CONTROL THEORY OF STOCHASTIC DISTRIBUTED PARAMETER SYSTEMS: RECENT PROGRESS AND OPEN PROBLEMS

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ABSTRACT

In recent years, important progresses have been made in the control theory for stochastic distributed parameter control systems (SDPSs for short). However, the theory is far from being complete. The primary difficulty is that many effective tools and methods for deterministic distributed parameter control systems and stochastic finite-dimensional control systems do not work anymore for SDPSs. One has to develop new mathematical tools, such as stochastic transposition method and stochastic Carleman estimate, even for some very simple SDPSs. The objectives of this paper are to provide some new results, to show some new phenomena, to explain the new difficulties, and to present some new methods for the control theory of SDPSs. We mainly focus on our works for the controllability for stochastic hyperbolic equations, and the Pontryagin-type maximum principle for controlled stochastic evolution equations. At last, a number of open questions and future directions of research are given.

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1. INTRODUCTION

Control theory was founded by N. Wiener in 1948. It is an interdiscipline among mathematics, engineering, and computer science. The early works in this field were mainly concerned with deterministic finite-dimensional control systems. Motivated by applications, numerous mathematicians and engineers put great effort to study control theory for more complex systems, such as distributed parameter control systems (typically governed by partial differential equations), stochastic finite-dimensional control systems (governed by stochastic differential equations), and SDPSs (typically governed by stochastic partial differential equations, SPDEs for short). These studies provide a rich source of complex mathematical problems, which have fundamental impact on the development of many areas in mathematics.

It is very surprising that the control theory for SDPSs is still in its infancy though it has been studied for around 60 years. Compared with other directions in mathematical control theory (including control theory for deterministic and stochastic finite-dimensional systems and that for distributed parameter systems), many aspects of control theory for SDPSs are much less understood or even still unknown. Nevertheless, one cannot, by no means, ignore its importance. On the one hand, the world is full of uncertainties. They enter the system through noise in sensing/actuation, external disturbances affecting the underlying system, and uncertain dynamics in the system (parameter errors, unmodeled effects, etc.). For lots of significant physical and biological systems, these uncertainties cannot be ignored, and the systems should be governed by SPDEs (e.g., [19]). This leads to a major requirement for the study of the control theory of SDPSs (e.g., [11, 21, 41]). On the other hand, control theory for deterministic finite-dimensional control systems is relatively mature now, and there is a huge list of publications for distributed parameter control systems and stochastic finite-dimensional control systems. The study of SDPSs is a natural development of the mathematical control theory. Then, what slows the pace of the control theory of SDPSs? In my opinion, it lies in the fact that the complexity of SDPSs introduces extreme difficulties. Firstly, the formulation of the control problems for SDPSs may differ from those for distributed parameter control systems or stochastic finite-dimensional control systems. Secondly, many powerful methods and tools developed for the latter two systems mentioned above cannot work for SDPSs. Thirdly, people know very little about SPDEs although much progress has been made in recent years. As a result, new notions and mathematical tools are required, even for some very simple SDPSs. We will demonstrate this by illustrative examples in Sections 2 and 3.

The most fundamental problem in control theory is to modify the behavior of the system by means of suitable "control" actions in an "optimal" way. This leads to the formation of *controllability* and *optimal control problems*. Roughly speaking, controllability involves finding one way to steer the state of the system to a desired target from a given starting point. Optimal control concerns finding the "best way," according to a given cost criterion, to achieve the desired goal. In this paper, we mainly focus on some recent progress on these two topics for SDPSs. We do not attempt to cover the whole field of these topics, which is virtually hopeless. Rather, with admitted bias, we choose subjects that are undergoing rapid change and require new approaches to meet the challenges and opportunities. No attempt will be made to provide an exhaustive list of all the papers in the corresponding topics, which would only tend to make the narrative very disjoint.

Although we will deal with SDPSs, it is helpful to introduce some fundamental ideas in a simpler setting, i.e., for finite-dimensional deterministic control systems. It can also help readers see the essential differences between the deterministic and stochastic problems.

Let $T > 0$. Consider the following control system:

$$\begin{cases} y_t(t) = Ay(t) + Bu(t), \quad \text{a.e. } t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ ($n, m \in \mathbb{N}$), y is the state, and $u \in L^2(0,T;\mathbb{R}^m)$ is the control.

Definition 1.1. The control system (1.1) is called *exactly controllable* at time T if for any $y_0, y_1 \in \mathbb{R}^n$, there is a control $u \in L^2(0,T;\mathbb{R}^m)$ such that the corresponding state y to (1.1) satisfies $y(T) = y_1$.

Remark 1.1. Definition 1.1 can be easily extended to more general control systems, for which the requirement $y(T) = y_1$ may be too restrictive and has to be relaxed. This leads to the notions of approximate/null/partial controllability, and so on.

The exact controllability problem of (1.1) can be regarded as a two-point boundary value problem. However, it is clearly ill-posed and cannot be solved by the classical well-posedness theory of ODEs. To study it, people introduce the adjoint equation of (1.1):[1]

$$\begin{cases} z_t(t) = -A^\top z(t), \quad t \in [0,T], \\ z(T) = z_T \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

and prove the following result:

Theorem 1.1. *The system* (1.1) *is exactly controllable at time* T *if and only if solutions to* (1.2) *satisfy*

$$|z_T|^2_{\mathbb{R}^n} \le \mathcal{C} \int_0^T \left|B^\top z(t)\right|^2_{\mathbb{R}^m} dt, \quad \forall z_T \in \mathbb{R}^n. \tag{1.3}$$

Here and henceforth, unless otherwise stated, we shall write $\mathcal{C}$ for a generic positive constant, which may vary from one place to another.

Remark 1.2. The inequality (1.3) is called an *observability estimate* for (1.2). Roughly speaking, it concerns whether the solution of (1.3) can be fully determined from the observation $B^\top z(t)$, $t \in [0,T]$. Usually, $B^\top$ is not of full row rank. Hence, one cannot solve for z_T from $B^\top z_T$ directly. In such a case, we do our observation on a time interval $[0,T]$. Besides the connection with controllability, observability has its own interest in control theory.

1 For any matrix D, denote by $D^\top$ the transpose of D.

Remark 1.3. Whether inequality (1.3) holds or not depends on A and B, where A decides the type of the control system and B reflects the way we control the system. A sufficient and necessary condition for (1.3) is that (A, B) fulfills the Kalman rank condition (e.g., [18]).

By Theorem 1.1, the controllability problem of (1.1) is reduced to an *a priori estimate* of its adjoint equation. This idea is greatly extended to different kinds of control systems. Most of the controllability results for linear control systems are proved by establishing suitable observability estimates for their adjoint equations (e.g., [16,21,40,49,50]). However, it is much more complicated to study the controllability problems for SDPSs. Indeed, as we will explain in Section 2, we have to handle the observability for backward SPDEs. Moreover, since one may put controls on both drift and diffusion terms in SDPSs (as we shall see in Section 2, sometimes it is necessary to introduce controls in such a way), the controls will affect each other. Further, compared with distributed parameter control systems, some new and unexpected phenomena are found for controllability problems of SDPSs:

(1) One may need stronger conditions to get the approximate controllability than the null controllability for SDPSs (e.g., [26]).

(2) Two controls are needed to get the exact controllability of stochastic Schrödinger equations and stochastic transport equations (e.g., [27,29]).

(3) The approximate/null controllability may be sensitive with respect to small perturbations of lower order terms (e.g., [8,23]).

(4) To get the exact controllability, the control may be very irregular (e.g., [31]).

(5) The reachable set is very "small" if there is no control in the diffusion term (e.g., [47]).

(6) A stochastic hyperbolic equation is not exactly controllable with controls acting on the whole domain where the equation evolves on (e.g., [37]).

Generally speaking, the controllability properties for different SDPSs are drastically different. Consequently, when studying controllability problems of SDPSs, we should consider concrete models of SDPSs. There are two prototypical equations needed to be understood first: the stochastic hyperbolic equation and the stochastic parabolic equation. Due to the limitation of space, we will focus on the former which possesses sufficient complexity to permit exposition of a wide variety of interesting questions and differs from the controllability of deterministic hyperbolic equations essentially. Readers are referred to [23,26,40,45] and the references therein for controllability of the latter equation.

Next, we present a typical optimal control problem. Fix a suitable function $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and a nonempty subset U of $\mathbb{R}^m$. Let

$$\mathcal{U} \stackrel{\Delta}{=} \big\{u : [0,T] \to U \mid u \text{ is Lebesgue measurable}\big\}.$$

Consider the following control system:

$$\begin{cases} y_t(t) = f\big(t, y(t), u(t)\big), & \text{a.e. } t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{1.4}$$

with a cost functional

$$\mathcal{J}(u) = \int_0^T g\big(t, y(t), u(t)\big)dt + h\big(y(T)\big), \quad u(\cdot) \in \mathcal{U}. \tag{1.5}$$

Here $y_0 \in \mathbb{R}^n$, y is the state, and u is the control, valued in $\mathbb{R}^n$ and U, respectively; g and h are suitable functions. The optimal control problem is as follows:

Problem (DOP). Find a $\bar{u} \in \mathcal{U}$ such that

$$\mathcal{J}(\bar{u}) = \inf_{u \in \mathcal{U}} \mathcal{J}(u). \tag{1.6}$$

Any control $\bar{u} \in \mathcal{U}$ satisfying (1.6) is called an *optimal control*, and the corresponding state, denoted by $\bar{y}$, is called an *optimal state*, and $(\bar{y}, \bar{u})$ is called an *optimal pair*.

