_3 is the 3×3 identity matrix.

Let $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$, where R is a positive constant such that $\Omega \subset\subset B_R$. Denote by Γ_R the boundary of B_R. In the domain $\mathbb{R}^3 \setminus B_R$, the solution of (3.1) has a series expansion in the spherical coordinates which may be used to derive the capacity operator T. In addition, it can be verified that the solution of (3.1) satisfies the transparent boundary condition

$$(\nabla \times E) \times \nu = \mathrm{i}\kappa T(E \times \nu) \quad \text{on } \Gamma_R,$$

where ν is the unit outward normal to Γ_R.

Define the boundary measurement in terms of the tangential trace of the electric field

$$\|E(\cdot,\kappa) \times \nu\|_{\Gamma_R}^2 = \int_{\Gamma_R} \left(\left|T\big(E(x,\kappa) \times \nu\big)\right|^2 + \left|E(x,\kappa) \times \nu\right|^2\right)\mathrm{d}\gamma(x).$$

Let J be the electric current density with the compact support Ω. The ISP of electromagnetic waves is to determine J from the tangential trace of the electric field $E(x,\kappa) \times \nu$ for $x \in \Gamma_R$.

The ISP for the fixed frequency case has been studied extensively. It is now well known that the problem is ill-posed with nonuniqueness and instability [1, 50, 69, 77, 82]. Due to the existence of infinitely many nonradiating fields, a source with extended support cannot be uniquely determined from surface measurements at a fixed frequency. Therefore, additional constraints need to be imposed in order to obtain a unique solution to the inverse problem. A usual choice is to find the source with a minimum energy norm. However, the difference between the minimum energy solution and the original source function could be significant. Another difficulty of the ISP at fixed frequency is the inherited instability due to exponential decay of the singular eigenvalues of the forward operator [34,35,72]. For the special cases of reconstruction for point sources, we refer to [6,9,38,39,107] for studies of the unique identifiability and stability of the problem.

The use of the multiple frequency data for the ISP provides an approach to circumvent the difficulties of nonuniqueness and instability presented at a fixed frequency. For the ISP of the Helmholtz equation, uniqueness and stability were established in [34] by multiple frequency measurements. The results indicate that the multifrequency ISP is not only uniquely solvable but also is Lipschitz stable when the highest wavenumber exceeds a certain real number.

In the rest of the section, we present our recent results on uniqueness and stability for the ISP of Maxwell's equations [30], and discuss the recent development on the inverse random source problems, where the current density is a random function.

3.2. Uniqueness and stability

Denote by $\mathbb{X}(B_R)$ the closure of the following set in the $L^2(B_R)^3$ norm:

$$\left\{E \in H(\mathrm{curl}, B_R) : \int_{B_R} \big((\nabla \times E) \cdot (\nabla \times \psi) - \kappa^2 E \cdot \psi\big)\mathrm{d}x = 0,\ \forall \psi \in C_0^\infty(B_R)^3\right\}.$$

We have the following orthogonal decomposition of $L^2(B_R)^3$ [1]:

$$L^2(B_R)^3 = \mathbb{X}(B_R) \oplus \mathbb{Y}(B_R),$$

where $\mathbb{Y}(B_R)$ is an infinite-dimensional subspace of $L^2(B_R)^3$ and the electric current densities in the subspace $\mathbb{Y}(B_R)$ are called nonradiating sources. It corresponds to finding a minimum norm solution when computing the component of the source in $\mathbb{X}(B_R)$.

The following two results characterize clearly the uniqueness and nonuniqueness of the ISP. The proofs can be found in [30].

Theorem 3.1. *Suppose $J \in \mathbb{Y}(B_R)$. Then J does not produce any tangential trace of electric fields on Γ_R and thus cannot be identified.*

Theorem 3.2. *Suppose $J \in \mathbb{X}(B_R)$, then J can be uniquely determined by the data $E \times \nu$ on Γ_R.*

Define a functional space

$$\mathbb{J}_M(B_R) = \left\{J \in \mathbb{X}(B_R) \cap H^m(B_R)^3 : \|J\|_{H^m(B_R)^3} \le M\right\},$$

where $m \ge d$ is an integer and $M > 1$ is a constant. The following theorem concerns the stability for the multifrequency ISP (3.1).

Theorem 3.3. *Let E be the solution of the source problem* (3.1)–(3.2) *corresponding to $J \in \mathbb{J}_M(B_R)$. Then*

$$\|J\|^2_{L^2(B_R)^3} \lesssim \varepsilon^2 + M^2\left(\frac{K^{\frac{2}{3}}|\ln \varepsilon|^{\frac{1}{4}}}{(R+1)(6m-15)^3}\right)^{5-2m},$$

where

$$\varepsilon = \left(\int_0^K \kappa^2 \| E(\cdot, \kappa) \times \nu \|_{\Gamma_R}^2 d\kappa \right)^{1/2}.$$

The stability result shows that as the highest frequency increases, the stability continues to improve and approaches the Lipschitz type.