Problem (DOP) can be regarded as an infinite-dimensional optimization problem. A principal approach to solve it is to derive necessary conditions satisfied by optimal solutions. Nevertheless, since $\mathcal{U}$ may be quite general, the classical variation technique cannot be applied to Problem (DOP) directly. In [43], L. S. Pontryagin's group employed the spike variation to derive the so-called *Pontryagin's Maximum Principle*, which states a necessary condition that any optimal pair must satisfy:

Theorem 1.2. *Let $(\bar{y}, \bar{u})$ be an optimal pair for Problem (DOP). Then, for a.e. $t \in [0,T]$,*

$$\mathbb{H}\big(t, \bar{y}(t), \bar{u}(t), z(t)\big) = \max_{u \in U} \mathbb{H}\big(t, \bar{y}(t), u, z(t)\big), \tag{1.7}$$

where $z : [0,T] \to \mathbb{R}^n$ solves

$$\begin{cases} z_t(t) = -f_y\big(t, \bar{y}(t), \bar{u}(t)\big)^\top z(t) + g_y\big(t, \bar{y}(t), \bar{u}(t)\big), & \text{a.e. } t \in [0,T], \\ z(T) = -h_y\big(\bar{y}(T)\big) \end{cases} \tag{1.8}$$

and

$$\mathbb{H}(t, y, u, p) \stackrel{\Delta}{=} \big\langle p, f(t, y, u)\big\rangle_{\mathbb{R}^n} - g(t, y, u), \quad (t, y, u, p) \in [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n.$$

The significance of Theorem 1.2 lies in that the infinite-dimensional optimization problem (1.6) is reduced to the finite-dimensional optimization problem (1.7) (in the pointwise sense). Particularly, in many cases, U is a finite set and (1.7) itself allows people to construct the optimal control.

Compared with Problem (DOP), there are new essential difficulties in establishing Pontryagin-type Maximum Principle for optimal control problems of SDPSs. The primary one is the well-posedness of the adjoint equation (a generalization of (1.8)), which is an operator-valued backward stochastic evolution equation. There is no suitable stochastic integration theory for general operator-valued stochastic processes. Hence, that equation cannot be understood as a stochastic integral equation and does not admit a mild or a weak solution.

To overcome this difficulty, we introduce a new notion, i.e., relaxed transposition solution and employ the stochastic transposition method to prove the well-posedness of that equation. More details are provided in Section 3.

In this paper, we consider control problems for SDPSs governed by Itô-type SPDEs. The system is completely observable (meaning that the controller is able to observe the system state completely) and the noise is a one-dimensional standard Brownian motion. For the optimal control problem, the cost functional is an integral over a deterministic time interval. The reasons for these settings are that we would like to show readers some fundamental structure and properties of control problems for SDPSs in a clean and clear way, and avoid technicalities caused by more complicated models.

The rest of this paper consists of three parts. The first (resp. second) one is devoted to controllability (resp. optimal control) problems for SDPSs. At last, in the third part, we provide some open problems for control theory of SDPSs.

2. EXACT CONTROLLABILITY OF STOCHASTIC HYPERBOLIC EQUATIONS

For the readers' convenience, we first recall some basic notations. Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ (with $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ being a filtration) be a complete filtered probability space. Denote by $\mathbb{F}$ the progressive σ-field (in $[0,T]\times\Omega$) with respect to $\mathbf{F}$. Let $\mathbf{X}$ be a Banach space. For any $p, q \in [1,\infty)$, write $L^p_{\mathcal{F}_t}(\Omega;\mathbf{X}) \stackrel{\Delta}{=} L^p(\Omega,\mathcal{F}_t,\mathbb{P};\mathbf{X})$ ($t\in[0,T]$), and define

$$
\begin{aligned}
&L^q_{\mathbb{F}}\big(0,T;L^p(\Omega;\mathbf{X})\big)\\
&\quad\stackrel{\Delta}{=}\left\{\varphi:(0,T)\times\Omega\to\mathbf{X}\mid \varphi(\cdot)\text{ is }\mathbf{F}\text{-adapted and }\int_0^T\big(\mathbb{E}|\varphi(t)|^p_{\mathbf{X}}\big)^{\frac{q}{p}}dt<\infty\right\}.
\end{aligned}
$$

Similarly, for $1\le p<\infty$, we may also define $L^\infty_{\mathbb{F}}(0,T;L^p(\Omega;\mathbf{X}))$, $L^p_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathbf{X}))$, and $L^\infty_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathbf{X}))$. In the sequel, we shall simply denote $L^p_{\mathbb{F}}(\Omega;L^p(0,T;\mathbf{X})\big)\equiv L^p_{\mathbb{F}}\big(0,T;L^p(\Omega;\mathbf{X}))$ by $L^p_{\mathbb{F}}(0,T;\mathbf{X})$. For any $p\in[1,\infty)$, set

$$
\begin{aligned}
C_{\mathbb{F}}\big([0,T];L^p(\Omega;\mathbf{X})\big)\stackrel{\Delta}{=}\big\{&\varphi:[0,T]\times\Omega\to\mathbf{X}\mid\varphi\text{ is }\mathbf{F}\text{-adapted and}\\
&\varphi:[0,T]\to L^p_{\mathcal{F}_T}(\Omega;\mathbf{X})\text{ is continuous}\big\}.
\end{aligned}
$$

Similarly, for any $k\in\mathbb{N}$, one can define the Banach space $C^k_{\mathbb{F}}([0,T];L^p(\Omega;\mathbf{X}))$. Also, we write $D_{\mathbb{F}}([0,T];L^p(\Omega;\mathbf{X}))$ for the Banach space of all X-valued, $\mathbf{F}$-adapted, stochastic processes X which are càdlàg in $L^p_{\mathcal{F}_T}(\Omega;X)$ and $|X|_{L^\infty_{\mathbb{F}}(0,T;L^p(\Omega;\mathbf{X}))}<\infty$, with the norm inherited from $L^\infty_{\mathbb{F}}(0,T;L^p(\Omega;\mathbf{X}))$.

Throughout this section, we assume that there is a 1-dimensional standard Brownian motion $W(\cdot)$ on $(\Omega,\mathcal{F},\mathbf{F},\mathbb{P})$ and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$.

Let $G\subset\mathbb{R}^n$ ($n\in\mathbb{N}$) be a bounded domain with a C^2 boundary Γ. Let $\Gamma_0\subset\Gamma$ be a nonempty subset satisfying suitable assumptions to be given later. Set $Q=(0,T)\times G$, $\Sigma=(0,T)\times\Gamma$, and $\Sigma_0=(0,T)\times\Gamma_0$. Let $(a^{jk})_{1\le j,k\le n}\in C^3(\overline{G};\mathbb{R}^{n\times n})$ be such that

$a^{jk} = a^{kj}$ $(j, k = 1, 2, \ldots, n)$ and, for some constant $s_0 > 0$,

$$\sum_{j,k=1}^{n} a^{jk}(x)\xi^j\xi^k \geq s_0|\xi|^2, \quad \forall (x, \xi) \stackrel{\Delta}{=} \big(x, \xi^1, \ldots, \xi^n\big) \in G \times \mathbb{R}^n.$$

Fix $a_1 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}(G; \mathbb{R}^n))$, $a_2, a_3, a_4 \in L^\infty_{\mathbb{F}}(0, T; L^\infty(G))$, and $a_5 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}_0(G))$.

2.1. Formulation of the problem

Consider the following controlled stochastic hyperbolic equation:

$$\begin{cases} dy_t - \displaystyle\sum_{j,k=1}^{n} (a^{jk} y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2 y + f)dt + (a_3 y + g)dW(t) & \text{in } Q, \\ y = h & \text{on } \Sigma, \\ y(0) = y_0, \ y_t(0) = y_1 & \text{in } G, \end{cases} \tag{2.1}$$

where the initial data $(y_0, y_1) \in L^2(G) \times H^{-1}(G)$, (y, y_t) is the state, and $f, g \in L^\infty_{\mathbb{F}}(0, T; H^{-1}(G))$ and $h \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma))$ are three controls. As we shall see in Section 2.2, equation (2.1) admits a unique *transposition solution*

$$y \in C_{\mathbb{F}}\big([0, T]; L^2\big(\Omega; L^2(G)\big)\big) \cap C^1_{\mathbb{F}}\big([0, T]; L^2\big(\Omega; H^{-1}(G)\big)\big).$$

Inspired by the definition of the exact controllability of deterministic hyperbolic equations and stochastic differential equations, we introduce the following notion.

Definition 2.1. We say that the control system (2.1) is exactly controllable at time T if for any $(y_0, y_1) \in L^2(G) \times H^{-1}(G)$ and $(y_0', y_1') \in L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$, one can find controls $(f, g, h) \in L^2_{\mathbb{F}}(0, T; H^{-1}(G)) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G)) \times L^2_{\mathbb{F}}(0, T; L^2(\Gamma))$ such that the corresponding state y to (2.1) satisfies that $(y(T), y_t(T)) = (y_0', y_1')$ a.s.

Remark 2.1. Compared with Definition 1.1, Definition 2.1 looks much more complex. This is due to the complexity of the control system. The two definitions share the same spirit, that is, using controls to steer the state of the system to the desired destination. Here and in what follows, we use adapted stochastic processes as controls according to two reasons:

(1) In stochastic control systems, "uncertainty" is critical, i.e., there is some possible variations in the system's behavior. The controls have to take different possibilities into account.

(2) We cannot use information from the future. Thus, the control at time t has to be measurable with respect to the σ-algebra $\mathcal{F}_t$, which reflects the information we can obtain at time t.

Three controls are applied in (2.1). One may expect the exact controllability to be correct. However, surprisingly enough, we have the following negative result.

Theorem 2.1 ([37, THEOREM 2.1]). *The system* (2.1) *is not exactly controllable for any* $T > 0$.

Remark 2.2. Both Theorem 2.1 and Theorem 2.2 below are negative results, which have their own interests. Indeed, one aspect of control theory that is particularly important is the exploration of fundamental limits of the control ability for a given control system, since trade-offs between the cost we pay for controls and the performance of the behavior of the system will be the primary design challenge for a control system.

The controls we introduce into (2.1) are the strongest possible ones. Theorem 2.1 shows that the controllability property of stochastic hyperbolic equations differs significantly from the well-known controllability property for deterministic hyperbolic equations (e.g., [50]). Motivated by this, we consider the following refined version of controlled stochastic hyperbolic equation:

$$\begin{cases} dy = \hat{y}dt + (a_4 y + f)dW(t) & \text{in } Q, \\ d\hat{y} - \displaystyle\sum_{j,k=1}^{n} (a^{jk} y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2 y + a_5 g)dt + (a_3 y + g)dW(t) & \text{in } Q, \\ y = \chi_{\Sigma_0} h & \text{on } \Sigma, \\ y(0) = y_0, \ \hat{y}(0) = \hat{y}_0 & \text{in } G. \end{cases} \tag{2.2}$$

Here $(y_0, \hat{y}_0) \in L^2(G) \times H^{-1}(G)$, $(y, \hat{y})$ is the state, and $f \in L^2_{\mathbb{F}}(0, T; L^2(G))$, $g \in L^2_{\mathbb{F}}(0, T; H^{-1}(G))$, and $h \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$ are controls. As we shall see in Section 2.2, the system (2.2) admits a unique *transposition solution* $(y, \hat{y}) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$. Readers are referred to [37] for the derivation of (2.2).

Remark 2.3. Usually, if we put a control in the diffusion term, it may affect the drift term in one way or another. Here we assume that the effect is in the form of "$a_5 g dt$" as that in the second equation of (2.2). One may consider a more general case, say, by adding a term like "$a_6 f dt$" (in which $a_6 \in L^\infty_{\mathbb{F}}(0, T; L^\infty(G))$) into the first equation of (2.2). However, except for $n = 1$, the corresponding controllability problem is still unsolved (e.g., [39]).

Definition 2.2. We say that the system (2.2) is exactly controllable at time T if for any $(y_0, \hat{y}_0) \in L^2(G) \times H^{-1}(G)$ and $(y_1, \hat{y}_1) \in L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$, one can find controls $(f, g, h) \in L^2_{\mathbb{F}}(0, T; L^2(G)) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G)) \times L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$ such that the corresponding solution $(y, \hat{y})$ to (2.2) satisfies that $(y(T), \hat{y}(T)) = (y_1, \hat{y}_1)$.

Under some assumptions, we can show that (2.2) is exactly controllable (see Theorem 2.3). Hence, from the viewpoint of controllability, (2.2) is a more reasonable model than (2.1).

2.2. Well-posedness of stochastic hyperbolic equations with boundary controls

Both (2.1) and (2.2) are SPDEs with nonhomogeneous boundary values. They may not have weak or mild solutions. Therefore, as the deterministic case (e.g., [22]), solutions to them are understood in the sense of a transposition solution. To this end, we need the following backward stochastic hyperbolic equation:

$$\begin{cases} dz = \hat{z}dt + ZdW(t) & \text{in } Q_\tau, \\ d\hat{z} - \displaystyle\sum_{j,k=1}^{n} (a^{jk} z_{x_j})_{x_k} dt = (b_1 \cdot \nabla z + b_2 z + b_3 Z + b_4 \hat{Z})dt + \hat{Z}dW(t) & \text{in } Q_\tau, \\ z = 0 & \text{on } \Sigma_\tau, \\ z(\tau) = z^\tau,\ \hat{z}(\tau) = \hat{z}^\tau & \text{in } G, \end{cases} \tag{2.3}$$

where $\tau \in (0, T]$, $Q_\tau \overset{\Delta}{=} (0,\tau) \times G$, $\Sigma_\tau \overset{\Delta}{=} (0,\tau) \times \Gamma$, $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G) \times L^2(G))$, $b_1 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}(G;\mathbb{R}^n))$, and $b_i \in L^\infty_{\mathbb{F}}(0, T; L^\infty(G))$ $(i = 2, 3, 4)$.

For any $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$, the system (2.3) admits a unique solution $(z, Z, \hat{z}, \hat{Z}) \in C_{\mathbb{F}}([0,\tau]; H_0^1(G)) \times L^2_{\mathbb{F}}(0,\tau; H_0^1(G)) \times C_{\mathbb{F}}([0,\tau]; L^2(G)) \times L^2_{\mathbb{F}}(0,\tau; L^2(G))$ (e.g., **[40, THEOREM 4.10]**), which satisfies the following hidden regularity:

Proposition 2.1 (**[37, PROPOSITION 3.1]**). *The solution $(z, \hat{z}, Z, \hat{Z})$ to (2.3) satisfies $\frac{\partial z}{\partial \nu}|_\Gamma \in L^2_{\mathbb{F}}(0,\tau; L^2(\Gamma))$ and*

$$\left|\frac{\partial z}{\partial \nu}\right|_{L^2_{\mathbb{F}}(0,\tau;L^2(\Gamma))} \le \mathcal{C}\left(|z^\tau|_{L^2_{\mathcal{F}_\tau}(\Omega;H_0^1(G))} + |\hat{z}^\tau|_{L^2_{\mathcal{F}_\tau}(\Omega;L^2(G))}\right), \tag{2.4}$$

where the constant $\mathcal{C}$ is independent of τ and $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$.

Definition 2.3. A stochastic process $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \cap C^1_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$ is called a *transposition solution* to (2.1) if for any $\tau \in (0, T]$ and $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$, it holds that

$$\begin{aligned} &\mathbb{E}\langle y_t(\tau), z^\tau\rangle_{H^{-1}(G),H_0^1(G)} - \mathbb{E}\langle y(\tau), \hat{z}^\tau\rangle_{L^2(G)} - \langle \hat{y}_0, z(0)\rangle_{H^{-1}(G),H_0^1(G)} + \langle y_0, \hat{z}(0)\rangle_{L^2(G)} \\ &\quad = \mathbb{E}\int_0^\tau \langle f, z\rangle_{H^{-1}(G),H_0^1(G)} dt + \mathbb{E}\int_0^\tau \langle g, Z\rangle_{H^{-1}(G),H_0^1(G)} dt - \mathbb{E}\int_0^\tau\int_{\Gamma_0} h\frac{\partial z}{\partial \nu} d\Gamma ds, \end{aligned}$$

where $(z, \hat{z}, Z, \hat{Z})$ solves (2.3) with $b_1 = -a_1$, $b_2 = -\mathrm{div} a_1 + a_2$, $b_3 = a_3$, and $b_4 = 0$.

A pair of stochastic processes $(y, \hat{y}) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$ is called a *transposition solution* to (2.2) if for any $\tau \in (0, T]$ and $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$, it holds that

$$\begin{aligned} &\mathbb{E}\langle \hat{y}(\tau), z^\tau\rangle_{H^{-1}(G),H_0^1(G)} - \mathbb{E}\langle y(\tau), \hat{z}^\tau\rangle_{L^2(G)} - \langle \hat{y}_0, z(0)\rangle_{H^{-1}(G),H_0^1(G)} + \langle y_0, \hat{z}(0)\rangle_{L^2(G)} \\ &\quad = -\mathbb{E}\int_0^\tau \langle f, \hat{Z}\rangle_{L^2(G)} dt + \mathbb{E}\int_0^\tau \langle g, a_5 z + Z\rangle_{H^{-1}(G),H_0^1(G)} dt - \mathbb{E}\int_0^\tau\int_{\Gamma_0} h\frac{\partial z}{\partial \nu} d\Gamma ds, \end{aligned}$$

where $(z, \hat{z}, Z, \hat{Z})$ solves (2.3) with $b_1 = -a_1$, $b_2 = -\mathrm{div} a_1 + a_2$, $b_3 = a_3$, and $b_4 = -a_4$.

Remark 2.4. By Proposition 2.1, the term "$\mathbb{E}\int_0^\tau\int_{\Gamma_0} h\frac{\partial z}{\partial \nu} d\Gamma ds$" makes sense. The above definitions of transposition solutions to (2.1) and (2.2) are the generalization of the transposition solution to deterministic hyperbolic equation (e.g., **[22]**).

Proposition 2.2 ([37, PROPOSITIONS 4.1 AND 4.2]). *The system* (2.1) *(*resp. (2.2)*) admits a unique transposition solution y (*resp. $(y, \hat{y})$*).*

2.3. The controllability results

We have introduced three controls (f, g, and h) in the system (2.2). At first glance, it seems unreasonable that especially the controls f and g in the diffusion terms of (2.2) are acting on the whole domain G. One may ask whether localized controls are enough or the boundary control can be dropped. However, the answers are "NO."