3.3. Inverse random source problems

Stochastic inverse problems refer to inverse problems that involve uncertainties due to the unpredictability of the model and incomplete knowledge of the system and measurements. Compared to deterministic counterparts, stochastic inverse problems have substantially more difficulties from randomness and uncertainties. New models and methodologies must be developed for solving stochastic inverse problems.

When the random source is modeled as the white noise, the stochastic ISP is considered for the Helmholtz equation [12, 15, 68, 98]. The goal is to reconstruct the statistical properties of the random source, such as the mean and variance, from boundary measurements of the radiated random wave field. Since the white noise has independent increments, Itô's calculus can be utilized to derive explicit formulas between the statistics of the wave field and the random source. Recently, the model of the microlocally isotropic Gaussian field is developed to handle stochastic processes with correlated increments [95, 97]. The stochastic inverse problem is to determine the microcorrelation strength in the principal symbol from some statistics of the random wave fields. More recently, a new model of the inverse random source problem has been proposed for the stochastic Helmholtz and Maxwell equations [99, 100], where the source is assumed to be driven by a fractional Gaussian field. The new model covers various stochastic processes and allows to deal with rougher sources.

4. INVERSE DIFFRACTION GRATING PROBLEM

For an IOP, the scattering object is a homogeneous obstacle with a given boundary condition. The inverse problem is to determine the obstacle from knowledge of the scattered field away from the obstacle. In this section, we consider the scattering of a time-harmonic electromagnetic plane wave by a (infinite) periodic structure (Figure 6), also known as a grating in diffractive optics, which may be regarded as a special class of the obstacle problem. The scattering problem in this setting is often referred to as the diffraction problem in the literature.

Due to important applications, especially in the design and fabrication of optical elements such as corrective lenses, antireflective interfaces, beam splitters, and sensors, the diffraction problems in periodic structures have been studied extensively. We refer to [17, 26] and references therein for the mathematical studies of the existence and uniqueness questions of the model problems. Numerical methods can be found in [14, 54, 63, 104] for either an integral equation approach or a variational approach. A comprehensive review can be

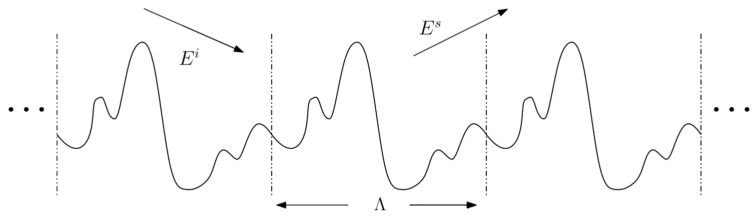

FIGURE 6
The inverse diffraction problem geometry. A plane wave E^i is incident on the surface with period Λ.

found in [16,105] on diffractive optics technology and its mathematical modeling, as well as computational methods.

This section is concerned with the inverse diffraction problem, which is to determine the periodic structure from a reflected field measured at a constant distance away from the structure corresponding to a given incident field. The inverse problem arises naturally in the study of optimal design problems in diffractive optics. The goal is to design a grating structure that gives rise to some specified far-field patterns [11,70].

The mathematical questions on uniqueness and stability for the inverse diffraction problem of both the two-dimensional Helmholtz equation and the three-dimensional Maxwell equations have been studied extensively in [3,10,18,47,49,52,84,92,110]. However, all of the above mentioned results are under fairly restrictive assumptions, or local in nature. A complete answer to the uniqueness question has been given in [47,48] for the determination of a three-dimensional polyhedral periodic diffraction structure by the scattered electromagnetic fields measured above the structure. The result indicates that the uniqueness by any given incident field fails for seven simple classes of regular polyhedral structures. Moreover, if a regular periodic polyhedral structure is not uniquely identifiable by a given incident field, then it belongs to a nonempty class of the seven classes whose elements generate the same total field as the original structure when impinged upon by the same incident field. Problems on global uniqueness or stability for the inverse diffraction problem are still open.

A number of numerical methods have been developed to solve these inverse problems [8,53,73,83,89]. Using a single-layer potential representation, we have presented in [29] an efficient RLM for solving the nonlinear inverse diffraction grating problem in a one-dimensional perfectly reflecting structure. The algorithm requires multifrequency data and the iterative steps are obtained by recursive linearization with respect to the wavenumber: at each step a nonlinear Landweber iteration is applied, with the starting point given by the output from the previous step at a lower wavenumber. Thus, at each stage an approximation to the grating surface filtered at a higher frequency is created. Starting from a reasonable initial guess, the RLM is shown to converge for a larger class of surfaces than the usual Newton's method using the same initial guess.