Theorem 2.2 ([37, THEOREM 2.3]). *For any open subset Γ_0 of Γ and open subset G_0 of G, the system* (2.2) *is not exactly controllable at any time $T > 0$, provided that one of the following three conditions is satisfied:*

(1) *$a_4 \in C_{\mathbb{F}}([0,T]; L^\infty(\Omega; L^\infty(G)))$, $G \setminus \overline{G_0} \neq \emptyset$, and f is supported in G_0;*

(2) *$a_3 \in C_{\mathbb{F}}([0,T]; L^\infty(\Omega; L^\infty(G)))$, $G \setminus \overline{G_0} \neq \emptyset$, and g is supported in G_0;*

(3) *$h = 0$.*

To get a positive controllability result for the system (2.2), the time T should be large enough due to the finite propagation speed of solutions to stochastic hyperbolic equations. On the other hand, noting that the deterministic wave equation is a special case of (2.2), by [2], we see that exact controllability of (2.2) is impossible without conditions on Γ_0 and $(a^{jk})_{1\leq j,k\leq n}$. Hence, to continue, we introduce the following assumptions:

Condition 2.1. *There exists a positive function $\varphi(\cdot) \in C^3(\overline{G})$ satisfying the following:*

(1) *For some constant $\mu_0 > 0$ and all $(x, \xi^1, \ldots, \xi^n) \in \overline{G} \times \mathbb{R}^n$,*

$$\sum_{j,k=1}^{n} \sum_{j',k'=1}^{n} \left[2a^{jk'}(a^{j'k}\varphi_{x_{j'}})_{x_{k'}} - a^{jk}_{x_{k'}} a^{j'k'}\varphi_{x_{j'}}\right]\xi^j \xi^k \geq \mu_0 \sum_{j,k=1}^{n} a^{jk}\xi^j\xi^k.$$

(2) *The function $\varphi(\cdot)$ has no critical point in $\overline{G}$, i.e., $|\nabla\varphi(x)| > 0$ for $x \in \overline{G}$.*

We shall choose the set Γ_0 as follows:

$$\Gamma_0 \triangleq \left\{ x \in \Gamma \;\middle|\; \sum_{j,k=1}^{n} a^{jk}\varphi_{x_j}(x)\nu^k(x) > 0 \right\}.$$

Also, write

$$R_1 \triangleq \sqrt{\max_{x\in\overline{G}} \varphi(x)}, \qquad R_0 \triangleq \sqrt{\min_{x\in\overline{G}} \varphi(x)}.$$

Clearly, if $\varphi(\cdot)$ satisfies Condition 2.1, then for any given constants $\alpha \geq 1$ and $\beta \in \mathbb{R}$, so does $\tilde{\varphi} = \alpha\varphi + \beta$ with μ_0 replaced by $\alpha\mu_0$. Therefore we may choose φ, μ_0, c_0, c_1 and T such that

Condition 2.2. *The following inequalities hold:*

(1) $\frac{1}{4}\sum_{j,k=1}^{n} a^{jk}(x)\varphi_{x_j}(x)\varphi_{x_k}(x) \geq R_1^2$, $\forall x \in \overline{G}$;

(2) $T > T_0 \triangleq 2R_1$;

(3) $(\frac{2R_1}{T})^2 < c_1 < \frac{2R_1}{T}$;

(4) $\mu_0 - 4c_1 - c_0 > c_0 + 2R_1(1 + |a_5|^2_{L^\infty_{\mathbb{F}}(0,T;L^\infty(G))})$.

Remark 2.5. As we have explained before Condition 2.2, this condition can always be satisfied. We put it here merely to emphasize the relationship among c_0, c_1, μ_0 and T.

Remark 2.6. To ensure that (4) in Condition 2.2 holds, c_1 and T depend on $|a_5|_{L^\infty_{\mathbb{F}}(0,T;L^\infty(G))}$. This seems to be reasonable because a_5 stands for the effect of the control in the diffusion term to the drift term. One needs time to get rid of such an effect. Nevertheless, this does not happen when $n = 1$ (e.g., [39]).

The exact controllability result for the system (2.2) is stated as follows:

Theorem 2.3 ([37, THEOREM 2.2]). *System* (2.2) *is exactly controllable at time T if Conditions* 2.1 *and* 2.2 *hold.*

Remark 2.7. Although it is necessary to put controls f and g on the whole domain G, one may suspect that Theorem 2.3 is trivial and give a possible “proof” of Theorem 2.3 as follows: Choosing $f = -a_4 y$ and $g = -a_3 y$, the system (2.2) becomes

$$\begin{cases} dy = \hat{y}dt & \text{in } Q, \\ d\hat{y} - \sum_{j,k=1}^{n} (a^{jk} y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2 y - a_5 a_3 y)dt & \text{in } Q, \\ y = \chi_{\Sigma_0} h & \text{on } \Sigma, \\ y(0) = y_0, \ \hat{y}(0) = \hat{y}_0 & \text{in } G. \end{cases} \tag{2.5}$$

This is a hyperbolic equation with random coefficients. If one regards the sample point ω as a parameter, then for every given $\omega \in \Omega$, there is a control $h(\cdot,\cdot,\omega)$ such that the solution to (2.5) fulfills $(y(T,x,\omega), \hat{y}(T,x,\omega)) = (y_1(x,\omega), \hat{y}_1(x,\omega))$. However, it is unclear whether the control constructed in this way is adapted to the filtration **F** or not. If it is not the case, then to determine the value of the control at present, one needs to use information from the future, which is meaningless in the stochastic framework.

In order to prove Theorem 2.3, by a standard duality argument, it suffices to establish the following observability estimate for the adjoint equation (2.3).

Theorem 2.4. *Under the assumptions of Theorem* 2.3, *all solutions to equation* (2.3) *with $\tau = T$ satisfy*

$$\begin{aligned} &\left|(z^T, \hat{z}^T)\right|_{L^2_{\mathcal{F}_T}(\Omega;H_0^1(G)\times L^2(G))} \\ &\quad \le \mathcal{C}\left(\left|\frac{\partial z}{\partial \nu}\right|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))} + |a_5 z + Z|^2_{L^2_{\mathbb{F}}(0,T;H_0^1(G))} + |\hat{Z}|^2_{L^2_{\mathbb{F}}(0,T;L^2(G))}\right). \end{aligned}$$

Remark 2.8. Although Theorem 2.4 is much more complex than Theorem 1.1, it has the same features in common with Theorem 1.1, that is, a solution of an equation can be fully determined by a suitable observation of the solution.

Remark 2.9. The proof of Theorem 2.4 is almost the same as that of [37, THEOREM 7.1]. We do not provide the explicit dependence of the constant $\mathcal{C}$ on the observation time T and the coefficients b_i $(1 \leq i \leq 4)$. Interested readers are referred to [37].

2.4. Carleman estimate

Theorem 2.4 is an observability estimate of equation (2.3). Generally speaking, there are three main approaches to establish the observability estimate for multidimensional deterministic hyperbolic equations.

The first is the multiplier techniques (e.g., [21]). Two key points for applying this method are the time reversibility of the equation and the time independence of the coefficients. Equation (2.3) does not fulfill the second property above.

The second approach is based on the microlocal analysis (e.g., [2]), which gives a sharp sufficient condition, i.e., the *Geometric Control Condition*, for the observability estimate of hyperbolic equations. It is interesting to generalize this method to study the observability estimate of equation (2.3).

The last one is the global Carleman estimate (e.g., [15, 49]). It has been generalized to study the observability estimate for stochastic hyperbolic equations recently (e.g., [28, 34, 48, 49]). Theorem 2.3 is also proved likewise. The key is the following identity.

Lemma 2.1 ([37, LEMMA 6.1]). *Let z be an $H^2(\mathbb{R}^n)$-valued Itô process and $\hat{z}$ be an $L^2(\mathbb{R}^n)$-valued Itô process such that for some $Z \in L^2_{\mathbb{F}}(0,T;H^1(\mathbb{R}^n))$, $dz = \hat{z}dt + ZdW(t)$ in $(0,T)\times\mathbb{R}^n$. Let $\ell, \Psi \in C^2((0,T)\times\mathbb{R}^n)$. Set $\theta = e^{\ell}$, $v = \theta z$ and $\hat{v} = \theta\hat{z} + \ell_t v$. Then, for a.e. $x \in \mathbb{R}^n$,*

$$
\begin{aligned}
&\theta\Bigg(-2\ell_t\hat{v} + 2\sum_{j,k=1}^n a^{jk}\ell_{x_j}v_{x_k} + \Psi v\Bigg)\Bigg[d\hat{z} - \sum_{j,k=1}^n \big(a^{jk}z_{x_j}\big)_{x_k}dt\Bigg] \\
&\quad + \sum_{j,k=1}^n \Bigg[\sum_{j',k'=1}^n \big(2a^{jk}a^{j'k'}\ell_{x_{j'}}v_{x_j}v_{x_{k'}} - a^{jk}a^{j'k'}\ell_{x_j}v_{x_{j'}}v_{x_{k'}}\big) \\
&\quad - 2\ell_t a^{jk}v_{x_j}\hat{v} + a^{jk}\ell_{x_j}\hat{v}^2 + \Psi a^{jk}v_{x_j}v - \frac{\Psi_{x_j}}{2}a^{jk}v^2 - \mathcal{A}a^{jk}\ell_{x_j}v^2\Bigg]_{x_k} \\
&\quad + d\Bigg[\ell_t\sum_{j,k=1}^n a^{jk}v_{x_j}v_{x_k} + \ell_t\hat{v}^2 - 2\sum_{j,k=1}^n a^{jk}\ell_{x_j}v_{x_k}\hat{v} - \Psi v\hat{v} + \Big(\mathcal{A}\ell_t + \frac{\Psi_t}{2}\Big)v^2\Bigg] \\
&= \Bigg\{\Bigg[\ell_{tt} + \sum_{j,k=1}^n \big(a^{jk}\ell_{x_j}\big)_{x_k} - \Psi\Bigg]\hat{v}^2 + \sum_{j,k=1}^n c^{jk}v_{x_j}v_{x_k} + \mathcal{B}v^2 \\
&\quad - 2\sum_{j,k=1}^n \big[\big(a^{jk}\ell_{x_k}\big)_t + a^{jk}\ell_{tx_k}\big]v_{x_j}\hat{v} + \Bigg(-2\ell_t\hat{v} + 2\sum_{j,k=1}^n a^{jk}\ell_{x_j}v_{x_k} + \Psi v\Bigg)^2\Bigg\}dt
\end{aligned}
$$

$$
\begin{aligned}
&+\ell_t(d\hat{v})^2-2\sum_{j,k=1}^{n}a^{jk}\ell_{x_j}dv_{x_k}d\hat{v}-\Psi dv d\hat{v}+\ell_t\sum_{j,k=1}^{n}a^{jk}(dv_{x_j})(dv_{x_k})\\
&+\mathcal{A}\ell_t(dv)^2-\Bigg\{\theta\Bigg(-2\ell_t\hat{v}+2\sum_{j,k=1}^{n}a^{jk}\ell_{x_j}v_{x_k}+\Psi v\Bigg)\ell_t Z\\
&-\Bigg[2\sum_{j,k=1}^{n}a^{jk}(\theta Z)_{x_k}\ell_{x_j}\hat{v}-\theta\Psi_t vZ+\theta\Psi\hat{v}Z\Bigg]\\
&+2\Bigg[\sum_{j,k=1}^{n}a^{jk}v_{x_j}(\theta Z)_{x_k}+\theta\mathcal{A}vZ\Bigg]\ell_t\Bigg\}dW(t),\quad a.s., \qquad (2.6)
\end{aligned}
$$

where $(dv)^2$ and $(d\hat{v})^2$ denote the quadratic variation processes of v and $\hat{v}$, respectively, and

$$
\begin{cases}
c^{jk}\triangleq\big(a^{jk}\ell_t\big)_t+\displaystyle\sum_{j',k'=1}^{n}\Big[2a^{jk'}\big(a^{j'k}\ell_{x_{j'}}\big)_{x_{k'}}-\big(a^{jk}a^{j'k'}\ell_{x_{j'}}\big)_{x_{k'}}\Big]+\Psi a^{jk},\\
\mathcal{A}\triangleq(\ell_t^2-\ell_{tt})-\displaystyle\sum_{j,k=1}^{n}\Big[a^{jk}\ell_{x_j}\ell_{x_k}-\big(a^{jk}\ell_{x_j}\big)_{x_k}\Big]-\Psi,\\
\mathcal{B}\triangleq\mathcal{A}\Psi+(\mathcal{A}\ell_t)_t-\displaystyle\sum_{j,k=1}^{n}\big(\mathcal{A}a^{jk}\ell_{x_j}\big)_{x_k}+\frac{1}{2}\Bigg[\Psi_{tt}-\sum_{j,k=1}^{n}\big(a^{jk}\Psi_{x_j}\big)_{x_k}\Bigg].
\end{cases}
$$

Remark 2.10. The derivation of (2.6) requires a fairly complex but elementary computation. Identities in the spirit of (2.6) are widely used to solve observability problems for deterministic and stochastic PDEs (e.g., [14, 15, 39, 40]).

Choosing $\ell(t,x)=\lambda[\varphi(x)-c_1(t-\frac{T}{2})^2]$ and $\Psi=\ell_{tt}+\sum_{j,k=1}^{n}(a^{jk}\ell_{x_j})_{x_k}-c_0\lambda$ in (2.6), integrating (2.6) in Q and taking the mathematical expectation, after some technical computations, one can prove Theorem 2.3.

The above not only gives a sketch of the proof of Theorem 2.3, but also presents a methodology of getting the observability estimates for SPDEs and backward SPDEs: indeed, one has to establish a suitable pointwise identity and choose a suitable weight function. Almost all observability estimates for SPDEs and backward SPDEs are obtained in this way (e.g., [14, 27–29, 34, 39, 40, 45, 48, 49]). That said, we do not mean that the proofs of these observability estimates are similar; rather we want to emphasize the common ground in the idea of the proofs.

3. PONTRYAGIN-TYPE STOCHASTIC MAXIMUM PRINCIPLE AND STOCHASTIC TRANSPOSITION METHOD

This section is devoted to the Pontryagin-type stochastic maximum principle (PMP for short) for optimal control problems of semilinear SDPSs. There is a long history for the study of this topic. We refer to [3] for a pioneering result and to [17, 44] and the references therein for subsequent results. These works addressed three special cases: (1) the diffusion term does not depend on the control variable; (2) U is convex; (3) the second-order derivatives of g and h with respect to y in (3.2) below are Hilbert–Schmidt operator-valued. On the one hand, under the first two assumptions (resp. the third assumption), the PMP and their proofs are similar to those of the distributed parameter control systems (resp. stochastic finite-dimensional control systems). On the other hand, when one puts a control in the drift term, it will affect the diffusion term, i.e., the control could influence the scale of uncertainty. Hence, it is important to study PMP for SDPSs with control-dependent diffusion terms and nonconvex control domains. This was done in [33] (some generalizations were given in [35, 36, 39]).

3.1. Formulation of the optimal control problem

Unlike in Section 2, we will formulate our system in an abstract framework. Throughout this section, $T > 0$, $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ (with $\mathbf{F} \stackrel{\Delta}{=} \{\mathcal{F}_t\}_{t\in[0,T]}$) is a fixed filtered probability space satisfying the usual conditions, on which a 1-dimensional standard Brownian motion $W(\cdot)$ is defined, and H is a separable Hilbert space. Denote by $\mathbb{F}$ the progressive σ-field (in $[0, T] \times \Omega$) with respect to $\mathbf{F}$.

Let A be a linear operator (with the domain $D(A) \subset H$), which generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H. Denote by A^* the adjoint operator of A, which generates the adjoint C_0-semigroup of $\{S(t)\}_{t\geq 0}$. Let U be a separable metric space. Put

$$\mathcal{U}[0,T] \stackrel{\Delta}{=} \{u : [0,T] \times \Omega \to U \mid u \text{ is } \mathbf{F}\text{-adapted}\}.$$

We assume the following condition:

(A1). *The maps $a, b : [0,T] \times H \times U \to H$ satisfy (for $\varphi = a, b$):* (i) *for any $(y,u) \in H \times U$, $\varphi(\cdot, y, u) : [0,T] \to H$ is Lebesgue measurable;* (ii) *for any $(t,y) \in [0,T] \times H$, $\varphi(t,y,\cdot) : U \to H$ is continuous; and* (iii) *there is a constant $\mathcal{C}_L > 0$ such that*

$$\begin{cases} |\varphi(t,y_1,u) - \varphi(t,y_2,u)|_H \leq \mathcal{C}_L |y_1 - y_2|_H, \\ |\varphi(t,0,u)|_H \leq \mathcal{C}_L, \end{cases} \quad \forall (t, y_1, y_2, u) \in [0,T] \times H \times H \times U.$$

Consider the following controlled stochastic evolution equation:

$$\begin{cases} dy(t) = [Ay(t) + a(t,y,u)]dt + b(t,y,u)dW(t), \quad \text{a.e. } t \in (0,T], \\ y(0) = \eta, \end{cases} \tag{3.1}$$

where $u \in \mathcal{U}[0,T]$ is control, y is state, and $\eta \in L^8_{\mathcal{F}_0}(\Omega; H)$. The control system (3.1) admits a unique mild solution $y \in C_{\mathbb{F}}([0,T]; L^8(\Omega; H))$ (e.g., [40, THEOREM 3.13]).

Remark 3.1. In (3.1), the diffusion term depends on the control. This means that the control could influence the scale of uncertainty (as is indeed the case in many practical systems, especially in the system of mesoscopic scale). In such a setting, the stochastic problems essentially differ from the deterministic ones.