An extension of the numerical method has been done in [28] for solving the inverse diffraction problem with phaseless data. By using multifrequency data, our algorithm is based on the RLM marching with respect to the wavenumber. With the starting point given by the output from the previous step at a lower wavenumber, a new approximation to the grating surface filtered at a higher frequency is updated by a Landweber iteration. The numerical results show that the continuation method cannot determine the location of the grating structure, but it can effectively reconstruct the grating shape from the phaseless data.

Another important extension of the method is to solve the inverse diffraction problem by a random periodic structure. Existing studies mostly assume that the periodic structure is deterministic and only the noise level of the measured data is considered for the inverse problem. In practice, however, there is a level of uncertainty of the scattering surface, e.g., the grating structure may have manufacturing defects or it may suffer other possible damages from regular usage. Therefore, in addition to the noise level of measurements, the random surface itself also influences the measured scattered fields. Surface roughness measurements are of great significance for the functional performance evaluation of machined parts and design of microoptical elements. Little is known in mathematics or computation about solving inverse problems of determining random surfaces. One challenge lies in the fact that the scattered fields depend nonlinearly on the surface, which makes the random surface reconstruction problem extremely difficult. Another challenge is to understand to what extend the reconstruction could be made. In other words, what statistical quantities of the profile could be recovered from the measured data? We have recently proposed an efficient numerical method in [36] to reconstruct the random periodic structure from multifrequency scattered fields measured at a constant height above the structure. We demonstrate that three critical statistical properties, namely the expectation, root mean square, and correlation length of the random structure may be reconstructed. Our method is based on a novel combination of the Monte Carlo technique for sampling the probability space, an RLM with respect to the wavenumber, and the Karhunen–Loève expansion of the random structure.

5. DISCUSSIONS AND FUTURE DIRECTIONS

This work is devoted to mathematical analysis and numerical methods for solving inverse scattering problems. On mathematical analysis, we have focused mainly on the stability analysis of the inverse problems. Numerically, we have discussed the recursive linearization approach. These results confirm that the spectral information is vital in stable solution of inverse scattering problems and whenever possible multiple frequency data should be taken and employed for reconstructions. There are tremendous research opportunities for mathematical analysis and numerical methods of inverse scattering problems to meet the continuous growing needs in science and engineering to explore the complex world, from the universe to the new materials, and to the cell. As the computing powers continue to increase and new fast algorithms are developed, inverse scattering problems will continue to contribute to the advancements of the relevant science and engineering.

In the following, we point out some future research directions in line of the research discussed in this work.

For inverse medium scattering problems, we present the stability estimates for the one-dimensional model. In the extreme case when all frequency data is attainable, the estimates have also been obtained in [45,46]. However, no stability estimate is available for the IMP in several dimensions. By taking into account the uncertainty principle, we conjecture that the reconstruction of the observable part of the scatterer is Lipschitz stable.

For the inverse diffraction problem, global uniqueness remains open. In the polyhedral structure cases, the problem was solved in [47,48] by using group symmetry properties of the structures and unique continuation properties. For the obstacle scattering problem including the diffraction problem, another interesting problem is to derive the stability estimate with explicit dependence on the wavenumber. The estimate will be particularly useful for convergence analysis of the numerical methods.

In computation, the multifrequency data-based RLM is shown to be stable and effective for solving inverse scattering problems. However, only limited progress has been made on convergence analysis [27,40,41,106]. It is expected that complete analysis should be done by combining the stability estimates and the uncertainty principle. Another difficulty is the incomplete data, including phaseless, limited aperture, or incomplete model. It is interesting to investigate how to employ computational inverse scattering problems to break the diffraction limit. In other words, how to balance the accuracy and resolution. Initial efforts were made on combining the RLM with near-field imaging techniques [31,32].

Another interesting direction is to study the stochastic inverse scattering problems. As discussed in Section 3.3, initial efforts have been made for solving inverse random source problems. However, little progress is made for inverse medium problems and inverse obstacle problems, where the scatterer and obstacle are respective random functions. It is of interest to consider the more challenging inverse random medium scattering problem. The medium is no longer deterministic and its randomness and uncertainty have to be modeled as well. Since the scattered field depends on the medium or the obstacle nonlinearly, as opposed to the linear dependence on the source, the scattering and inverse scattering problems become much more challenging. In particular, new mathematical and computational frameworks are in demand for solving these problems.

Finally, although the scope of this work is limited to inverse scattering problems in acoustic and electromagnetic waves, we believe many of the methods and techniques discussed here could apply to inverse scattering problems in other wave models. Other emerging topics which beyond the scope of this work but could change the future landscape of solving inverse scattering problems include deep learning type methods [43,91] and optimal transport methods [75].

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