Also, we need the following condition:

(A2). *The maps* $g(\cdot,\cdot,\cdot):[0,T]\times H\times U\to\mathbb{R}$ *and* $h(\cdot):H\to\mathbb{R}$ *satisfy:* (i) *for any* $(y,u)\in H\times U$, $g(\cdot,y,u):[0,T]\to\mathbb{R}$ *is Lebesgue measurable;* (ii) *for any* $(t,y)\in[0,T]\times H$, $g(t,y,\cdot):U\to\mathbb{R}$ *is continuous; and* (iii) *there is a constant* $\mathcal{C}_L>0$ *such that*

$$\begin{cases} \big|g(t,y_1,u)-g(t,y_2,u)\big|_H+\big|h(y_1)-h(y_2)\big|_H\le\mathcal{C}_L|y_1-y_2|_H,\\ \big|g(t,0,u)\big|_H+\big|h(0)\big|_H\le\mathcal{C}_L,\end{cases}$$
$$\forall (t,y_1,y_2,u)\in[0,T]\times H\times H\times U.$$

Define a cost functional $\mathcal{J}(\cdot)$ (for the control system (3.1)) as follows:

$$\mathcal{J}(u)\overset{\Delta}{=}\mathbb{E}\bigg[\int_0^T g\big(t,y(t),u(t)\big)dt+h\big(y(T)\big)\bigg],\quad \forall u\in\mathcal{U}[0,T],\tag{3.2}$$

where y is the state of (3.1) corresponding to u. Consider an optimal control problem:

Problem (OP). *Find a* $\bar u\in\mathcal{U}[0,T]$ *such that*

$$\mathcal{J}(\bar u)=\inf_{u\in\mathcal{U}[0,T]}\mathcal{J}(u).\tag{3.3}$$

Any $\bar u$ satisfying (3.3) is called an *optimal control*. The corresponding state $\bar y$ is called an *optimal state*, and $(\bar y,\bar u)$ is called an *optimal pair*.

3.2. Transposition solution and relaxed transposition solution to backward stochastic evolution equation

We first recall that the key idea in the proof of Theorem 1.2 is as follows: One first perturbs an optimal control by means of the spike variation, then considers the first-order term in a sort of Taylor expansion with respect to this perturbation. By sending the perturbation to zero, one obtains a kind of variational inequality. The Pontryagin's maximum principle then follows from a duality argument. When applying this idea to study PMP for Problem (OP), one encounters an essential difficulty, which, roughly speaking, is that the Itô stochastic integral $\int_t^{t+\varepsilon} r\,dW(s)$ is only of order $\sqrt{\varepsilon}$ (rather than ε as with the Lebesgue integral). To overcome this difficulty, we should study both the first and second order terms in the Taylor expansion of the spike variation. In such case, inspired by [42], we need to introduce two adjoint equations. The first is

$$\begin{cases} dz=-A^*z\,dt+F(t,z,Z)dt+Z\,dW(t) & \text{in } [0,T),\\ z(T)=z_T.\end{cases}\tag{3.4}$$

In (3.4), $F:[0,T]\times H\times H\to H$ is Lebesgue measurable with respect to t and Lipschitz continuous with respect to z and Z.

Neither the usual natural filtration condition nor the quasi-left continuity is assumed for the filtration $\mathbf{F}$, and the operator A is only assumed to generate a general C_0-semigroup. Hence, equation (3.4) may not have a weak or mild solution. Similar to equation (2.1), we should introduce new notion of solution to (3.4). To this end, consider the following stochastic evolution equation:

$$\begin{cases} d\varphi = (A\varphi + \psi)ds + \tilde{\psi}dW(s) & \text{in } (t,T], \\ \varphi(t) = \zeta, \end{cases} \tag{3.5}$$

where $t \in [0,T)$, $\psi \in L^1_{\mathbb{F}}(t,T;L^2(\Omega;H))$, $\tilde{\psi} \in L^2_{\mathbb{F}}(t,T;H)$, and $\zeta \in L^2_{\mathcal{F}_t}(\Omega;H)$. Equation (3.5) admits a unique (mild) solution $\varphi \in C_{\mathbb{F}}([t,T];L^2(\Omega;H))$ (e.g., [40, THEOREM 3.13]).

Definition 3.1. We call $(z,Z) \in D_{\mathbb{F}}([0,T];L^2(\Omega;H)) \times L^2_{\mathbb{F}}(0,T;H)$ a transposition solution to (3.4) if for any $t \in [0,T]$, $\psi \in L^1_{\mathbb{F}}(t,T;L^2(\Omega;H))$, $\tilde{\psi} \in L^2_{\mathbb{F}}(t,T;H)$, $\zeta \in L^2_{\mathcal{F}_t}(\Omega;H)$, and the corresponding solution $\varphi \in C_{\mathbb{F}}([t,T];L^2(\Omega;H))$ to (3.5), it holds that

$$\begin{aligned} &\mathbb{E}\langle \varphi(T), z_T\rangle_H - \mathbb{E}\int_t^T \langle \varphi(s), f(s,z(s),Z(s))\rangle_H ds \\ &\quad = \mathbb{E}\langle \zeta, z(t)\rangle_H + \mathbb{E}\int_t^T \langle \psi(s), z(s)\rangle_H ds + \mathbb{E}\int_t^T \langle \tilde{\psi}(s), Z(s)\rangle_H ds. \end{aligned} \tag{3.6}$$

Remark 3.2. On the one hand, if (3.4) admits a strong solution $(z,Z) \in [C_{\mathbb{F}}([0,T]; L^2(\Omega;H)) \cap L^2_{\mathbb{F}}(0,T;D(A))] \times L^2_{\mathbb{F}}(0,T;H)$, then, we can get (3.6) by Itô's formula (e.g., [40, THEOREM 2.142]). On the other hand, (3.6) can be used to get the PMP for Problem (OP). These are the reasons for introducing Definition 3.1. The main idea of this definition is to interpret the solution to a less understood equation by means of another well-understood one.

Theorem 3.1 ([33, THEOREM 3.1]). *Equation* (3.4) *has a unique transposition solution* (z,Z) *and*

$$\begin{aligned} &\big|(z,Z)\big|_{D_{\mathbb{F}}([0,T];L^2(\Omega;H))\times L^2_{\mathbb{F}}(0,T;H)} \\ &\quad \le \mathcal{C}\big(\big|F(\cdot,0,0)\big|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;H))} + |z_T|_{L^2_{\mathcal{F}_T}(\Omega;H)}\big). \end{aligned}$$

The proof of Theorem 3.1 is based on a Riesz-type representation theorem obtained in [31].

The second adjoint equation is[2]

$$\begin{cases} dP = \big[-(A^*+J^*)P - P(A+J) - K^*PK - (K^*Q+QK) + F\big]dt + QdW(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T), \\ P(T) = P_T. \end{cases} \tag{3.7}$$

where $F \in L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))$, $P_T \in L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))$, and $J,K \in L^4_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H)))$.

2 In this paper, for any operator-valued process R, we denote by R^* its pointwise dual operator-valued process.

Equation (3.7), as written, is a rather formidable operator-valued backward stochastic evolution equation. When $H = \mathbb{R}^n$, (3.7) is an $\mathbb{R}^{n\times n}$ matrix-valued backward stochastic differential equation, and therefore, the desired well-posedness follows from that of an $\mathbb{R}^{n^2}$ (vector)-valued backward stochastic differential equation. However, in the infinite-dimensional setting, although $\mathcal{L}(H)$ is still a Banach space, it is neither reflexive nor separable even if H itself is separable. There exists no stochastic integration/evolution equation theory that can be employed to treat the well-posedness of (3.7) even if the filtration **F** is generated by $W(\cdot)$ (e.g., [46]). Hence, we should employ the stochastic transposition method again and define the solution to (3.7) in the transposition sense. To this end, we need the following stochastic evolution equation:

$$\begin{cases} d\varphi = (A+J)\varphi ds + \psi ds + K\varphi dW(s) + \tilde{\psi} dW(s) \quad \text{in } (t,T], \\ \varphi(t) = \xi. \end{cases} \tag{3.8}$$

Here $\xi \in L^4_{\mathcal{F}_t}(\Omega; H)$ and $\psi, \tilde{\psi} \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$. Also, we should introduce the solution space for (3.7). Write

$$\begin{aligned} \mathcal{P}[0,T] \stackrel{\Delta}{=} \Big\{ P : [0,T]\times\Omega \to \mathcal{L}(H) \;\big|\; & |P|_{\mathcal{L}(H)} \in L^\infty_{\mathbb{F}}\big(0,T; L^2(\Omega)\big) \text{ and for every} \\ & t \in [0,T] \text{ and } \xi \in L^4_{\mathcal{F}_t}(\Omega; H),\ P\xi \in D_{\mathbb{F}}\big([t,T]; L^{\frac{4}{3}}(\Omega; H)\big) \text{ and} \\ & |P\xi|_{D_{\mathbb{F}}([t,T]; L^{\frac{4}{3}}(\Omega;H))} \le \mathcal{C}|\xi|_{L^4_{\mathcal{F}_t}(\Omega;H)} \Big\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{Q}[0,T] \stackrel{\Delta}{=} \Big\{ \big(Q^{(\cdot)}, \hat{Q}^{(\cdot)}\big) \;\big|\; & \text{for any } t \in [0,T],\ \text{both } Q^{(t)} \text{ and } \hat{Q}^{(t)} \text{ are bounded linear} \\ & \text{operators from } L^4_{\mathcal{F}_t}(\Omega; H) \times L^2_{\mathbb{F}}\big(t,T; L^4(\Omega; H)\big) \times L^2_{\mathbb{F}}\big(t,T; L^4(\Omega; H)\big) \text{ to} \\ & L^2_{\mathbb{F}}\big(t,T; L^{\frac{4}{3}}(\Omega; H)\big) \text{ and } Q^{(t)}(0,0,\cdot)^* = \hat{Q}^{(t)}(0,0,\cdot) \Big\}. \end{aligned}$$

Definition 3.2. We call $(P(\cdot), Q^{(\cdot)}, \hat{Q}^{(\cdot)}) \in \mathcal{P}[0,T] \times \mathcal{Q}[0,T]$ a relaxed transposition solution to (3.7) if for any $t \in [0,T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, and $\psi_1, \psi_2, \tilde{\psi}_1, \tilde{\psi}_2 \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$, it holds that

$$\begin{aligned} & \mathbb{E}\big\langle P_T\varphi_1(T), \varphi_2(T)\big\rangle_H - \mathbb{E}\int_t^T \big\langle F(s)\varphi_1(s), \varphi_2(s)\big\rangle_H ds \\ & = \mathbb{E}\big\langle P(t)\xi_1, \xi_2\big\rangle_H + \mathbb{E}\int_t^T \big\langle P(s)\psi_1(s), \varphi_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)\varphi_1(s), \psi_2(s)\big\rangle_H ds \\ & \quad + \mathbb{E}\int_t^T \big\langle P(s)K(s)\varphi_1(s), \tilde{\psi}_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)\tilde{\psi}_1(s), K(s)\varphi_2(s) + \tilde{\psi}_2(s)\big\rangle_H ds \\ & \quad + \mathbb{E}\int_t^T \big\langle \tilde{\psi}_1(s), \hat{Q}^{(t)}(\xi_2, \psi_2, \tilde{\psi}_2)(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle Q^{(t)}(\xi_1, \psi_1, \tilde{\psi}_1)(s), \tilde{\psi}_2(s)\big\rangle_H ds, \end{aligned}$$

Here, for $j = 1,2$, φ_j solves (3.8) with ξ, ψ, and $\tilde{\psi}$ replaced by ξ_j, ψ_j, and $\tilde{\psi}_j$, respectively.

Remark 3.3. Due to the very weak characterization of Q, a relaxed transposition solution is more like a half-measure rather than the natural solution to (3.7). We believe that a more suitable definition should be as follows:

Let $\hat{\mathcal{Q}}[0,T] \stackrel{\Delta}{=} \big\{ Q : [0,T]\times\Omega \to \mathcal{L}(H) \;\big|\; |Q|_{\mathcal{L}(H)} \in L^2_{\mathbb{F}}\big(0,T; L^2(\Omega)\big)\big\}$.

We call $(P(\cdot), Q(\cdot)) \in \mathcal{P}[0,T] \times \hat{\mathcal{Q}}[0,T]$ a *transposition solution* to (3.7) if for any $t \in [0,T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, and $\psi_1, \psi_2, \tilde{\psi}_1, \tilde{\psi}_2 \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$, it holds that

$$
\begin{aligned}
&\mathbb{E}\langle P_T\varphi_1(T), \varphi_2(T)\rangle_H - \mathbb{E}\int_t^T \langle F(s)\varphi_1(s), \varphi_2(s)\rangle_H ds \\
&\quad = \mathbb{E}\langle P(t)\xi_1, \xi_2\rangle_H + \mathbb{E}\int_t^T \langle P(s)\psi_1(s), \varphi_2(s)\rangle_H ds + \mathbb{E}\int_t^T \langle P(s)\varphi_1(s), \psi_2(s)\rangle_H ds \\
&\qquad + \mathbb{E}\int_t^T \langle P(s)K(s)\varphi_1(s), \tilde{\psi}_2(s)\rangle_H ds + \mathbb{E}\int_t^T \langle P(s)\tilde{\psi}_1(s), K(s)\varphi_2(s) + \tilde{\psi}_2(s)\rangle_H ds \\
&\qquad + \mathbb{E}\int_t^T \langle Q(s)\tilde{\psi}_1(s), \varphi_2(s)\rangle_H ds + \mathbb{E}\int_t^T \langle Q(s)\varphi_1(s), \tilde{\psi}_2(s)\rangle_H ds.
\end{aligned}
$$

Here, for $j = 1, 2$, φ_j solves (3.8) with ξ, ψ, and $\tilde{\psi}$ replaced by ξ_j, ψ_j, and $\tilde{\psi}_j$, respectively. If (3.7) admits a transposition solution, then it has a relaxed transposition solution (e.g., [40, REMARK 12.11]). Until now, we have no idea how to prove the existence of a transposition solution to (3.7). In such a case, sometimes, we introduce another kind of solution, namely, the V-transposition solution to (3.7), as a substitute (e.g., [12, 32, 38, 39]).

Remark 3.4. Only the first term P of the solution to (3.7) appears in the PMP for Problem (OP). Nevertheless, the characterization of Q has its own interest. On the one hand, Q is used to get higher-order necessary conditions and to solve operator-valued backward stochastic Riccati equations (e.g., [12, 32, 38, 39]). On the other hand, the information about the whole solution helps us understand the first part of the solution.

Theorem 3.2 ([33, THEOREM 6.1]). *Suppose that $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$ is separable. Then equation (3.7) admits a unique relaxed transposition solution $(P(\cdot), Q^{(\cdot)}, \hat{Q}^{(\cdot)})$. Furthermore,*

$$
|P|_{\mathcal{P}[0,T]} + \left|(Q^{(\cdot)}, \hat{Q}^{(\cdot)})\right|_{\mathcal{Q}[0,T]} \le \mathcal{C}\left(|F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))} + |P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))}\right).
$$

3.3. Pontryagin-type maximum principle

Let us assume a further condition:

(A3). *For any $(t,u) \in [0,T] \times U$, the maps $a(t,\cdot,u)$, $b(t,\cdot,u)$, $g(t,\cdot,u)$, and $h(\cdot)$ are C^2, such that for $\varphi = a, b$, and $\psi = g, h$, $\varphi_x(t,x,\cdot)$, $\psi_x(t,x,\cdot)$, $\varphi_{xx}(t,x,\cdot)$, and $\psi_{xx}(t,x,\cdot)$ are continuous for any $(t,x) \in [0,T] \times H$. Moreover, there exists a constant $\mathcal{C}_L > 0$ such that*

$$
\begin{cases}
|\varphi_x(t,x,u)|_{\mathcal{L}(H)} + |\psi_x(t,x,u)|_H \le \mathcal{C}_L, \\
|\varphi_{xx}(t,x,u)|_{\mathcal{L}(H,H;H)} + |\psi_{xx}(t,x,u)|_{\mathcal{L}(H)} \le \mathcal{C}_L,
\end{cases}
\quad \forall (t,x,u) \in [0,T] \times H \times U.
$$

Remark 3.5. Condition (A3) is a little restrictive. When the C_0-semigroup $\{S(t)\}_{t\ge 0}$ enjoys some smoothing effect, it can be relaxed (e.g., [37]). Due to (A3), Theorem 3.3 cannot be applied to stochastic linear quadratic optimal control problems for SDPSs directly. Nevertheless, following the proof of Theorem 3.3, we can get the PMP for that problem (e.g., [35]).

Let $\mathbb{H}(t,x,\rho,k_1,k_2) \triangleq \langle k_1, a(t,x,\rho)\rangle_H + \langle k_2, b(t,x,\rho)\rangle_H - g(t,x,\rho)$ for $(t,x,\rho,k_1,k_2) \in [0,T] \times H \times U \times H \times H$.

Theorem 3.3. *Suppose that* $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$ *is separable and (A1)–(A3) hold. Let* $(\bar{y}(\cdot),\bar{u}(\cdot))$ *be an optimal pair of Problem (OP),* $(z(\cdot),Z(\cdot))$ *be the transposition solution to* (3.4) *with* $F(t,z,Z) = -a_y(t,\bar{y}(t),\bar{u}(t))^*z - b_y(t,\bar{y}(t),\bar{u}(t))^*Z + g_y(t,\bar{y}(t),\bar{u}(t))$, $z_T = -h_y(\bar{y}(T))$*, and* $(P(\cdot),Q^{(\cdot)},\hat{Q}^{(\cdot)})$ *be the relaxed transposition solution to* (3.7) *with*

$$\begin{cases} P_T = -h_{yy}(\bar{y}(T)), & J(t) = a_y(t,\bar{y}(t),\bar{u}(t)), \\ K(t) = b_y(t,\bar{y}(t),\bar{u}(t)), & F(t) = -\mathbb{H}_{yy}(t,\bar{y}(t),\bar{u}(t),z(t),Z(t)). \end{cases}$$

Then, for a.e. $(t,\omega)\in[0,T]\times\Omega$ *and for all* $\rho\in U$,

$$\begin{aligned} &\mathbb{H}\big(t,\bar{y}(t),\bar{u}(t),z(t),Z(t)\big) - \mathbb{H}\big(t,\bar{y}(t),\rho,z(t),Z(t)\big) \\ &\quad - \frac{1}{2}\big\langle P(t)\big[b\big(t,\bar{y}(t),\bar{u}(t)\big) - b\big(t,\bar{y}(t),\rho\big)\big], b\big(t,\bar{y}(t),\bar{u}(t)\big) - b\big(t,\bar{y}(t),\rho\big)\big\rangle_H \geq 0. \end{aligned}$$

Remark 3.6. Compared with Theorem 1.2, the main difference in Theorem 3.3 is the appearance of the term P. This reflects that, in the stochastic situation, the controller has to balance the scale of control and the degree of uncertainty if the control affects the volatility of the system. If b is independent of u, then we do not need P and one adjoint equation, say (3.4), is enough to get the PMP for Problem (OP) (e.g., **[40, THEOREM 12.4]**).

PMP is a necessary condition for optimal controls, which gives a minimum qualification for the candidates of optimal controls. It is natural to ask whether it is also sufficient. To this end, let us introduce the following assumption.

(A4). *The control domain* U *is a convex subset with a nonempty interior of a separable Hilbert space* $\tilde{H}$. *The maps* a,b, *and* g *are locally Lipschitz in* u, *and their derivatives in* x *are continuous in* (x,u).

Theorem 3.4. *Suppose the assumptions of Theorem* 3.3 *and (A4) hold. Let* $u\in\mathcal{U}[0,T]$ *and* y *be the corresponding state of* (3.1). *Let* (z,Z) *be the transposition solution to* (3.4) *with* $F(t,z,Z) = -a_y(t,y(t),u(t))^*z - b_y(t,y(t),u(t))^*Z + g_y(t,y(t),u(t))$, $z_T = -h_y(y(T))$*, and* $(P(\cdot),Q^{(\cdot)},\hat{Q}^{(\cdot)})$ *be the relaxed transposition solution to* (3.7) *with*

$$\begin{cases} P_T = -h_{yy}(y(T)), & J(t) = a_y(t,y(t),u(t)), \\ K(t) = b_y(t,y(t),u(t)), & F(t) = -\mathbb{H}_{yy}(t,y(t),u(t),z(t),Z(t)). \end{cases}$$

Suppose that $h(\cdot)$ *is convex,* $\mathbb{H}(t,\cdot,\cdot,z(t),Z(t))$ *is concave for all* $t\in[0,T]$ *a.s., and*

$$\begin{aligned} &\mathbb{H}\big(t,y(t),u(t),z(t),Z(t)\big) - \mathbb{H}\big(t,y(t),\rho,z(t),Z(t)\big) \\ &\quad - \frac{1}{2}\big\langle P(t)\big[b\big(t,y(t),u(t)\big) - b\big(t,y(t),\rho\big)\big], b\big(t,y(t),u(t)\big) - b\big(t,y(t),\rho\big)\big\rangle_H \geq 0 \end{aligned}$$

for all $\rho\in U$, *then* $(y(\cdot),u(\cdot))$ *is an optimal pair of Problem (OP).*

4. OPEN PROBLEMS

SDPSs offers challenges and opportunities for the study of the mathematical control theory. There are many interesting problems in this topic. Some of them are listed below, which is by no means an exhaustive list and only reflects our research taste. We believe that new mathematical results and even fundamentally new approaches will be required.

(1) Null and approximate controllability of stochastic hyperbolic equations. We have shown that the system (2.1) is not exactly controllable for any $T > 0$ and $\Gamma_0 \subset \Gamma$. It is natural to ask whether it is null/approximately controllable. Of course, for these problems, fewer controls should be employed. The difficulty to do that lies in proving suitable observability estimate of equation (2.3), in which Z and $\hat{Z}$ do not appear in the right-hand side.

(2) Exact controllability for stochastic wave-like equations with more regular controls. Is the system (2.2) exactly controllable when $g \in L^2_{\mathbb{F}}(0, T; L^2(G))$? The desired controllability is equivalent to the following observability estimate:

$$\begin{aligned} &\left|(z^T, \hat{z}^T)\right|_{L^2_{\mathcal{F}_T}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_T}(\Omega; L^2(G))} \\ &\quad \leq \mathcal{C}\left(\left|\frac{\partial z}{\partial \nu}\right|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))} + |a_5 z + Z|_{L^2_{\mathbb{F}}(0,T;L^2(G))} + |\hat{Z}|_{L^2_{\mathbb{F}}(0,T;L^2(G))}\right), \end{aligned} \tag{4.1}$$

where $(z, Z, \hat{z}, \hat{Z})$ is the solution to (2.3) with $\tau = T$ and final datum $(z^T, \hat{z}^T)$. But one cannot mimic the method in [37] to prove (4.1).

(3) Null/approximate controllability for stochastic parabolic equations with one control. One needs two controls to get the null/approximate controllability for stochastic parabolic equations (e.g., [45]). We believe that one control is enough. However, except for some special cases (e.g., [23,26]), we have no idea on how to prove that.

(4) The cost for the approximate controllability for SDPSs. It is shown in [45] that stochastic parabolic equations are approximately controllable. But it does not give any estimate for the cost of the control. Can one generalize the results in [10] to stochastic parabolic equations? Furthermore, it deserves to study the cost of the approximate controllability for general SDPSs.

(5) Controllability for semilinear SDPSs. In [9], based on sharp estimates on the dependence of controls for the underlying linear equation perturbed by a potential and fixed point arguments, it was proved that semilinear parabolic and hyperbolic equations are null controllable with nonlinearities that grow slower than $s \log(s)^{\frac{3}{2}}$. Whether such results can be obtained for semilinear stochastic parabolic/hyperbolic equations is open. On the other hand, for nonlinearities growing at infinity as $s \log(s)^p$ with $p > 2$, one cannot get the null controllability due to the blow-up of solutions. However, this does not exclude controllability for some particular classes of nonlinear terms (e.g., [7]). More generally, there are lots of interesting results for controllability of semilinear distributed parameter systems (e.g., [6]). So a systematic study of controllability problems for semilinear SDPSs deserves attention.

(6) Stabilization of SDPSs. Stabilization for distributed parameter control systems is a well-studied area. In recent years, some progresses were obtained for SDPSs (e.g., [1,4]). However, this problem is far from being well understood. For example, as far as we know, there is no result for the stabilization of stochastic hyperbolic equations with localized damping.

(7) Optimal control problems for SDPSs with endpoint/state constraints. For some special constraints, such as $y(T)$ belonging to some nonempty open subset of $L^2_{\mathcal{F}_T}(\Omega; H)$, one can use the Ekeland variational principle to establish a Pontryagin-type maximum principle with nontrivial Lagrange multipliers. Nevertheless, for the general case, one does need some further conditions to obtain nontrivial results. For deterministic optimal control problems, people introduce the so-called finite codimensionality condition to guarantee the nontriviality of the Lagrange multiplier (e.g., [20, 25]). There are some attempts to generalize this condition to the stochastic framework (e.g., [24]). Another way is to use some tools from the set-valued analysis (e.g., [12]). However, the existing results are still not satisfactory so far.

(8) Well-posedness of (3.7) in the sense of transposition solution. It would be quite important for some optimal control problems to prove that equation (3.7) admits a unique transposition solution. So far this is only done for a very special case (e.g., [33, THEOREM 4.1]).

(9) Higher-order necessary conditions for optimal controls. Similar to calculus, in addition to the first-order necessary conditions (PMP), sometimes higher-order necessary conditions should be established to distinguish optimal controls from the candidates satisfying the first-order necessary conditions trivially. Some results in this direction for SDPSs can be found in [12, 13, 32]. However, these results were obtained only under very strong assumptions which should be relaxed. To this end, we believe one should first show the existence of a transposition solution to equation (3.7).

(10) Existence of optimal controls. We have discussed the necessary conditions for optimal controls without proving the existence of an optimal control, which is a very difficult problem. There are two general approaches available to study it. One is to prove the verification theorem, the other is to show that a minimizing sequence of controls is compact. Both methods have not been developed well for SDPSs. Except for some trivial cases, such as

- U is a closed and convex subset of a reflective Banach space V, and the functionals g and h are convex and for some δ, $\mu > 0$,
  $$g(x,u,t) \geq \delta |u|_V - \mu, \quad h(x) \geq -\mu, \quad \forall (x,u,t) \in H \times V \times [0,T];$$
- U is a closed, convex and bounded subset of a reflective Banach space V, and the functionals g and h are convex;

there is no further result for that problem.

(11) The relationship between PMP and dynamic programming for SDPSs. PMP and dynamic programming serve as two of the most important tools in solving optimal control problems. Both of them provide some necessary conditions for optimal controls. There should exist a basic link between them. This link is established for finite dimensional stochastic control systems (e.g., [47]). A possible relationship unavoidably involves the derivatives of the value functions, which could be nonsmooth in even very simple cases (e.g., [5]).

(12) The connection between controllability and optimal control. The survey divides itself naturally into two parts—controllability and optimal control. There should be a close relationship between these two topics. Some initial findings are given in [24], in which a new link between (finite-codimensional exact) controllability and optimal control problems for SDPSs with endpoint state constraints is presented. However, lots of things are to be done, which are by no means easy tasks.

(13) Numerics of the controllability and optimal control problems for SDPSs. By generalizing J.-L. Lions' HUM (e.g., [16]), one can find the numerical solution to controllability problems of SDPSs by solving suitable adjoint equations numerically (e.g., [40, SECTION 7.4]). On the other hand, by Theorem 3.4, one can obtain an optimal control by solving suitable forward–backward stochastic evolution equation. Unfortunately, the numerical approximation of the equations mentioned above can be quite cumbersome. We refer the readers to [30] and references therein for some recent works on this. There are lots of things to be done.

(14) What can we benefit from the uncertainty? From Sections 2 and 3, we see that the uncertainty in SDPSs places many disadvantages for controlling the systems. Nevertheless, sometimes, surprisingly, it provides advantages (e.g., [34,39]). What can we benefit from the uncertainty in SDPSs is far from being understood. We believe that the study for that problem will lead to new insights into uncertainty.

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QI LÜ

School of Mathematics, Sichuan University, Chengdu 610065, Sichuan Province, China, lu@scu.edu.cn